Elastic deformation of a cylindrical hole in ice

Based on Aadnøy (1987), "A complete elastic model for fluid-induced and in-situ generated stresses with the presence of a borehole." *Energy Sources* vol. 9 pp. 239–259.

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Aadnøy is interested in the stresses surrounding a borehole through porous, fluid-filled rock, for reasons of exploration safety. He derives an analytic solution for the stress field near a cylindrical borehole through a solid (non-porous) medium, then goes on but we can stop there. From the stress solution, it is straightforward to get the resulting strains using an elastic constitutive relation (Hooke's Law of Elasticity).

The Aadnøy paper is in the Dropbox folder. (link)

Aadnøy's setup and stress solutions

Aadnøy finds the stress field around the borehole by summing the independent stress contributions from three sources: hydrostatic stress (including water pressure P_w), deviatoric stresses (σ_x and σ_y), and shear stress (τ_{xy}). The solution is in cylindrical coordinates (r, θ, z). Borehole radius is a.

$$\begin{aligned}
\sigma_r &= \frac{\sigma_x + \sigma_y}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_x - \sigma_y}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta + \tau_{xy}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\sin 2\theta - \frac{a^2}{r^2}P_w \\
\sigma_\theta &= \frac{\sigma_x + \sigma_y}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_x - \sigma_y}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta - \tau_{xy}\left(1 + \frac{3a^4}{r^4}\right)\sin 2\theta - \frac{a^2}{r^2}P_w \\
\sigma_z &= -2\nu\left(\sigma_x - \sigma_y\right)\frac{a^2}{r^2}\cos 2\theta - 4\nu\tau_{xy}\frac{a^2}{r^2}\sin 2\theta
\end{aligned} \tag{1}$$

This assumes plane strain in z, i.e., the bottom of the ice has ϵ_z=0 (no vertical deformation). The stresses in z are arranged such that this is possible (see the appearance of elastic parameter ν (Poisson's ratio) in the equation for σ_z).

These can be plugged into Hooke's Law to get the corresponding strain at any point in the domain. Hooke's Law is just a linear combination of the three stresses in Eqn. 1:

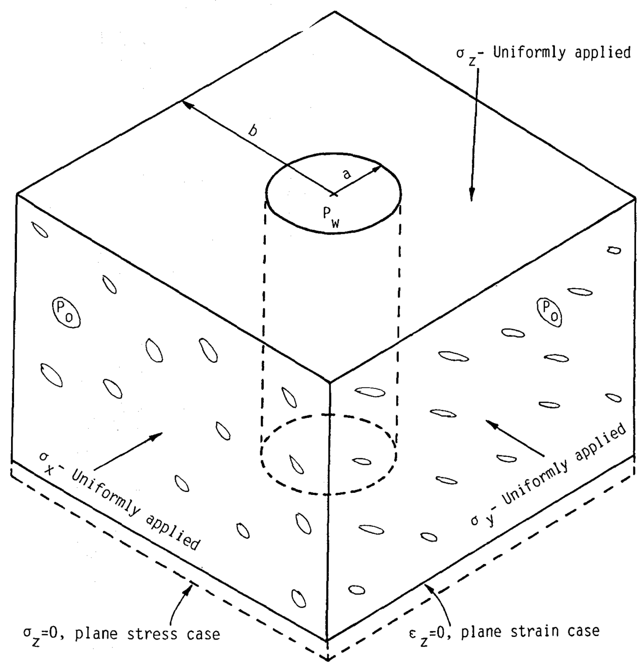

Figure 1: *Setup, from Aadnøy (1987). His model is for a borehole in a porous rock medium, through which water can percolate. That is why it has freckles. We ignore pores. We also use the plane strain case.*

$$
\begin{aligned}
\epsilon_r &= E^{-1}\left(\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right) \\
\epsilon_\theta &= E^{-1}\left(\sigma_\theta - \nu\left(\sigma_r + \sigma_z\right)\right) \\
\epsilon_z &= E^{-1}\left(\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right)
\end{aligned}
\tag{2}
$$

where E is Young's modulus ($\sim$1 GPa) and ν is Poisson's ratio ($\sim$0.3, unitless).

Integrated elastic deformation

We are interested in the total deformation of the borehole wall, or a spatially integrated version of ϵ_r. Deformation will be greatest at the borehole wall ($r = a$) and will fall off to zero as $r \to \infty$. We would like to sum all these strains over the infinite distance of the ice block in order to get the total elastic deformation experienced from the borehole's perspective. Integrating Eqn. 2 over $r|_\infty^a$ entails integrating each stress from Eqn. 1 over the same limits, then summing them together with some constants involved. So, we would like to integrate all the *r-dependent* terms in those stresses over these limits. We can (and must) ignore any constant terms in Eqn. 1 because these do not contribute to spatially varying deformation. Also, integrating a constant over infinite limits makes an infinite result.

Eqn. 1 with the constants removed are as follows:

$$
\begin{aligned}
\sigma_r^* &= \frac{\sigma_x+\sigma_y}{2}\left(-\frac{a^2}{r^2}\right) + \frac{\sigma_x-\sigma_y}{2}\left(\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta + \tau_{xy}\left(\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\sin 2\theta - \frac{a^2}{r^2}P_w \\
\sigma_\theta^* &= \frac{\sigma_x+\sigma_y}{2}\left(\frac{a^2}{r^2}\right) - \frac{\sigma_x-\sigma_y}{2}\left(\frac{3a^4}{r^4}\right)\cos 2\theta - \tau_{xy}\left(\frac{3a^4}{r^4}\right)\sin 2\theta - \frac{a^2}{r^2}P_w \\
\sigma_z^* &= -2\nu\left(\sigma_x-\sigma_y\right)\frac{a^2}{r^2}\cos 2\theta - 4\nu\tau_{xy}\frac{a^2}{r^2}\sin 2\theta
\end{aligned}
\tag{3}
$$

(It is just taking out all the $+1$s in the terms in parentheses.)

Indefinite integrals of Eqn. 3 are provided by Wolfram Alpha:

$$
\begin{aligned}
\int \sigma_r^* dr &= -\frac{a^2}{2r^3}\left[\left(a^2-4r^2\right)\left(\sigma_x-\sigma_y\right)\cos 2\theta + 2\tau_{xy}\left(a^2-4r^2\right)\sin 2\theta + r^2\left(2P_w-\sigma_x-\sigma_y\right)\right] \\
\int \sigma_\theta^* dr &= -\frac{a^2}{2r^3}\left[a^2\left(\sigma_x-\sigma_y\right)\cos 2\theta + 2a^2\tau_{xy}\sin 2\theta - r^2\left(\sigma_x+\sigma_y+2P_w\right)\right] \\
\int \sigma_z^* dr &= -\frac{2a^2\nu}{r}\left[\left(\sigma_x-\sigma_y\right)\cos 2\theta + 2\tau_{xy}\sin 2\theta\right]
\end{aligned}
\tag{4}
$$

These should all have dimensions of Pa·m.

Definite integrals of Eqn. 4 go pretty smoothly, with everything in the $r \to \infty$ expressions going to zero. Also, we don't care about tangential variations (coordinate θ), so we will replace all $\cos 2\theta$ or $\sin 2\theta$ with its average absolute value, $\frac{1}{2}$. We are left with

$$
\begin{aligned}
\int_\infty^r \sigma_r^* dr &= -\frac{3a}{4}\left(\sigma_x-\sigma_y+2\tau_{xy}\right) + \frac{a}{2}\left(\sigma_x+\sigma_y-2P_w\right) \\
\int_\infty^r \sigma_\theta^* dr &= \frac{a}{4}\left(\sigma_x-\sigma_y\right) + \frac{a}{2}\left(\tau_{xy}-\sigma_x-\sigma_y-2P_w\right) \\
\int_\infty^r \sigma_z^* dr &= a\nu\left[\left(\sigma_x-\sigma_y\right)+2\tau_{xy}\right]
\end{aligned}
\tag{5}
$$

So, finally we can take Eqn. 5 and sum them all together with the appropriate elastic constants from Hooke's Law (Eqn. 2) to get the deformation in the r, θ, and z directions. We actually only care about r, so that's all I'll do here.

$$
\begin{aligned}
\int_\infty^a \epsilon_r dr &= E^{-1}\left[\int_\infty^a \sigma_r^* dr - \nu\left(\int_\infty^a \sigma_\theta dr + \int_\infty^a \sigma_z dr\right)\right] \\
&= \left[a\frac{3-\nu}{4} - a^2\nu\right]\left(\sigma_x-\sigma_y\right) + a\left[\frac{1+\nu}{2}\right]\left(\sigma_x+\sigma_y\right) + P_w a\left[\nu-\frac{1}{2}\right] + a\left[\frac{3}{4}-\frac{\nu}{2}-2\nu^2\right]\tau_{xy}
\end{aligned}
\tag{6}
$$

With typical ice sheet values (a ∼1 meter; σ_x, σ_y and τ_{xy} ∼100 kPa; P_w equivalent to ∼500 meters of water), the total elastic deformation comes to a few millimeters. Since it is elastic, this deformation happens essentially instantly. In the model, it should happen inside a single timestep. In the next timestep, a new moulin radius a will be provided (based on the sums of the elastic deformation, creep deformation, sidewall melting, other things...?).

I think it is just fine that a will vary with depth (i.e., it is not a cylinder but a janky pipe of many diameters).