Elastic deformation around a cylindrical hole in ice

Supporting Information **S1** for "Controls on Greenland moulin geometry and evolution from the Moulin Shape model", *The Cryosphere*.

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S1.1 Introduction

Here we describe the derivation of the elastic deformation component of the MouSh model. It is based on Aadnøy [1987].

Bernt Aadnøy, a petroleum engineer, derived expressions for the stresses surrounding a borehole (wellbore) through competent rock [Aadnøy, 1987]. He applied the Kirsch [1898] solutions for a circular hole in a plate, stacking many plates to achieve a borehole. He derives an analytic solution for the stress field near a cylindrical borehole through a uniform, solid (non-porous) medium. From the stress solution, we derive the resulting strains using an elastic constitutive relation (Hooke's Law) and integrate the strains to get the total elastic deformation at the borehole wall. We take this borehole through rock as a direct analogue to a moulin through ice.

We treat the moulin as a stack of independent plates, each with a hole in them, of radius a. The radius of the hole in each plate (equivalently, at each z) is independent of the radius in the plate above and below, but generally, $a(z)$ is smoothly varying because the forces at each z are smoothly varying.

S1.2 Aadnøy's setup and stress solutions

Aadnøy [1987] finds the stress field around a borehole by summing the independent stress contributions from three sources: hydrostatic stress ($P = P_w - P_i = \rho_w g(h_w - z) - \rho_i g(H_i - z)$), deviatoric stresses ($\sigma_x$ and σ_y), and shear stress (τ_{xy}). The sign of the pressure P is "positive outward", i.e., net water pressure ($P_w > P_i$) opens the moulin and net ice pressure ($P_i > P_w$) closes the moulin.

S1.2.1 Assumptions

The Aadnøy [1987] solution is based on the Kirsch [1898] equations, which describe the stresses around a hole when the rock is subject to deviatoric stress in one direction, but elaborates from them by adding a second deviatoric stress, a shear stress, and pressure. The Kirsch [1898] and Aadnøy [1987] equations assume that the rock (ice) is a competent linear elastic material. The Kirsch [1898] solution is appropriate for a material stressed below its elastic limit, or roughly one half its compressive strength [Goodman, 1989]. The compressive strength of ice is 3–10 MPa [Fransson, 2009], making the elastic limit 1–5 MPa. This is equivalent to the cryostatic pressure in an empty borehole in ice 100–500 m thick, or the cryo/hydrostatic pressure in a borehole in ice 1–6 km thick that is water-filled to flotation. Because moulin water levels are typically $>\sim 50\%$ the flotation level, moulins meet these requirements. We note that toward the beginning or end of the melt season (when water levels are lowest), and in thick ice ($>\sim$1000 m), the ice surrounding the moulin likely approaches or may exceed the elastic limit.

Aadnøy [1987] assumes plane strain in z, i.e., ϵ_z=0 (no vertical deformation anywhere). This is consistent with the assumptions of our overall MouSh model and is the most basic formulation in solid mechanics. The total absence of vertical deformation in the face of finite horizontal deformation can be accommodated by an effective infinite domain in the cross-sectional plane of the moulin (xy). We happen to make this assumption anyway by summing elastic deformation from the point at infinity to the moulin wall (Sect. S1.3).

Alternately, Aadnøy [1987] also presents a plane stress solution. Plane strain is appropriate for thin plates with free surfaces (the top and bottom, z-facing surfaces), which differs from our "stack of plates" domain because our stacked plates have no free surfaces (excepting the topmost and bottommost plates). Aadnøy [1987]'s plane stress solution differs by a factor of $\frac{1+\nu}{1+\nu+\nu^2}$ from the plane strain solution [Goodman, 1989]; for $\nu = 0.3$, this is a change of 7%. The difference is small and plane stress is a less appropriate formulation than plane strain.

S1.2.2 Solution

The Aadnøy [1987] solution is in cylindrical coordinates (r, θ, z). The radius of the hole is a. Figure S1 shows the problem geometry.

The Kirsch [1898] equations for stresses around a hole in an infinite plate made of an elastic material are as follows:

$$\begin{aligned}
\sigma_r &= \frac{\sigma_x+\sigma_y}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{\sigma_x-\sigma_y}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta+\tau_{xy}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\sin 2\theta+\frac{a^2}{r^2}\Delta P\\
\sigma_\theta &= \frac{\sigma_x+\sigma_y}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{\sigma_x-\sigma_y}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta-\tau_{xy}\left(1+\frac{3a^4}{r^4}\right)\sin 2\theta-\frac{a^2}{r^2}\Delta P\\
\sigma_z &= \sigma_{zz}-2\nu\left(\sigma_x-\sigma_y\right)\frac{a^2}{r^2}\cos 2\theta-4\nu\tau_{xy}\frac{a^2}{r^2}\sin 2\theta
\end{aligned} \tag{S1}$$

Here, ΔP is the change in pressure around the borehole. In the rock mechanics example, ΔP is equivalent to just P, the pressure, because it assumes the borehole was very recently drilled. For the moulin case, where the water level fluctuates by the minute, ΔP is the change in pressure over the time interval in question.

Applying Hooke's Law to these equations yields the corresponding strain at any point in the domain. Hooke's Law is just a linear combination of the three stresses in Eqn. S1:

$$\begin{aligned}
\epsilon_r &= E^{-1}\left(\sigma_r-\nu\left(\sigma_\theta+\sigma_z\right)\right)\\
\epsilon_\theta &= E^{-1}\left(\sigma_\theta-\nu\left(\sigma_r+\sigma_z\right)\right)\\
\epsilon_z &= E^{-1}\left(\sigma_z-\nu\left(\sigma_r+\sigma_\theta\right)\right)
\end{aligned} \tag{S2}$$

where E is Young's modulus (~1 GPa) and ν is Poisson's ratio (~0.3 for ice; unitless).

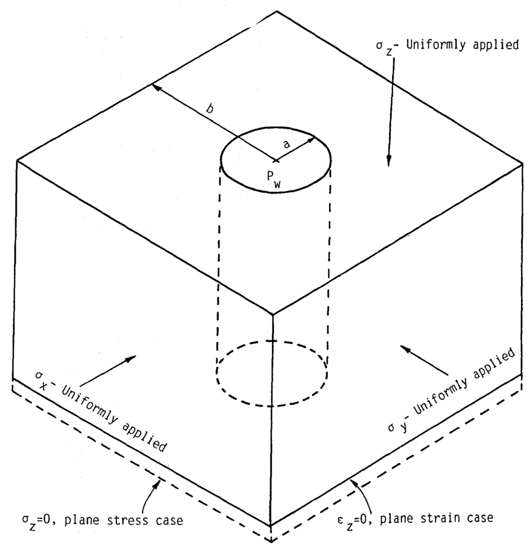

Figure S1: *Problem setup, adapted from Aadnøy [1987], of a borehole in a rock medium. We adapt this to a cylindrical moulin through ice. We use the plane strain case, although the plane stress case is equivalent within 7%.*

S1.3 Integrated elastic deformation

To calculate the radial expansion or contraction of moulin size, we must know the total elastic deformation of the moulin wall. This is the spatial integral of ϵ_r, from the borehole wall ($r = a$) to the end of the domain ($r = \infty$). Deformation will be greatest at the borehole wall ($r = a$) and will fall off to zero as $r \to \infty$.

Integrating Eqn. S2 over $r|_\infty^a$ entails integrating each stress from Eqn. S1 over the same limits, then summing them together with the appropriate constants involved. So, must simply integrate all the *r-dependent* terms in the Eqn. S1 stresses over $r|_{\infty}^{a}$. We ignore any constant (r-independent) terms in Eqn. S1 because these do not contribute to spatially varying deformation.

Eqn. S1 with the constants removed are as follows:

$$
\begin{aligned}
\sigma_r^* &= \quad \left(\Delta P - \frac{\sigma_x + \sigma_y}{2}\right)\left(\frac{a^2}{r^2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)\left(\frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) \\
\sigma_\theta^* &= -\left(\Delta P - \frac{\sigma_x + \sigma_y}{2}\right)\left(\frac{a^2}{r^2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)\left(\frac{3a^4}{r^4}\right) \\
\sigma_z^* &= \left(-2\nu\left(\sigma_x - \sigma_y\right)\cos 2\theta - 4\nu\tau_{xy}\sin 2\theta\right)\left(\frac{a^2}{r^2}\right)
\end{aligned}
\tag{S3}
$$

Indefinite integrals of Eqn. S3 are as follows:

$$
\begin{aligned}
\int \sigma_r^* dr &= \left(2\Delta P - (\sigma_x + \sigma_y)\right)\left(\frac{a^2}{2r}\right) + \left((\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta\right)\left(\frac{2a^2}{r} - \frac{3a^4}{2r^3}\right) \\
\int \sigma_\theta^* dr &= -\left(2\Delta P - (\sigma_x + \sigma_y)\right)\left(\frac{a^2}{2r}\right) + \left((\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta\right)\left(\frac{3a^4}{2r^3}\right) \\
\int \sigma_z^* dr &= 2\nu\left(\frac{a^2}{r}\right)\left((\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta\right)
\end{aligned}
\tag{S4}
$$

These all have dimensions of Pa·m.

Next, we evaluate definite integrals of Eqn. S4, over $r|_{\infty}^{a}$. Every term in the $r \to \infty$ expressions go to zero. Similarly, all tangential variations (coordinate θ) do not affect moulin size, so we replace all $\cos 2\theta$ or $\sin 2\theta$ with its average absolute value, $\frac{1}{2}$. This gives

$$
\begin{aligned}
\int_\infty^r \sigma_r^* dr &= a\left(\quad \Delta P - \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{4}(\sigma_x - \sigma_y) + \frac{1}{2}\tau_{xy}\right) \\
\int_\infty^r \sigma_\theta^* dr &= a\left(-\Delta P + \frac{1}{2}(\sigma_x + \sigma_y) + \frac{3}{4}(\sigma_x - \sigma_y) + \frac{3}{4}\tau_{xy}\right) \\
\int_\infty^r \sigma_z^* dr &= \nu a\left((\sigma_x - \sigma_y) + 2\tau_{xy}\right)
\end{aligned}
\tag{S5}
$$

Finally, we take a linear combination of Eqns. S5: a sum with the appropriate coefficients from Hooke's Law (Eqn. S2) to get the strain in the r, θ, and z directions, although we discard strain in the θ or z directions. We thus obtain u_r, the total radial deformation in r, by $u_r = \int_\infty^a \epsilon_r dr$.

$$u_r = \int_{\infty}^{a} \epsilon_r dr = E^{-1} \left[\int_{\infty}^{a} \sigma_r^* dr - \nu \left(\int_{\infty}^{a} \sigma_\theta dr + \int_{\infty}^{a} \sigma_z dr \right) \right]$$
$$= \frac{a}{E} \left[(1+\nu) \left(\Delta P - \frac{1}{2}(\sigma_x + \sigma_y) \right) + \frac{1}{4}(\sigma_x - \sigma_y)(1 - 3\nu - 4\nu^2) + \frac{1}{4}\tau_{xy}(2 - 3\nu - 8\nu^2) \right] \quad \text{(S6)}$$
$$\text{for } \Delta P = \rho_w g(\Delta h_w - z) - \rho_i g(\Delta H_i - z) = \Delta P_w - \Delta P_i$$

In the moulin model, we assume that all pressure changes ΔP are due to changes in water level Δh_w and that the ice thickness H_i stays constant. Thus, $\Delta P = \rho_w g(\Delta h_w - z) = \Delta P_w$.

As a check, the integrated displacement u_r (Eq. S6) increases with moulin radius a. This makes sense as a tighter radius of curvature (low a) is more difficult to deform radially (low u_r). Inward deformation (moulin closure) will have negative u_r and outward deformation (moulin expansion) will have positive u_r.

For a typical Greenland Ice Sheet moulin with radius $a \sim$1 meter, the pressure change associated with $\Delta h_w \sim$1 meter will induce elastic deformation of a few micrometers. This water level change would typically occur over many minutes to a few hours, yielding elastic deformation of up to some 10^{-4} meters per day.

S1.4 Simplest case: Zero deviatoric and shear stresses

The deviatoric and shear stresses σ_x, σ_y and τ_{xy} are generally not well know and/or are variable from place to place in the ablation zone. Thus, we make a further simplification that $\sigma_x = \sigma_y = \tau_{xy} = 0$, which yields the most basic expression for radial elastic deformation u_r:

$$u_r = \frac{a}{E}(1+\nu)\Delta P \quad \text{(S7)}$$

This is the same equation as is commonly used for dilatometer testing in rock mechanics [Goodman, 1989, page 190].

S1.5 Instantaneous elastic deformation and calculated deformation rates

Elastic displacement is an instantaneous process that occurs in reaction to a change in stress. In the case of a moulin during the melt season, the water level in the moulin changes essentially continuously, which induces continuous changes in pressure (ΔP), which drives continuous elastic deformation (Eqs. S6–S7), although we calculate it only once per timestep. To compare elastic deformation (instantaneous) to viscous deformation (occurring over a time interval), we assume the deformation rate occurs over the entire timestep:

$$\text{elastic deformation rate} = \frac{u_r}{\Delta t} \tag{S8}$$

This is analogous to how we calculate a viscous deformation rate or a rate of refreezing.

More precisely, one could express this in terms of the rate of pressure change, $\frac{\Delta P}{\Delta t}$:

$$\frac{u_r}{\Delta t} = \frac{a}{E}(1+\nu)a\frac{\Delta P}{\Delta t} \tag{S9}$$

This approach assumes that the water pressure varies smoothly over the time interval in question. This is generally true: we run the model at 5-minute timesteps, and the most common discontinuous variations in pressure are likely sourced from rain storms or other sudden melt events (time scales of hours).

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