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A Complete Elastic Model for Fluid-Induced and In-Situ Generated Stresses with the Presence of a Borehole

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Abstract *This paper develops a complete linear elastic model of a square plate with a circular hole in the middle. The model takes into account two horizontal in-situ stresses, the axial stress and stresses due to fluid flow in the borehole and inside the porous plate. The model contains solutions for both plane strain and plane stress conditions.*

The model has many applications in applied mechanics. In rock mechanics, it gives the total stress field, including the effects of the fluid pressure inside the rock. It may also be used to model the mechanical response of chemical absorption processes like clay swelling or thermal effects. Assuming no porosity or chemical effects inside the material, the model reduces to a set of equations for stress concentration around a drilled hole in a plate.

One particular application of the model is in wellbore stability problems as applied to the petroleum and mining industry. Here, the knowledge of the stress fields is of utmost importance for a safe and economical operation.

Introduction

In the science of rock mechanics as applied to borehole stability and fracturing of the wellbore, there is an ever increasing need to develop mathematical tools to simulate the physical problems. Today, in particular, the oil wells are being drilled at deeper depths and in deeper waters and with increased hole deviation from vertical. Under such conditions the borehole mechanics may themselves be limiting factors. Proper understanding of the borehole, on the other hand, may provide the necessary insight to ensure safety and reliability under these extreme conditions.

Lubinsky (1954) solved the problem of radial flow in a thick-walled cylinder using a poroelastic-thermoelastic analogy method. However, the thick-walled cylinder approach may not be a suitable tool for formations where the two horizontal in-situ stresses are different or for deviated wellbores. Lubinsky also solved for the plane stress case only, which may not be a good assumption for real boreholes. Bradley (1979) used a set of equations developed by Fairhurst for a square plate with a circular hole in the middle. Porosity or fluid flow effects were not included in this model. Furthermore, it was not clearly defined whether it was a plane stress or plane strain solution.

Hiramatsu et al. (1968) derived some of the resultant stress equations. Again,

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their solution did not include a complete plane stress and plane strain analysis, and fluid flow inside the rock was not taken into account. Haimson et al. (1967) also give similar equations, while Geertsma (1974) gives a general outline of the problem.

This paper develops a complete mathematical model for a square plate with a circular hole in the middle. The model takes into account stresses due to fluid flow inside the pores of the rock as well as different in-situ stresses. Such a model is versatile and useful in modelling real oil fields. The model assumes either no porosity (solid rock case) or a strong pore structure. For the last case, the fluid pressure distribution as a function of radius from the wellbore can be found from the radial flow equation. The model can also be used for low-permeability rocks, provided the pressure distribution is known. Other limitations are that the model assumes linear elasticity, and homogeneous and isotropic rock properties, and that plasticity effects are not present. Finally, the plate is assumed relatively long compared to its width in order to avoid end effects.

The model is shown in Figure 1. In the development of the model, two cases were concurrently evaluated, namely, the plain strain and the plane stress case. The plain strain case assumes no displacement along the axis of the borehole and will generally be recommended for field use.

An important application of this model is in wellbore stability studies of deviated wellbores. Aadnøy (1985) discusses in length the criteria and mechanisms of wellbore collapse and fracturing and gives a complete set of equations for rotation of the wellbore to any angle. These two reports should form a basis for wellbore stability evaluation in a real field. The most important input parameters are the three in-situ stresses and the fluid pressure inside the rock.

Before proceeding with the solution, note the following reasoning used in the derivations. The total stressfield consists of following components:

1. The stressfield in the undisturbed or virgin formation,
2. The stressfield created by drilling the hole,
3. The stressfield imposed by boundary conditions and by fluid flow inside the rock.

Equilibrium Equations

The stress components in a continuous body that is in equilibrium must satisfy the equilibrium equations, which in cylindrical coordinates are as follows (Lekhnitskii 1968):

$$\begin{gathered}\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{(\sigma_r-\sigma_\theta)}{r}+R=0\\ \frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}+\theta=0\\ \frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}+Z=0\end{gathered}\tag{1}$$

In the equations above, R, θ, and Z designate the body forces referred to a unit volume in directions r, θ, and z.

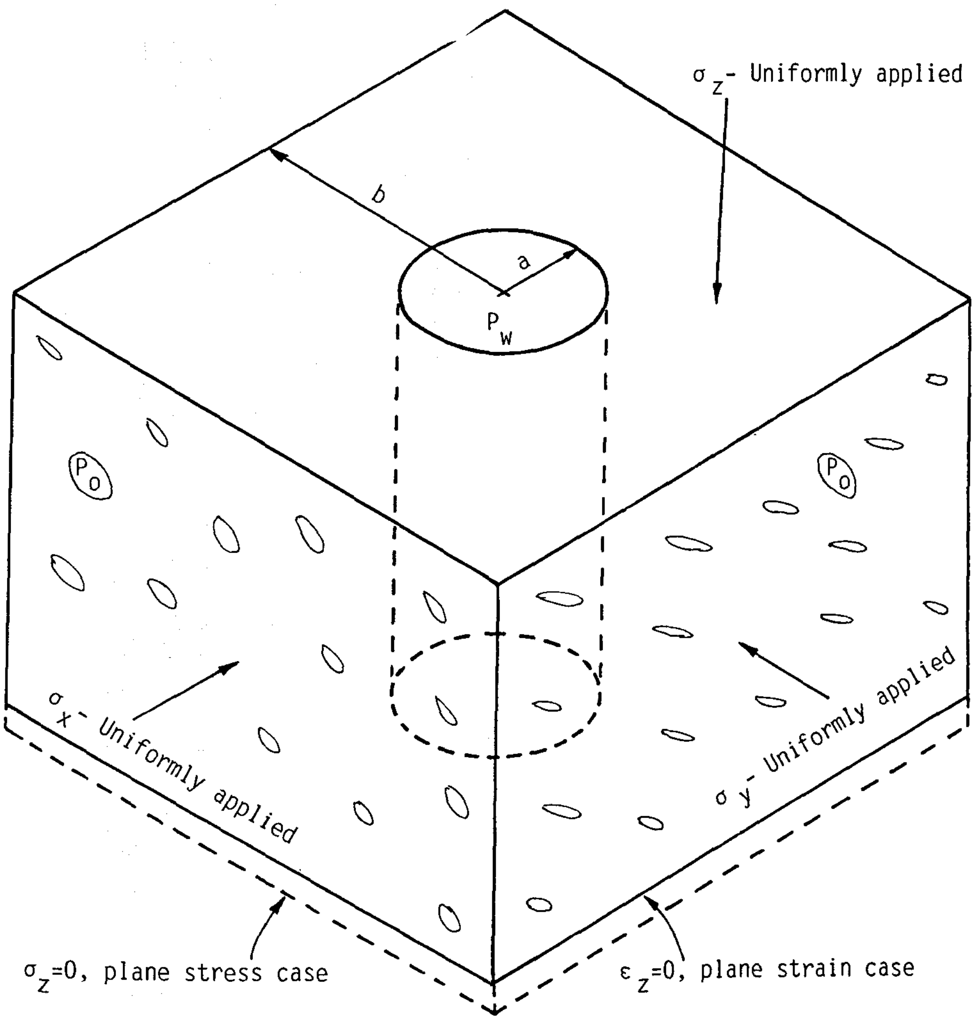


Figure 1. Definition of problem.

Equations of motion of a continuous medium differ from the equilibrium equations above only in that they have inertial terms at the right side of Equation 1 instead of zeroes. The inertial terms are the product of the material density and the accellerations in the coordinate directions. The accelerations are usually expressed in the theory of elasticity by the displacements of the body particles in the coordinate directions, namely, they are equal to the second derivatives of displacements with respect to time. We designate displacements in directions r, θ, and z by u_r, u_θ, and u_z, respectively.

The state of deformation in the neighborhood of a given point in a continuous body is characterized by six components of deformation: three relative elongations, each designated by ϵ with an appropriate subscript, and three relative shears, each designated by γ with two subscripts.

The components of deformation for cylindrical coordinates will be ϵ_r, ϵ_θ, and ϵ_z, and $\gamma_{\theta z}$, γ_{rz}, and $\gamma_{r\theta}$, respectively.

In the case of small deformations the relative strains can be defined in following manner:

$$\epsilon_r = \frac{\partial u_r}{\partial r} \qquad \gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \tag{2}$$

$$\epsilon_z = \frac{\partial u_z}{\partial z} \qquad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

The above formulas and equations are true for any continuous body, both elastic and inelastic. In the wellbore problem, axial symmetry may be assumed. This implies that the boundary loads are applied along and normal to the hole axis, and results in:

$$\tau_{rz} = \tau_{\theta z} = \gamma_{rz} = \gamma_{\theta z} = 0 \tag{3}$$

The equilibrium equations will then reduce to:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + \theta = 0 \tag{4}$$

$$\frac{\partial \tau_z}{\partial_z} + Z = 0$$

Furthermore, if the rotational symmetry is assumed, Equation 4 reduces further to:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial \sigma_z}{\partial z} + Z = 0 \tag{5}$$

Equations 1 and 5 will in the following be applied to a complete model of stresses in porous bodies; first, the equations for the external loads on the rock itself will be derived, then the solution for the stresses due to the flow of fluids inside the rock will be derived. Finally the two solutions will be superimposed. The superposition principle is valid here because Equation 1 and the constitutive relations consists of linear equations.

In-Situ Stresses

To model the in-situ stresses an ideal model of a square plate with a circular hole in the middle will be used (Timoshenko and Goodier 1951). The reason for choosing such a model is to be able to use different values for the three in-situ stresses.

If the two horizontal in-situ stresses are of equal magnitude, the solution reduces to the so-called "thick-walled cylinder" model.

Every plane of an isotropic body is a plane of elastic symmetry, and every direction is a principal direction. The generalized Hooke's law for an isotropic body is:

$$\begin{aligned}
\epsilon_r &= \{\sigma_r - \nu(\sigma_\theta + \sigma_z)\}/E; &\quad \gamma_{\theta z} &= \tau_{\theta z}/G \\
\epsilon_\theta &= \{\sigma_\theta - \nu(\sigma_r + \sigma_z)\}/E; &\quad \gamma_{rz} &= \tau_{rz}/G \\
\epsilon_z &= \{\sigma_z - \nu(\sigma_r + \sigma_\theta)\}/E; &\quad \gamma_{r\theta} &= \tau_{r\theta}/G
\end{aligned} \tag{6}$$

This model will be used by assuming that the in-situ loads are fixed and then the wellbore stresses will be observed around the borehole. If applied for deviated wellbores, it is assumed regardless of the position of the borehole that the plane strain conditions applies, that is axial strain due to the presence of the borehole and the imposed boundary conditions is zero. Using this information, the normal strains in Equation 6 can be written as follows:

$$\begin{aligned}
\sigma_r &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\epsilon_r + \frac{\nu}{1-\nu}\epsilon_\theta\right\} \\
\sigma_\theta &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\epsilon_\theta + \frac{\nu}{1-\nu}\epsilon_r\right\} \\
\sigma_z &= \frac{\nu E}{(1+\nu)(1-2\nu)}\{\epsilon_r + \epsilon_\theta\}, \; \tau_{r\theta} = G\gamma_{r\theta}
\end{aligned} \tag{7}$$

First we will derive the "thick-walled cylinder" model. Inserting Equation 7 into Equation 5 and introducing the definitions for strain from Equation 4, the following result is obtained if body forces are neglected:

$$\begin{aligned}
\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} &= 0 \\
\frac{d\sigma_z}{dz} &= 0
\end{aligned} \tag{8}$$

The second term implies that the axial stress is constant throughout the borehole section under consideration. For horizontally oriented wellbores one can justify such an assumption. For vertical wellbores the overburden has a gradient in the order of 1 psi/ft. The problem is controversial, but it is avoided by assuming a relatively short section of the wellbore is considered at a time. In other words, a wellbore is analyzed a section at a time, assuming the axial stress within each section to be constant. Equation 8 can then be written as:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = 0 \tag{9}$$

Integrating this equation two times yields the general solution:

$$u_r = C_1r + C_2/r \tag{10}$$

Using the definitions of strain from Equation 2 and inserting Equation 10 into Equation 7, the stresses in a plate with circular hole can be expressed as follows:

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left\{C_1 + \frac{C_2}{r^2}(1-2\nu)\right\}$$
$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left\{C_1 - \frac{C_2}{r^2}(1-2\nu)\right\} \tag{11}$$
$$\sigma_z = \frac{2\nu E C_1}{(1+\nu)(1-2\nu)}$$

The constants C_1 and C_2 can be evaluated by appealing to the boundary conditions. If we want to solve for the stress distribution of a cylinder with external and internal pressure, the procedure is simply to apply the boundary conditions to Equation 11. However, in the case of a plate with uneven in-situ stresses applied, shear stresses do not vanish anymore and therefore the compatibility criteria must be applied. In the following section this solution will be derived.

Compatibility Equation

Compatibility means that strains must be compatible with stresses, that is, physically permissible. Mathematically, the strain functions must be continuous and must possess sufficient continuous partial derivatives to satisfy the following (for plane problems in polar coordinates):

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left[\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right] = 0 \tag{12}$$

If the body forces are set equal zero a function that satisfies Equation 12 is Airy's stress function:

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$
$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} \tag{13}$$
$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial F}{\partial \theta}\right]$$

Expanding Equation 12 results in the so-called Euler differential equation:

$$\frac{d^4 F}{dr^4} + \frac{2}{r}\frac{d^3 F}{dr^3} - \frac{1}{r^2}\frac{d^2 F}{dr^2} + \frac{1}{r^3}\frac{dF}{dr} = 0 \tag{14}$$

Now look at Figure 2. At point A we assume that the stress field is the same as it would be without the hole. The stresses on the element under consideration is

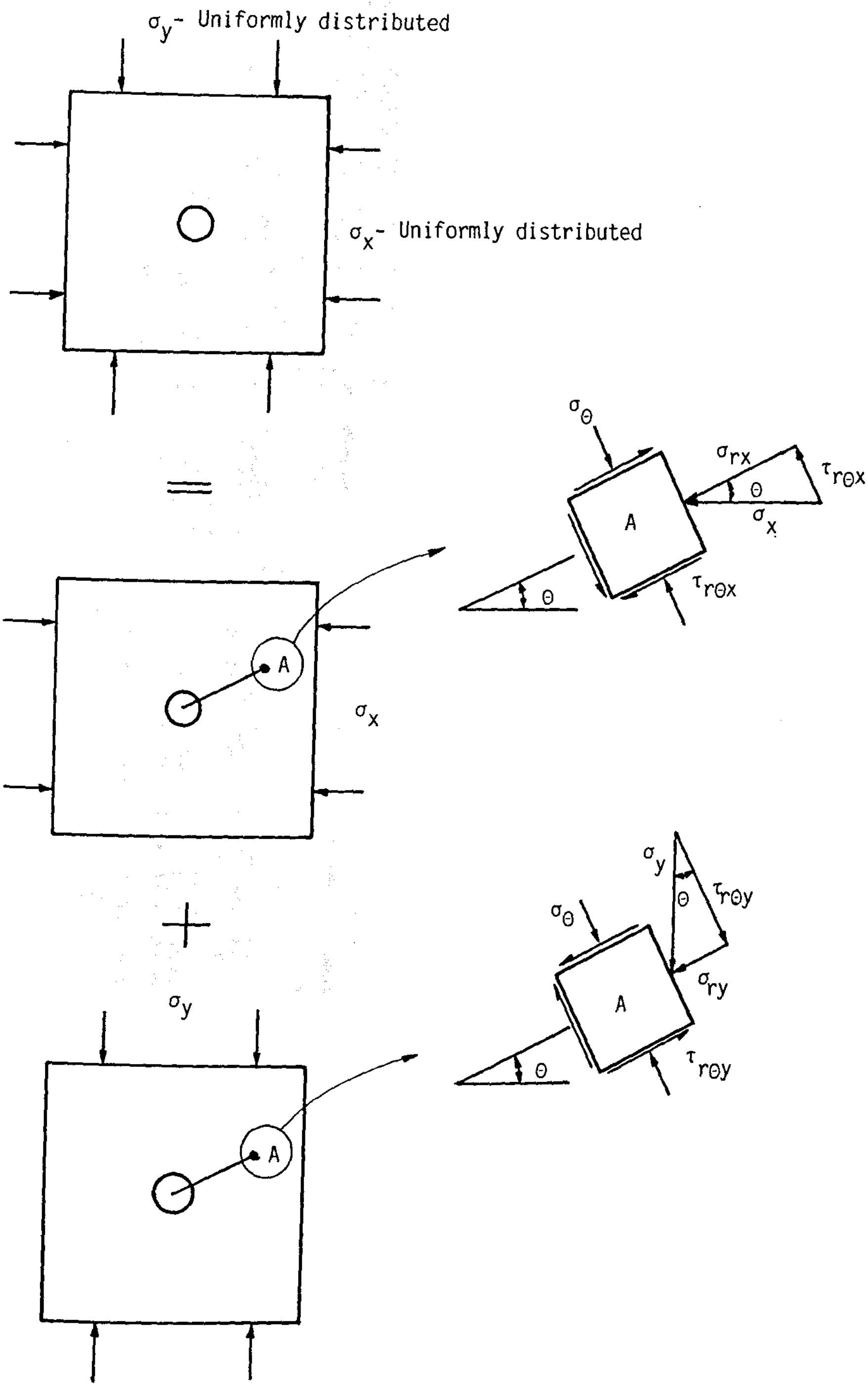


Figure 2. Superposition of the in-situ stresses.

at radius A:

$$\begin{aligned}
\sigma_{rx} &= \sigma_x \cos^2\theta = \sigma_x(1 + \cos 2\theta)/2 \\
\tau_{r\theta x} &= \sigma_x \sin\theta \cos\theta = +\sigma_x(\sin 2\theta)/2 \\
\sigma_{ry} &= \sigma_y \sin^2\theta = \sigma_y(1 - \cos 2\theta)/2 \\
\tau_{r\theta y} &= \sigma_y \sin\theta \cos\theta = +\sigma_y(\sin 2\theta)/2
\end{aligned} \tag{15}$$

Since the in-situ stresses are all dependent on the rotational angle 2θ, a stress function is defined as:

$$F = F(r, \theta) = F(r) \cos 2\theta \tag{16}$$

The general solution of Equation 12 with Equation 16 is:

$$F(r) = C_1r^2 + C_2r^4 + C_3/r^2 + C_4 \tag{17}$$

Inserted into Equation 13 the stresses become:

$$\begin{aligned}
\sigma_r &= -\left\{2C_1 + \frac{6C_3}{r^4} + \frac{4C_4}{r^2}\right\} \cos 2\theta \\
\sigma_\theta &= \left\{2C_1 + 12C_2r^2 + \frac{6C_3}{r^4}\right\} \cos 2\theta \\
\tau_{r\theta} &= \left\{2C_1 + 6C_2r^2 - \frac{6C_3}{r^4} - \frac{2C_4}{r^2}\right\} \sin 2\theta
\end{aligned} \tag{18}$$

Remembering that the principle of superposition applies, a specific procedure will be used in solving for the stresses. The total stresses acting on the plate are as follows:

Inside the hole: P_w

On outer boundary (Equation 15):

$$\sigma_r = \sigma_{rx} + \sigma_{ry} = \{(\sigma_x + \sigma_y) + (\sigma_x - \sigma_y) \cos 2\theta\}/2 \tag{19}$$

Axial load (overburden for vertical hole): σ_{zz}

The procedure is as follows: Noting that the outer radial stress on the plate is the sum of a constant term and a term that is a function of the angle 2θ; the two parts can be solved separately and the solutions superimposed.

Hydrostatic Stress

First, the thickwalled cylinder solution (Equation 11) is applied for following boundary conditions:

$$\begin{aligned}
\sigma_r &= P_w & \text{at: } r &= a \\
\sigma_r &= (\sigma_x + \sigma_y)/2 & \text{at: } r &= b \\
\sigma_z &= \sigma_z = \sigma_{zz} = \sigma_{zz} & \text{at: } r &= b
\end{aligned}$$

Inserting these conditions into Equation 11 and solving for the constants results in following stresses:

$$\sigma_{r1} = \frac{1}{2}(\sigma_x + \sigma_y)\left(1 - \frac{a^2}{r^2}\right) + \frac{a^2}{r^2} p_w$$

$$\sigma_{\theta 1} = \frac{1}{2}(\sigma_x + \sigma_y)\left(1 + \frac{a^2}{r^2}\right) - \frac{a^2}{r^2} p_w$$

$$\tau_{r\theta} = 0$$

For plane strain condition: $\epsilon_z = 0$ (20)

$$\sigma_{z1} = \sigma'_{zz} + \nu(\sigma_{ri} + \sigma_{\theta 1} = \sigma'_{zz} + \nu)(\sigma_x + \sigma_y) = \sigma_{zz}$$

For plane stress condition:

$$\sigma_{z1} = \sigma_{zz}$$

The assumption above is that $a/b = 0$, i.e., that the plate has infinite extent. The axial stress due to overburden could be superimposed directly, but by using the definition stated in the boundary condition it is seen that for the plane strain case, hydrostatic loading conditions create no additional axial stress. The sum of the radial and tangential components is constant for any radius.

Deviatoric Stress

Next, Equation 18 is evaluated with following boundary conditions:

$$\sigma_r = 0 \qquad \text{at: } r = a$$

$$\sigma_r = \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta \qquad \text{at: } r = b$$

$$\tau_{r\theta} = 0 \qquad \text{at: } r = a$$

The results of these derivations are:

$$\sigma_{r2} = \frac{1}{2}(\sigma_x - \sigma_y)\left\{1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right\}\cos 2\theta$$

$$\sigma_{\theta 2} = -\frac{1}{2}(\sigma_x - \sigma_y)\left\{1 + 3\frac{a^4}{r^4}\right\}\cos 2\theta$$

$$\tau_{r\theta 2} = -\frac{1}{2}(\sigma_x - \sigma_y)\left\{1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right\}\sin 2\theta \qquad (21)$$

For plane strain conditions:

$$\sigma_{z2} = -2\nu\frac{a^2}{r^2}(\sigma_x - \sigma_y)\cos 20$$

For plane stress: $\sigma_{z2} = 0$

Shear Stress

Next the shearforces must be solved for. Referring to Figure 2, one can see that the resultant shear force is:

$$\tau_{r\theta} = \tau_{r\theta x} - \tau_{r\theta y} = +(\sigma_x - \sigma_y)(\sin 2\theta)/2 = +\tau_{xy} \sin 2\theta \tag{22}$$

Now, using a stress function of the form: $F = F(r) \sin 2\theta$, the general solution of Equation 12 is similar to that of Equation 17. However, the stress equation is identical to Equation 16 except for the trigonometric parts.

Applying following boundary conditions:

$$\begin{aligned} \tau_{r\theta} &= 0 && \text{at: } r = a \\ \tau_{r\theta} &= -\tau_{xy} \sin 2\theta && \text{at: } r = b \\ \sigma_r &= 0 && \text{at: } r = a \end{aligned}$$

The shear stresses results in:

$$\begin{aligned} \sigma_{r3} &= \tau_{xy} \left\{1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right\} \sin 2\theta \\ \sigma_{\theta 3} &= -\tau_{xy} \left\{1 + 3\frac{a^4}{r^4}\right\} \sin 2\theta \\ \tau_{r\theta 3} &= \tau_{xy} \left\{1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right\} \cos 2\theta \end{aligned} \tag{23a}$$

For plane strain conditions:

$$\sigma_{z3} = \nu(\sigma_{r3} + \sigma_{\theta 3}) = -4\nu\tau_{xy}\frac{a^2}{r^2} \sin 2\theta$$

For plane stress: $\sigma_{z3} = 0$

Two shear stresses are not accounted for in the previous derivation because they were assumed to vanish in Equation 3. These are τ_{xz} and τ_{yz}. Transforming these stresses into cylindrical coordinates, the following result is obtained:

$$\begin{aligned} \tau_{rz} &= \tau_{xz} \cos\theta + \tau_{yz} \sin\theta \\ \tau_{\theta z} &= -\tau_{xz} \sin\theta + \tau_{yz} \cos\theta \end{aligned}$$

τ_{rz} must have the value given above near the outer ends of the plate but vanish at the wellbore. Assuming a radial distribution similar to Equation 20, the following result is obtained as given by Bradley (1979):

$$\begin{aligned} \tau_{rz} &= (\tau_{xz} \cos\theta + \tau_{yz} \sin\theta)\left\{1 - \frac{a^2}{r^2}\right\} \\ \tau_{\theta z} &= (-\tau_{xz} \sin\theta + \tau_{yz} \cos\theta)\left\{1 + \frac{a^2}{r^2}\right\} \end{aligned} \tag{23b}$$

The last set of equations are not derived, but it can be shown that they satisfy the third of Equation 1 and the compatibility conditions for strain, which are not given here.

Total Stresses

The total stresses acting on the plate is the sum of Equations 20, 21, and 23:

$$\sigma_r = \frac{1}{2}(\sigma_x + \sigma_y)\left(1 - \frac{a^2}{r^2}\right) + \frac{1}{2}(\sigma_x - \sigma_y)\left\{1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right\}\cos 2\theta + \tau_{xy}\left\{1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right\}\sin 2\theta + \frac{a^2}{r^2}P_w$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y)\left(1 + \frac{a^2}{r^2}\right) - \frac{1}{2}(\sigma_x - \sigma_y)\left\{1 + 3\frac{a^4}{r^4}\right\}\cos 2\theta - \tau_{xy}\left\{1 + 3\frac{a^4}{r^4}\right\}\sin 2\theta - \frac{a^2}{r^2}P_w$$

Plane strain:

$$\sigma_z = \sigma_{zz} - 2\nu(\sigma_x - \sigma_y)\frac{a^2}{r^2}\cos 2\theta - 4\nu\tau_{xy}\frac{a^2}{r^2}\sin 2\theta \tag{24}$$

Plane stress:

$$\sigma_z = \sigma_{zz}$$

$$\tau_{r\theta} = \left[\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta\right]\left\{1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right\}$$

$$\tau_{rz} = (\tau_{xz}\cos\theta + \tau_{yz}\sin\theta)\left\{1 - \frac{a^2}{r^2}\right\}$$

$$\tau_{\theta z} = (-\tau_{xz}\sin\theta + \tau_{yz}\cos\theta)\left\{1 + \frac{a^2}{r^2}\right\}$$

At the borehole ($r = a$) Equation 24 reduces to:

$$\sigma_r = P_w$$

$$\sigma_\theta = (\sigma_x + \sigma_y - P_w) - 2(\sigma_x - \sigma_y)\cos 2\theta - 4\tau_{xy}\sin 2\theta$$

Plane strain:

$$\sigma_z = \sigma_{zz} - 2\nu(\sigma_x - \sigma_y)\cos 2\theta - 4\nu\tau_{xy}\sin 2\theta \tag{25}$$

Plane stress: $\sigma_z = \sigma_{zz}$

$$\tau_{r\theta} = 0$$

$$\tau_{rz} = 0$$

$$\tau_{\theta z} = 2(-\tau_{xz}\sin\theta + \tau_{yz}\cos\theta)$$

Porous Model

Stresses Due to Fluid Flow

Lubinsky (1954) showed that a thermoelastic-poroelastic analogy can be used to calculate stresses due to body forces inside a material. Fluid flow in a porous medium is one type of body force. Another widely used variant of the thermal analogy method is derived by Biot (see Geerstma 1966 for details). In his example, Lubinsky superimposed the solutions of stresses due to the load on the rock matrix (in our case the weight of the overburden and the horizontal in-situ stresses), the hydrostatic fluid pressure at any location in the rock, and the body force due to the flow of the fluids. When comparing thermal stress problems and corresponding problems for fluid flow in porous media, Lubinsky applied the thermal solutions by replacing temperature with pressure. The thermal expansion coefficient was replaced by: $K = (1 - \beta)(1 - 2\nu)/E$. When proceeding with the derivation of this solution, radially symmetric pressure distribution is assumed, i.e., the thick-walled cylinder approach is applied. All shear stresses and shear strains vanish and the stress equilibrium equation can be used without invoking compatibility. For a porous medium with fluid pressure inside, Equation 6 can be written:

$$
\begin{aligned}
\epsilon_r - KP &= \{\sigma_r - \nu(\sigma_\theta + \sigma_z)\}/E \\
\epsilon_\theta - KP &= \{\sigma_\theta - \nu(\sigma_r + \sigma_z)\}/E \\
\epsilon_z - KP &= \{\sigma_z - \nu(\sigma_r + \sigma_\theta)\}/E
\end{aligned}
\tag{26}
$$

Equation 26 can now be solved for the stresses:

$$
\begin{aligned}
\sigma_r &= \left\{\epsilon_r \frac{(1-\nu)}{\nu} + \epsilon_\theta + \epsilon_z\right\} \frac{\nu E}{(1-2\nu)(1+\nu)} - (1-\beta)P \\
\sigma_\theta &= \left\{\epsilon_\theta \frac{(1-\nu)}{\nu} + \epsilon_r + \epsilon_z\right\} \frac{\nu E}{(1-2\nu)(1+\nu)} - (1-\beta)P \\
\sigma_z &= \left\{\epsilon_z \frac{(1-\nu)}{\nu} + \epsilon_r + \epsilon_\theta\right\} \frac{\nu E}{(1-2\nu)(1+\nu)} - (1-\beta)P
\end{aligned}
\tag{27}
$$

Letting the solution take the form of plane strain, that is, $\epsilon_z = 0$, and utilizing the strain definitions from Equation 2, inserting Equation 26 into Equation 5 results in the equivalent of Equation 9 when body force is included:

$$
\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = K\frac{1+\nu}{1-\nu}\frac{dP}{dr}
\tag{28}
$$

Integration of this equation yields:

$$
u_r = \frac{(1+\nu)K}{(1-\nu)\,r}\int_a^r Pr\,dr + C_1 r + \frac{C_2}{r}
$$

$$
\sigma_r = \frac{-KE}{1-\nu}\frac{1}{r^2}\int_a^r Pr\,dr + \frac{E}{(1+\nu)(1-2\nu)}\left\{C_1 - \frac{C_2}{r^2}(1-2\nu)\right\}
$$

$$\sigma_\theta = \frac{KE}{1-\nu}\frac{1}{r^2}\int_a^r \mathrm{Pr\,dr} - \frac{KEP}{1-\nu} + \frac{E}{(1+\nu)(1-2\nu)}\left\{C_1 + \frac{C_2}{r^2}(1-2\nu)\right\} \quad (29)$$

$$\sigma_z = \frac{-KEP}{1-\nu} + \frac{2\nu EC_1}{(1-2\nu)(1+\nu)}$$

Normal force distributed according to σ_z must be applied to the ends of the cylinder in order to keep $\epsilon_z = 0$ throughout.

The inner radius of the cylinder being a and the outer radius of the cylinder being b, the constants C_1 and C_2 in Equation 29 are determined so that σ_r will be zero at these two radii. Also, we will assume that the outer radius b goes towards infinity. The evaluation of the boundary conditions then result in:

$$C_1 = C_2 = 0$$

To make the analysis consistent with the sign convention chosen for the rest of the analysis, the sign is now reversed so that compression is defined as positive. Substituting these values into Equation 29, the following result is obtained:

$$\sigma_r = (1-\beta)\frac{1-2\nu}{1-\nu}\left\{\frac{1}{r^2}\int_a^r \mathrm{Pr\,dr}\right\}$$

$$\sigma_\theta = -(1-\beta)\frac{1-2\nu}{1-\nu}\left\{\frac{1}{r^2}\int_a^r \mathrm{Pr\,dr} - P\right\} \quad (30)$$

$$\sigma_z = (1-\beta)\frac{1-2\nu}{1-\nu}P$$

Before proceeding further, let us define following stress definitions as given in Serafim (1968). Effective stress is equal to the total stress with the pore pressure multiplied by a coefficient η subtracted, or:

$$\sigma(\mathrm{eff}) = \sigma(\mathrm{tot}) - \eta P \quad (31)$$

The pore pressure coefficient η needs some clarification. Lubinsky (1954) defined it equal to the porosity, f. This is not accepted today. Nur and Byerbee (1974) derived it analytically, and found: $\eta = (1-\beta)$. (For definition of β, see the nomenclature.) Using this definition, the pore pressure coefficient is in the order of $\eta = 0.8$ for many rocks. However, in geotechnical engineering and rock mechanics it is common practice to define: $\eta = 1$. It is well-known that in the failure criteria for brittle rocks the coefficient η is equal to one, or very nearly so. Also, what is termed total stresses here is also defined as global stresses by some authors. Next, we want to find the pore pressure variation inside the rock and let $\eta = 1$, and introduce a finite radius of influence for the flow b which can take any value. Equation 30 with the radial flow equation inserted becomes:

$$\sigma_r = (1-\beta)\frac{1-2\nu}{1-\nu}\frac{P_w - P_o}{2}\left[\left(1-\frac{a^2}{r^2}\right)\left\{1+\frac{1}{2\log b/a}\right\} - \frac{\log r/a}{\log b/a}\right]$$

$$\sigma_\theta = (1-\beta)\frac{1-2\nu}{1-\nu}\frac{P_w - P_o}{2}\left[\left(1-\frac{a^2}{r^2}\right)\left\{1+\frac{1}{2\log b/a}\right\} - 2 + \frac{\log r/a}{\log b/a}\right] \quad (32)$$

$$\sigma_z = (1-\beta)\frac{1-2\nu}{1-\nu}(P_w - P_o)\left\{1 - \frac{\log r/a}{\log b/a}\right\}$$

Considering the stresses at the wellbore, that is, letting $r = a$ these equations simplify to:

$$\sigma_r = 0$$

$$\sigma_\theta = -(1 - \beta)\frac{1 - 2\nu}{1 - \nu}(P_w - P_o) \tag{33}$$

$$\sigma_z = -(1 - \beta)\frac{1 - 2\nu}{1 - \nu}(P_w - P_o)$$

Before proceeding further, let us look at the axial stress equation. Lubinsky (1954) based his solution on Timoshenko's (1951) derivation for the thermal stress problem. Timoshenko used a plane strain solution to solve a plane stress problem. This could be done because of symmetry and because of uniform axial stress. The reasoning used was to superimpose a uniform axial stress $\sigma_z = C_3$ so that the resultant force on the end of the cylinder was zero. The self-equilibrating distribution remaining on each end will, by Saint-Venants principle, give rise only to local effects at the ends. The constant C_3 is determined as follows:

$$\int_a^b C_3 2\pi r\,dr = -\frac{2\pi KE}{1-\nu}\int_a^b Pr\,dr$$

Performing the integration and inserting the result in σ_z in Equation 29 gives the final result:

$$C_3 = 0$$

We find that the plane stress solution for axial stress is identical to the plane strain solution. All stresses and displacements will be the same for the two cases.

Studying the case of flow between wellbore and the formation obeying the radial flow equation:

$$P = P_w - (P_w - P_o)\frac{\log r/a}{\log b/a}P - O$$

Evaluation of the Porous Model

In this section an analysis will be performed using Equation 32. Only the stresses due to fluid flow will be examined. Later, the complete model including the in-situ stresses will be put together. The objective of this section is to evaluate the effect of the wellbore pressure and the effect of transient flow behavior in the formation. Note that compressive stress is defined as positive here. Three pressure profiles are used. One uses a ratio: b:a = 100, and this can be called a steady-state solution. If this ratio is increased toward infinity, practically the same stress distributions are obtained. The transient pressure profile uses b:a = 2, and an intermediate profile using b:a = 10 is also studied. This is an approximate transient solution. An exact solution would include the time-dependent terms in Equation 2. The idea behind the ratio's above is that once a flow is initiated and sustained, the pressure will propagate from the wellbore towards an infinite radius, approaching steady-state conditions.

For the pressure profiles in Figures 3, 4, and 5 to be valid, the constant terminal rate condition at the wellbore requires that the flow rate for the transient case is larger than for the steady-state case. (All other parameters are assumed to be equal.) This can be seen from the slope of the pressure profile at the wellbore. Two conditions of wellbore pressure are evaluated, namely, a pressure buildup case and a pressure drawdown case.

The pressure buildup case assumes a wellbore pressure of twice the in-situ reservoir pressure and should be representative for a wellbore pressurized towards fracturing. Figure 3 shows the pressure profile and the stress distributions for the steady-state case. The radial stress is zero at the wellbore, but a maximum positive value just a small distance from the wellbore wall is observed. Since compressive stresses in general improve wellbore stability, the radial stress is considered to have a positive effect.

The tangential stress component, on the other hand, is tensile and strongly reduces wellbore stability by increasing the total tensile stress for steady-state flow conditions, and is therefore detrimental for wellbore stability.

The flow of fluids into the formation causes an axial tensile stress. Unless the formation is shallow (horizontal fracture), however, this may not be of much concern.

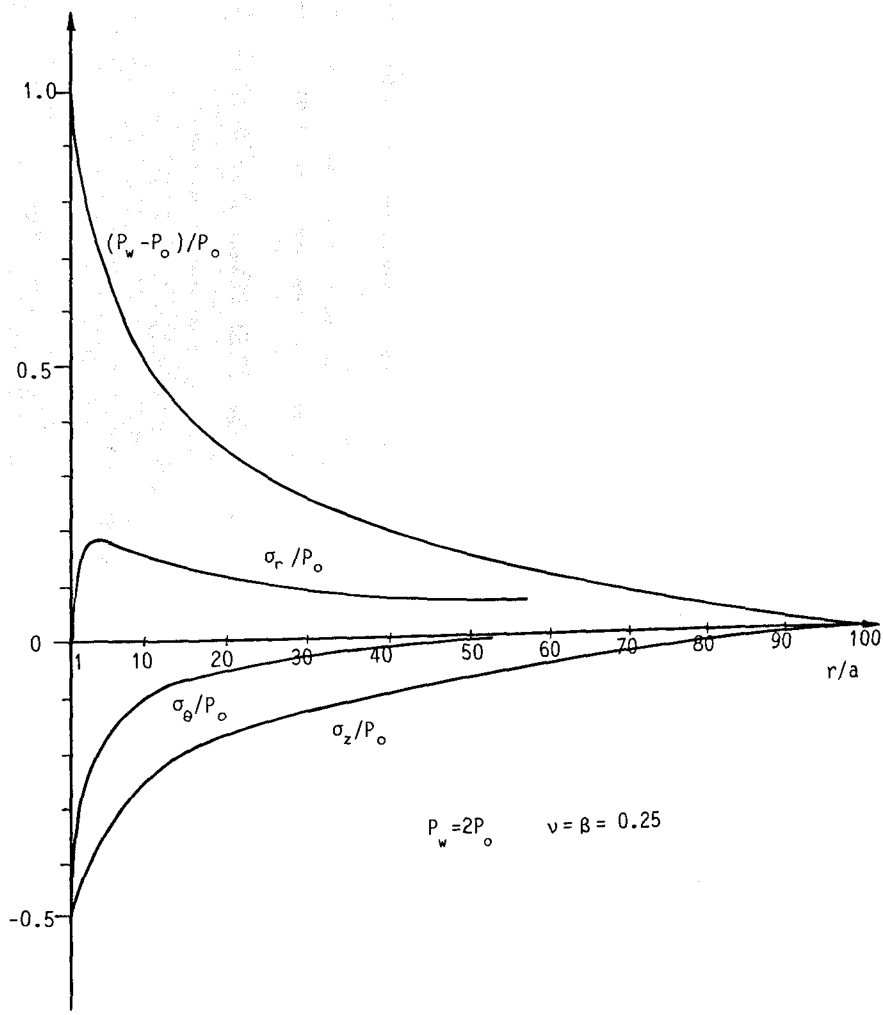


Figure 3. Pressure buildup, steady-state case.

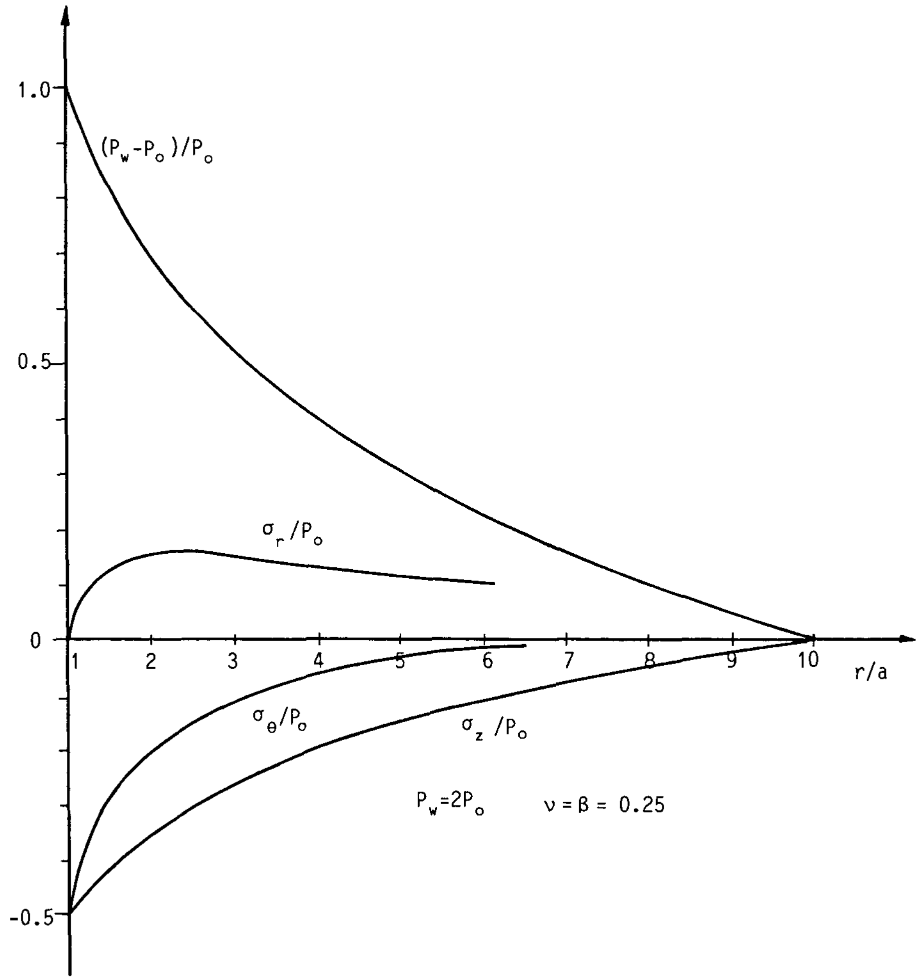


Figure 4. Pressure buildup.

Comparing Figures 3, 4, and 5, we see that the axial and tangential stresses have equal magnitude at the wellbore regardless of flow conditions. The radial stresses decrease in magnitude with increased transient behavior.

For pressure drawdown cases, the curves have the same shape; only the sign is reversed. Therefore, the radial stresses will become critical, and may cause spalling of the formation (Aadnøy 1985).

Summary

For pressure buildup cases, wellbore stability is reduced due to lowering of the tangential stress is significantly lower when fluid is flowing into the formation. Radial stresses do not affect stability very much, and the axial stress may decrease the stability for shallow formations.

For pressure drawdown cases, fluid flow slightly improves stress fields of axial and tangential stresses. For the radial stress component may reduce the stability of the borehole, with spalling of the holewalls as a potential end result.

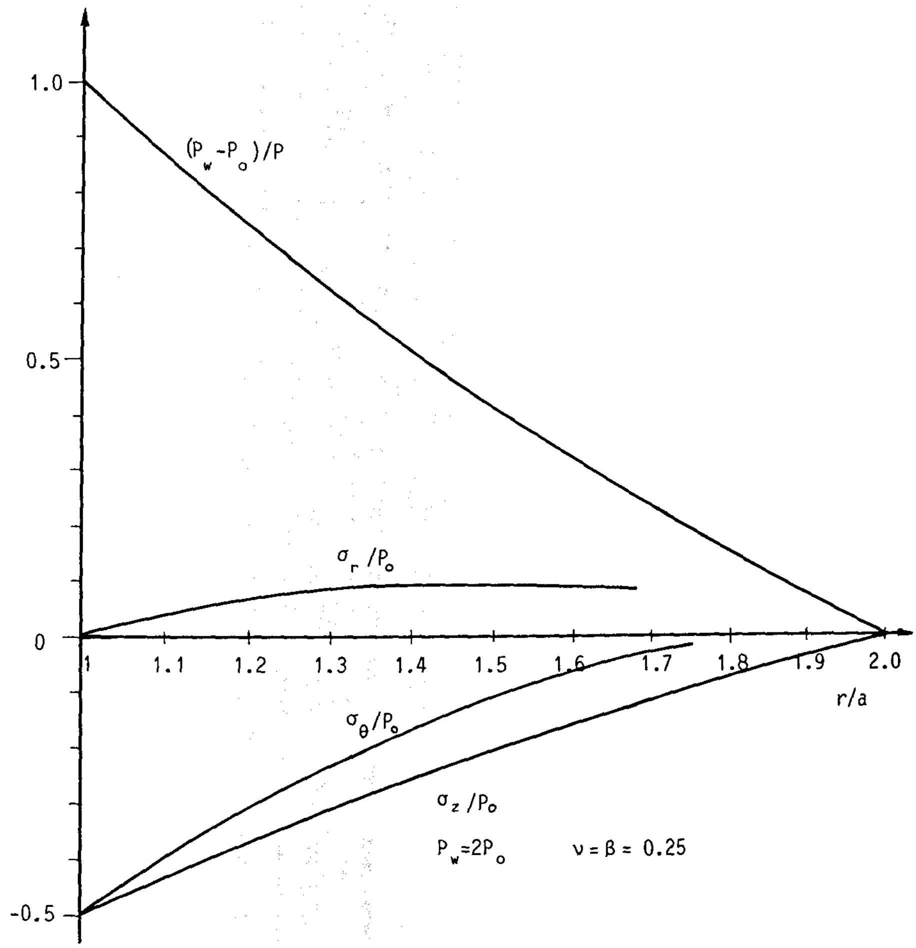


Figure 5. Pressure buildup, transient case.

The effects are proportional to the wellbore formation pressure difference. Only the radial stress component increases from transient to steady state conditions.

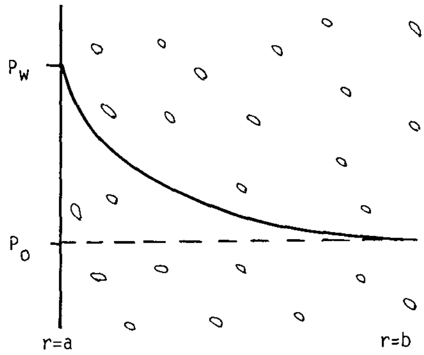
The Complete Model

Matching Boundary Conditions

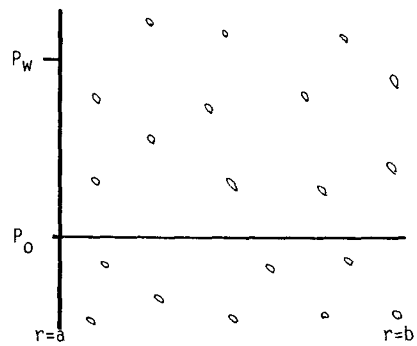
Before the solutions for the solid rock model and the fluid flow model can be superimposed, the boundary conditions must be discussed. Two cases are constructed that are believed to be representative for many real-life applications.

The first case assumes communication between the wellbore and the formation. Both pressure drawdown and buildup fall under this category. The second case assumes a perfect mudcake that acts as a step function, on one side the pressure is P_w, on the other side P_o. These cases are illustrated in Figure 6.

First, the case with open communication between wellbore and formation. For this case, there is pressure continuity across the wellbore wall and Equations 24 and 32 can be added directly. The stresses that are contributed by the fluid are



a) Communication wellbore/formation.



b) Effective mudcake.

Figure 6. Main wellbore/formation conditions.

then for Equation 25 at the wellbore: ($\eta = 1$)

$$\sigma_r(\text{eff}) = 0, \qquad \sigma_\theta(\text{eff}) = -2P_w$$

The second case is a little different. Here it is seen that:

$$\sigma_r(\text{eff}) = P_w - P_o, \qquad \sigma_\theta(\text{eff}) = -(P_w + P_o)$$

It is seen that the mudcake case gives different stresses at the wellbore than the other case. The above definitions are in agreement with Haimson and Fairhurst (1967) in their analysis of hydraulic fracturing.

The Complete Model

Now the total model can be put together. Defining a factor: $M = \eta P_w - n(P_w - P_o)\dfrac{\log r/a}{\log b/a}$ and $N = 1$, if communication between formation and borehole exist; and $M = \eta P_o$, $N = O$ if a pressure differential builds up across the mudcake.

The total set of equations for the effective stresses can be written:

$$
\begin{aligned}
\sigma_r(\text{eff}) &= \frac{1}{2}(\sigma_x + \sigma_y)\left\{1 - \frac{a^2}{r^2}\right\} + \frac{1}{2}(\sigma_x - \sigma_y)\left\{1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right\}\cos 2\theta \\
&\quad + \tau_{xy}\left\{1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right\}\sin 2\theta + \frac{a^2}{r^2}P_w \\
&\quad + N(1-\beta)\frac{1-2\nu}{1-\nu}\frac{P_w - P_o}{2}\left[\left(1 - \frac{a^2}{r^2}\right)\left\{1 + \frac{1}{2\log(b/a)}\right\} - \frac{\log r/a}{\log b/a}\right] - M \\
\sigma_\theta(\text{eff}) &= \frac{1}{2}(\sigma_x + \sigma_y)\left\{1 + \frac{a^2}{r^2}\right\} - \frac{1}{2}(\sigma_x - \sigma_y)\left\{1 + 3\frac{a^4}{r^4}\right\}\cos 2\theta \\
&\quad - \tau_{xy}\left\{1 + 3\frac{a^4}{r^4}\right\}\sin 2\theta - \frac{a^2}{r^2}P_w \qquad (34) \\
&\quad + N(1-\beta)\frac{1-2\nu}{1-\nu}\frac{P_w - P_o}{2}\left[\left(1 - \frac{a^2}{r^2}\right)\left\{1 + \frac{1}{2\log(b/a)}\right\} - 2 + \frac{\log r/a}{\log b/a}\right] - M \\
\tau_{r\theta} &= \left\{\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta\right\}\left\{1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right\} \\
\tau_{rz} &= (\tau_{xz}\cos\theta + \tau_{yz}\sin\theta)\left\{1 - \frac{a^2}{r^2}\right\} \\
\tau_{\theta z} &= (-\tau_{xz}\sin\theta + \tau_{yz}\cos\theta)\left\{1 + \frac{a^2}{r^2}\right\}
\end{aligned}
$$

Plane strain conditions:

$$
\begin{aligned}
\sigma_z(\text{eff}) &= \sigma_{zz} - 2\nu(\sigma_x - \sigma_y)\frac{a^2}{r^2}\cos 2\theta - 4\nu\tau_{xy}\frac{a^2}{r^2}\sin 2\theta \\
&\quad - N(1-\beta)\frac{1-2\nu}{1-\nu}(P_w - P_o)\left\{1 - \frac{\log(r/a)}{\log(b/a)}\right\} - M
\end{aligned}
$$

Plane stress conditions:

$$
\sigma_z(\text{eff}) = \sigma_{zz} - N(1-\beta)\frac{1-2\nu}{1-\nu}(P_w - P_o)\left\{1 - \frac{\log(r/a)}{\log(b/a)}\right\} - M
$$

At the wellbore ($r = a$) these equations simplify to:

$$
\begin{aligned}
\sigma_r(\text{eff}) &= P_w - M \\
\sigma_\theta &= (\sigma_x + \sigma_y) - 2(\sigma_x - \sigma_y)\cos 2\theta - 4\tau_{xy}\sin 2\theta - P_w \\
&\quad - N(1-\beta)\frac{1-2\nu}{1-\nu}(P_w - P_o) - M \\
\tau_{r\theta} &= 0 \\
\tau_{rz} &= 0 \\
\tau_{\theta z} &= 2(-\tau_{xz}\sin\theta + \tau_{yz}\cos\theta)
\end{aligned}
$$

Plane strain:

$$\sigma_z = \sigma_{zz} - 2\nu(\sigma_x - \sigma_y)\cos 2\theta - 4\nu\tau_{xy}\sin 2\theta$$
$$- N(1-\beta)\frac{1-2\nu}{1-\nu}(P_w - P_o) - M \tag{35}$$

Plane stress:

$$\sigma_z = \sigma_{zz} - N(1-\beta)\frac{1-2\nu}{1-\nu}(P_w - P_o) - M$$

Nomenclature

σ_r, σ_θ, σ_z = Radial, tangential, and axial stresses.
$\tau_{r\theta}$, $\tau_{\theta z}$, τ_{rz} = Shear stresses.
ϵ_r, ϵ_θ, ϵ_z = Strains.
$\gamma_{\theta z}$, γ_{rz}, $\gamma_{r\theta}$ = Shear strains.
R, θ, Z = Body forces.
σ_x, σ_y = The horizontal in-situ stresses.
u_r, u_θ, u_z = Displacements in the respective directions.
x, y, z = Cartesian coordinates.
r = Radius from the center of the wellbore.
θ = Angle around wellbore with respect to x-axis.
z = Axial borehole coordinate.
a = Borehole radius.
b = Outer radius to end of plate.
σ_{zz} = Weight of the overburden.
E = Young's elastic modulus.
ν = Poisson's ratio.
G = Shear modulus.
F(r) = Stress function that satisfies compatibility equation.
P_w = Wellbore pressure.
P_o = Pore pressure in undisturbed formation.
P = Fluid pressure.
τ_{xy} = $(\sigma_x - \sigma_y)/2$ = Plane shear stress in undisturbed rock.
K = $(1-\beta)(1-2\nu)/E$ = Pore pressure expansion coefficient.
β = Compressibility ratio interpore:porous matter.
= $\dfrac{E/(1-2\nu)}{E_i/(1-2\nu_i)}$ (index i refers to interpore material).
f = Porosity of the rock.
η = Effective stress coefficient.

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