EECS 16A Designing Information Devices and Systems I
Fall 2024 Homework 6

This homework is due Friday, October 11th, 2024, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned).

1. Mechanical Gram-Schmidt

For each of the following sets of vectors, use Gram-Schmidt to construct an orthonormal basis that shares their span.

(a) $D = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) $C = \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Hint: Recall that we define the complex inner product between $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ *as* $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\top \mathbf{b}^*$

2. Functional Gram-Schmidt

In this question, we will be demonstrating an example of the Gram Schmidt process, with functions!

For this question, we will define the inner product between two functions f and g as follows:

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx$$

Intuitively, we can think of the inner product of functions as similar to the vector inner product! Instead of taking the sum of the products of each pair of terms, we are taking the following:

$$f(a)g(a)dx + f(a+dx)g(a+dx)dx + \cdots + f(b)g(b)dx = \int_{a}^{b} f(x)g(x)dx, \text{ as } dx \to 0$$

Note that the norm of a function, $\|f(x)\|$, is defined as $\sqrt{\langle f(x), f(x) \rangle} = \sqrt{\int_{-1}^{1} f(x)^2 dx}$.

Construct an **orthonormal** basis that spans $\mathscr{P}_1$, the set of all polynomials of at most degree 1.

Hint: The basis $\{1, x\}$ *spans* $\mathscr{P}_1$.

3. Types of Matrices

For the following matrices, answer each of the following questions:

(a) Must the matrix be square?

(b) Must the matrix be upper triangular?

(c) Must the matrix be lower triangular?

(d) Must the matrix be diagonal?

(e) Must the matrix be a zero matrix?

(f) Must the matrix be an identity matrix?

(g) Must the matrix be symmetric?

For each question, you should indicate if the matrix *must*, *must not*, or *may or may not* belong to each of the classes.

(a) Matrix $\mathbf{I}$ is an identity matrix.

(b) Matrix $\mathbf{Z}$ is a zero matrix.

(c) Matrix $\mathbf{A}$ is upper triangular and lower triangular.

(d) Matrix $\mathbf{B}$ is the transpose of a lower triangular matrix.

(e) Matrix $\mathbf{C}$ is not square.

(f) Matrix $\mathbf{D}$ is not diagonal, but is the valid sum of some matrix $\mathbf{D}'$ with $(\mathbf{D}')^\top$.

(g) Matrix $\mathbf{E}$ is the valid sum of zero matrix $\mathbf{Z}'$ with $(\mathbf{Z}')^T$.

4. Braincells

After making your way through the last homework assignment, your remaining braincells are in disarray. Let's work together to get them back in order!

For all subparts of this problem, let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be an *adjacency matrix* representing the connections between each of your braincells with the other braincells. In particular, entry $\mathbf{B}_{ij}$ represents the speed of communication between braincell j and braincell i, specifically $j \to i$, where every $\mathbf{B}_{ij} = 0$ if there is no communication between braincell j and braincell i at all. Note that braincells do not communicate with themselves, so all diagonal entries should be 0.

(a) At the very beginning, all of your braincells are disconnected from one another. Denote this adjacency matrix as $\mathbf{A}$. What kind of matrix is $\mathbf{A}$?

(b) After a good night's rest, all of your brain connections are beginning to come back. In particular, the speed of communication between every pair of braincells has increased by 1. Assuming that the adjacency matrix before going to sleep was represented by $\mathbf{B}$, express this new adjacency matrix, $\mathbf{B}'$, in terms of $\mathbf{B}$ and constant matrices, as well as any scalar multiples.

Note: Recall in the problem statement that braincells do not communicate with themselves. Your new $\mathbf{B}'$ *should preserve this property.*

(c) After consuming your favorite source of caffeine, the communication speed between all of your braincells quadruples. Express this new adjacency matrix, $\mathbf{C}$, in terms of the adjacency matrix before caffeine, $\mathbf{B}$, and a constant matrix, as well as any scalar multiples.

(d) All ready to go, you attend your favorite TA's discussion. After collaborating with your peers on the new course content, the communication speed between 2 braincells from k to braincell l increases by 10. The rest of the connections stay the same. Express the new adjacency matrix, $\mathbf{D}$, in terms of the adjacency matrix before discussion, $\mathbf{B}$, and a constant matrix, as well as any scalar multiples.

Note: If you cannot provide a closed-form mathematical expression for $\mathbf{D}$*, a qualitative expression is also acceptable.*

(e) For this subpart, imagine that the speed of communication between braincells i and j is equal in both directions. In other words, consider the case where $\mathbf{B}_{ij} = \mathbf{B}_{ji} \; \forall \, i, j \in \{1, 2, \ldots, n\}$. In this case, what kind of matrix is $\mathbf{B}$?

5. Alice and Bob's Game (Adapted from 2008 William Lowell Putnam Competition)

Alice and Bob are playing a game, where each of them puts a finite, real value into an open spot in a $2N \times 2N$ matrix, where N is an integer. Once all values are in the matrix, if the columns of the matrix are linearly independent, Alice wins. Otherwise, Bob wins.

Alice goes first. Which of the two, when playing optimally, will win? We will approach this question in subparts.

(a) Let's look at a simpler problem for Alice and Bob. Say we have a $1 \times 2N$ empty vector. Again, Alice and Bob take turns placing real values into the vectors. Once all values are filled, if they sum to 0, Bob wins. Otherwise, Alice wins. What is a winning strategy for Bob, assuming Alice goes first?

Hint: To approach these kinds of questions, define how Bob would "respond" to Alice's turn. For example, if Alice puts a value x at index i, how should Bob respond?

(b) Let's say that for an $N \times N$ matrix, the values in each row sum to 0. Show that this matrix has linearly dependent columns.

Hint: Think of each column of the matrix as vectors in our N-dimensional space. Can you show that these vectors are linearly dependent?

(c) Using your answers to parts (a) and (b), define a winning strategy for Bob.

6. Correctness of the Gram-Schmidt Algorithm

Suppose we take a list of real vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ in $\mathbb{R}^k$ and run the following Gram-Schmidt algorithm on it to perform orthonormalization. It produces the vectors $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$.

Algorithm 1 The Gram-Schmidt Algorithm

for $i = 1$ up to n **do** ▷ Iterate through the vectors
  $\mathbf{r}_i = \mathbf{a}_i - \sum_{j<i} \left(\mathbf{q}_j^\top \mathbf{a}_i\right) \mathbf{q}_j$ ▷ Find the amount of $\mathbf{a}_i$ that remains after we project
  if $\mathbf{r}_i = \mathbf{0}$ **then**
    $\mathbf{q}_i = \mathbf{0}$
  else
    $\mathbf{q}_i = \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|}$ ▷ Normalize the vector.
  end if
end for

In this problem, we will prove the correctness of the Gram-Schmidt algorithm by showing that the following three properties hold on the vectors output by the algorithm:

(a) If $\mathbf{q}_i \neq \mathbf{0}$, then $\mathbf{q}_i^\top \mathbf{q}_i = \|\mathbf{q}_i\|^2 = 1$ (i.e. the $\mathbf{q}_i$ have unit norm whenever they are nonzero).

(b) For all $i \neq j$, $\mathbf{q}_i^\top \mathbf{q}_j = 0$ (i.e. $\mathbf{q}_i$ and $\mathbf{q}_j$ are orthogonal).

(c) For all $1 \leq \ell \leq n$, $\operatorname{span}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell) = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_\ell)$.

(a) First, use the construction from the if/then/else statement in the algorithm to show that the first property holds. In particular, show that $\|\mathbf{q}_i\| = 1$ if $\mathbf{q}_i \neq \mathbf{0}$.

(b) Next, we establish orthogonality between every pair of vectors in $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$. Essentially, every time we generate a new $\mathbf{q}_j$, we want to show that it is orthogonal to the vectors generated so far. Formally, we want to show that for $j < k$, $\mathbf{q}_j^\top \mathbf{q}_k = 0$ for all $2 \leq k \leq n$.

Fix k and assume that for all previously generated vectors $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{k-1}\}$, each pair of distinct vectors are orthogonal to each other.

Under this assumption, show that for all $i \leq k$, that $\mathbf{q}_j^\top \mathbf{q}_i = 0$ for all $j < i$.

This implies that $\{\mathbf{q}_1, \mathbf{q}_2\}$ are orthogonal to each other, which implies that each distinct pair in $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ are orthogonal to each other, and so on until we have shown that every pair of distinct vectors in $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ are orthogonal to each other.

Hint: The cases $i \leq k-1$ are already covered by our assumption, so we can focus specifically on $i = k$. Notice that the case $\mathbf{q}_k = \mathbf{0}$ is also true, since the inner product of any vector with $\mathbf{q}_k = \mathbf{0}$ is $\mathbf{0}$. As a result, focus on the case $\mathbf{q}_k \neq \mathbf{0}$ and expand what you know about $\mathbf{q}_k$.

Note: This proof technique is formally known as strong induction, but its specific mechanics and correctness are not within the scope of this class. You do not need to understand it to solve this problem.

(c) (OPTIONAL) Lastly, we show the final property by considering each ℓ from 1 to n, and showing the statement that $\operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\} = \operatorname{span}\{\mathbf{q}_1, \ldots, \mathbf{q}_\ell\}$. This statement is true when $\ell = 1$ since the algorithm produces $\mathbf{q}_1$ as a scaled version of $\mathbf{a}_1$. Now assume that this statement is true for $\ell = k-1$. Under this assumption, show that the spans are the same for $\ell = k$.

This implies that because $\operatorname{span}\{\mathbf{a}_1\} = \operatorname{span}\{\mathbf{q}_1\}$, then so too is $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\} = \operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2\}$, and so forth, until we get that $\operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \operatorname{span}\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$.

Hint: What you need to show is that if there exist $\alpha_1, \alpha_2, ..., \alpha_k$ *such that* $\mathbf{y} = \sum_{j=1}^{k} \alpha_j \mathbf{a_j}$*, then there exist* $\beta_1, \beta_2, ..., \beta_k$ *such that* $\mathbf{y} = \sum_{j=1}^{k} \beta_j \mathbf{q_j}$*. This shows the forward direction; you should also show the reverse direction.*
Try writing $\mathbf{a}_k$ *in terms of* $\mathbf{q}_k$ *and earlier* $\mathbf{q}_j$*, and use the condition for* $\ell = k-1$ *to show the condition for* $\ell = k$*. Don't forget the case that* $\mathbf{q}_k = \mathbf{0}$*.*

7. Homework Process and Study Group

Whom did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.