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JEE-MAIN MATHEMATICS - COMPLEX NUMBERS 2022-23

Time : 90 Min	Maths : Complex Numbers	Marks : 120

Hints and Solutions

01) Ans: **B)** 1,1

Sol: $(1+\omega)^7 = (1+\omega)(1+\omega)^6 \quad = (1+\omega)(-\omega^2)^6 = 1+\omega$

$\Rightarrow A + B\omega = 1+\omega \qquad \Rightarrow A = 1, B=$

02) Ans: **A)** - 4

Sol: $(3+5\omega+3\omega^2)^2 + (3+3\omega+5\omega^2)^2$

$= (3+3\omega+3\omega^2+2\omega)^2 + (3+3\omega+3\omega^2+2\omega^2)^2$

$(1+\omega+\omega^2 = 0, \ \omega^3 = 1)$

$= (2\omega)^2 + (2\omega^2)^2 = 4\omega^2 + 4\omega^4 = 4(-1) = -4$

03) Ans: **A)** $||z_1|-|z_2||$

Sol: $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

As $\arg z_1 - \arg z_2 = 0$

$\therefore |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$

$= (|z_1|-|z_2|)^2 \Rightarrow |z_1 - z_2| = ||z_1|-|z_2||$

04) Ans: **C)** $\pi - 2\tan^{-1} x$

Sol: Let $z = i\log\left(\frac{x-i}{x+i}\right) \Rightarrow \frac{z}{i} = \log\left(\frac{x-i}{x+i}\right)$

$\Rightarrow \frac{z}{i} = \log\left[\frac{x-i}{x+i} \times \frac{x-i}{x-i}\right] = \log\left[\frac{x^2-1-2ix}{x^2+1}\right]$

$\Rightarrow \frac{z}{i} = \log\left[\frac{x^2-1}{x^2+1} - i\frac{2x}{x^2+1}\right] \qquad \text{......(i)}$

Since, $\log(a+ib) = \log(re^{i\theta}) = \log r + i\theta$

$= \log\sqrt{a^2+b^2} + i\tan^{-1}(b/a)$

$\therefore$ From equation (i),

$$\frac{z}{i} = \log\sqrt{\left(\frac{x^2-1}{x^2+1}\right)^2 + \left(\frac{-2x}{x^2+1}\right)^2} + i\tan^{-1}\left(\frac{-2x}{x^2-1}\right)$$

$$\frac{z}{i} = \log\frac{\sqrt{x^4+1-2x^2+4x^2}}{(x^2+1)^2} + i\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$= \log 1 + i(2\tan^{-1} x) \quad = 0 + i(2\tan^{-1} x)$

$\therefore z = i^2 2\tan^{-1} x = -2\tan^{-1} x$

$z = \pi - 2\tan^{-1} x$

05) Ans: **A)** $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

Sol: $x^2 - \sqrt{3}x + 1 = 0 \Rightarrow x = \frac{\sqrt{3} \pm \sqrt{3-4}}{2}$

$\Rightarrow x = \frac{\sqrt{3} \pm i}{2} = \frac{\sqrt{3}}{2} \pm \frac{i}{2}$

$\Rightarrow x = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$ [By taking +ve sign]

06) Ans: **A)** the hyperbola $x^2 - y^2 = 1$.

Sol: Let, $z = (x+iy) \Rightarrow z^2 = x^2 - y^2 + 2ixy$

$\Rightarrow \mathrm{Re}(z^2) = 1 \Rightarrow x^2 - y^2 = 1$, which is the equation of hyperbola.

07) Ans: **A)** $\frac{10}{\sqrt{2}}(1+i)$

Sol: $\frac{4(\cos 75^o + i\sin 75^o)}{0.4(\cos 30^o + i\sin 30^o)}$

$= 10(\cos 75^o + i\sin 75^o)(\cos 30^o - i\sin 30^o)$

$= 10(\cos 45^o + i\sin 45^o) = \frac{10}{\sqrt{2}}(1+i)$

08) Ans: **A)** $|a|^2 > b$

Sol: By adding $a\bar{a}$ on both the sides of

$z\bar{z} + a\bar{z} + \bar{a}z = -b$

we get, $(z+a)(\bar{z}+\bar{a}) = a\bar{a} - b$

$\Rightarrow |z+a|^2 = |a|^2 - b, \quad \{\because z\bar{z} = |z|^2\}$

This equation will represent a circle with center $z = -a$, if $|a|^2 - b > 0$ i.e. $|a|^2 > b$ because $|a|^2 = b$ represents point circle only.

09) Ans: **D)** x = 4n, where n is any positive integer.

Sol: $\left(\frac{1+i}{1-i}\right)^x = 1 \Rightarrow \left[\frac{(1+i)^2}{1-i^2}\right]^x = 1$

$\Rightarrow \left(\frac{1+i^2+2i}{1+1}\right)^x = 1 \Rightarrow i^x = 1 \qquad \therefore x = 4n, n \in I^+$

10) Ans: **A)** 0

Sol: Here, $i + i^2 + i^3 + ...$ up to 1000 terms

$= \frac{i(1-i^{1000})}{1-i} = \frac{i(1-(i^4)^{250})}{1-i} = \frac{i(1-1)}{1-i} = 0$

11) Ans: **B)** $\frac{-1+i\sqrt{3}}{2}$

Sol: The cube roots of unity are

$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$.

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12) Ans: **A)** $i\cot\frac{\theta}{2}$

Sol: $a=\cos\theta+i\sin\theta.$

$\therefore \frac{1+a}{1-a}=\frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}.$

Rationalization of denominator, we get

$$\frac{1+a}{1-a}=\frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}\times\frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta}$$

$$=\frac{(1+\cos\theta)(1-\cos\theta)+(1+\cos\theta)i\sin\theta+(1-\cos\theta)i\sin\theta-i^2\sin^2\theta}{(1-\cos\theta)^2-(i\sin\theta)^2}$$

$$=\frac{1-\cos^2\theta+i\sin\theta+i\sin\theta\cos\theta+i\sin\theta-i\sin\theta\cos\theta-\sin^2\theta}{1+\cos^2\theta-2\cos\theta+\sin^2\theta}$$

$$=\frac{1-(\cos^2\theta+\sin^2\theta)+2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)-2\cos\theta}$$

$$=\frac{2i\sin\theta}{2(1-\cos\theta)}=\frac{i.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=i\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=i\cot\frac{\theta}{2}$$

13) Ans: **A)** $\pi/2$

Sol: Let, $z=\frac{1+\sqrt{3}i}{\sqrt{3}-i}=\frac{1+\sqrt{3}i}{\sqrt{3}-i}\times\frac{\sqrt{3}+i}{\sqrt{3}+i}$

$=\frac{\sqrt{3}+i+3i-\sqrt{3}}{3+1}=\frac{4i}{4}=i$

$\therefore \text{amp}(z)=\pi/2 \quad [\because \tan\theta=b/a]$

14) Ans: **B)** $2\cos n\theta$

Sol: $x+\frac{1}{x}=2\cos\theta$

$\Rightarrow x^2-2x\cos\theta+1=0 \Rightarrow x=\cos\theta\pm i\sin\theta$

$\Rightarrow x^n=\cos n\theta\pm i\sin n\theta \Rightarrow \frac{1}{x}=\frac{1}{\cos\theta\pm i\sin\theta}$

$\Rightarrow \frac{1}{x}=\cos\theta\mp i\sin\theta \Rightarrow \frac{1}{x^n}=\cos n\theta\mp i\sin n\theta$

$\therefore x^n+\frac{1}{x^n}=2\cos n\theta$

15) Ans: **D)** 729

Sol: $(1-2\omega+\omega^2)^6=(1+\omega^2-2\omega)^6$

$=(-\omega-2\omega)^6=(-3\omega)^6$

$=(-3)^6(\omega^3)^2 \quad [\text{As } 1+\omega+\omega^2=0, \omega^3=1]$

=729.

16) Ans: **D)** - 2

Sol: $z=x-iy, z^{1/3}=p+iq$

$\Rightarrow z=(p+iq)^3=p^3-iq^3+3p^2qi-3pq^2$

$\Rightarrow z=(p^3-3pq^2)+i(3p^2q-q^3)$

On equating real and imaginary part, we get

$x=p^3-3pq^2, y=-(3p^2q-q^3)$

$x=p(p^2-3q^2), \quad y=q(q^2-3p^2)$

$\frac{x}{p}=p^2-3q^2 \quad .....(i)$

$\frac{y}{q}=q^2-3p^2 \quad ......(ii)$

Now, adding (i) and (ii),

$\frac{x}{p}+\frac{y}{q}=p^2+q^2-3(q^2+p^2)=-2p^2-2q^2$

$\frac{x}{p}+\frac{y}{q}=-2(p^2+q^2). \quad \therefore \frac{\frac{x}{p}+\frac{y}{q}}{p^2+q^2}=-2$

17) Ans: **A)** $4x-3$

Sol: $x+iy=\frac{3}{2+\cos\theta+i\sin\theta}$

$=\frac{3(2+\cos\theta-i\sin\theta)}{(2+\cos\theta)^2+\sin^2\theta}=\frac{6+3\cos\theta-3i\sin\theta}{4+\cos^2\theta+4\cos\theta+\sin^2\theta}$

$=\left[\frac{6+3\cos\theta}{5+4\cos\theta}\right]+i\left[\frac{-3\sin\theta}{5+4\cos\theta}\right]$

$\Rightarrow x=\frac{3(2+\cos\theta)}{5+4\cos\theta}, y=\frac{-3\sin\theta}{5+4\cos\theta}$

$\therefore x^2+y^2=\frac{9}{(5+4\cos\theta)^2}[4+\cos^2\theta+4\cos\theta+\sin^2\theta]$

$=\frac{9}{5+4\cos\theta}=4\left[\frac{6+3\cos\theta}{5+4\cos\theta}\right]-3=4x-3$

18) Ans: **A)** $-\sin\theta$

Sol: Suppose, $z=e^{e^{-i\theta}}=e^{\cos\theta-i\sin\theta}=e^{\cos\theta}e^{-i\sin\theta}$

$z=e^{\cos\theta}[\cos(\sin\theta)-i\sin(\sin\theta)]$

$z=e^{\cos\theta}\cos(\sin\theta)-ie^{\cos\theta}\sin(\sin\theta)$

Now, $\text{amp}(z)=\tan^{-1}\left[-\frac{e^{\cos\theta}\sin(\sin\theta)}{e^{\cos\theta}\cos(\sin\theta)}\right]$

$=\tan^{-1}[\tan(-\sin\theta)]=-\sin\theta$

19) Ans: **B)** $\frac{\pi}{2}$

Sol: This is one of the fundamental concepts.

20) Ans: **B)** $\alpha+\beta-\pi$

Sol: We know that principal argument of a complex number lie between $-\pi$ and π, but $\alpha+\beta>\pi$.

Thus, principal

$\arg(z_1z_2)=\arg(z_1)+\arg(z_2)=\alpha+\beta$ is given by $\alpha+\beta-\pi$.

21) Ans: **A)** 0

Sol: The complex cube roots of unity are $1, \omega, \omega^2$

Let $\alpha=\omega, \beta=\omega^2$; Then $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}$

$=\omega^4+(\omega^2)^4+(\omega^{-1})(\omega^2)^{-1}=\omega+\omega^2+1=0$

22) Ans: **A)** 5

Sol: $z_1=1+i\Rightarrow z_1=(1,1)$

$z_2=-2+3i\Rightarrow z_2=(-2,3)$

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$z_3 = \frac{ai}{3} \Rightarrow z_3 = (0, a/3)$

Since, z_1, z_2 and z_3 are collinear,

$$\begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & a/3 & 1 \end{vmatrix} = 0 \Rightarrow -\frac{a}{3}(1+2) + 1(3+2) = 0$$

$\Rightarrow -a + 5 = 0 \Rightarrow a = 5$

23) Ans: **B)** 1 - i

Sol: Suppose, $z = 1 + i \Rightarrow \bar{z} = 1 - i$

24) Ans: **A)** 0

Sol: Let,

$$1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2e^{i\pi/3}$$

$\therefore (1 + i\sqrt{3})^9 = (2e^{i\pi/3})^9 = 2^9.e^{i(3\pi)}$

$= 2^9(\cos 3\pi + i \sin 3\pi) = -2^9$

$\therefore a + ib = (1 + i\sqrt{3})^9 = -2^9 \qquad \therefore b = 0$

25) Ans: **B)** $3\cos(\alpha + \beta + \gamma)$

Sol: $\cos\alpha + \cos\beta + \cos\gamma = 0$ and

$\sin\alpha + \sin\beta + \sin\gamma = 0$

Let $a = \cos\alpha + i\sin\alpha$; $b = \cos\beta + i\sin\beta$ and

$c = \cos\gamma + i\sin\gamma$.

$\therefore a+b+c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0 = 0$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$ or

$(\cos\alpha + i\sin a)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$= (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$

$= 3[\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

i.e. $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$.

26) Ans: **A)** 0

Sol: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1 + z_2| = [(r_1\cos\theta_1 + r_2\cos\theta_2)^2$

$+ (r_2\sin\theta_1 + r_2\sin\theta_2)^2]^{1/2}$

$= [r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)]^{1/2} = [(r_1 + r_2)^2]^{1/2}$

$\because |z_1 + z_2| = |z_1| + |z_2|$

$\therefore \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$

Hence, $\arg(z_1) - \arg(z_2) = 0$

27) Ans: **3.23** Sol:

$$\left||z| - \left|\frac{4}{z}\right|\right| \le \left|z - \frac{4}{z}\right| \Rightarrow \left||z| - \frac{4}{|z|}\right| \le 2 \Rightarrow \left|r - \frac{4}{r}\right| \le 2$$

$\Rightarrow -2 \le r - \frac{4}{r} \le 2$ where $r = |z|$.

$-2 \le r - \frac{4}{r} \Rightarrow r^2 + 2r - 4 \ge 0.$

The corresponding roots are $r = \frac{-2 + \sqrt{20}}{2} = -1 \pm \sqrt{5}$

$r > 0, r^2 + 2r - 4 \ge 0 \Rightarrow r \ge \sqrt{5} - 1 \rightarrow (1)$

$\Rightarrow r - \frac{4}{r} \le 2 \Rightarrow r^2 - 2r - 4 \le 0.$

The corresponding roots are $r = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$

$\Rightarrow r^2 - 2r - 4 \le 0 \Rightarrow 1 - \sqrt{5} \le r \le 1 + \sqrt{5}$

$0 \le r \le \sqrt{5} + 1 \rightarrow (2)$

From (1) and (2) $\sqrt{5} - 1 \le r \le \sqrt{5} + 1$.

Therefore, the greatest value is $\sqrt{5} + 1$ =3.23

28) Ans: **-2** Sol:

$$|i + z|^2 - |i - z|^2 = \left|i + a - \frac{i}{2}\right|^2 - \left|i - a + \frac{i}{2}\right|^2$$

$$= \left|a + \frac{i}{2}\right|^2 - \left|-a + \frac{3i}{2}\right|^2 = \left(a^2 + \frac{1}{4}\right) - \left(a^2 + \frac{9}{4}\right) = -2$$

29) Ans: **0.32** Sol:

$$\left|\frac{1}{(2+i)^2} - \frac{1}{(2-i)^2}\right| = \left|\frac{(2-i)^2 - (2+i)^2}{(4-i^2)^2}\right|$$

$$= \left|\frac{-8i}{25}\right| = \frac{8}{25} = 0.32$$

30) Ans: **3** Sol: Let $z = x + iy$, thus given equation becomes

$(x + iy)(x - iy) + (2 - 3i)(x + iy) + (2 + 3i)(x - iy) + 4 = 0$

$x^2 + y^2 + 2x + 3y - 3ix + 2iy + 2x - 2iy + 3ix + 3y + 4$

$= 0 \Rightarrow x^2 + y^2 + 4x + 6y + 4 = 0$

Given equation represents a circle with

Radius $= \sqrt{2^2 + 3^2 - 4} = \sqrt{4 + 9 - 4} = \sqrt{9} = 3$

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