Studentpad

Set theory, relations and functions 2022-23

Time : 90 Min	Maths : Set Theory, Relations and Functions	Marks : 120

Hints and Solutions

01) Ans: **C)** ≥ 0, for all real θ

Sol: Given, $f(\theta)=\sin\theta(\sin\theta+\sin 3\theta)$

$=(\sin\theta+3\sin\theta-4\sin^3\theta)\sin\theta$

$=(4\sin\theta-4\sin^3\theta)\sin\theta=\sin^2\theta(4-4\sin^2\theta)$

$=4\sin^2\theta\cos^2\theta=(2\sin\theta\cos\theta)^2 \quad =(\sin 2\theta)^2 \geq 0$

which is true for all θ.

02) Ans: **A)** [0, 1]

Sol: $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$

Clearly $f(x)$ is defined, if $4+x\geq 0 \Rightarrow x\geq -4$

$4-x\geq 0 \Rightarrow x\leq 4$

$x(1-x)\geq 0 \Rightarrow x\geq 0$ and $x\leq 1$

$\therefore$ Domain of $f=(-\infty,4]\cap[-4,\infty)\cap[0,1]=[0,1]$

03) Ans: **A)** odd function.

Sol: $f(x)=\sin\left(\log(x+\sqrt{1+x^2})\right)$

$\Rightarrow f(-x)=\sin[\log(-x+\sqrt{1+x^2})]$

$\Rightarrow f(-x)=\sin\log\left((\sqrt{1+x^2}-x)\dfrac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow f(-x)=\sin\log\left[\dfrac{1}{(x+\sqrt{1+x^2})}\right]$

$\Rightarrow f(-x)=\sin\left[\log(x+\sqrt{1+x^2})^{-1}\right]$

$\Rightarrow f(-x)=\sin\left[-\log(x+\sqrt{1+x^2})\right]$

$\Rightarrow f(-x)=-\sin\left[\log(x+\sqrt{1+x^2})\right] \Rightarrow f(-x)=-f(x)$

$\therefore f(x)$ is odd function.

04) Ans: **B)** $39\leq x\leq 63$

Sol: Suppose A represents the set of Indians who like mangoes while B represents the set of Indians who like apples.

Consider population of India be 100.

Then, $n(A)=63, n(B)=76$

Now, $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$=63+76-n(A\cap B)$

$\therefore n(A\cup B)+n(A\cap B)=139$

$\Rightarrow n(A\cap B)=139-n(A\cup B)$

But $n(A\cup B)\leq 100$

$\therefore -n(A\cup B)\geq -100$

$\therefore 139-n(A\cup B)\geq 139-100=39$

$\therefore n(A\cap B)\geq 39 \Rightarrow 39\leq n(A\cap B)$ .....(i)

Again, $A\cap B\subseteq A, A\cap B\subseteq B$

$\therefore n(A\cap B)\leq n(A)=63$ and $n(A\cap B)\leq n(B)=76$

$\therefore n(A\cap B)\leq 63$ .....(ii)

$\therefore 39\leq n(A\cap B)\leq 63 \Rightarrow 39\leq x\leq 63$

05) Ans: **B)** Onto but not one-one

Sol: **PLAN :** To check nature of function.

(i) One-one To check one-one, we must check whether $f'(x)>0$ or $f'(x)<0$ in given domain.

(ii) Onto To check onto, we must check Range=Codomain

Description of situation To find range in given domain $[a,b]$, put $f'(x)=0$ and find

$x=\alpha_1,\alpha_2,\ldots,\alpha_n\in[a,b]$

Now, find $\{f(a),f(\alpha_1),f(\alpha_2),\ldots,f(\alpha_n),f(b)\}$ its greatest and least values gives you range.

Now, $f:[0,3]\to[1,29]$ $f(x)=2x^3-15x^2+36x+1$

$\therefore f'(x)=6x^2-30x+36 \quad =6(x^2-5x+6)$

$=6(x-2)(x-3)$

For given domain $[0,3]$, $f(x)$ is increasing as well as decreasing $\Rightarrow$ many-one

Now, put $f'(x)=0 \quad \Rightarrow x=2,3$

So, for range $f(0)=1, f(2)=29, f(3)=28$

$\Rightarrow$ Range $\in[1,29]$

Hence, onto but not one-one.

06) Ans: **B)** $\left[-\dfrac{\pi}{4},\dfrac{\pi}{4}\right]$

Sol: By definition of composition of function, $g(f(x))=(\sin x+\cos x)^2-1$, is invertible (i.e. bijective)

$\Rightarrow g\{f(x)\}=\sin 2x$ is bijective.

We know, $\sin x$ is bijective, only when $x\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$.

Therefore, $g\{f(x)\}$ is bijective, if $-\dfrac{\pi}{2}\leq 2x\leq\dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{4}\leq x\leq\dfrac{\pi}{4}$

07) Ans: **B)** {- 2, - 1, 0, 1, 2}

Sol: $\because R=\{(x,y)\,|\,x,y\in Z, x^2+y^2\leq 4\}$

$\therefore R=\{(-2,0),(-1,0),(-1,1),(0,-1)(0,1),$
$(0,2),(0,-2),(1,0),(1,1),(2,0)\}$

Thus Domain of $R=\{-2,-1,0,1,2\}$

08) Ans: **B)** $\pm\sqrt{n\pi}, n \in \{1,2,...\}$

Sol: $f(x) = x^2, g(x) = \sin x \quad (gof)(x) = \sin x^2$

$go(gof)(x) = \sin(\sin x^2)$

$(fogogof)(x) = (\sin(\sin x^2))^2 \qquad ...(i)$

Again, $(gof)(x) = \sin x^2$

$(gogof)(x) = \sin(\sin x^2) \qquad ...(ii)$

Here, $(fogogof)(x) = (gogof)(x)$

$\Rightarrow (\sin(\sin x^2))^2 = \sin(\sin x^2)$

$\Rightarrow \sin(\sin x^2)\{\sin(\sin x^2) - 1\} = 0$

$\Rightarrow \sin(\sin x^2) = 0$ or $\sin(\sin x^2) = 1$

$\Rightarrow \sin x^2 = 0$ or $\sin x^2 = \frac{\pi}{2}$

$\Rightarrow x^2 = n\pi \left[\sin x^2 = \frac{\pi}{2} \text{ is not possible as } -1 \le \sin\theta \le 1\right]$

$\therefore x = \pm\sqrt{n\pi}$

09) Ans: **A)** $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

Sol: In $f(x) = \frac{a^x + 1}{a^x - 1}$,

$f(-x) = \frac{a^{-x}+1}{a^{-x}-1} = \frac{1+a^x}{1-a^x} = -\frac{a^x+1}{a^x-1} = -f(x)$

$\therefore$ It is an odd function.

In $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$,

$f(-x) = (-x)\frac{a^{-x}-1}{a^{-x}+1} = -x\frac{1-a^x}{1+a^x} = x\frac{a^x-1}{a^x+1} = f(x)$

$\therefore$ It is an even function.

In $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$, $f(x) = \frac{a^{-x} - a^x}{a^{-x} + a^x} = -f(x)$

$\therefore$ It is an odd function.

In $f(x) = \sin x$, $f(-x) = \sin(-x) = -\sin x = -f(x)$

$\therefore$ It is an odd function.

10) Ans: **C)** 3

Sol: As $A \subseteq B$, $\therefore A \cap B = A$

$\therefore n(A \cap B) = n(A) = 3$

11) Ans: **C)** 6, 4

Sol: As $2^m - 2^n = 48 = 16 \times 3 = 2^4 \times 3$

$\Rightarrow 2^n(2^{m-n} - 1) = 2^4 \times 3 \quad \Rightarrow n = 4$ and

$2^{m-n} = 4 = 2^2$

$\Rightarrow m - n = 2 \quad \Rightarrow m - 4 = 2 \quad \Rightarrow m = 6$

$\Rightarrow m = 6, n = 4$

12) Ans: **C)** 0

Sol: Given that, $f(x) = \cos(\log x)$

Now, let $y = f(x).f(4) - \frac{1}{2}\left[f\left(\frac{x}{4}\right) + f(4x)\right]$

$\Rightarrow y = \cos(\log x).\cos(\log 4)$

$-\frac{1}{2}\left[\cos\log\left(\frac{x}{4}\right) + \cos(\log 4x)\right]$

$\Rightarrow y = \cos(\log x)\cos(\log 4)$

$-\frac{1}{2}[\cos(\log x - \log 4) + \cos(\log x + \log 4)]$

$\Rightarrow y = \cos(\log x)\cos(\log 4) - \frac{1}{2}[2\cos(\log x)\cos(\log 4)]$

$\Rightarrow y = 0$

13) Ans: **C)** $R \cap S$ is an equivalence relation on A

Sol: Given, R and S are relations on set A.

$\therefore R \subseteq A \times A$ and $S \subseteq A \times A \Rightarrow R \cap C \subseteq A \times A$

$\Rightarrow R \cap S$ is also a relation on A.

Reflexivity: Suppose a be an arbitrary element of A.

Then, $a \in A \Rightarrow (a, a) \in R$ and $(a, a) \in S$

[$\because$ R and S are reflexive]

$\Rightarrow (a, a) \in R \cap S$

$\therefore (a, a) \in R \cap S$ for all $a \in A$.

So, $R \cap S$ is a reflexive relation on A.

Symmetry: Suppose $a, b \in A$ such as $(a, b) \in R \cap S$.

Then, $(a, b) \in R \cap S \Rightarrow (a, b) \in R$ and $(a, b) \in S$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$, [$\because$ R and S are symmetric]

$\Rightarrow (b, a) \in R \cap S \qquad \therefore (a, b) \in R$

$\Rightarrow (b, a) \in R \cap S$ for all $(a, b) \in R \cap S$.

So, $R \cap S$ is symmetric on A.

Transitivity: Suppose $a, b, c \in A$ such as $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$.

Then, $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$

$\Rightarrow \{((a, b) \in R \text{ and } (a, b) \in S)\}$ and $\{((b, c) \in R \text{ and } (b, c) \in S\}$

$\Rightarrow \{(a, b) \in R, (b, c) \in R\}$ and $\{(a, b) \in S, (b, c) \in S\}$

$\Rightarrow (a, c) \in R$ and $(a, c) \in S$

As R and S are transitive,

$\therefore (a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

$(a, b) \in S$ and $(b, c) \in S \Rightarrow (a, c) \in S$

$\Rightarrow (a, c) \in R \cap S$

Thus, $(a, b) \in R \cap S$ and $(b, c) \in R \cap S \Rightarrow (a, c) \in R \cap S$.

$\therefore R \cap S$ is transitive on A.

Hence, R is an equivalence relation on A.

14) Ans: **C)** $f(x) = \tan^{-1}x - \frac{x}{\sqrt{1+x^2}}$

Sol: Both $\tan^{-1}x$ and $\frac{x}{\sqrt{1+x^2}}$ are bounded functions.

Thus, $\tan^{-1}x - \frac{x}{\sqrt{1+x^2}}$ is also bounded.

15) Ans: **D)** $\sqrt{x}-1, x \ge 0$
Sol: It is only to find the inverse.
Consider $y = f(x) = (x+1)^2$, for $x \ge -1$
$\pm\sqrt{y} = x+1, \quad x \ge -1$
$\Rightarrow \sqrt{y} = x+1 \quad \Rightarrow y \ge 0,\ x+\ 1\ge \quad \Rightarrow x = \sqrt{y}\ -1$
$\Rightarrow f^{-1}(y) = \sqrt{y}-1 \quad \Rightarrow f^{-1}(x) = \sqrt{x}-1 \ \Rightarrow x \ge$

16) Ans: **B)** $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
Sol: $f(x) = \sin x + \cos x, g(x) = x^2 - 1$
or $g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$
Clearly, $g(f(x))$ is invertible in $-\frac{\pi}{2} \le 2x \le \frac{\pi}{2}$
($\because \sin\theta$ is invertible when $-\pi/2 \le \theta \le \pi/2$)
or $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

17) Ans: **C)** Symmetric and transitive
Sol: The void relation R on A is not reflexive because $(a,a) \notin R$ for any $a \in A$. The void relation is symmetric and transitive.

18) Ans: **D)** all of above

19) Ans: **A)** $(A - B) \cap (A - C)$
Sol: According to De' Morgans law.

20) Ans: **B)** $f[F(x)]$
Sol: $F[f(x)] = F(\log_a x) = a^{\log_a x} = x$
$f[F(x)] = f(a^x) = \log_a a^x = x \log_a a = x$

21) Ans: **A)** $\{(1, 2), (5, 1), (3, 1)\}$
Sol: $(A, B) \in R \Leftrightarrow (B, A) \in R^{-1}$
$\therefore R^{-1} = \{(3, 1), (5, 1), (1, 2)\}$

22) Ans: **D)** 2^9
Sol: $A = \{2, 4, 6\}$; $B = \{2, 3, 5\}$
$\therefore A \times B$ contains $3 \times 3 = 9$ elements
$\therefore$ No. of relations from A to $B = 2^9$

23) Ans: **A)** N_{12}
Sol: $N_3 \cap N_4 = \{3, 6, 9, 12, 15, ...\} \cap \{4, 8, 12, 16, 20, ...\}$
$= \{12, 24, 36......\} = N_{12}$

24) Ans: **B)** $\{a, b, c\}$
Sol: $B \cup C = \{a, b, c, d, e\}$
$\therefore A \cap (B \cup C) = \{a, b, c\} \cap \{a, b, c, d, e\} = \{a, b, c\}$

25) Ans: **C)** Y
Sol: As $4^n - 3n - 1 = (3+1)^n - 3n - 1$
$= 3^n + {}^nC_1 3^{n-1} + {}^nC_2 3^{n-2} + ... + {}^nC_{n-1}3 + {}^nC_n - 3n - 1$

$= {}^nC_2 3^2 + {}^nC_3 . 3^3 + ... + {}^nC_n 3^n$,
$({}^nC_0 = {}^nC_n, \quad {}^nC_1 = {}^nC_{n-1}$ etc)
$= 9[{}^nC_2 + {}^nC_3(3) + ... + {}^nC_n 3^{n-1}]$
$\therefore 4^n - 3n - 1$ is a multiple of 9 for $n \ge 2$.
For $n = 1, \ 4^n - 3n - 1 = 4 - 3 - 1 = 0$,
For $n = 2, \ 4^n - 3n - 1 = 16 - 6 - 1 = 9$
$\therefore 4^n - 3n - 1$ is a multiple of 9 for all $n \in N$
Here X contains elements which are not multiples of 9 and clearly Y contains all multiples of 9.
$\therefore X \subseteq Y$ i.e. $X \cup Y = Y$

26) Ans: **250** Sol:
$n(H \cup B) = 1000, n(H) = 750, n(B) = 400$
$n(H \cup B) = n(H) + n(B) - n(H \cap B)$
$\therefore 1000 - 750 + 400 - n(H \cap B) \quad \Rightarrow n(H \cap B) = 15$
i.e. 15 person can speack both Hindi and Bengali.
Only Hindi: $n(H) - n(H \cap B) = 750 - 150 = 600$
Only Bengali: $n(B) - n(H \cap B) = 400 - 150 = 250$.

27) Ans: **6** Sol: There are two possibilities that either $A \subset B$ or A an B are disjoint.
If $A \subset B$ then $A \cup B = B \quad \therefore n(A \cup B) = n(B)$
If A and B are disjoint i.e. $A \cap B = \phi$
then $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 0 = 9$.
Thus the minimum number of elements will be when $A \subset B$ and in that case the number of elements is 6.

28) Ans: **7** Sol: For R add $(1,1), (2,2), (3,3)3$
For S add $(2,1), (3,2)2$
For T add $(1,3), (3,1)2$
So the minimum number of ordered pairs to be added is 7.

29) Ans: **41** Sol: $S = \{1, 2, 3, 4\}$, clearly each element can be put in 3 ways either in subsets or we dont put in any subset
Therefore, total number of ordered
pairs $= \frac{3 \times 3 \times 3 \times 3 - 1}{2} + 1 = 41$

30) Ans: **20** Sol:
$n(U) = 200, n(M) = 120, n(P) = 90, n(C) = 70$
$n(M \cap P) = 40, n(P \cap C) = 30, n(C \cap M) = 50$
$n(M \cup P \cup C)' = 20$
$x \in A \cup B \Rightarrow x \in A$ or $x \in B$ or $x \in$ both
$x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$
i.e. $x \in (A \cup B)' \Rightarrow x \notin A$ and $x \notin B$.
Here we have to find the number od students who study the three subjects i.e. $n(M \cap P \cap C)$.
we are given $n(M \cup P \cup C)' = 20$
$\therefore n(U) - n(M \cup P \cup C) = 20$
$\therefore n(M \cup P \cup C) = 200 - 20 = 180$
or $n(M) + n(P) + n(C) - [n(M \cap P) + n(P \cap C) + n(C \cap M)]$
$+ n(M \cap P \cap C) = 180$
or $(120 + 90 + 70) - [40 + 30 + 50] + n(M \cap P \cap C) = 180$

$280 - 120 + n(M \cap P \cap C) = 180$
$\therefore n(M \cap P \cap C) = 180 - 160 = 20$.
Thus 20 studies all the three subjects.