Studentpad

JEE-MAIN MATHEMATICS - QUADRATIC EQUATIONS 2022-23

Time : 90 Min	Maths : Quadratic Equations	Marks : 120

Hints and Solutions

01) Ans: **A)** 40

Sol: Considering α as a common root, then

$\alpha^2 + a\alpha + 10 = 0$ .....(i)

and $\alpha^2 + b\alpha - 10 = 0$ .....(ii)

Now, form (i) - (ii),

$(a-b)\alpha + 20 = 0 \quad \Rightarrow \alpha = -\dfrac{20}{a-b}$

Substituting the value of α in (i), we get,

$$\left(-\frac{20}{a-b}\right)^2 + a\left(-\frac{20}{a-b}\right) + 10 = 0$$

$\Rightarrow 400 - 20\,a(a-b) + 10(a-b)^2 = 0$

$\Rightarrow 40 - 2a^2 + 2ab + a^2 + b^2 - 2ab = 0$

i.e. $a^2 - b^2 = 40$

02) Ans: **A)** - b

Sol: Let α, α^n be the two roots.

Then $\alpha + \alpha^n = -b/a, \alpha\alpha^n = c/a$

Now eliminating α, we get,

$$\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = -\frac{b}{a}$$

$$\Rightarrow a.a^{-\frac{1}{n+1}}.c^{\frac{1}{n+1}} + a.a^{-\frac{n}{n+1}}.c^{\frac{n}{n+1}} = -b$$

$$\Rightarrow (a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} = -b$$

03) Ans: **A)** $1:2:3$

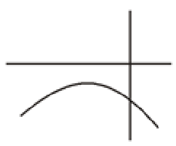
Sol: Given equation are $x^2 + 2x + 3 = 0$ ...(i)

and $ax^2 + bx + c = 0$ ...(ii)

As, Eq. (i) has imaginary roots, so Eq. (ii) will also have both roots same as Eq. (i).

So, $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3}$

Therefore, $a:b:c$ is $1:2:3$



04) Ans: **B)**

Sol:

	(a)	(b)	(c)	(d)
a (using concavity)	>0	<0	<0	>0
c (using f(0))	>0	<0	>0	<0
b(using abscissa of vertex (-b/2a))	>0	<0	<0	<0
abc	>0	<0	>0	>0

05) Ans: **A)** $x^3 - 64 = 0$

Sol: Let $y = x^2 \quad \Rightarrow x = \sqrt{y}$

$\therefore x^3 + 8 = 0 \quad \Rightarrow y^{3/2} + 8 = 0$

$\Rightarrow y^3 = 64 \quad \Rightarrow y^3 - 64 = 0$

Hence, the equation having roots α^2, β^2 and γ^2 is $x^3 - 64 = 0$.

06) Ans: **C)** $a^2x^2 + a(c-b)x - b\ c = 0$

Sol: $(a^2 + b^2)x^2 - 2b(a+c)x + b^2 + c^2 = 0$

$D = 4b^2(a+c)^2 - 4(a^2+b^2)(b^2+c^2)$

$= -4(b^4 - 2b^2ac + a^2c^2) \;\; = -4(b^2 - ac)^2$

For real roots, $D \geq 0$

$\Rightarrow -4(b^2 - ac)^2 \geq 0 \;\; \Rightarrow b^2 - ac = 0$

Roots are real and equal.

Therefore, roots are $\dfrac{2b(a+c) \pm 0}{2(a^2+b^2)} = \dfrac{b(a+c)}{a^2+ac} = \dfrac{b}{a}$

This root satisfies option (c).

07) Ans: **B)** a = b = c, a = -2b = -2c

Sol: Since $f(x) = x^3 - 3b^2 + 2c^3$ is divisible by x-a and x-b,

$\therefore f(a) = 0 \quad \Rightarrow a^3 - 3b^2 a + 2c^3 =$ ....(i)

and $f(b) = 0 \quad \Rightarrow b^3 - 3b^3 + 2c^3 = 0$ ....(ii)

from (ii), b = c

From (i), $a^3 - 3ab^2 + 2b^3 = 0$ (Putting b = c)

$\Rightarrow (a-b)(a^2 + ab - 2b^2) = 0$

$\Rightarrow a = b$ or $a^2 + ab = 2b^2$

$\therefore a = b = c$ or $a^2 + ab = 2b^2$ and $b = c$

$a^2 + ab = 2b^2$ is satisfied by $a = -2b$. But b = c

$\therefore a^2 + ab - 2b^2$

and b = c is equivalent to $a = -2b = -2c$.

08) Ans: **A)** -3, -4

Sol: Let the correct equation be $x^2 + px + q = 0$ .....(i)

Roots found by the first student are 6 and 2.

Their sum $= 6 + 2 = 8 = -p$

and the product $= 6 \times 2 = 12 = q$

$\therefore$ (i) reduces to $x^2 - 8x + 12 = 0$ .....(ii)

But he has committed mistake only in the coefficient of x i.e. in p. Thus, q remains equal to 12 while p in the actual equation has been taken wrongly by the first student.

Now roots found by the second student are 2 and -9.
Their sum $= -9+2 = -7 = -p$
and the product = -9×2 = -18 = q i.e. p = 7 and $q = 18$ in (i)
But he has committed mistake only in the constant term i.e. in q. Thus p remains equal to 7.
$\therefore$ The correct equation from (i) is $x^2+7x+12=0$
$\Rightarrow (x+4)(x+3)=0 \quad \Rightarrow x=-4,-3$.

09) Ans: **B)** both b^2-ac and $b_1{}^2-a_1c_1$ must be perfect squares
Sol: Given equations are:
$ax^2+2bx+c=0 \qquad \ldots(1)$
and $a_1x^2+2b_1x+c_1=0 \qquad \ldots(2)$
As equation (1) and (2) have only one common root and a, b, c, a_1, b_1, c_1 are rational, therefore, common root cannot be imaginary or irrational (as irrational roots occur in conjugate pair when coefficients are rational, also complex roots always occur in conjugate pain).
Hence common root must be rational.
Therefore, both roots of equations (1) and (2) will be rational.
Therefore, $4\left(b^2-ac\right)$ and $4\left(b_1^2-a_1c_1\right)$ must be perfect squares (squares of rational numbers).
Therefore, b^2-ac and $b_1^2-a_1c_1$ must be perfect squares.

10) Ans: **D)** $2\{p^2-2q+p'^2-2q'-pp'\}$
Sol: From given $\alpha+\beta=p$,
$\alpha\beta=q, \alpha'+\beta'=p', \alpha'\beta'=q'$
Now, $(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$
$=2(\alpha^2+\beta^2)+2(\alpha'^2+\beta'^2)-2\alpha'(\alpha+\beta)-2\beta'(\alpha+\beta)$
$=2\left\{(\alpha+\beta)^2-2\alpha\beta+(\alpha'+\beta')^2-2\alpha'\beta'-(\alpha+\beta)(\alpha'+\beta')\right\}$
$=2\left\{p^2-2q+p'^2-2q'-pp'\right\}$

11) Ans: **B)** 32
Sol: Here $\alpha^2+\beta^2=-4, \alpha^3+\beta^3=-16$
$\therefore (\alpha^2+\beta^2)(\alpha^3+\beta^3)=\alpha^5+\beta^5+\alpha^2\beta^2(\alpha+\beta)$
$\Rightarrow (-4)(-16)=\alpha^5+\beta^5+(16)(2) \quad \Rightarrow \alpha^5+\beta^5=32$

12) Ans: **B)** $x>0$
Sol: Here, $x+2>\sqrt{x+4} \quad \Rightarrow (x+2)^2>(x+4)$
$\Rightarrow x^2+4x+4>x+4 \quad \Rightarrow x^2+3x>0$
$\Rightarrow x(x+3)>0 \quad \Rightarrow x<-3$ or $x>0 \quad \Rightarrow x>$

13) Ans: **B)** 4
Sol: If α, β, γ are the roots of the equation
$x^3-px^2+qx-r=0$.
$\therefore (\alpha+\beta)^{-1}+(\beta+\gamma)^{-1}+(\gamma+\alpha)^{-1}=\dfrac{p^2+q}{pq-r}$
Now, from given $p=0, q=4, r=-1$
$\Rightarrow \dfrac{p^2+q}{pq-r}=\dfrac{0+4}{0+1}=4$

14) Ans: **B)** $p\in(0,\pi)$
Sol: $(\cos p-1)x^2+(\cos p)x+\sin p=0$
Its discriminant $D\ge 0$ as roots are real.
$\Rightarrow \cos^2 p-4(\cos p-1)\sin p\ge 0$
$\Rightarrow \cos^2 p-4\cos p\sin p+4\sin p\ge 0$
$\Rightarrow (\cos p-2\sin p)^2-4\sin^2 p+4\sin p\ge 0$
$\Rightarrow (\cos p-2\sin p)^2+4\sin p(1-\sin p)\ge 0 \qquad \ldots..(i)$
Now, $(1-\sin p)\ge 0$ for all real p, $\sin p>0$ for $0<p<\pi$.
$\therefore 4\sin p(1-\sin p)\ge 0$, when $0<p<\pi$ or $p\in(0,\pi)$

15) Ans: **A)** $b^2<c$
Sol: $x^2+2bx+c=(x+b)^2+c-b^2$
Since $(x+b)^2$ is a perfect square, hence the given expression is positive, if $c-b^2>0$ or $b^2<c$.

16) Ans: **A)** one root in $(-\infty,a)$ and the other in $(b,+\infty)$.
Sol: $(x-a)(x-b)=1 \quad \Rightarrow x^2-(a+b)x+ab-1$
$\therefore$ Discriminant $=(a+b)^2-4(ab-1)=(b-a)^2+4>0$
$\therefore$ Both roots are real.
Let they are α,β where
$\alpha=\dfrac{(a+b)-\sqrt{(b-a)^2+4}}{2}$, $\beta=\dfrac{(a+b)+\sqrt{(b-a)^2+4}}{2}$
Clearly, $\alpha<\dfrac{(a+b)-\sqrt{(b-a)^2}}{2}=\dfrac{(a+b)-(b-a)}{2}=a$
$(\because b>a)$
and $\beta>\dfrac{(a+b)+\sqrt{(b-a)^2}}{2}=\dfrac{a+b+b-a}{2}=b$
So, one root α is less than a and the other root β is greater than b.

17) Ans: **B)** 1
Sol: We have $\sqrt{2x-4}=1+\sqrt{x+5}$
Squaring, $2x-4=1+(x+5)+2\sqrt{x+5}$
$\Rightarrow x-10=2\sqrt{x+5} \quad \Rightarrow x^2+100-20x=4x+20$
$\Rightarrow x^2-24x+80=0 \Rightarrow x=4,20$
Putting $x=4$, we get $\sqrt{4}-\sqrt{9}=1$, which is not possible.
Putting $x=20$, we get $\sqrt{36}-\sqrt{25}=1$
Thus, x = 20 is the only solution.

18) Ans: **D)** $5,-4,\dfrac{1\pm 5\sqrt{-3}}{2}$
Sol: Here, $x^4-2x^3+x-380=0$
By using remainder theorem, we get,
$(x-5)(x+4)(x^2-x+19)=0$
$x-5=0$, $x-4=0$ and $x^2-x+19=0$

i.e. $x = 5$, $x = -4$ and $x = \dfrac{-1 \pm 5\sqrt{-3}}{2}$

19) Ans: **A)** $b^2pr = q^2ac$

Sol: Given that α_1, α_2 are the roots of $ax^2 + bx + c = 0$

$\therefore \alpha_1 + \alpha_2 = -\dfrac{b}{a}$ and $\alpha_1\alpha_2 = \dfrac{c}{a}$

Now β_1, β_2 are the roots of $px^2 + qx + r = 0$

$\therefore \beta_1 + \beta_2 = -\dfrac{q}{p}$ and $\beta_1\beta_2 = \dfrac{r}{p}$

Given system is $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$

$\Rightarrow \dfrac{\alpha_1}{\beta_1} = \dfrac{\alpha_2}{\beta_2}$

Now, we have $\dfrac{\alpha_1\,\alpha_2}{\beta_1\,\beta_2} = \dfrac{c/a}{r/p}$ or $\dfrac{\alpha_1}{\beta_1}.\dfrac{\alpha_2}{\beta_2} = \dfrac{cp}{ar}$

As $\dfrac{\alpha_1}{\beta_1} = \dfrac{\alpha_2}{\beta_2}$ $\Rightarrow \dfrac{\alpha_1}{\alpha_1} = \dfrac{\beta_1}{\beta_2}$ or $\dfrac{\alpha_1^2}{\alpha_2^2} = \dfrac{\beta_1^2}{\beta_2^2}$

$\Rightarrow \dfrac{\alpha_1^2 + \alpha_2^2}{\alpha_2^2} = \dfrac{\beta_1^2 + \beta_2^2}{\beta_2^2}$ (adding 1 on both side)

$\Rightarrow \dfrac{\alpha_2^2}{\beta_2^2} = \dfrac{\alpha_1^2 + \alpha_2^2}{\beta_1^2 + \beta_2^2} = \dfrac{(\alpha_1 + \alpha_2)^2 - 2\alpha_1\alpha_2}{(\beta_1 + \beta_2)^2 - 2\beta_1\beta_2}$

Now, putting these values, we get,

$\dfrac{cp}{ar} = \dfrac{b^2/a^2 - 2(c/a)}{q^2/p^2 - 2(r/p)} = \dfrac{(b^2 - 2ac)p^2}{(q^2 - 2pr)a^2}$

$\Rightarrow \dfrac{c}{r} = \dfrac{pb^2 - 2acp}{q^2a - 2apr}$

$\Rightarrow b^2rp - 2acpr = q^2ac - pr2ac$ i.e. $b^2pr = q^2ac$

20) Ans: **B)** $-3 \le x \le \dfrac{3}{2}$

Sol: Given, $2x^2 + 3x - 9 \le 0 \Rightarrow 2x^2 + 6x - 3x - 9 \le 0$

$\Rightarrow 2x(x+3) - 3(x+3) \le 0$ $\Rightarrow (2x - 3)(x + 3)$

i.e. $-3 \le x \le 3/2$

21) Ans: **B)** 1, -8

Sol: $x^{2/3} + x^{1/3} - 2 = 0$ $\Rightarrow (x^{1/3})^2 + 1(x^{1/3}) - 2$

Let $a = x^{1/3}$,

$\therefore a^2 + a - 2 = 0$ $\Rightarrow a = 1, -2$

$\therefore x = 1, -8$ (by $a = x^{1/3}$)

22) Ans: **B)** b = - c

Sol: Let α, β be the roots of the given equation $ax^2 + bx + c = 0$

$\therefore \alpha + \beta = -\dfrac{b}{a};\ \alpha\beta = \dfrac{c}{a}$

New roots of equation $2x^2 + 8x + 2 = 0$ are $\alpha - 1,\ \beta - 1$

Their sum $= \alpha + \beta - 2 = -\dfrac{b}{a} - 2 = -\dfrac{8}{2} = -4 \Rightarrow \dfrac{b}{a} = 2$

Their product

$= (\alpha - 1)(\beta - 1) = \alpha\beta - (\alpha + \beta) + 1 = \dfrac{c}{a} + \dfrac{b}{a} + 1 = 1$

[$\therefore$ New equation is $2x^2 + 8x + 2 = 0$]

$\Rightarrow c + b = 0 \Rightarrow b = -c.$

23) Ans: **D)** 0

Sol: Let $\tan x = t \in R$

$f(x) = t^2 - 3pt + 2p^2 + 1,\ t \in R$

Graph should not lie below x - axis.

$\therefore D \le 0$ $\Rightarrow 9p^2 - 4(2p^2 + 1) \le 0 \Rightarrow p \in [-2, 2]$

24) Ans: **A)** $x^2 - 2bx + 4 = 0$

Sol: As α, β are roots of

$x^2 + bx + 1 = 0,\ \alpha + \beta = -b,\ \alpha\beta = 1$

We have

$\left(-\alpha - \dfrac{1}{\beta}\right) + \left(-\beta - \dfrac{1}{\alpha}\right) = -(\alpha + \beta) - \left(\dfrac{1}{\beta} + \dfrac{1}{\alpha}\right)$

$= -(\alpha + \beta) - \dfrac{(\alpha + \beta)}{\alpha\beta} = b + b = 2b$

and $\left(-\alpha - \dfrac{1}{\beta}\right)\left(-\beta - \dfrac{1}{\alpha}\right) = \alpha\beta + 2 + \dfrac{1}{\alpha\beta}$ $= 1 + 2 + 1 = 4$

Therefore, the equation whose roots are

$-\alpha - \dfrac{1}{\beta}$ and $-\beta - \dfrac{1}{\alpha}$ is $x^2 - x(2b) + 4 = 0.$

25) Ans: **D)** $-\dfrac{9}{4}$

Sol: As, a and b are the roots of the equation $x^2 + ax + b = 0$

Therefore, a + b = -a and ab = b

Now, $ab = b \Rightarrow (a - 1)b = 0 \Rightarrow a = 1$

$(\because b \ne 0)$

Putting $a = 1$ in a + b = - a, we get b = - 2

$\Rightarrow x^2 + ax + b = x^2 + x - 2 \Rightarrow (x + 1/2)^2 - 1/4 - 2$

$= (x + 1/2)^2 - 9/4$, which has minimum value $-9/4$

26) Ans: **24** Sol: Let x,y be the digits of the number in ten's place and unit's place respectively.

Given that $10x + y = 4(x + y) \to (1)$ and

$10x + y = 3xy \to (2)$

$(1) \Rightarrow 10x + y = 4x + 4y \Rightarrow 6x = 3y \Rightarrow y = 2x$

$(2) \Rightarrow 10x + 2x = 3x(2x) \Rightarrow 12x = 6x^2$

$\Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 2 (\because x \ne 0)$

$x = 2 \Rightarrow y = 4.$

Therefore, the required number is 24.

27) Ans: **9** Sol: Arithmetic mean $\ge$ Geometric mean $\Rightarrow \dfrac{a + b + c}{3} \ge (abc)^{1/3}$,

$\dfrac{1/a + 1/b + 1/c}{3} \ge \left(\dfrac{1}{a}.\dfrac{1}{b}.\dfrac{1}{c}\right)^{1/3}$

$$\Rightarrow a+b+c \geq 3(abc)^{1/3},\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{(abc)^{1/3}}$$

$$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3.(abc)^{1/3}.\frac{3}{(abc)^{1/3}}$$

$$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

28) Ans: **5** Sol: Let x be the number of points marked on the plane.
The number of line segments formed by connecting pairwise $= {}^{x}C_2$
Give that ${}^{x}C_2 = 10 \Rightarrow x(x-1)/2 = 10 \Rightarrow x^2 - x = 20$
$\Rightarrow x^2 - x - 20 = 0 \Rightarrow (x-5)(x+4) = 0 \Rightarrow x = 5$ [x cannot be negative]

29) Ans: **2** Sol: **Case I**: $x^2+4x+3>0$
$\Rightarrow x^2+4x+3+2x+5=0 \ \Rightarrow x^2+6x+8=0$
$\Rightarrow (x+2)(x+4) \Rightarrow x=-2,-4$
$x=-2$ is not satisfying the condition $x^2+4x+3>0$, thus $x=-4$ is the only solution of the given equation.
Case II: $x^2+4x+3<0$
$\Rightarrow -(x^2+4x+3)+2x+5=0$
$\Rightarrow -x^2-2x+2=0 \ \Rightarrow x^2+2x-2=0$
$\Rightarrow (x+1+\sqrt{3})(x+1-\sqrt{3})=0$
$\Rightarrow x=-1+\sqrt{3},-1-\sqrt{3}$
Thus, $x=-(1+\sqrt{3})$ satisfies the given condition $x^2+4x+3<0$, whereas $x=-1+\sqrt{3}$ is not satisfying the condition.
Therefore, number of real solutions are two.

30) Ans: **3** Sol: α, β are the roots of $x^2-6x-2=0$
$\Rightarrow \alpha+\beta=6, \alpha\beta=-2..$

$$\Rightarrow \frac{a_{10}-2a_8}{2a_9} = \frac{\alpha^{10}-\beta^{10}-2(\alpha^8-\beta^8)}{2(\alpha^9-\beta^9)}$$

$$= \frac{\alpha^{10}-\beta^{10}+\alpha\beta(\alpha^8-\beta^8)}{2(\alpha^9-\beta^9)} \Rightarrow \frac{\alpha(\alpha^9-\beta^9)+\beta(\alpha^9-\beta^9)}{2}$$

$$= \frac{\alpha+\beta}{2} = \frac{6}{2} = 3$$