# Chapter 6

# Constrained Optimization Algorithmic Methods

**In this chapter we discuss how we can solve certain constrained problems by aplying the algorithmic methods developed for unconstrained problems.**

## 6.1 Exterior Penalty Methods

**In exterior penalty methods we replace the constrained problem by a sequence of unconstrained problems. The solutions of the unconstrained problems generallyviolate the constraints of the original problem while giving the penalized functional a lower value than the constrained functional. The idea is that the sequence of solutions of the unconstrained problems converge to the solution of the constrained problem.**

**Consider the constrained problem**

$$\begin{aligned} &\min\ f(x) \\ &\textbf{subject to} \\ &g(x) \le 0 \qquad\qquad (\mathcal{P}_1) \\ &h(x) = 0 \\ &x \in X \end{aligned}$$

**where** $X$ **is a nonempty open subset of** $\mathbb{R}^n$**,** $g(x) = (g_1(x), \cdots, g_m(x))^T$**,** $h(x) = (h_1(x), \cdots, h_k(x))^T$**.**

**Define** $\phi, \psi : \mathbb{R} \longrightarrow \mathbb{R}$ **as follows:**

$$\begin{cases} \phi(t) = 0 & \textbf{if } t \le 0 \\ \phi(t) > 0 & \textbf{if } t > 0 \end{cases}$$

$$\begin{cases} \psi(t) = 0 & \textbf{if } t = 0 \\ \psi(t) > 0 & \textbf{if } t \ne 0 \end{cases}$$

**Correponding to** $(\mathcal{P}_1)$ **and** $\mu \ge 0$ **we consider the unconstrained penalized problem** $(\mathcal{P}_{1\mu})$

$$(\mathcal{P}_{1\mu}) = \min\{f(x) + \mu\, \alpha(x)\}$$

**where**

$$\alpha(x) = \sum_{i=1}^{m} \phi\left(g_i(x)\right) + \sum_{i=1}^{k} \psi\left(h_i(x)\right)$$

**We make the standing assumptions that** $f, g, h, \phi$ **and** $\psi$ **are continuous.**

**We solve** $(\mathcal{P}_{1\mu})$ **for increasing values of** $\mu \ge 0$ **and consider the convergence property of** $\{x_\mu\}_{\mu \ge 0}$ **as** $\mu \nearrow \infty$**, where** $x_\mu$ **is a solution of** $(\mathcal{P}_{1\mu})$**. The function** $f(x) + \mu\, \alpha(x)$ **is called a penalized functional and** $\phi\left(g_i(x)\right)$ **and** $\psi\left(h_i(x)\right)$ **are called penalty terms.**

**Lemma 6.1.1** ***Let*** $\theta(\mu) = \inf \{f(x) + \mu\, \alpha(x) \mid x \in X\}$ ***and*** $x_\mu \in X$ ***be such that*** $\theta(x_\mu) = f(x_\mu) + \mu\, \alpha(x_\mu)$***. Then,***

**(i)** $\inf \{f(x) \mid x \in X, g(x) \leq 0, h(x) = 0\} \geq \sup_{\mu \geq 0}\ \theta(\mu)$

**(ii)** ***The function*** $\mu \mapsto f(x_\mu)$ ***and*** $\mu \mapsto \theta(\mu)$ ***are nondecreasing while the function*** $\mu \mapsto \alpha(x_\mu)$ ***is nonincreasing.***

*Proof* **: For any** $x \in X$ **such that** $g(x) \leq 0$**,** $h(x) = 0$**, and any** $\mu \geq 0$ **we have**

$$f(x) = f(x) + \mu\, \alpha(x) \geq \inf\{f(x) + \mu\, \alpha(x) | x \in X\} = \theta(\mu)$$

**Thus,**

$$\begin{aligned} \theta(\mu) &\leq f(x), \quad \mu \geq 0 \\ \sup\ \theta(\mu) &\leq f(x) \\ \mu &\geq 0 \end{aligned}$$

**Thus,**

$$\sup_{\mu \geq 0}\ \theta(\mu) \leq \inf\ \{f(x) | x \in X, g(x) \leq 0, h(x) = 0\}$$

**proving (i).**

**Next, we suppose** $0 \leq \mu < \lambda$**, and**

$$\theta(\lambda) = f(x_\lambda) + \lambda\, \alpha(x_\lambda) = \min\ \{f(x) + \lambda\, \alpha(x) | x \in X\} \tag{6.1}$$

$$\theta(\mu) = f(x_\mu) + \mu\, \alpha(x_\mu) = \min\ \{f(x) + \mu\, \alpha(x) | x \in X\} \tag{6.2}$$

**We have**

$$\theta(\mu) = f(x_\mu) + \mu\, \alpha(x_\mu) \leq f(x_\lambda) + \mu\, \alpha(x_\lambda) \leq f(x_\lambda) + \lambda\, \alpha(x_\lambda) = \theta(\lambda)$$

**Thus,** $\mu \mapsto \theta(\mu)$ **is nondecreasing.**

$$f(x_\lambda) + \lambda\, \alpha(x_\lambda) \leq f(x_\mu) + \lambda\, \alpha(x_\mu) \tag{6.3}$$

$$f(x_\mu) + \mu\, \alpha(x_\lambda) \leq f(x_\lambda) + \mu\, \alpha(x_\lambda) \tag{6.4}$$

$$\tag{6.5}$$

**From (6.3) we have**

$$\lambda\, [\alpha(x_\lambda) - \alpha(x_\mu)] \leq f(x_\mu) - f(x_\lambda) \tag{6.6}$$

**From (6.4) we have**

$$f(x_\mu) - f(x_\lambda) \leq \mu\ [\alpha(x_\lambda) - \alpha(x_\mu)] \tag{6.7}$$

**From (6.7) we have**

$$-\mu\ [\alpha(x_\lambda) - \alpha(x_\mu)] \leq f(x_\lambda) - f(x_\mu) \tag{6.8}$$

**Adding inequalities (6.6) and (6.8) we have**

$$(\lambda - \mu)\ [\alpha(x_\lambda) - \alpha(x_\mu)] \leq 0 \tag{6.9}$$

**Since** $\lambda - \mu > 0$ **we have**

$$\begin{aligned} \alpha(x_\lambda) - \alpha(x_\mu) &\leq 0 \\ \alpha(x_\lambda) &\leq \alpha(x_\mu). \end{aligned}$$

**Thus,** $\mu \mapsto \alpha(x_\mu)$ **is nonincreasing.**

**From (6.4) we have**

$$f(x_\mu) - f(x_\lambda) \leq \mu\ [\alpha(x_\lambda) - \alpha(x_\mu)] \tag{6.10}$$

**Since $\mu \geq 0$ and $\alpha(x_\lambda) - \alpha(x_\mu) \leq 0$, (6.10) gives**

$$f(x_\mu) - f(x_\lambda) \leq 0.$$

**Thus,**

$$f(x_\mu) \leq f(x_\lambda)$$

**showing that $\mu \mapsto \alpha(x_\mu)$ is nondecreasing.** ◻

**Theorem 6.1.1** ***Suppose that the feasible set for $(\mathcal{P}_1)$ is nonempty. Assume that there exists $x_\mu \in X$ such that $\theta(\mu) = f(x_\mu) + \mu\ \alpha(x_\mu)$ for each $\mu \geq 0$. Assume that $\{x_\mu | \mu \geq 0\}$ is a compact subset of $X$. Then, there exists $x^* \in X$ such that***

$$\begin{aligned} & \inf\{f(x) | x \in X, g(x) \leq 0, h(x) = 0\} \\ = & \min\{f(x) | x \in X, g(x) \leq 0, h(x) = 0\} \\ = & f(x^*) \\ = & \sup_{\mu \geq 0} \theta(\mu) = \lim_{\mu \to \infty} \theta(\mu) \end{aligned}$$

*Proof* **: Let $\Lambda = \sup_{\mu \geq 0}\ \theta(\mu)$. By Lemma 4.1.1 $\theta$ is a nondecreasing function of $\mu \geq 0$. Thus,**

$$\lim_{\mu \to \infty}\ \theta(\mu) = \Lambda.$$

**We must show that $\alpha(x_\mu) \to \infty$ as $\mu \to \infty$. Suppose $\inf_{\mu \geq 0} \alpha(x_\mu) = \delta > 0$. Then,**

$$\begin{aligned} & \inf\{f(x) | x \in X, g(x) \leq 0, h(x) = 0\} \\ \geq & \theta(\mu) = f(x_\mu) + \mu\ \alpha(x_\mu) \\ \geq & f(x_\mu) + \mu\ \delta \end{aligned}$$

**Since $\mu \mapsto f(x_\mu)$ is nondecreasing $f(x_\mu) + \mu\ \delta \mapsto \infty$. Thus, since**

$$\inf\{f(x) | x \in X, g(x) \leq 0, h(x) = 0\} < \infty$$

**due to the fact that the feasible set is nonempty it follows that $\inf_{\mu \geq 0}\ \alpha(x_\mu) = 0$. Since $\mu \mapsto \alpha(x_\mu)$ is nonincreasing, we must conclude $\lim_{\mu \to \infty} \alpha(x_\mu) = 0$.**

**Since $\{x_\mu | \mu \geq 0\}$ is a compact subset of $X$ there exist $\mu_1 < \mu_2 < \cdots < \mu_n < \cdots$ and $x^* \in X$ such that**

$$\lim_{n \to \infty} x_{\mu_n} = x^*.$$

**By continuity of $\alpha$ we have $\alpha(x^*) = 0$. Thus, $x^*$ is feasible. Since $f, g, h, \phi, \psi$ are continuous**

$$\lim_{n \to \infty} \theta(\mu_n) = \Lambda,$$

**and**

$$\inf\{f(x) | x \in X, g(x) \leq 0, h(x) = 0\} \geq \sup_{\mu \geq 0} \theta(\mu) = \lim_{n \to \infty} \theta(\mu_n) = f(x^*).$$

**Since $x^*$ is feasible it follows that**

$$\inf\{f(x) | x \in X, g(x) \leq 0, h(x) = 0\} = f(x^*).$$

◻

**Remark 6.1.1** ***Trying to find $x_\mu$ can be computationally difficult. To illustrate this point we assume that the auxiliary function $f_\mu(x)$ is twice continuously differentiable. In the case we have only equality constraints***

$$f_\mu^{''} = f^{''}(x) + \sum_{i=1}^{k} \mu\ \psi^{'}\ (h_i(x))\ \ h_i^{''}(x) + \mu\ \sum_{i=1}^{k} \mu\ \psi^{''}\ (h_i(x))\ \ h_i^{'}(x)\ h_i^T(x).$$

***The Hessian can be ill conditioned as $\mu \to \infty$.***

**Example 6.1.1**

**Consider the problem**

$$\begin{aligned}&\min\ f(x) = x_1^2 + x_2^2\\ &\textbf{subject to}\\ &h(x) = x_1 + x_2 - 1 = 0\end{aligned}$$

**Set**

$$f_\mu(x) = f(x) + \mu\ (h(x))^2$$

**Then,**

$$\begin{aligned}\frac{\partial}{\partial\, x_1} f_\mu(x) &= 2\,(1+\mu)\,x_1 + 2\,\mu\,(x_2 - 1)\\ \frac{\partial}{\partial\, x_2} f_\mu(x) &= 2\,(1+\mu)\,x_2 + 2\,\mu\,(x_1 - 1)\\ \frac{\partial^2}{\partial\, x_1^2} f_\mu(x) &= \frac{\partial^2}{\partial\, x_2^2} f_\mu(x) = 2\,(1+\mu)\\ \frac{\partial^2}{\partial\, x_2\,\partial\, x_1} f_\mu(x) &= 2\,\mu\end{aligned}$$

$$f_\mu^{''}(x) = 2\begin{pmatrix} 1+\mu & \mu \\ \mu & 1+\mu \end{pmatrix}$$

**Thus, the hessian of $f_\mu$ is $2\,(1+\mu) > 0$. We note the system**

$$\begin{aligned}\frac{\partial}{\partial\, x_1} f_\mu(x) &= 0\\ \frac{\partial}{\partial\, x_1} f_\mu(x) &= 0\end{aligned}$$

**has a solution**

$$x_1(\mu) = x_2(\mu) = \frac{\mu}{1+2\,\mu}.$$

**We note $x_1(\mu)$ and $x_2(\mu)$ converge to $\frac{1}{2}$ as $\mu \to \infty$.**

$$h\,(x_1(\mu), x_2(\mu)) = -\frac{1}{1+2\,\mu} < 0$$

**Thus, $x_1(\mu)$ and $x_2(\mu)$ are infeasible and**

$$h^2\,(x_1(\mu), x_2(\mu)) = \frac{1}{(1+2\,\mu)^2} \longrightarrow 0 \quad \textbf{as} \quad \mu \longrightarrow \infty.$$

**Next,**

$$f_\mu(x_\mu) = \frac{\mu}{1+2\,\mu} < \frac{1}{2}$$

**and**

$$\lim_{\mu\to\infty} f_\mu(x_\mu) = \frac{1}{2}$$

**We note that $f_\mu(x_\mu)$ increases to $\frac{1}{2}$ as $\mu \to \infty$. We can verify that $x_1 = x_2 = \frac{1}{2}$ provides the solution to the problem.**

## 6.2 Methods of Feasible Directions

### 6.2.1 Zoutendijk's Method

**Consider the problem**

$$\begin{aligned}&\min f(x)\\&\textbf{subject to}\quad g(x) \leq 0\end{aligned}$$

**Algorithm**

**(i) Start with an initial feasible point $x_1$, and small numbers $\varepsilon_1, \varepsilon_2, \cdots$ for convergence criteria. Evaluate $f(x_1)$, $g_j(x_1)$, $\quad j = 1, \cdots, m$. Set the iteration number $i = 1$.**

**(ii) If $g_j(x_i) < 0, \quad j = 1, \cdots, m$ set the current search direction as $S_i = -\nabla f(x_i)$. Normalize $S_i$ in a suitable manner, and go to step (v). If at least one $g_j(x_i) = 0$, go to step (iii).**

**(iii) Find a feasible direction $S$ by solving the problem:**

$$\begin{aligned}&\min \alpha\\&\textbf{subject to}\\&S^T \nabla g_j(x_i) + \theta\, \alpha \leq 0, \quad j = 1, 2, \cdots, p\\&S^T \nabla f(x_i) + \alpha \leq 0\\&-1 \leq S_i \leq 1, \quad i = 1, 2, \cdots, m\end{aligned}$$

**where $S_i$ is the $i$th component of $S$, and where the first $p$ of the $m$ constraints are assumed to be active at the point $x_i$, The $\theta_j$'s can be taken to be 1.**

**(iv) If the value of $\alpha$ found in step (iii) is nearly equal to zero, i.e., $\alpha \leq \varepsilon_i$ terminate the iteration and set $x_{opt} = x_i$. If $\alpha > \varepsilon_i$ go to step (v) by taking $S_i = S$.**

**(v) Find a suitable step length $\lambda_i$ along the direction $S_i$ and obtain a new point $x_{i+1}$ as**

$$x_{i+1} = x_i + \lambda_i\, S_i$$

**(vi) Evaluate the objective function $f(x_{i+1})$.**

**(vii) Test for the convergence of the method. If**

$$\left| \frac{f(x_i) - f(x_{i+1}}{f(x_i)} \right| \leq \varepsilon_2 \quad \textbf{and} \quad \|x_i - x_{i+1}\| \leq \varepsilon_3$$

**terminate the iteration process taking $x_{opt} = x_{i+1}$. Otherwise, go to step (viii).**

**(viii) Set the new iteration number as $i = i + 1$, and repeat from step (ii) onwards.**

**Remark 6.2.1**

$$\begin{aligned}&\min -\alpha\\&\textbf{\textit{subject to}}\\&S^T \nabla g_j(x_i) + \theta_j\, \alpha \leq 0, \quad j = 1, 2, \cdots, p\\&S^T \nabla f(x_i) + \alpha \leq 0\\&-1 \leq S_i \leq 1, \quad i = 1, 2, \cdots, n\end{aligned}$$

***is a linear programming problem. To put it in standard form we may proceed as follows:***

$$\begin{aligned}
-1 \leq S_i \leq 1 \quad &\Longrightarrow \quad 0 \leq S_i + 1 \leq 2 \\
S^T \nabla g_j(x_i) \quad &= \quad (S_1, \cdots, S_n) \cdot \begin{pmatrix} \partial_1 g_j(x_i) \\ \partial_2 g_j(x_i) \\ \vdots \\ \partial_n g_j(x_i) \end{pmatrix} \\
&= \quad (S_1 + 1, \cdots, S_n + 1) \cdot \begin{pmatrix} \partial_1 g_j(x_i) \\ \partial_2 g_j(x_i) \\ \vdots \\ \partial_n g_j(x_i) \end{pmatrix} - (1, \cdots, 1) \cdot \begin{pmatrix} \partial_1 g_j(x_i) \\ \partial_2 g_j(x_i) \\ \vdots \\ \partial_n g_j(x_i) \end{pmatrix}
\end{aligned}$$

**Thus, if we set $t_i = S_{i+1}$, then $0 \leq t_i \leq 2$, and**

$$S^T \nabla g_j(x_i) = t_i \, \partial_1 g_j(x_1) + \cdots + t_n \, \partial_n g_j(x_i) - \sum_{k=1}^{n} \partial \, g_j(x_i).$$

**Thus, the above linear programming problem can be written as**

$$\begin{aligned}
&\min -\alpha \\
&\textbf{subject to} \\
&\sum_{k=1}^{n} t_k \, \partial_k \, g_j(x_i) + \theta_j \, \alpha \leq \sum_{k=1}^{n} \partial_k \, g_j(x_i) \\
&\sum_{k=1}^{n} t_k \, \partial_k \, f(x_i) + \alpha \leq \sum_{k=1}^{n} \partial_k \, f(x_i), \qquad 0 \leq t_i \leq 2
\end{aligned}$$

**Finally,**

$$\begin{aligned}
&\min -\alpha \\
&\textbf{subject to} \\
&\sum_{k=1}^{n} t_k \, \partial_k \, g_j(x_i) + \theta_j \, \alpha + y_j = \sum_{k=1}^{n} \partial_k \, g_j(x_i) \\
&\sum_{k=1}^{n} t_k \, \partial_k \, f(x_i) + \alpha + y_{p+1} = \sum_{k=1}^{n} \partial_k \, f(x_i) \\
&t_1 + y_{p+2} = 2 \\
&t_2 + y_{p+3} = 2 \\
&\vdots \\
&t_1 + y_{p+n+1} = 2 \\
&t_1 \geq 0 \\
&t_2 \geq 0 \\
&\vdots \\
&t_n \geq 0 \\
&y_1 \geq 0 \\
&\vdots \\
&y_{p+n+1} \geq 0 \\
&\alpha \geq 0
\end{aligned}$$

## 6.3 Barrier Methods

**Barrier methods generate a sequence of strictly feasible iterates that converge to a solution of the problem from the interior of the feasible region. For this reason they are called interior point methods. They are not appropriate for equality constraints.**

**Consider the nonlinear inequality constrained problem**

$$(P_2) \qquad \begin{cases} \min f(x) \\ \textbf{subject to} \\ g_i(x) \geq 0, \quad i = 1, \cdots, m \end{cases}$$

**Assume that** $f$**,** $g_i, i = 1, \cdots, m$ **are continuous, and the set**

$$\{x \in \mathbb{R}^n \mid g_i(x) > 0, i = 1, \cdots, m\} \text{ is not empty}.$$

**Let**

$$\varphi_1(x) = -\sum_{i=1}^{m} \ln\left(g_i(x)\right).$$

**Note that if** $g_i(x) \longrightarrow 0^+$ **then** $\varphi_1(x) \longrightarrow \infty$**.**

**Next, let**

$$\varphi_2(x) = \sum_{i=1}^{m} \frac{1}{g_i(x)}$$

**If** $g_i(x) \longrightarrow 0^+$ **then** $\varphi_2(x) \longrightarrow \infty$**.**

**Let**

$$S^\circ = \{x \mid g_i(x) > 0, i = 1, \cdots, m\}$$

**Assume that** $S^\circ \neq \emptyset$**.**

$$S^{\bar{circ}} = \{x \mid g_i(x) \geq 0, \quad i = 1, \cdots, m\}.$$

**Then, to solve the problem** $(P_2)$ **we consider barrier functions. For example, we consider**

$$f_\mu(x) = f(x) + \mu\, \varphi_1(x), \quad \mu > 0$$

**or**

$$f_\mu(x) = f(x) + \mu\, \varphi_2(x)$$

**for** $\mu$ **values tending to zero. These two barrier functions are widely used. The parameter** $\mu$ **is called the barrier parameter of the function** $f_\mu$**.**

**The barrier method solves a sequence of unconstrained problems**

$$\min_x f_{\mu_k}(x)$$

**where** $\mu_1 > \mu_2 > \mu_3 > \cdots \searrow 0$**.**

**Theorem 6.3.1** ***Consider the problem***

$$(P_2) \qquad \begin{cases} \min f(x) \\ \textbf{\textit{subject to}} \\ g_i(x) \geq 0, \quad i = 1, \cdots, m \end{cases}$$

***Let***

$$S^\circ = \{x | g_i(x) > 0, i = 1, \cdots, m\}$$

$$S = \{x | g_i(x) \geq 0, i = 1, \cdots, m\}$$

***Assume***

**(i)** $f, g_i, i = 1, \cdots, m$ ***are continuous on*** $\mathbb{R}^n$***.***

**(ii)** ***The set*** $\{x : x \in S, f(x) \leq \alpha\}$ ***is bounded for every*** $\alpha > 0$***.***

**(iii)** $S^\circ \neq \emptyset$

**(iv)** $\bar{S^\circ} = S$

***Suppose*** $x_\mu \in S^\circ$ ***is such that***

$$f_\mu\left(x(\mu)\right) = \min\left\{f(x) + \mu\, \varphi(x)\right\}$$

***Then,***

**(a)** $f_{\mu_{k+1}}(x_{\mu_{k+1}}) \leq f_{\mu_k}(x_{\mu_k}$

**(b)** $\varphi(x_{\mu_{k+1}}) > \varphi(x_{\mu_k})$

**(c)** $\{x_{\mu_k}\}$ ***has a convergent subsequence*** $\{\mu_{k_\ell}\}$

**(c)** ***If*** $\{\mu_{k_\ell}\} \longrightarrow x^*$ ***then*** $x^*$ ***is a solution of problem*** *($P_2$)*

*Proof* **:**

$$f(x_{\mu_{n+1}}) + \mu_{n+1}\varphi(x_{\mu_{n+1}}) \leq f(x_{\mu_n}) + \mu_{n+1}\varphi(x_{\mu_n}) \tag{6.11}$$

$$f(x_{\mu_n}) + \mu_n\varphi(x_{\mu_n}) \leq f(x_{\mu_{n+1}}) + \mu_n\varphi(x_{\mu_{n+1}}) \tag{6.12}$$

**Multiply inequality (6.11) by** $\mu_n$ **and inequality (??) by** $\mu_{n+1}$ **to obtain**

$$\mu_n\ f(x_{\mu_{n+1}}) + \mu_n\ \mu_{n+1}\varphi(x_{\mu_{n+1}}) \leq \mu_n\ f(x_{\mu_n}) + \mu_n\ \mu_{n+1}\varphi(x_{\mu_n}) \tag{6.13}$$

$$\mu_{n+1}\ f(x_{\mu_n}) + \mu_{n+1}\ \mu_n\varphi(x_{\mu_n}) \leq \mu_{n+1}\ f(x_{\mu_{n+1}}) + \mu_{n+1}\ \mu_n\varphi(x_{\mu_{n+1}}) \tag{6.14}$$

**Adding inequalities (6.13) ad (6.14) we obtain**

$$\mu_n\ f(x_{\mu_{n+1}})+\mu_{n+1}\ f(x_{\mu_n})+\mu_n\ \mu_{n+1}[\varphi(x_{\mu_{n+1}})+\varphi(x_{\mu_n})] \leq \mu_n\ f(x_{\mu_n})+\mu_{n+1}\ f(x_{\mu_{n+1}})+\mu_n\ \mu_{n+1}[\varphi(x_{\mu_n})+\varphi(x_{\mu_{n+1}})] \tag{6.15}$$

**Then, from (3.6) it follows that**

$$(\mu_{n+1} - \mu_n)\ f(x_{\mu_n}) \leq (\mu_{n+1} - \mu_n)\ f(x_{\mu_{n+1}}) \tag{6.16}$$

**Thus, since** $\mu_{n+1} - \mu_n < 0$**,**

$$f(x_{\mu_{n+1}}) \leq f(x_{\mu_n}) \tag{6.17}$$

**That is,**

$$\{f(x_{\mu_k})\}_{n=1}^{\infty}$$

**is a nonincreasing sequence.**

**From (6.12) we have**

$$\varphi(x_{\mu_n}) \leq \varphi(x_{\mu_{n+1}}) \tag{6.18}$$

**Thus, the sequence** $\{\varphi(x_{\mu_n})\}_{n=1}^{\infty}$ **is a nondecreasing sequence.**

**By assumption (ii)** $\{x_{\mu_n}\}$ **has a sbsequence** $\{x_{\mu_{n_\ell}}\}$ **which is convergent. Thus, there exist** $x^*$ **feasible such that**

$$\lim_{\ell\to\infty} x_{\mu_{n_\ell}} = x^*.$$

**since** $f$ **is continuous** $f(x_{\mu_{n_\ell}}) \longrightarrow f(x^*)$**. Next,**

$$\begin{aligned} f(x_{\mu_{k+1}}) + \mu_{k+1}\varphi(x_{\mu_{k+1}}) &\leq f(x_{\mu_k}) + \mu_{k+1}\varphi(x_{\mu_k}) \\ &\leq f(x_{\mu_k}) + \mu_k\varphi(x_{\mu_k}) \end{aligned}$$

**Thus, the sequence** $\{f(x_{\mu_{k+1}}) + \mu_{k+1}\varphi(x_{\mu_{k+1}})\}_{k=1}^{\infty}$ **is a nonincreasing sequence, and**

$$\lim_{\ell\to\infty}\left[f(x_{\mu_{k_\ell}}) + \mu_{k_\ell}\varphi(x_{\mu_{k_\ell}})\right] = f(x^*) + \lim_{\ell\to\infty}\mu_{k_\ell}\varphi(x_{\mu_{k_\ell}}) \geq f(x^*)$$

**Suppose** $\hat{x}$ **is a solution to** $(P_2)$**. Then, for every** $\varepsilon > 0$**, there exists** $x_\varepsilon$ **such that** $|\hat{x} - x_\varepsilon| < \varepsilon$ **and** $g_i(x_\varepsilon) > 0$**,** $i = 1, \cdots, m$**. Then, there exists** $k_0$ **such that for** $k \geq k_0$

$$f(x_{\mu_k}) + \mu_k \ \varphi(x_{\mu_k}) \leq f(x_\varepsilon) + \mu_k \ \varphi(x_\varepsilon) \tag{6.19}$$

**Thus,**

$$\begin{aligned} f(x^*) &\leq \lim_{\ell \to \infty} \left[ f(x_{\mu_{k_\ell}}) + \mu_{k_\ell} \varphi(x_{\mu_{k_\ell}}) \right] \\ &\leq \lim_{\ell \to \infty} \left[ f(x_\varepsilon) + \mu_{k_\ell} \varphi(x_\varepsilon) \right] \\ &= f(x_\varepsilon) \end{aligned}$$

**Thus,**

$$f(x^*) \leq f(\hat{x}).$$

**Since**

$$f(\hat{x}) \leq f(x^*),$$

**we have**

$$\begin{aligned} f(x^*) &\leq \lim_{\ell \to \infty} \left[ f(x_{\mu_{k_\ell}}) + \mu_{k_\ell} \varphi(x_{\mu_{k_\ell}}) \right] \\ &\leq f(\hat{x}) \\ &\leq f(x^*) \end{aligned}$$

**Thus,** $x^*$ **is a solution to** $(P_2)$ **and**

$$\lim_{\ell \to \infty} \mu_{k_\ell} \ \varphi(x_{\mu_{k_\ell}}) = 0.$$

□