Bayesian Central Limit Theorem Approximation

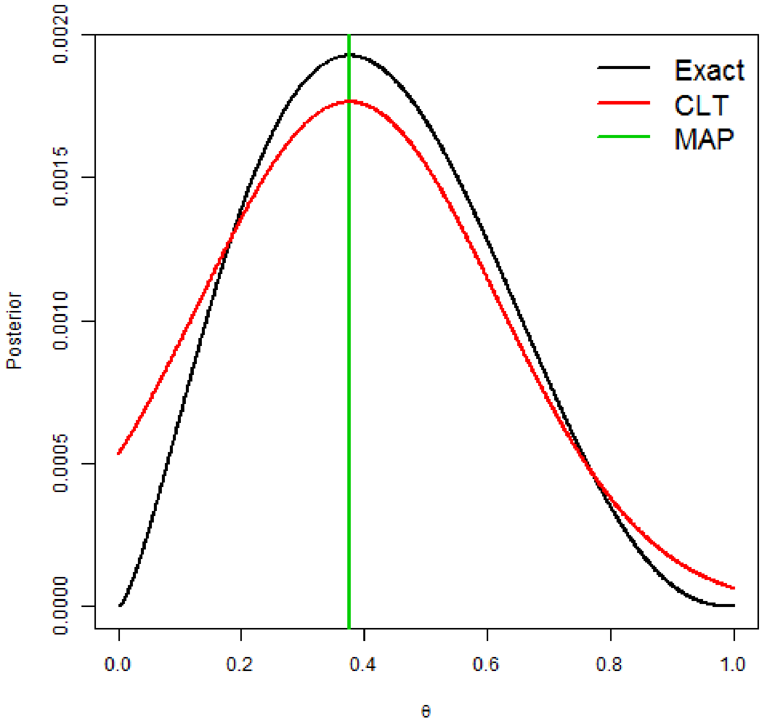
Chapter 3.1.3: Bayesian Central Limit Theorem (CLT)

The model is $Y|\theta \sim \text{Binomial}(n,\theta)$ and $\theta \sim \text{Beta}(½,½)$. The exact posterior is $\theta|Y \sim \text{Beta}(Y+½, n+½)$. Below we compare the exact posterior with a Gaussian approximation to the posterior for small, medium, and large datasets (as determined by n).

The Gaussian approximation is centered around the maximum a posteriori (MAP) estimate $\hat{\theta} = A/(A+B)$ where $A = Y - ½$ and $B = n - Y - ½$. This is found by taking the derivative of the log posterior with respect to θ, setting it to zero, and solving for θ. The posterior variance is approximated as $\frac{1}{\frac{A}{\hat{\theta}^2} + \frac{B}{(1-\hat{\theta})^2}}$. This is a function of the second derivative of the log posterior evaluated at $\hat{\theta}$.

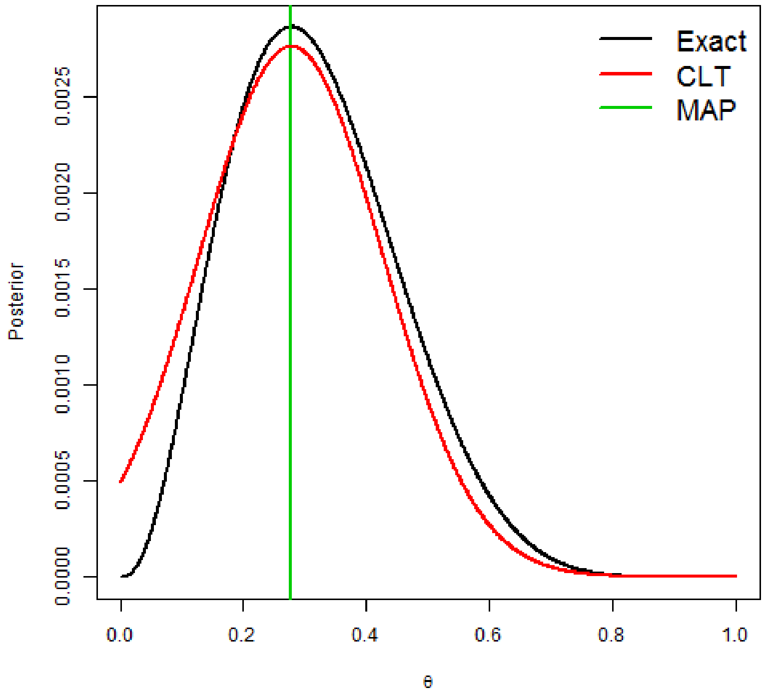
Approximation in a small sample case

```r
theta <- seq(0.001,0.999,.001) # Grid of thetas for plotting
Y     <- 2                     # The data
n     <- 5

# Compute the posterior mean and Fisher information matrix

A         <- Y-0.5
B         <- n-Y-0.5
theta_MAP <- A/(A+B)
Info      <- A/theta_MAP^2+B/(1-theta_MAP)^2

# Plot the true and approximate posteriors

post1 <- dbinom(Y,n,theta)*dbeta(theta,0.5,0.5)
post1 <- post1/sum(post1)
post2 <- dnorm(theta,theta_MAP,sqrt(1/Info))
post2 <- post2/sum(post2)

plot(theta,post1,type="l",lwd=2,
     xlab=expression(theta),ylab="Posterior")
abline(v=theta_MAP,col=3,lwd=2)
lines(theta,post2,col=2,lwd=2)
legend("topright",c("Exact","CLT","MAP"),bty="n",col=1:3,cex=1.5,lwd=2)
```



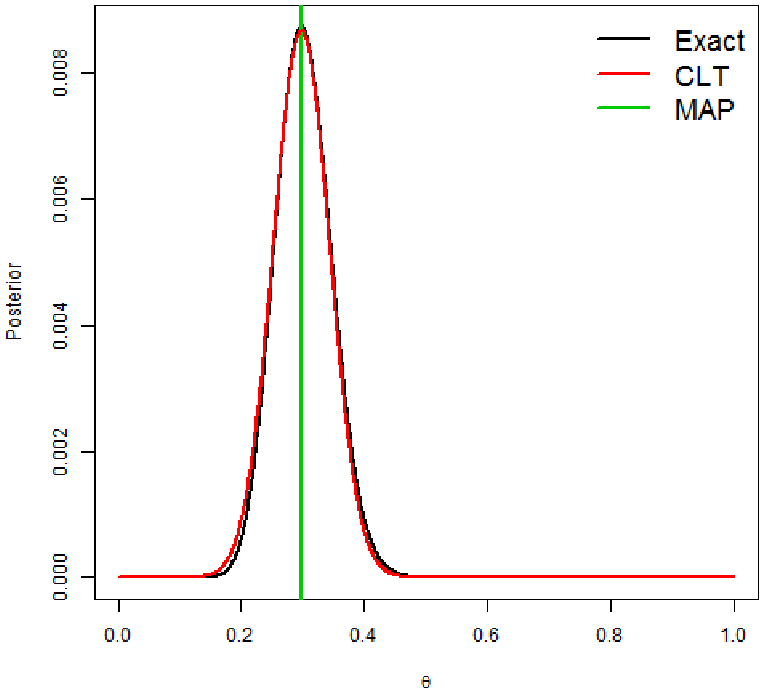
Approximation in a medium sample case

```
theta <- seq(0.001,0.999,.001) # Grid of thetas for plotting
Y     <- 3                     # The data
n     <- 10

# Compute the posterior mean and Fisher information matrix

A         <- Y-0.5
B         <- n-Y-0.5
theta_MAP <- A/(A+B)
Info      <- A/theta_MAP^2+B/(1-theta_MAP)^2

# Plot the true and approximate posteriors

post1 <- dbinom(Y,n,theta)*dbeta(theta,0.5,0.5)
post1 <- post1/sum(post1)
post2 <- dnorm(theta,theta_MAP,sqrt(1/Info))
post2 <- post2/sum(post2)

plot(theta,post1,type="l",lwd=2,
     xlab=expression(theta),ylab="Posterior")
abline(v=theta_MAP,col=3,lwd=2)
lines(theta,post2,col=2,lwd=2)
legend("topright",c("Exact","CLT","MAP"),bty="n",col=1:3,cex=1.5,lwd=2)
```



Approximation in a large sample case

```
theta <- seq(0.001,0.999,.001) # Grid of thetas for plotting
Y     <- 30                    # The data
n     <- 100

# Compute the posterior mean and Fisher information matrix

A         <- Y-0.5
B         <- n-Y-0.5
theta_MAP <- A/(A+B)
Info      <- A/theta_MAP^2+B/(1-theta_MAP)^2

# Plot the true and approximate posteriors

post1 <- dbinom(Y,n,theta)*dbeta(theta,0.5,0.5)
post1 <- post1/sum(post1)
post2 <- dnorm(theta,theta_MAP,sqrt(1/Info))
post2 <- post2/sum(post2)

plot(theta,post1,type="l",lwd=2,
     xlab=expression(theta),ylab="Posterior")
abline(v=theta_MAP,col=3,lwd=2)
lines(theta,post2,col=2,lwd=2)
legend("topright",c("Exact","CLT","MAP"),bty="n",col=1:3,cex=1.5,lwd=2)
```



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