ST 437/537: Applied Multivariate and Longitudinal Data Analysis

Comparing mean vectors from multiple independent populations: Part II

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Two-way MANOVA

Consider the extrusion of **[plastic film data] (data/T6-4.dat)** shown in Johnson and Wichern textbook, Table 6.4. The description of the data (as provided in the textbook) is given below:

“The optimum conditions for extruding plastic film have been examined using a technique called Evolutionary Operations. In the course of the study, three responses X_1-tear resistance, X_2-gloss, and X_3-opacity, were measured at two levels of the factors *rate of extrusion* and *amount of an additive*. The measurements were repeated five times at each combinations of the factor levels.”

```
data <- read.table("data/T6-4.dat", header = F)
colnames(data) <- c("rate", "additive" ,"Tear", "Gloss", "Opacity")
data
```

```
##    rate additive Tear Gloss Opacity
## 1     0        0  6.5   9.5     4.4
## 2     0        0  6.2   9.9     6.4
## 3     0        0  5.8   9.6     3.0
## 4     0        0  6.5   9.6     4.1
## 5     0        0  6.5   9.2     0.8
## 6     0        1  6.9   9.1     5.7
## 7     0        1  7.2  10.0     2.0
## 8     0        1  6.9   9.9     3.9
## 9     0        1  6.1   9.5     1.9
## 10    0        1  6.3   9.4     5.7
## 11    1        0  6.7   9.1     2.8
## 12    1        0  6.6   9.3     4.1
## 13    1        0  7.2   8.3     3.8
## 14    1        0  7.1   8.4     1.6
## 15    1        0  6.8   8.5     3.4
## 16    1        1  7.1   9.2     8.4
## 17    1        1  7.0   8.8     5.2
## 18    1        1  7.2   9.7     6.9
## 19    1        1  7.5  10.1     2.7
## 20    1        1  7.6   9.2     1.9
```

```
x = as.matrix(data[,3:5])
rate = as.factor(data[,1])
additive = as.factor(data[,2])
```

Our goal is to evaluate main effects of the two factors and their interaction on the three response variables.

Univariate two-way ANOVA

In general, suppose there are g **levels of factor 1** and b **levels of factor 2**. We observe n independent observations for each of the gb combinations of factor levels

Denote the rth observation at level ℓ of factor 1 and level k of factor 2 by $X_{\ell kr}$

Population mean at level ℓ of factor 1 and level k of factor 2 is denoted by $\mu_{\ell k}$

We consider the following decomposition:

$$\underbrace{\mu_{\ell k}}_{\text{Mean of }(\ell,k)\text{-th group}} = \underbrace{\mu}_{\text{Overall mean}} + \underbrace{\tau_\ell}_{\text{effect of factor 1}} + \underbrace{\beta_k}_{\text{effect of factor 2}} + \underbrace{\gamma_{\ell k}}_{\text{interaction effect}},$$

where μ is the overall mean, τ_ℓ is the fixed effect of factor 1, β_k is the fixed effect of factor 2, and $\gamma_{\ell k}$ is the interaction between the two factors.

Typically We put constraints

$$\sum_{\ell=1}^{g} \tau_\ell = \sum_{k=1}^{b} \beta_k = \sum_{\ell=1}^{g} \gamma_{\ell k} = \sum_{k=1}^{b} \gamma_{\ell k} = 0.$$

The sum of squares for the factor effects and their interaction are shown below.

Source of variation	**Sum of squares (SS)**	**Degrees of freedom**
Factor 1	SSA = $\sum_{\ell=1}^{g} bn(\bar{x}_{\ell\cdot} - \bar{x})^2$	$g-1$
Factor 2	SSB = $\sum_{k=1}^{b} gn(\bar{x}_{\cdot k} - \bar{x})^2$	$b-1$
Interaction	SSAB = $\sum_{\ell=1}^{g}\sum_{k=1}^{b} n(x_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})^2$	$(g-1)(b-1)$
Residual	SSE = $\sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x}_{\ell k})^2$	$gb(n-1)$

The test statistics to test for main effect of factor 1; of factor 2; interaction between the two factors are shown below:

$$F_A = \frac{\text{SSA}/(g-1)}{\text{SSE}/\{gb(n-1)\}}, \quad F_B = \frac{\text{SSB}/(b-1)}{\text{SSE}/\{gb(n-1)\}}, \quad F_{AB} = \frac{\text{SSAB}/\{(g-1)(b-1)\}}{\text{SSE}/\{gb(n-1)\}}$$

We reject H_0 (no effect) if the corresponding test statistic is larger than an appropriate F critical value. Typically, we **test for interaction first**; if we conclude that there is no interaction effect, only then we test for main effects of each factor.

Let us consider only the `tear resistance` variable, and perform two-way anova.

```
tear <- x[, 1]
res <- lm(tear ~ rate*additive)
out <- anova( res )
```

The formula `tear ~ rate*additive` specifies that the response is `tear` and `rate` and `additive` are the covariates (factors). The term `rate*additive` specifies that the model should include the main effect i=of `rate`, main effect of `additive` and their interaction effect. Another way to specify the same model is `rate + additive + rate:additive`, where `rate:additive` specifies just the interaction term.

```
out
```

```
## Analysis of Variance Table
##
## Response: tear
##               Df Sum Sq Mean Sq F value   Pr(>F)
## rate           1 1.7405 1.74050 15.7868 0.001092 **
## additive       1 0.7605 0.76050  6.8980 0.018330 *
## rate:additive  1 0.0005 0.00050  0.0045 0.947143
## Residuals     16 1.7640 0.11025
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We start by looking at the interaction term: the p-value indicates that there is no interaction effect. Thus we are free to examine the main effects of the factors. Both factor-effects have significant (< 0.05) p-values, indicating that the response mean differs among the various groups.

Since there is no significant interaction effect, we might want to refit the model with only the main effects:

```
res.add <- lm(tear ~ rate + additive)
out.add <- anova( res.add )
out.add
```

```
## Analysis of Variance Table
##
## Response: tear
##           Df Sum Sq Mean Sq F value    Pr(>F)
## rate       1 1.7405 1.74050  16.769 0.0007549 ***
## additive   1 0.7605 0.76050   7.327 0.0149597 *
## Residuals 17 1.7645 0.10379
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Both factor effects have significant (< 0.05) p-values, indicating that the response mean differs among the various groups.

Two-way MANOVA

Two-way MANOVA proceeds in a similar way as the univariate case, however, we replace the sum of squares by the corresponding cross-product matrices. Each factor effect and their interaction can be tested using Wilks lambda statistic.

In our data set, we perform a two-way MANOVA as follows.

```
# Perform  two-way MANOVA with interaction
res <- lm(x ~ rate * additive)
fit <- manova(res)
summary(fit, test="Wilks")
```

```
##               Df   Wilks approx F num Df den Df   Pr(>F)
## rate           1 0.38186   7.5543      3     14 0.003034 **
## additive       1 0.52303   4.2556      3     14 0.024745 *
## rate:additive  1 0.77711   1.3385      3     14 0.301782
## Residuals     16
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We reach very similar conclussions as belore: there is no evidence of an intercation effect, but main effects of both factors are significant.

The `Manova()` function in the `car` library gives more detailed output.

```
library(car)
```

```
## Loading required package: carData
```

```
res <- lm(x ~ rate * additive)
summary( Manova( res ) )
```

```
##
## Type II MANOVA Tests:
##
## Sum of squares and products for error:
##           Tear  Gloss Opacity
## Tear     1.764  0.020  -3.070
## Gloss    0.020  2.628  -0.552
## Opacity -3.070 -0.552  64.924
##
## ------------------------------------------
##
## Term: rate
##
## Sum of squares and products for the hypothesis:
##            Tear   Gloss Opacity
## Tear     1.7405 -1.5045  0.8555
## Gloss   -1.5045  1.3005 -0.7395
## Opacity  0.8555 -0.7395  0.4205
##
## Multivariate Tests: rate
##                  Df test stat approx F num Df den Df   Pr(>F)
## Pillai            1 0.6181416 7.554269      3     14 0.003034 **
## Wilks             1 0.3818584 7.554269      3     14 0.003034 **
## Hotelling-Lawley  1 1.6187719 7.554269      3     14 0.003034 **
## Roy               1 1.6187719 7.554269      3     14 0.003034 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## ------------------------------------------
##
## Term: additive
##
## Sum of squares and products for the hypothesis:
##           Tear  Gloss Opacity
## Tear    0.7605 0.6825  1.9305
## Gloss   0.6825 0.6125  1.7325
## Opacity 1.9305 1.7325  4.9005
##
## Multivariate Tests: additive
##                  Df test stat approx F num Df den Df   Pr(>F)
## Pillai            1 0.4769651 4.255619      3     14 0.024745 *
## Wilks             1 0.5230349 4.255619      3     14 0.024745 *
## Hotelling-Lawley  1 0.9119183 4.255619      3     14 0.024745 *
## Roy               1 0.9119183 4.255619      3     14 0.024745 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## ------------------------------------------
##
## Term: rate:additive
##
## Sum of squares and products for the hypothesis:
##           Tear  Gloss Opacity
## Tear    0.0005 0.0165  0.0445
## Gloss   0.0165 0.5445  1.4685
## Opacity 0.0445 1.4685  3.9605
##
## Multivariate Tests: rate:additive
##                  Df test stat approx F num Df den Df  Pr(>F)
## Pillai            1 0.2228942 1.338522      3     14 0.30178
## Wilks             1 0.7771058 1.338522      3     14 0.30178
## Hotelling-Lawley  1 0.2868261 1.338522      3     14 0.30178
## Roy               1 0.2868261 1.338522      3     14 0.30178
```

The top block in the out put above, marked `Sum of squares and products for error` gives the within cross-product matrix. The subsequent blocks give results for each factor effects and their interaction.

We can also extract univariate ANOVA results for individual variables:

```
summary.aov(res)
```

```
##  Response Tear :
##                Df Sum Sq Mean Sq F value   Pr(>F)
## rate            1 1.7405 1.74050 15.7868 0.001092 **
## additive        1 0.7605 0.76050  6.8980 0.018330 *
## rate:additive   1 0.0005 0.00050  0.0045 0.947143
## Residuals      16 1.7640 0.11025
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##  Response Gloss :
##                Df Sum Sq Mean Sq F value  Pr(>F)
## rate            1 1.3005 1.30050  7.9178 0.01248 *
## additive        1 0.6125 0.61250  3.7291 0.07139 .
## rate:additive   1 0.5445 0.54450  3.3151 0.08740 .
## Residuals      16 2.6280 0.16425
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##  Response Opacity :
##                Df Sum Sq Mean Sq F value Pr(>F)
## rate            1  0.421  0.4205  0.1036 0.7517
## additive        1  4.901  4.9005  1.2077 0.2881
## rate:additive   1  3.960  3.9605  0.9760 0.3379
## Residuals      16 64.924  4.0578
```

Some Discussion

- Individual tests ignore correlation among the p variables, and may give misleading results. Thus a single multivariate test is often preferable over p univariate tests. The result determines whether one should look closer (individual variables or groups)
- Non-normality and unequal covariance matrices: For large sample sizes, non-normality has little effect on the tests. If the sample sizes are equal in each group, some differences in covariance matrices across groups can also be ignored.
- The different test statistics (Wilks' lambda, Lawley-Hotelling trace, Pillai's trace, Roy's largest square root) are nearly equivalent for very large sample sizes. For moderate sample sizes, the first three tests behave similarly.

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