1.10

$$m_t = \sum_{k=0}^{p} c_k t^k$$

$$\nabla m_t = m_t - m_{t-1}$$
$$= \sum_{k=0}^{p} c_k t^k - \sum_{k=0}^{p} c_k (t-1)^k$$
$$= \sum_{k=0}^{p} c_k \left(t^k - (t-1)^k\right) = \sum_{k=0}^{p} c_k \underbrace{\left(- \sum_{j=0}^{k-1} \binom{k}{j} t^j (-1)^{k-j}\right)}_{deg = k-1 \leq p-1}$$

Do đó ∇m_t là đa thức bậc $p-1$.

Theo nguyên lý quy nạp, ta có $\nabla^p m_t = const$ là đa thức bậc 0

$$\Rightarrow \nabla^{p+1} m_t = \nabla^p m_t - \nabla^p m_{t-1} = 0$$

1.11

(a) $a_j = \frac{1}{2q+1}$, $-q \leq j \leq q$

$$\sum_{j=-q}^{q} a_j m_{t-j} = \frac{1}{2q+1} \sum_{j=-q}^{q} \left(c_0 + c_1 (t-j)\right)$$
$$= \frac{1}{2q+1} \left(c_0 (2q+1) + c_1 \sum_{j=-q}^{q} (t-j)\right)$$
$$= \frac{1}{2q+1} \left(c_0 (2q+1) + c_1 t (2q+1)\right) = c_0 + c_1 t = m_t$$

(b) $$E[A_t] = E\left[\sum_{j=-q}^{q} a_j z_{t-j}\right] = \sum_{j=-q}^{q} a_j E[z_{t-j}] = 0$$

$$\mathrm{Var}[A_t] = \mathrm{Var}\left[\sum_{j=-q}^{q} a_j Z_{t-j}\right]$$

$$= \sum_{j=-q}^{q} a_j^2 \,\mathrm{Var}[Z_{t-j}] = \frac{1}{(2q+1)^2}\sum_{j=-q}^{q}\sigma^2 = \frac{\sigma^2}{2q+1}$$

1.12

(a) $$\sum_j a_j m_{t-j} = \sum_j a_j \sum_{r=0}^{k} c_r (t-j)^r$$

$$= \sum_j a_j \sum_{r=0}^{k} c_r \sum_{i=0}^{r} \binom{r}{i} t^r (-j)^{r-i}$$

$$= \sum_{r=0}^{k} c_r \left(\sum_{i=0}^{r}\binom{r}{i} t^r \sum_j a_j (-j)^{r-i}\right) \quad (1)$$

Ta cần có $\sum_j a_j m_{t-j} = m_t = \sum_{r=0}^{k} c_r t^r \quad (2)$

Từ (1), (2) $\Rightarrow$ $t^r = \sum_{i=0}^{r}\binom{r}{i} t^r \sum_j a_j(-j)^{r-i}, \ \forall r \in \{0,1,\dots,k\}$

$\Leftrightarrow$ $1 = \sum_{i=0}^{r}\binom{r}{i}\sum_j a_j(-j)^{r-i}, \ \forall r\in\{0,1,\dots,k\}$

$r=0 \Rightarrow 1 = \sum_j a_j$

$r=1 \Rightarrow 1 = \sum_{i=0}^{1}\binom{1}{i}\sum_j a_j(-j)^{1-i} = \underbrace{\sum_j a_j}_{1} + \sum_j a_j(-j)$

$\Leftrightarrow \sum_j j a_j = 0$

Tương tự với $r = 2,3,\dots,k$, ta được $\sum_j j^r a_j = 0$

(b) $[a_0, a_1, \dots, a_7] = \frac{1}{320}[74, 67, 46, 21, 3, -5, -6, -3]$

$$\left.\begin{array}{l} \bullet\ \sum_{j=0}^{7} a_j = 1 \\ \bullet\ \sum_{j=0}^{7} j^r a_j = 0,\ \forall r\in\{1,2,3\}\end{array}\right\} \Rightarrow \text{đpcm}$$

1.13

$m_t = c_0 + c_1 t$, c_0, c_1 arbitary

Without distortion $\Leftrightarrow$ $m_t = (1 + \alpha B + \beta B^2 + \gamma B^3) m_t$

$$\Leftrightarrow \alpha m_{t-1} + \beta m_{t-2} + \gamma m_{t-3} = 0$$

$$\Leftrightarrow \alpha(c_0 + c_1(t-1)) + \beta(c_0 + c_1(t-2)) + \gamma(c_0 + c_1(t-3)) = 0$$

$$\Leftrightarrow (c_0 + c_1 t)(\alpha + \beta + \gamma) - c_1(\alpha + 2\beta + 3\gamma) = 0 \ , \ \forall c_0, c_1, t$$

$$\Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha + 2\beta + 3\gamma = 0 \end{cases} \qquad (1)$$

Eliminate arbitary seasonal components of $d = 2$

$$\Leftrightarrow \begin{cases} (1 + \alpha B + \beta B^2 + \gamma B^3) s_t = 0 \ , \ \forall t \\ s_{t+2} = s_t \ ; \ s_1 + s_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} s_t + \alpha s_{t-1} + \beta s_{t-2} + \gamma s_{t-3} = 0 \\ s_{t+2} = s_t \ ; \ s_1 + s_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} s_t(1+\beta) + s_{t-1}(\alpha + \gamma) = 0 \\ s_{t+2} = s_t \ ; \ s_1 + s_2 = 0 \end{cases}$$

$$\Leftrightarrow 1 + \beta = \alpha + \gamma \quad (2)$$

$$(1), (2) \Rightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha + 2\beta + 3\gamma = 0 \\ \alpha - \beta + \gamma = 1 \end{cases} \Leftrightarrow (\alpha, \beta, \gamma) = \left(\frac{1}{4} ; \frac{-1}{2} ; \frac{1}{4}\right)$$

1.14

$$[a_{-2}, a_{-1}, a_0, a_1, a_2] = \frac{1}{9}[-1, 4, 3, 4, -1]$$

$$\left.\begin{array}{l} \bullet \ \sum_{j=-2}^{2} a_j = 1 \\ \bullet \ \sum_{j=-2}^{2} j^r a_j = 0, \ \forall r \in \{1,2,3\} \end{array}\right\} \Rightarrow \text{Passes third degree polynomial trends without distortion}$$

Seasonal component with $d = 3$: $\begin{cases} s_{t+3} = s_t, \ \forall t \\ s_1 + s_2 + s_3 = 0 \end{cases}$

$$\sum_{j=-2}^{2} a_j s_{t-j} = \frac{1}{9}\left(-s_{t-2} + 4s_{t-1} + 3s_t + 4s_{t+1} - s_{t+2}\right)$$

$$= \frac{1}{9}\left(-s_{t+1} + 4s_{t-1} + 3s_t + 4s_{t+1} - s_{t+2}\right)$$

$$= \frac{1}{3}\left(s_t + s_{t+1} + s_{t+2}\right) = 0$$

$\Rightarrow$ Eliminates seasonal components with period 3.

1.15

(a) $\nabla\nabla_{12} X_t = \nabla\nabla_{12} m_t + \nabla\nabla_{12} s_t + \nabla\nabla_{12} Y_t$

$$\nabla\nabla_{12} m_t = \underbrace{\nabla\nabla_{12} a}_{0} + b\nabla\nabla_{12} t$$

$$= b(1-B)(1-B^{12})t$$

$$= b(1-B-B^{12}+B^{13})t$$

$$= b\left(t-(t-1)-(t-12)+(t-13)\right) = 0$$

$$\nabla\nabla_{12} s_t = (1-B-B^{12}+B^{13})s_t$$

$$= s_t - s_{t-1} - s_{t-12} + s_{t-13} = 0$$

$$\Rightarrow \nabla\nabla_{12} X_t = \underbrace{\nabla\nabla_{12} Y_t}_{\text{stationary}}$$

$$\gamma_{\nabla\nabla_{12}X}(h) = Cov(\nabla\nabla_{12}X_{t+h}, \nabla\nabla_{12}X_t)$$

$$= Cov(\nabla\nabla_{12}y_{t+h}, \nabla\nabla_{12}y_t)$$

$$= Cov(y_{t+h} - y_{t+h-1} - y_{t+h-12} + y_{t+h-13}\ ,\ y_t - y_{t-1} - y_{t-12} + y_{t-13})$$

$$= Cov(y_{t+h},\ y_t - y_{t-1} - y_{t-12} + y_{t-13}) -$$
$$- Cov(y_{t+h-1},\ y_t - y_{t-1} - y_{t-12} + y_{t-13}) -$$
$$- Cov(y_{t+h-12},\ y_t - y_{t-1} - y_{t-12} + y_{t-13}) +$$
$$+ Cov(y_{t+h-13},\ y_t - y_{t-1} - y_{t-12} + y_{t-13})$$

$$= \sigma^2(\chi_{\{0\}} - \chi_{\{-1\}} - \chi_{\{-12\}} + \chi_{\{-13\}})(h) -$$
$$- \sigma^2(\chi_{\{1\}} - \chi_{\{0\}} - \chi_{\{-11\}} + \chi_{\{-12\}})(h) -$$
$$- \sigma^2(\chi_{\{12\}} - \chi_{\{11\}} - \chi_{\{0\}} + \chi_{\{-1\}})(h) +$$
$$+ \sigma^2(\chi_{\{13\}} - \chi_{\{12\}} - \chi_{\{1\}} + \chi_{\{0\}})(h)$$

$$= \sigma^2(\chi_{\{-13\}} - 2\chi_{\{-12\}} + \chi_{\{-11\}} - 2\chi_{\{-1\}} + 4\chi_{\{0\}} -$$
$$- 2\chi_{\{1\}} + \chi_{\{11\}} - 2\chi_{\{12\}} + \chi_{\{13\}})(h)$$

$$= \begin{cases} \sigma^2 & \text{nếu } h = \pm 13 ; \pm 11 \\ -2\sigma^2 & \text{nếu } h = \pm 12 ; \pm 1 \\ 4\sigma^2 & \text{nếu } h = 0 \\ 0 & \text{nếu thái lại} \end{cases}$$

(b) $\nabla_{12}^2 X_t = \nabla_{12}^2 (a+bt)s_t + \nabla_{12}^2 y_t$

$$\nabla_{12}^2 (a+bt)s_t = (1-B^{12})^2\{(a+bt)s_t\}$$
$$= (1-2B^{12}+B^{24})\{(a+bt)s_t\}$$
$$= (a+bt)s_t - 2(a+b(t-12))s_{t-12} + (a+b(t-24))s_{t-24}$$
$$= ((a+bt) - 2(a+b(t-12)) + (a+b(t-24)))s_t = 0$$

$$\Rightarrow \nabla_{12}^2 X_t = \underbrace{\nabla_{12}^2 y_t}_{\text{stationary}}$$

$$\gamma_{\nabla_{12}^2 X}(h) = Cov(\nabla_{12}^2 X_{t+h}, \nabla_{12}^2 X_t)$$

$$= Cov(\nabla_{12}^2 Y_{t+h}, \nabla_{12}^2 Y_t)$$

$$= Cov(Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24})$$

$$\begin{aligned} =\ & Cov(Y_{t+h}, Y_t - 2Y_{t-12} + Y_{t-24}) - \\ -\ & 2Cov(Y_{t+h-12}, Y_t - 2Y_{t-12} + Y_{t-24}) + \\ +\ & Cov(Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24}) \end{aligned}$$

$$\begin{aligned} =\ & \sigma^2(\chi_{\{0\}} - 2\chi_{\{-12\}} + \chi_{\{-24\}})(h) - \\ -\ & 2\sigma^2(\chi_{\{12\}} - 2\chi_{\{0\}} + \chi_{\{-12\}})(h) + \\ +\ & \sigma^2(\chi_{\{24\}} - 2\chi_{\{12\}} + \chi_{\{0\}})(h) \end{aligned}$$

$$= \sigma^2(\chi_{\{-24\}} - 4\chi_{\{-12\}} + 6\chi_{\{0\}} - 4\chi_{\{12\}} + \chi_{\{24\}})(h)$$

$$= \begin{cases} \sigma^2 & \text{nếu } h = \pm 24 \\ -4\sigma^2 & \text{nếu } h = \pm 12 \\ 6\sigma^2 & \text{nếu } h = 0 \\ 0 & \text{nếu trái lại} \end{cases}$$