1.6

a, $Var\left(\frac{1}{4}\sum_{k=1}^{4} X_k\right) = \frac{1}{16} Var\left(\sum_{k=1}^{4} X_k\right)$

$= \frac{1}{16}\left(\sum_{k=1}^{4} var(X_k) + 2\sum_{1\le k<l\le 4} Cov(X_k, X_l)\right)$

$= \frac{1}{16}\left(4\gamma_X(0) + 2(3\phi\gamma_X(0) + 2\phi^2\gamma_X(0) + \phi^3\gamma_X(0))\right)$

Thay $\phi = 0,9$:

$\frac{1}{16}\left(4 + 2(3 \cdot 0,9 + 2 \cdot 0,9^2 + 0,9^3)\right) = 0,8811$

b, $Var\left(\frac{1}{4}\sum_{k=1}^{4} X_k\right) = \frac{1}{16}\left(4 + 2(-3 \cdot 0,9 + 2 \cdot 0,9^2 - 0,9^3)\right)$

$= 0,0239 < 0,8811$

1.8

a, $X_t \sim WN(0,1)$

$E(X_t) = \begin{cases} E(Z_t) = 0 & t \text{ chẵn} \\ E\left(\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(E(Z_{t-1}^2) - 1\right) & t \text{ lẻ} \end{cases}$

Ta sẽ tính $\gamma_X(t, t+h)$

• $h = 0$: $\gamma_X(t,t) = \begin{cases} E(Z_t^2) = 1 & , t \text{ chẵn} \\ E\left(\frac{(Z_{t-1}^2 - 1)^2}{2}\right) = \frac{1}{2} E(Z_{t-1}^4 - 2Z_{t-1}^2 + 1) = 1, & t \text{ lẻ} \end{cases}$

$h = 1$:

$$\gamma_X(t+1,t) = \begin{cases} E\left(\frac{Z_{t-1}^2}{\sqrt{2}} \cdot Z_t\right) = \frac{1}{\sqrt{2}} E(Z_t^3 - Z_t) = 0 & t \text{ chẵn} \\ E\left(Z_{t+1} \cdot \frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right) = E(Z_{t+1}) \cancel{\times} \frac{E(Z_{t-1}^2) - 1}{\sqrt{2}} = 0 & t \text{ lẻ} \end{cases}$$

$h \geq 2$: hiển nhiên $\gamma_X(t+h,t) = 0$

$\Rightarrow X_t \sim WN(0,1)$

X_t ko phải IID $N(0,1)$ do X_t và X_{t-1} ko độc lập

b)

$$E(X_{n+1} \mid X_1, \dots, X_n) = \begin{cases} E(Z_{n+1} \mid Z_0, Z_2, \dots, Z_{n-1}) = E(Z_{n+1}) = 0 & n \text{ lẻ} \\ E\left(\frac{Z_n^2 - 1}{\sqrt{2}} \middle| Z_0, Z_2, \dots, Z_n\right) = \frac{Z_n^2 - 1}{\sqrt{2}} = \frac{X_n^2 - 1}{\sqrt{2}}, & n \text{ chẵn} \end{cases}$$

1.9

(a)

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t = \frac{1}{n}\sum_{t=1}^{n}(a + bt) = a + \frac{b(n+1)}{2}$$

Thứ ngày

$$n\hat{\gamma}(h) = \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x})$$

$$= \sum_{t=1}^{n-|h|} \left(a + b(t+|h|) - a - \frac{b(n+1)}{2}\right)\left(a + bt - a - \frac{b(n+1)}{2}\right)$$

$$= \sum_{t=1}^{n-|h|} b^2 \left(t + |h| - \frac{n+1}{2}\right)\left(t - \frac{n+1}{2}\right)$$

$$= b^2 \sum_{t=1}^{n-|h|} \left(t^2 - t(n+1-|h|) + \frac{(n+1)^2}{4}\right)$$

$$= b^2 \left(\frac{1}{6}(n-|h|)(n+1-|h|)(2n+1-2|h|) + \frac{1}{2}(n-|h|)(n+1-|h|)^2 + \frac{1}{4}(n-|h|)(n+1)^2\right)$$

$$\overset{n\to+\infty}{\sim} b^2\left(\frac{1}{6}n.n2n + \frac{1}{2}n.n^2 + \frac{1}{4}n.n^2\right) = \frac{13}{12}b^2n^3$$

$$\Rightarrow \hat{\gamma}(h) \sim \frac{13}{12}b^2n^2 \quad \forall h \in \mathbb{Z}$$

$$\Rightarrow \lim_{n\to\infty} \hat{\rho}(h) = \lim_{n\to\infty} \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = 1$$

b, Ko mất tính tổng quát,

cho $c = 1$ và $h > 0$

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} \cos(kx) = \frac{\sin\left(\frac{nx}{2}\right)\cos\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$\forall x \in \mathbb{R}, n \in \mathbb{N}^*$

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t = \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t)$$

$$= \frac{\sin\left(\frac{n\omega}{2}\right)\cos\left(\frac{(n+1)\omega}{2}\right)}{n\sin\left(\frac{\omega}{2}\right)} \xrightarrow{n\to+\infty}$$

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h}-\bar{x})(x_t-\bar{x})$$

$$= \frac{1}{n}\sum_{t=1}^{n-h}\left(x_{t+h}x_t - \bar{x}(x_{t+h}+x_t) + \bar{x}^2\right)$$

Biến đổi:

$$\frac{1}{n}\sum_{t=1}^{n-h} x_{t+h}x_t = \frac{1}{n}\sum_{t=1}^{n-h}\cos((t+h)\omega)\cos(t\omega)$$

$$= \frac{1}{2n}\sum_{t=1}^{n-h}\left(\cos((2t+h)\omega) + \cos(\omega h)\right)$$

$$= \frac{1}{2n}(n-h)\cos(\omega h) + \frac{1}{2n}\sum_{t=1}^{n-h}\cos((2t+h$$

$$\sim \frac{1}{2}\cos(\omega h)$$

$$\frac{1}{n}\sum_{t=1}^{n-h}\bar{x}(x_{t+h}+x_t) = \bar{x}\cdot\frac{1}{n}\sum_{t=1}^{n-h}\left(\cos((t+h)\omega)+\cos(t\right.$$

$$\to 0$$

Do đó, $\hat{\gamma}(h) \overset{n\to\infty}{\sim} \frac{1}{2}\cos(\omega h)$

$h=0 \Rightarrow \hat{\gamma}(0) = \frac{1}{2}$

$$\lim \hat{\rho}(h) = \lim_{n\to\infty}\frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = \cos(\omega h)$$