2.9.

a, $E(Y_t) = E(X_t + W_t) = E(X_t)$

$= E(\phi X_{t-1} + Z_t)$

$= E\left(\sum_{n=0}^{t} \phi^n Z_{t-n}\right) = 0$

$Cov(Y_t, Y_{t+h}) = Cov(X_t + W_t, X_{t+h} + W_{t+h})$

$= Cov(X_t, X_{t+h}) = \gamma_X(h) = \dfrac{\sigma^2 \phi^h}{1-\phi^2}$, $h>0$

b,

$U_t = Y_t - \phi Y_{t-1}$. U_t and U_s are independent whenever $|s-t|>1$

$\gamma_U(h) = 0$, $|h|>1$

$E(U_t) = E(Y_t - \phi Y_{t-1}) = 0$

$\gamma_U(h) = Cov(Y_t - \phi Y_{t-1}, Y_{t+h} - \phi Y_{t+h-1})$

$= Cov(Y_t, Y_{t+h}) - \phi Cov(Y_{t-1}, Y_{t+h}) - \phi Cov(Y_t, Y_{t+h-1})$

$+ \phi^2 Cov(Y_{t-1}, Y_{t+h-1})$

$= \gamma_Y(h) - \phi\gamma_Y(h+1) - \phi\gamma_Y(h-1) + \phi^2\gamma_Y(h)$

$= \dfrac{\sigma^2\phi^h}{1-\phi^2} - \phi\dfrac{\sigma^2\phi^{h+1}}{1-\phi^2} - \phi\dfrac{\sigma^2\phi^{h-1}}{1-\phi^2} + \phi^2\dfrac{\sigma^2\phi^h}{1-\phi^2}$

$= \sigma^2\left[\dfrac{1}{1-\phi^2}\left(\phi^h - \phi^{h+2} - \phi^h + \phi^{h+2}\right)\right] = 0$ ~~$\forall h \geq 1$~~ $\forall h>1$

~~$\gamma_U(0) = E(Y_t^2) - \phi E(Y_{t-1}, Y_t) - \phi($~~

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$\Rightarrow \{U_t\}$ is 1-correlated

c) ARMA(1,1)

$$\cancel{X_t - \phi X_{t-1} = Z_t + \phi Z_{t-1}}$$

$$\cancel{= Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}}$$

It is ~~to~~ ~~deduce~~ from that U_t is a MA(1) process

$$\Rightarrow Y_t - \phi Y_{t-1} = Z_t + \theta Z_{t-1}$$

Since $\{Y_t\}$ is stationary, it follows that $\{Y_t\}$ is ARMA(1,1)

2.10

$$X_t - 0{,}5 X_{t-1} = Z_t + 0{,}5 Z_{t-1} \quad , \{Z_t\} \sim WN(0, \sigma^2)$$

$$\phi(z) = \cancel{1 - 0{,}5 X_t} \; 1 - 0{,}5z$$

Zeros of $\phi(z)$ have $|z| = 2$

$\Rightarrow$ all zeros of $\phi(z)$ stay outside of the unit circle

$\Rightarrow \phi(z) \neq 0 \quad \forall |z| < 1$ which means $\{X_t\}$ is casual

$$\Rightarrow X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

$$\psi_0 = 1, \quad \psi_1 = \theta_1 + \psi_0 \phi_1 = 0{,}5 - 1.0{,}5 = 0$$

$$\psi_2 = \theta_2 + \psi_1 \phi_1 + \psi_0 \phi_2 = 0 + 0 + 0 = 0$$

$$\psi_j = 0 \quad \forall j \geq 1$$

$$\Rightarrow X_t = Z_t$$

The MA polynomial $\theta(z) = 1 + 0{,}5z$ which also has zero at $z = 2$ which is outside of unit circle. $\Rightarrow$ ~~invertib~~ invertible

$$\pi_0 = 1, \quad \pi_1 = -\phi_1 - \theta_1 \pi_0 = +0{,}5 - 0{,}5 = 0, \quad \pi_j = 0 \; \forall j \geq 1$$

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2.11.

An approximate 95% confidence interval for μ is:

$$\left(\bar{X}_n - 1{,}96\,\frac{v^{1/2}}{\sqrt{n}},\ \bar{X}_n + 1{,}96\,\frac{v^{1/2}}{\sqrt{n}}\right), \quad v = \sum_{|h|<\infty} \gamma(h)$$

$$\hat{v} = \sum_{|h|<\sqrt{n}} \left(1 - \frac{|h|}{\sqrt{n}}\right)\hat{\gamma}(h)$$

$$\cancel{\Rightarrow v = (1+2\sum}$$

for AR(1): $\gamma(h) = \sigma^2 \dfrac{\phi^{|h|}}{1-\phi^2}$

$$\Rightarrow v = \left(1 + 2\sum_{h=1}^{\infty} \phi^h\right)\sigma^2/(1-\phi^2) = \frac{\sigma^2}{(1-\phi)^2} = \frac{2}{(1-0{,}6)^2} = 12{,}5$$

$$\Rightarrow \bar{X}_n \pm 1{,}96\, n^{-1/2}/(1-\phi) = 0{,}271 \pm \frac{1{,}96}{10(1-0{,}6)}$$

$$= 0{,}271 \pm 0{,}49 = (-0{,}219;\ 0{,}716)$$

Since $0 \in (-0{,}219,\ 0{,}716)$ ~~this~~, the data are compatible with the hypothesis that $\mu = 0$

2.12.

For MA(1) process, $\gamma(h) = Cov(X_t, X_{t+h})$

$$= Cov(Z_t + \theta Z_{t-1},\ Z_{t+h} + \theta Z_{t+h-1})$$

$$= \begin{cases} \sigma^2(1+\theta^2), & h=0 \\ \sigma^2\theta, & h=\pm 1 \\ 0, & |h|>1 \end{cases}$$

$$\Rightarrow v = \sigma^2(1+\theta)^2 = (1-0{,}6)^2 = 0{,}16$$

An approximate 95% confidence interval for μ therefore is

$$\left(0{,}157 - 1{,}96 \cdot \frac{\sqrt{916}}{\sqrt{100}}, 0{,}157 + 1{,}96 \frac{\sqrt{916}}{\sqrt{100}}\right)$$

$$= (0{,}0786, 0{,}2354)$$

$\Rightarrow$ The data are not compatible with the hypothesis $\mu = 0$

2.13

a) $$w_{ij} = \sum_{k=1}^{\infty} \{\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\} \times \{\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)\}$$

a) For AR(1) process,

$$w_{ii} = (1-\phi^{2i})(1+\phi^2)(1-\phi^2)^{-1} - 2i\phi^{2i}$$

The 95% confidence bounds for $\hat{\rho}(i)$ are: $\hat{\rho}(i) \pm 1{,}96 n^{-1/2} w_{ii}^{1/2}$

$\Rightarrow$ 95% cb for $\rho(1)$: $\hat{\rho}(1) \pm 1{,}96 . 10^{-1} . \left(\frac{297}{235}\right)$

$$= 0{,}438 \pm \frac{1{,}96}{10} \cdot \frac{9}{25} = (0{,}36744, 0{,}50856)$$

95% cb for $\rho(2)$: $\hat{\rho(2)} \pm 1{,}96 n^{-1/2} w_{22}^{1/2}$

$$= 0{,}145 \pm \frac{1{,}96}{10}\left(\frac{657}{625}\right) = (-0{,}061035;$$
$$0{,}351035)$$

We also have that $\rho(1) = \cancel{\frac{\phi}{1-\phi^2}} = \phi^1 = 0{,}8$

$$\rho(2) = 0{,}8^2 = 0{,}64$$

$\Rightarrow$ The data are not compatible b, For MA(1) process with $\theta = 0{,}6$

$$w_{ii} = \begin{cases} 1 - 3\rho^2(1) + 4\rho^4(1) & , i = 1 \\ 1 + 2\rho^2(1) & , i > 1 \end{cases}$$

The 95% confidence interval for $\rho(1)$: $\left(\rho(1) = \frac{\theta}{1+\theta^2} = \frac{15}{34}\right)$

$$\hat{\rho}(1) \pm \frac{1{,}96}{\sqrt{n}} w_{11} = 0{,}438 \pm \frac{1{,}96}{10} \cdot 0{,}45397$$

$$= (0{,}34902 \; ; \; 0{,}526979) \ni \frac{15}{34}$$

The 95% confidence interval for $\rho(2)$: $(\rho(2) = 0)$

$$\hat{\rho}(2) \pm \frac{1{,}96}{\sqrt{n}} = 0{,}145 \pm \frac{1{,}96}{10} = (-0{,}051 ; 0{,}341) \ni 0$$

$\Rightarrow$ The data are compatible

2.14 $\quad X_t = A\cos(\omega t) + B\sin(\omega t), t = 0, \pm 1, \ldots$

$\gamma(h) = \cos(\omega h)$

a, Find $P_1 X_2$

The best linear predictor of X_{n+1} in terms of $\{1, X_n, \ldots, X_1\}$

is $P_n X_{n+1} = a_n' X_n$

where $X_n = (X_n, \ldots, X_1)$ and:

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & & & \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(n) \end{bmatrix}$$

$$\begin{pmatrix} 1 & \cos(\omega) & & \cos((n-1)\omega) \\ \cos(\omega) & 1 & & \cos((n-2)\omega) \\ & & & \\ \cos((n-1)\omega) & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \\ \\ a_n \end{pmatrix} = \begin{pmatrix} \cos(\omega) \\ \cos(2\omega) \\ \vdots \\ \cos(n\omega) \end{pmatrix}$$

$$\Rightarrow a = [\cos(\omega) \quad 0 \quad 0 \quad]'$$

and hence the best linear predictor of X_{n+1} in terms of $\{X_1, \ldots, X_n\}$

is $P_n X_{n+1} = a_n' X_n = \cos(\omega) X_n$

with mean square error

$$E(X_{n+1} - P_n X_{n+1})^2 = \cancel{E}\, X_{n+1} -$$

$$= \gamma(0) - a_n' \gamma_n(1) = 1 - \cos^2(\omega) = \sin^2(\omega)$$

$$\Rightarrow \begin{cases} P_1 X_2 = \cos(\omega) X_1 \\ P_2 X_3 = \cos(\omega) X_2 \end{cases}$$

c)

$$\sum_{j=1}^{\infty} \cos((i-j)\omega)\, a_j = \underline{\cos(\omega i)} \quad , i = 1, \ldots$$

$\cos(\omega i)(a_j - 1) + a_x \cos(i-j)\ldots$

~~$a_1 + a_2 (0 - 1)\cos(\omega)$~~ $X_3 = a_2 X_1 + a_1 X_2$

2.14. $X_{n+1} = a_1 X_{n} + a_2 X_{n-1} + \dots a_n X_1$

a, $\gamma(0)^{-1}\gamma(1) = \cos(\omega)$

$$\Rightarrow \begin{cases} P_1 X_2 = a_1 X_1 = \cos(\omega) X_1 \\ E(X_2 - P_1 X_2)^2 = \gamma(0) - a_1\gamma(1) = 1 - \cos^2(\omega) = \sin^2(\omega) \end{cases}$$

b, $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}^{-1} \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos(\omega) \\ \cos(\omega) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \cos(\omega) \\ \cos(2\omega) \end{bmatrix} = \begin{bmatrix} 2\cos(\omega) \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} P_2 X_3 = -X_1 + 2\cos(\omega) X_2 \\ E(X_3 - P_2 X_3)^2 = \gamma(0) - (a_1 \; a_2) \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix} \end{cases}$$

$$= 1 - [2\cos(\omega) \; -1] \begin{bmatrix} \cos(\omega) \\ \cos(2\omega) \end{bmatrix} = 0$$

c, $$\tilde{P}_n X_{n+1} = \sum_{j=1}^{+\infty} a_j X_{n+1-j}$$

trong đó $a = [a_1, a_2, \dots]^T$ là no của hệ phương trình

$$\sum_{j=1}^{+\infty} a_j \gamma(i-j) = \gamma(h+i-1) \quad , \forall i = 1,2$$

$$\Leftrightarrow \Gamma_\infty a_\infty = \gamma_\infty(h)$$

$$h=1 \rightarrow \Gamma_\infty a_\infty = \gamma_\infty(1) \Leftrightarrow \begin{cases} \gamma(0)a_1 + \gamma(1)a_2 + \gamma(2)a_3 + \dots = \gamma(1) \\ \gamma(1)a_1 + \gamma(0)a_2 + \dots = \gamma(2) \\ \vdots \end{cases}$$

~~2.15~~

$$\Leftrightarrow \begin{cases} a_1 + \cos(\omega)a_2 + \cos(2\omega)a_3 + \dots = \cos(\omega) \\ \cos(\omega)a_1 + a_2 + \cos(2\omega)a_3 + \dots = \cos(2\omega) \\ \vdots \end{cases}$$

2.15

$$X_{n+1} = Z_{n+1} + \sum_{j=1}^{p} \phi_j X_{n+1-j}$$

$$\Rightarrow P_n X_{n+1} = \underbrace{P_n Z_{n+1}}_{0} + \sum_{j=1}^{p} \phi_j P_n X_{n+1-j} = \sum_{j=1}^{p} \phi_j X_{n+1-j}$$

2.18.

MA(1) : $X_t = Z_t - \theta Z_{t-1}$, $Z_t \sim WN(0, \sigma^2)$

$$Z_t = (1-\theta B)^{-1} X_t$$

$$= \sum_{j=0}^{\infty} (\theta B)^j X_t = \sum_{j=0}^{\infty} \theta^j X_{t-j} \quad , Z_t \sim WN(0,\sigma^2)$$

Tác động toán tử $\tilde{P}_n$ vào hai vế, ta có:

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$$\tilde{P}_n Z_{n+1} = \sum_{j=0}^{\infty} \theta^j \tilde{P}_n X_{n+1-j}$$

$$\Leftrightarrow\ 0 = \tilde{P}_n X_{n+1} + \sum_{j=1}^{\infty} \theta^j X_{n+1-j}$$

$$\Leftrightarrow\ \tilde{P}_n X_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j}$$

Trung bình bình phương sai số :

$$E(X_{n+1} - \tilde{P}_n X_{n+1})^2 = E\left(X_{n+1} + \sum_{j=1}^{\infty} \theta^j X_{n+1-j}\right)^2$$

$$= E(Z_{n+1})^2 = \sigma^2$$

2.21:

$$X_t = Z_t + \theta Z_{t-1} \quad , \{Z_t\} \sim WN(0, \sigma^2)$$

a, $a_n = (a_1\ a_2)$ $W = (X_{\cancel{2}}, X_{\cancel{1}})$, $\gamma_{\cancel{n}} = [\gamma(\cancel{1}), \gamma(\cancel{2})]$

$\Gamma_n a_n = \gamma$

$$\begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$

$$\begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \theta \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{(1+\theta^2)^2 - \theta^2} \begin{bmatrix} 1+\theta^2 & -\theta \\ -\theta & 1+\theta^2 \end{bmatrix} \begin{bmatrix} \theta \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{1+\theta^2+\theta^4} \begin{bmatrix} \theta + \theta^3 \\ -\theta^2 \end{bmatrix}$$

b, $Y = X_3$, $W = (X_4, X_5)$ $\gamma = [\gamma(1), \gamma(2)]$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{bmatrix} \cancel{\gamma(1)} & \cancel{\gamma(2)} \\ \cancel{\gamma(2)} & \cancel{\gamma(1)} \end{bmatrix} \qquad \frac{1}{|A|}\begin{bmatrix} d & -b \\ -b & a \end{bmatrix}$$

$$\begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \theta \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{1+\theta^2+\theta^4} \begin{bmatrix} \theta - \theta^2 + \theta^3 \\ \theta - \theta^2 + \theta^3 \end{bmatrix}$$

c,

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \gamma(2) \\ \gamma(1) \\ \gamma(1) \end{bmatrix}$$

$Y = X_2$

$W(X_1\ X_3)$

$\gamma_2 = \begin{bmatrix} \gamma(1) \\ \gamma(1) \end{bmatrix}$

$$\begin{bmatrix} 1+\theta^2 & \theta & 0 \\ \theta & 1+\theta^2 & \theta \\ 0 & \theta & 1+\theta^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta \\ \theta \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \dfrac{-\theta^4 + \theta^3 - \theta^2}{1+\theta^2+\theta^4+\theta^6} \\ \dfrac{\theta^3 - \theta^2 + \theta}{\theta^4 + 1} \\ \dfrac{\theta^8 - \theta^4 + \theta^3 - \theta^2 + \theta}{1+\theta^2+\theta^4+\theta^6} \end{bmatrix}$$

$\text{Error} = \cancel{\text{Var}(X} \; E(X_3 - P(X_3 | W))^2 = \cancel{\theta} \; \sigma^2(1+\theta^2)$

$a'\gamma$

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