Home work

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2.1 Suppose that $X_1, X_2, \ldots,$ is a stationary time series with mean μ and ACF $\rho(\cdot)$. Show that the best predictor of X_{n+h} of the form $aX_n + b$ is obtained by choosing $a = \rho(h)$ and $b = \mu(1 - \rho(h))$.

$$\hat{X}_{n+h} = aX_n + b$$

$$E\left[(\hat{X}_{n+h} - X_{n+h})^2\right] \to \min$$

$$= E\left[(aX_n + b - X_{n+h})^2\right]$$

$$= E\left[a^2X_n^2 + b^2 + X_{n+h}^2 + 2abX_n - 2aX_nX_{n+h} - 2bX_{n+h}\right]$$

$$= a^2E(X_n^2) + b^2 + E(X_{n+h}^2) + 2abE(X_n) - 2aE(X_nX_{n+h}) - 2bE(X_{n+h})$$

$$= a^2E(X_n^2) + b^2 + E(X_{n+h})^2 + 2ab\mu - 2aE(X_nX_{n+h}) - 2b\mu = m(a,b)$$

$$= [a\ b]\begin{bmatrix} E(X_n^2) & \mu \\ \mu & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + [-2E(X_nX_{n+h})\ \ -2\mu]\begin{bmatrix} a \\ b \end{bmatrix} = m(a,b)$$

$$= x^TAx + b^Tx$$

$$\frac{\partial m}{\partial x} = 2Ax + b = 0 \Rightarrow x = -A^{-1}b = \begin{bmatrix} E(X_n^2) & \mu \\ \mu & 1 \end{bmatrix}^{-1}\begin{bmatrix} E(X_nX_{n+h}) \\ \mu \end{bmatrix}$$

$$= \frac{1}{E(X_n^2) - \mu^2}\begin{bmatrix} 1 & -\mu \\ -\mu & E(X_n^2) \end{bmatrix}\begin{bmatrix} E(X_nX_{n+h}) \\ \mu \end{bmatrix}$$

$$= \frac{1}{\gamma(0)}\begin{bmatrix} E(X_nX_{n+h}) - \mu^2 \\ -\mu E(X_nX_{n+h}) + \mu E(X_n^2) \end{bmatrix} = \frac{1}{\gamma(0)}\begin{bmatrix} \gamma(h) \\ \mu(\gamma(0) - \gamma(h)) \end{bmatrix}$$

$$= \begin{bmatrix} \rho(h) \\ \mu(1 - \rho(h)) \end{bmatrix}$$

2.2 Show that the process

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t = 0, \pm 1, \ldots$$

(where A and B are uncorrelated random variables with mean 0 and variance 1 and ω is a fixed frequency in the interval $[0, \pi]$), is stationary and find its mean and autocovariance function. Deduce that the function $\kappa(h) = \cos(\omega h)$, $h = 0, \pm 1, \ldots$, is nonnegative definite.

$$E(X_t) = E(A\cos(\omega t) + B\sin(\omega t))$$

$$= \cos(\omega t)E(A) + \sin(\omega t)E(B)$$

$$= 0 \quad (\text{not depend on } t)$$

$$Var(X_t) = E(X_t^2) = Cov(X_t, X_t)$$

$$= Cov(A\cos(\omega t) + B\sin(\omega t))$$

$$= \cos^2(\omega t)Var(A) + \cos(\omega t)\sin(\omega t)Cov(A,B)$$

$$+ \cos(\omega t)\sin(\omega t)Cov(B,A) + B\sin^2(\omega t)Var(B)$$

$= 1$ (not depend on t)

$\Rightarrow \{X_t\}$ is a stationary process

The sequence ACVF;

$\gamma(h) = Cov(X_t, X_{t+h})$

$= Cov[A\cos(\omega t) + B\sin(\omega t), A\cos(\omega t + h\omega) + B\sin(\omega t + h\omega)]$

$= \cos(\omega t)\cos(\omega t + h\omega) + \sin(\omega t)\sin(\omega t + h\omega)$

$= \cos(\omega t - \omega t - \omega h) = \cos(\omega h)$

$\Rightarrow \kappa(h) = \cos(\omega h)$ is nonnegative define due to theorem 2.1.1

2.3 a. Find the ACVF of the time series $X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}$, where $\{Z_t\} \sim$ WN(0, 1).

b. Find the ACVF of the time series $Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}$, where $\{\tilde{Z}_t\} \sim$ WN(0, 0.25). Compare with the answer found in (a).

a, $\gamma(h) = Cov(X_t, X_{t+h}) = Cov(Z_t + 0{,}3Z_{t-1} - 0{,}4Z_{t-2}), Z_{t+h} + 0{,}3Z_{t+h-1} - 0{,}4Z_{t+h-2})$

$= \chi_{\{0\}} + 0{,}3\chi_{\{1\}} - 0{,}4\chi_{\{2\}}$

$+ 0{,}3\chi_{\{-1\}} + 0{,}3^2\chi_{\{0\}} - 0{,}3.0{,}4\chi_{\{1\}}$

$- 0{,}4\chi_{\{-2\}} - 0{,}4\,0{,}3\chi_{\{-1\}} + 0{,}4^2\chi_{\{0\}}$

$$\Rightarrow \gamma(h) = \begin{cases} 1 + 0{,}3^2 + 0{,}4^2 & , h = 0 \\ 0{,}3 - 0{,}3.\,0{,}4 & , h = 1, -1 \\ -0{,}4 & , h = \pm 2 \end{cases}$$

b, $\gamma(h) = Cov(Y_t, X_{t+h}) = (\chi_{\{0\}} - 1{,}2\chi_{\{1\}} - 1{,}6\chi_{\{2\}}$

$- 1{,}2\chi_{\{-1\}} + 1{,}2^2\chi_{\{0\}} + 1{,}2.1{,}6\chi_{\{1\}}$

$- 1{,}6\chi_{\{-2\}} + 1{,}2.1{,}6\chi_{\{-1\}} + 1{,}6^2\chi_{\{0\}}).\,0{,}25$

$$\Rightarrow \gamma(h) = \begin{cases} 0{,}25(1 + 1{,}2^2 + 1{,}6^2) & , h = 0 \\ (-1{,}2 + 1{,}2.1{,}6)\,0{,}25 & , h = \pm 1 \\ -1{,}6 & , h = \pm 2 \end{cases}$$

2.4 It is clear that the function $\kappa(h) = 1$, $h = 0, \pm 1, \ldots$, is an autocovariance function, since it is the autocovariance function of the process $X_t = Z$, $t = 0, \pm 1, \ldots$, where Z is a random variable with mean 0 and variance 1. By identifying appropriate sequences of random variables, show that the following functions are also autocovariance functions:

a. $\kappa(h) = (-1)^{|h|}$

b. $\kappa(h) = 1 + \cos\left(\frac{\pi h}{2}\right) + \cos\left(\frac{\pi h}{4}\right)$

c. $\kappa(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0.4, & \text{if } h = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$

$Var(X_t) = 1$

$X_t = (-1)^t$

a) $Cov(X_t, X_{t+h}) = (-1)^{|h|} = E(X_t X_{t+h}) - E(X_t)E(X_{t+h})$

$= \cos(\pi h)$

$\Rightarrow Z_t = A\cos(\pi t) + B\sin(\pi t)$

where A and B are random uncorrelated variables with mean 0 and variance 1

b) $\kappa(h) = 1 + \cos\left(\frac{\pi h}{2}\right) + \cos\left(\frac{\pi h}{4}\right)$

$= 1 + 2\cos\left(\frac{3\pi h}{4}\right)\cos\left(\frac{\pi h}{4}\right)$

c) $\kappa(h) = \gamma(0) + 0{,}4\,\gamma(1) + 0{,}4\,\gamma(-1)$

$= Cov(aX_t + bX_{t-1}, aX_{t+h} + bX_{t+h-1})$

$= a^2\gamma(0) + ab\,\gamma(1) + ab\,\gamma(-1) + b^2\gamma(0)$

$\Rightarrow \begin{cases} a^2+b^2 = 1 \\ ab = 0{,}4 \end{cases} \Rightarrow \begin{cases} (a+b)^2 = 1{,}8 \\ ab = 0{,}4 \end{cases}$

$\Rightarrow \begin{cases} a = \arccos\left(\frac{2}{\sqrt{5}}\right) \\ b = \arcsin\left(\frac{1}{\sqrt{5}}\right) \end{cases}$

2.5 Suppose that $\{X_t, t = 0, \pm 1, \ldots\}$ is stationary and that $|\theta| < 1$. Show that for each fixed n the sequence

$$S_m = \sum_{j=1}^{m} \theta^j X_{n-j}$$

is convergent absolutely and in mean square (see Appendix C) as $m \to \infty$.

$E(S'_m - S'_n)^2 = E\left(\sum_{m<i<n} |\theta^i X_{n-i}|\right)^2$

$S'_m = \sum_{j=1}^{\infty} |\theta^j X_{n-j}|$

$\leq E\left(\sum_{m<i<n} |\theta^i|^2 \sum_{m\leq i\leq n} |X_{n-i}|^2\right)$

$= \sum_{m<i<n} |\theta^i|^2 \, E\left(\sum |X_{n-i}|^2\right) \to 0$

when $m,n \to \infty$

since $\sum |\theta|^{2i}$

converges

$\Rightarrow S_m$ absolutely converges in mean squares

$$W_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j},$$

where $k = j - k'$ $\quad j = 2k$, $k = j - k$

$$j - k = k' \quad \psi_j = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{j-k} = \sum_{k=-\infty}^{\infty} \beta_k \alpha_{j-k}. \quad \alpha_k = \alpha_{j-k} \qquad (2.2.6)$$

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$$\sum_{k=-\infty}^{+\infty} \alpha_k \beta_{j-k} = \sum_{k'=-\infty}^{+\infty} \beta_{k'} \alpha_{j-k'}$$

$k' = j - k$ is a bijective from $[-\infty, +\infty] \to [-\infty, +\infty]$

$\Rightarrow \forall k, \exists! k' : k' = j - k$

2.7 Show, using the geometric series $1/(1-x) = \sum_{j=0}^{\infty} x^j$ for $|x| < 1$, that $1/(1-\phi z) = -\sum_{j=1}^{\infty} \phi^{-j} z^{-j}$ for $|\phi| > 1$ and $|z| \geq 1$.

$$\sum_{j=1}^{+\infty} (\phi z)^{-j} = \sum_{j=0}^{+\infty} (\phi z)^{-j} - 1 = \frac{1}{1 - \frac{1}{\phi z}} - 1$$

$$= \frac{\phi z}{\phi z - 1} - 1$$

$$= \frac{1}{\phi z - 1} \quad (\text{dfcm})$$

2.8 Show that the autoregressive equations

$$X_t = \phi_1 X_{t-1} + Z_t, \quad t = 0, \pm 1, \ldots,$$

$$\sum \frac{1}{\phi z^j} = \frac{1}{1 - \frac{1}{\phi z}} - 1 = \frac{\phi z}{\phi z - 1} - 1 \quad \frac{1}{\phi z - 1}$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $|\phi| = 1$, have *no stationary solution*. HINT: Suppose there does exist a stationary solution $\{X_t\}$ and use the autoregressive equation to derive an expression for the variance of $X_t - \phi_1^{n+1} X_{t-n-1}$ that contradicts the stationarity assumption.

Suppose that there exists a stationary solution to the equation

We denote that solution as $\{X_t\}$

That means:

$$Z_t = X_t - \phi X_{t-1}$$

$$\mathrm{Var}(X_t - \phi^{n+1} X_{t-n-1}) = \mathrm{Var}(Z_t + \phi Z_{t-1} + \ldots + \phi^n Z_{t-n})$$

$$\Rightarrow Var(X_t) + Var(X_{t-n-1}) = \sigma^2 (1 + \phi^2 + \dots + \phi^{2n})$$

$$\Rightarrow Var(X_t) + Var(X_{t-n-1}) = (n+1)\sigma^2$$

$$\Rightarrow Var(X_t) \to \infty \quad \text{when} \quad t \to \infty$$

$$\Rightarrow E(X_t^2) \to \infty \quad \text{when} \quad t \to \infty$$