- Chapter 6 exercises

Chapter 6 exercises

6.1 Suppose that $\{X_t\}$ is an ARIMA(p, d, q) process satisfying the difference equations

$$\phi(B)(1-B)^d X_t = \theta(B) Z_t, \quad \{Z_t\} \sim \mathrm{WN}\left(0, \sigma^2\right).$$

Show that these difference equations are also satisfied by the process $W_t = X_t + A_0 + A_1 t + \cdots + A_{d-1} t^{d-1}$, where $A_0, \ldots, A_{d-1}$ are arbitrary random variables.

$$\phi(B) = 1 + \phi_1 B + \phi_2 B^2 + \ldots + \phi_p B^p$$

and

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$$

We have

$$\begin{aligned} \phi(B)(1-B)^d W_t &= \phi(B)(1-B)^d (X_t + A_0 + A_1 t + \ldots + A_{d-1} t^{d-1}) & (1) \\ &= \theta(B) Z_t + \phi(B)(1-B)^d (A_0 + A_1 t + \ldots + A_{d-1} t^{d-1} & (2) \end{aligned}$$

We know that if we apply the backward operator on a polynolial of d degree, we will receive the one of $d-1$ degree which results in the vanishing of the second term in the (2) equation.

6.2 Verify the representation given in (6.3.4).

$$X_t - \mu = \phi_1 (X_{t-1} - \mu) + \cdots + \phi_p \left(X_{t-p} - \mu\right) + Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$

This model can be rewritten as (see Problem 6.2)

$$\nabla X_t = \phi_0^* + \phi_1^* X_{t-1} + \phi_2^* \nabla X_{t-1} + \cdots + \phi_p^* \nabla X_{t-p+1} + Z_t, \qquad (6.3.4)$$

where $\phi_0 = \mu\left(1 - \phi_1 - \cdots - \phi_p\right)$, $\phi_1^* = \sum_{i=1}^p \phi_i - 1$, and $\phi_j^* = -\sum_{i=j}^p \phi_i$, $j = 2, \ldots, p$. If the autoregressive polynomial has a unit root at 1, then $0 = \phi(1) = -\phi_1^*$,

We have the following model:

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + ... + \phi_p(X_{t-p} - \mu) + Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

Subtract both sides by X_{t-1},

$$\begin{gathered} X_t - X_{t-1} = \mu - X_{t-1} + \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - X_{t-1} + X_{t-1} - \mu) \\ +... + \phi_p(X_{t-p} - X_{t-p+1} + X_{t-p+1} - ... - X_{t-1} + X_{t-1}) + Z_t \end{gathered}$$

which is equivalent to

$$\begin{gathered} \nabla X_t = \mu - X_{t-1} + \phi_1(X_{t-1} - \mu) + \phi_2(-\nabla X_{t-1} + X_{t-1} - \mu) + .. + \\ +\phi_p(-\nabla X_{t-p} - \nabla X_{t-p+1} - ... - \nabla X_{t-1} + X_{t-1} - \mu) + Z_t \end{gathered}$$

by a simple algebraic transformation

$$\begin{gathered} \nabla X_t = \mu(1 - \phi_1 - ... - \phi_p) + (\phi_1 + \phi_2 + ... + \phi_p - 1)X_{t-1} + \\ (-phi_1 - \phi_2 - ... - \phi_p)\nabla X_{t-1} + (-\phi_2 - \phi_3 - ... - \phi_p)\nabla X_{t-2} ... + (-\phi_p)\nabla X_{t-p} + Z_t \end{gathered}$$

and we have $(6.3.4)$ representation.