probability starts with a sample space that describes possible outcomes in an experiment.

— coin tossing — two possible $\{S, F\}$

$\Omega = \{H, T\}$. (one toss) — medicine to subject.

— # email msg received in a week

$\Omega = \{0, 1, 2, \cdots\}$

Sample spaces are sets. If Ω is a sample space and $A \subset \Omega$, we call A an event.

— $A = \{$ three heads when a coin 6 times $\}$.

In an experiment w/ two outcomes. S and F, which is performed n times

$A = \{ k$ successes were obseved $\}$

$k = \{0, 1, 2, \cdots n\}$.

— A and B are events.

$$A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}$$
$$A \setminus B = \{x \in \Omega : x \in A \text{ and } x \notin B\}$$
$$A^c = \{x \in \Omega : x \notin A\}$$
$$A \Delta B = \{x \in \Omega : A \setminus B \cup B \setminus A\}$$

Symmetric diff of A and B.

ϕ : empty set

Definition. A probability is an assignment of a value to each event in a sample space, the value must lie in $[0, 1]$. and is $P(A)$. denotes the value assigned to A and $A_1, A_2, \cdots A_n$ are mutually disjoint events then.

$$P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} P(A_j) \quad \text{and } P(\Omega) = 1$$

$$P(\phi) = 0 \quad 1 = P(A \cup A^c) = P(A) + P(A^c)$$

$$\boxed{P(A^c) = 1 - P(A)}$$

counting (when the probability at each outcome in a sample space is the same)

$$P(A) = \frac{\#A}{\#\Omega}$$

Multiplication principle.

If experiment 1 has m possible outcomes and experiment 2 has n possible outcomes, then exp1. and 2 have mn possible outcomes

$\underset{job1}{\underline{47}}\ \underset{job2}{\underline{46}}\ \underset{job3}{\underline{45}}$ # of ways to form an ordered. Sample w/out replacement from a group of 47

$\underline{47}\ \underline{47}\ \underline{47}$ # ways to select an ordered sample w/ replacement from a group of 47

How many ways can a comittee of size 5 be selected from a group of 47.

$$\frac{\underline{47}\ \underline{46}\ \underline{45}\ \underline{44}\ \underline{43}}{5!}$$

overcounted

ways ~~to~~ arrange 5.

permutations.

A permutation is an ordering of distinguishable objects.

— How many ways to select r objects w/out replacement from a group of n objects?

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

p(full house)

$$\frac{13\binom{4}{3}\,12\binom{4}{2}}{\binom{52}{5}}$$

p(flush)

$$\frac{4\binom{13}{5}}{\binom{52}{5}}$$

Store n objects in r closets w/ n_1 objects going in closet 1,
$n_2 \to$ closet 2, $\cdots$ $n_r \to$ closet r. $\sum_{i=1}^{r} n_i = n$.

How many ways ?

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2\cdots}{n_r} = \binom{n}{n_1\, n_2 \cdots n_r}$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_r!}$$ multinomial coefficient.

Multinomial Theorem

$$(X_1+X_2+\cdots+X_r)^n = \sum \binom{n}{n_1\, n_2 \cdots n_r} X_1^{n_1} X_2^{n_2} \cdots X_r^{n_r}$$

$$\overline{\ \ }_1\ \overline{\ \ }_2\ \overline{\ \ }_3\ \overline{\ \ }_4\ \overline{\ \ }_5\ \overline{\ \ }_6\ \overline{\ \ }_7 \qquad \binom{52}{5\,5\,5\,5\,5\,5\,22} = \frac{52!}{(5!)^6 \cdot 22!}$$

Conditional probability & Independence

If $A, B \subset \Omega$ and $P(B) > 0$, then $P(A|B) = \frac{P(AB)}{P(B)}$

$$P(A \cap B) = P(A|B) \cdot P(B).$$

Law of total probability

If $B_1, B_2 \cdots B_n$ are mutually disjoint, $\bigcup_{i=1}^{n} B_i = \Omega$ $P(B_i) > 0$

$$P(A) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i).$$

Bayes Rule

Under same conditions as Law of Total prob.

$$P(B_j|A) = \frac{P(A|B_j)\, P(B_j)}{\sum_{i=1}^{n} P(A|B_i) P(B_i)}.$$

Independence

A and B are called independent if

$$P(A \cap B) = P(A) P(B)$$

If $P(B) > 0$ this is equivalent to

$$P(A|B) = P(A)$$

Random variables

Job Doob
Willam Feller

$X. \ \Omega \to \mathbb{R}$ by $X(\omega)$. for $\omega \in \Omega$.
is a function.

e.g. # of success in n trails.

distribution function of X $\quad F(x) = P(X \le x) \quad P\{\omega \in \Omega \ X(\omega) \le x\}$

i) $0 \le F(x) \le 1$

ii) $F(x)$ is a nondecreasing function

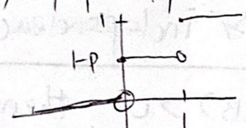
iii) F is right continuous.

discrete r.v.s dist'n has jumps interupted by flat spots.

continuous r.v.s. dist'n is continuous

Ex 1. Bernoulli dist'n.

PMF $\quad P(X=0) = 1-p \quad P(X=1) = p \quad F(x) = P(X \le x)$



$F(x) = \sum_{y \le x} P(y)$

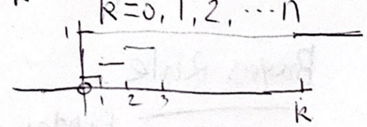
Ex 2. Binomial R.V.

n. independent Bernoulli trials performed in an exp.

with prob. p of success, $1-p$ of failure and

X = # successes in n trails. $\quad Bi(n, p)$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, 2, \cdots n$



Ex 3. Geometric R.V.

perform ind. Bernoulli trails until a success occurs

X = # of trails on which 1st success occurs.

$P(X=k) = (1-p)^{k-1} p \qquad k = 1, 2, 3, \cdots$

$P(X < \infty) = \sum_{k<\infty} P(k) = \sum_{k<\infty} (1-p)^{k-1} p = p \sum_{k<\infty} (1-p)^{k-1} = p \sum_{j=0}^{\infty} (1-p)^j$

$= \frac{p}{1-(1-p)} = 1.$

Ex 4. Negative Binomial.

perform iid. Bernoulli trails until r successes occur.

$X=$ # of trails on which r^{th} success occured.

$$p(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$$

$$F(X)=\sum_{k\le X}p(k)$$

Ex 5. poisson R.V. $P(\lambda)$ $\lambda>0$

$$p(X=k)=\frac{\lambda^k}{k!}e^{-\lambda} \quad k=0,1,2,\cdots$$

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}=e^{-\lambda}\cdot e^{\lambda}=1$$

Let X_n be $B_i(n,P_n)$ with $\lim_{n\to\infty}nP_n=\lambda$ $(0,\infty)$

$$p(X_n=0)=(1-P_n)^n\doteq\left(1-\frac{\lambda}{n}\right)^n\to e^{-\lambda}$$

$$\frac{p(X_n=k+1)}{p(X_n=k)}=\frac{\binom{n}{k+1}P_n^{k+1}(1-P_n)^{n-k-1}}{\binom{n}{k}P_n^k(1-P_n)^{n-k}}=\frac{1}{k+1}(n-k)P_n(1-P_n)^{-1}$$

$$\downarrow n\to\infty$$

$$\frac{1}{k+1}\cdot\lambda=\frac{\lambda}{k+1}$$

$$\lim_{n\to\infty}P_n(1)=\lambda e^{-\lambda}$$

$$\lim_{n\to\infty}P_n(2)=\frac{\lambda^2}{2}e^{-\lambda}$$

$$\lim_{n\to\infty}P_n(3)=\lim_{n\to\infty}\frac{P_n(3)}{P_n(2)}\cdot P_n(2)=\frac{\lambda^3}{3!}e^{-\lambda}$$

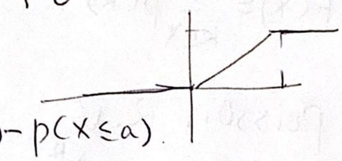
$n\to\infty$. P_n small $=\frac{\lambda}{n}$ poisson $\Leftrightarrow$ n. Bernoulli

Continuous R.V.S. · If F has a derivative $F' = f$, then (mostly true)

$$F(x) = \int_{-\infty}^{x} f(y)\,dy \qquad f: \text{density of } X$$

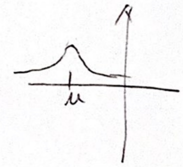
Ex.1. Uniform R.V. on $[0,1]$

density: $f(x) = \begin{cases} 1, & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$



$$p(a < X \le b) = p(X \le b) - p(X \le a)$$
$$= \int_a^b f(x)\,dx$$

Ex 2. $X \sim N(\mu, \sigma^2)$. normal dist.



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

If $X \sim N(0,1)$ · $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

$$1 \overset{?}{=} p(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx.$$
even function
$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}}\,dx$$

proof: $\frac{2}{\pi}\left(\int_0^{\infty} e^{-\frac{x^2}{2}}dx\right)^2 = \frac{2}{\pi}\int_0^{\infty} e^{-\frac{x^2}{2}}dx \int_0^{\infty} e^{-\frac{y^2}{2}}dy = \frac{2}{\pi}\iint e^{-\frac{x^2+y^2}{2}}\,dx\,dy$

polar space

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-\frac{r^2}{2}}\, r\,dr\,d\theta = \int_0^{\infty} r e^{-\frac{r^2}{2}}\,dr = 1$$

$$\left\{ p\left(\frac{X}{\sqrt{n}} \le x\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy. \right\}$$

Bernoulli $\rightarrow$ Normal. $p = \frac{1}{2}$.

— X Geometric r.v. w/ parameter p

$$P(X>n+k-1 \mid X>n-1) = \frac{P(X>n+k-1)}{P(X>n-1)} = \frac{1-P(X\le n+k-1)}{1-P(X\le n-1)}$$

numerator
$$= \sum_{k+1}^{\infty}(1-p)^{j-1}p = p\sum_{k+1}^{\infty}(1-p)^{j-1} = p\sum_j p(1-p)^k\sum_{j=0}^{\infty}(1-p)^j = (1-p)^k$$

$$= \frac{(1-p)^{n+k-1}}{(1-p)^{n-1}} = (1-p)^k$$ no memory

— If $\lambda>0$. X is called an exponential r.v. w/ parameter λ

$$P(X\le x) = \begin{cases} 1-e^{-\lambda x} & x\ge 0 \\ 0 & x<0\end{cases}$$ (waiting time)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x\ge 0 \\ 0 & x<0\end{cases}$$

$$P(y<X\le x) = F(x)-F(y) = \int_y^x f(t)\,dt$$

$$P(X>s+t \mid X>t) = P(X>s)$$ e.g. light bulb burned out

— The gamma function is $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, $\alpha>0$

$$\Gamma(\alpha) = -x^{\alpha-1}e^{-x}\Big|_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx = (\alpha-1)\int_0^\infty x^{(\alpha-1)-1}e^{-x}dx$$

$$\left.\begin{aligned}\Gamma(\alpha) &= (\alpha-1)\Gamma(\alpha-1) \\ \Gamma(1) &= 1\end{aligned}\right\} \Gamma(n) = (n-1)! \quad = (\alpha-1)\Gamma(\alpha-1)$$

$$\int_0^\infty x^{\alpha-1}\cdot e^{-\lambda x}dx = \frac{1}{\lambda}\lambda^{1-\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy = \lambda^{-\alpha}\Gamma(\alpha) = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-\lambda x}dx$$ is a density.

X is a $\Gamma(\alpha,\lambda)$ r.v. if its density is above.

traffic waiting time (5^{th} car)

Functions of r.v.s

strictly inc

X dist'n F. what is dist'n of $g(X)$?

$$P(g(X)\le x) = P(X \le g^{-1}(x)) = F(g^{-1}(x))$$

— X uniform dist'd on $[0,1]$. $g(x) = -\frac{1}{\lambda}\ln(1-x)$. $\leftrightarrow$ inc. function

$$F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Uniform $\longrightarrow$ exponential

$$P(X \le \underbrace{1-e^{-\lambda x}}_{[0,1]}) = 1-e^{-\lambda x}$$

— $X \sim N(\mu, \sigma^2)$. $Y = aX+b$. what is dist'n of Y?

$$P(Y \le x) = P(aX+b \le x) = P(X \le \frac{x-b}{a}) \quad \text{if } a>0$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\frac{x-b}{a}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\, dy$$

$$\boxed{\frac{d}{dx}\int_{-\infty}^{h(x)} f(y)\,dy \qquad f(h(x))\cdot h'(x)}$$

The density of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\left(\frac{x-b}{a}-\mu\right)^2}{2\sigma^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{2\pi\sigma^2 a^2}} \exp\left\{-\frac{(x-(b+a\mu))^2}{2\sigma^2 a^2}\right\}$$

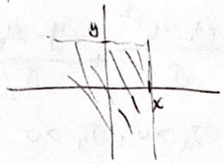
$$\sim N(a\mu+b,\ a^2\sigma^2)$$

$\boxed{X \sim (\mu, \sigma^2) \Rightarrow X \sim N(0,1)}$

$a = \frac{1}{\sigma}$ $\cancel{a=\frac{1}{\sigma}}$, $b = -\frac{\mu}{\sigma}$ $\boxed{X \sim N(0,1) \Rightarrow \sigma X + \mu \sim N(\mu, \sigma^2)}$

Jointly distributed r.v. s

(X, Y) has joint dist'n F if $P(X \le x, Y \le y) = F(x, y)$



$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds$$

$$P(X \le x, Y \le y) = \sum_{u \le x, v \le y} p(u,v)$$

~~Ex 1~~ From Joint dist'n $F(x, y)$ of (X, Y), we can retrieve the marginal dist'n

$$F_X(x) = \begin{cases} \sum_{u \le x} \sum_{v} f(u,v) & \text{discrete case} \\ \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,v)\,dv\,du & \text{cont. case} \end{cases}$$

$$f_X(x) = \sum_{v} f(u,v) \quad \text{discrete case}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,v)\,dv \quad \text{cont. case}$$

Ex: $f(x, y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x \le y, \\ 0 & \text{otherwise} \end{cases}$ $\lambda > 0$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,v)\,dv = \lambda^2 \int_{x}^{\infty} e^{-\lambda v}\,dv = -\lambda e^{-\lambda v}\Big|_{x}^{\infty} \quad x > 0.$$

$$= \lambda e^{-\lambda x} \quad x > 0.$$

exponential.

$$f_Y(y) = \lambda^2 \int_0^y e^{-\lambda y}\,du = \lambda^2 y e^{-\lambda y} \sim \Gamma(2, \lambda).$$

conditional densities

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

$$P_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)}$$

Joint Normal Distribution

(X,Y) has joint normal dist if its density is

$$f(x,y)=\frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2(1-\rho^2)}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

$\rho>0 \quad \sigma_x>0,\ \sigma_y>0$

Independence

$$\begin{cases} F(x,y)=F_X(x)\cdot F_Y(y) \\ f(x,y)=f_X(x)\cdot f_Y(y) \qquad p(x,y)=p_X(x)\cdot p_Y(y) \\ f_{X|Y}(x,y)=f_X(x) \qquad p_{X|Y}(x|y)=p_X(x) \end{cases}$$

$\rho=0$

$$\frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$=\frac{1}{\sqrt{2\pi\sigma_x^2}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}\cdot\frac{1}{\sqrt{2\pi\sigma_y^2}}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

$$f(x,y)=\begin{cases}\frac{1}{\pi r^2}, & \text{if } x^2+y^2<r^2 \\ 0 & \text{otherwise}\end{cases}$$

X and Y ind ?

~~Marginal~~

(X,Y) joint pmf $p(x,y)$, what is pmf of $X+Y$?

$$p(X+Y=z)=\sum_x p(x,z-x)$$

if X & Y ind

$$\sum_x p_X(x)\,p_Y(z-x)$$

Convolution.

Ex1 $X\sim P(\lambda)$, $Y\sim P(\mu)$. X & Y ind.

discrete

$$p(X+Y=z)=\sum_{x=0}^{z}p_X(x)\,p_Y(z-x)=\sum_{x=0}^{z}\frac{\lambda^x}{x!}e^{-\lambda}\cdot\frac{\mu^{z-x}}{(z-x)!}e^{-\mu}$$

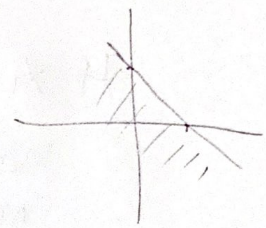
$$=\frac{1}{z!}e^{-(\lambda+\mu)}\sum_{x=0}^{z}\frac{z!}{x!(z-x)!}\lambda^x\cdot\mu^{z-x}=\frac{1}{z!}e^{-(\lambda+\mu)}\underbrace{\sum_{x=0}^{z}\binom{z}{x}\lambda^x\mu^{z-x}}_{(\lambda+\mu)^z}$$

$$=\frac{(\lambda+\mu)^z}{z!}e^{-(\lambda+\mu)}$$

poisson.

Ex 2 cont.

$$P(X+Y \le z) = \iint_{\{(x,y),\, x+y\le z\}} f(x,y)\,dx\,dy$$



$$F_{X+Y}(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dy\,dx \;\cancel{dydx}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z} f(x, v-x)\,dv\,dx$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx \overset{\text{if ind}}{=} \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

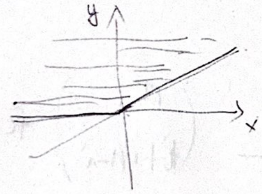
X & Y are ind. $N(0,1)$

what is dist'n of $X+Y$?

$$\boxed{\frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}}} \quad \sim N(0,2)$$

Ex 3 $\frac{X}{Y} = h(x,y)$. $P(X/Y \le z) = P(X/Y \le z,\ Y>0) + P(X/Y \le z,\ Y<0)$

$= P(X \le yz, Y>0) + P(X \ge zY, Y<0)$



both invertable transformation

$u = g_1(x,y)$. $v = g_2(x,y)$ $\quad (x,y) \to (u,v)$

$$J(x,y) = J = \det\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} \ne 0 \quad \text{assume}$$

$$f_{UV}(u,v) = f(h_1(u,v), h_2(u,v))\,\left|J^{-1}(h_1(u,v), h_2(u,v))\right|$$

Order Statistics

If $X_1, X_2, \cdots X_n$ are iid w/. dist'n $F(x) = P(X_i \le x)$. $i = 1, 2 \cdots n$

maximum: U

minimum V

$F_U(u) = P(U \le u) = P(X_1 \le u)P(X_2 \le u) \cdots P(X_n \le u) = [F(u)]^n$

$\therefore f_U(u) = n f(u) \cdot F(u)^{n-1}$

$F_V(v) = P(V \le v) = \cancel{P(X_1 \le v)P(X_2 \le v) \cdots P(X_n \le v)}$

$= \cancel{(1 - P(X_1 > v))^n = 1}$

$= 1 - P(V > v) = 1 - P(X_1 > v)P(X_2 > v) \cdots P(X_n > v)$

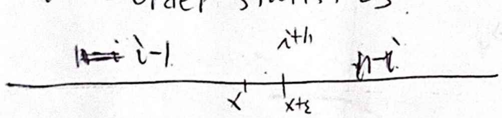
$= 1 - [1 - F(v)]^n$

$f_V(v) = n f(v) \cdot [1 - F(v)]^{n-1}$

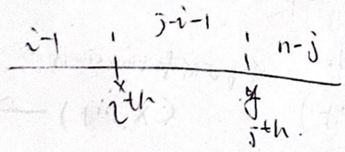
Ex. If $X_1, X_2, \cdots X_n$ are iid $\mathcal{E}(\lambda)$. Find the dist'n of ~~max~~ min.

$f_X(x) = n\lambda e^{-\lambda x}(e^{-\lambda x})^{n-1} = n\lambda e^{-n\lambda x} \sim \mathcal{E}(n\lambda)$.

Ex. i^{th}-order statistics



$\binom{n}{i-1,\ 1,\ n-i} f(x) F(x)^{i-1} (1 - F(x))^{n-i}$

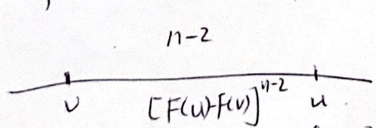


$F(x \le X \le x+\varepsilon) = \int_x^{x+\varepsilon} f(x)\,dx$

Joint dist'n of order statistics

Joint density of the min and max.

$f(u, v) = n(n-1) f(u) f(v) [F(u) - F(v)]^{n-2} \quad u \ge v$



Expectations

discrete $E[X] = \sum_x x\,p(x)$

— $X \sim P(\lambda)$. $E(X) = \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda.$

— Geometric $E[X] = \sum_{x=1}^{\infty} x\,p(x) = \sum_{x=1}^{\infty} x \cdot (1-p)^{x-1} \cdot p = p \sum_{x=1}^{\infty} x (1-p)^{x-1}$

$$= -p \sum_{x=0}^{\infty} \frac{d(1-p)^x}{dp} = -p \frac{d}{dp} \sum_{x=0}^{\infty} (1-p)^x = -p \cdot \frac{d}{dp} \frac{1}{1-(1-p)} = \frac{1}{p}$$

Continuous $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$

— Uniform $E[X] = \int_0^1 x\,dx = \frac{1}{2}$

$X \sim \Gamma(\alpha, \lambda)$. $f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad x > 0$

— $E[X] = \int_0^{\infty} x f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \underbrace{\int_0^{\infty} x^{\alpha+1-1} e^{-\lambda x} dx}_{\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}.$

$$E[g(X)] = \begin{cases} \sum_x g(x) p_X(x) \\ \int_{-\infty}^{\infty} g(x) f_X(x)\,dx. \end{cases}$$

EX. $X \sim N(\mu, \sigma^2)$

compute $E(X)$, $E[(X-\mu)^2]$, $E[e^{tX}]$

$$E[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{\text{odd function}} + \frac{\mu}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= 0 + \mu = \mu$$

$$E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy.$$

$\left(y = \frac{x-\mu}{\sigma},\ dy = \frac{1}{\sigma} dx \right)$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \underbrace{y}_{u} \cdot \underbrace{y e^{-\frac{y^2}{2}} dy}_{dv} = \frac{2\sigma^2}{\sqrt{2\pi}} \left[-y \cdot e^{-\frac{y^2}{2}} \Big|_0^{\infty} + \underbrace{\int_0^{\infty} e^{-\frac{y^2}{2}} dy}_{\sqrt{2\pi}} \right]$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} = \sigma^2$$

variance

$\int_0^{\infty} -y \cdot d e^{-\frac{y^2}{2}}$

$$E[e^{tx}] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{x^2-(2\mu+2\sigma^2 t)x+\mu^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx \cdot e^{\frac{(\mu+\sigma^2 t)^2-\mu^2}{2\sigma^2}}$$

$$= e^{\mu t + \frac{\sigma^2}{2}t^2}$$

$\boxed{\text{Moment Generation function}}$

· one-to-one mapping between $M_X(t)$ and $f_X(x)$

$$M_X(t) = E[e^{tX}] \doteq \begin{cases} \sum_x e^{tx} p_X(x) \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx \end{cases}$$

Ex. $X \sim \Gamma(\alpha, \lambda)$. $f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $x>0$

$$M_X(t) = E[e^{tX}] \doteq \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{tx} x^{\alpha-1} e^{-\lambda x} dx \qquad t<\lambda.$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$$

Ex. $X_1 \sim \Gamma(\alpha, \lambda)$, $X_2 \sim \Gamma(\beta, \lambda)$ what is dist'n of X_1+X_2 ?

$$E[e^{t(X_1+X_2)}] \overset{\text{iid}}{=} E[e^{tX_1}]E[e^{tX_2}] = \left(\frac{\lambda}{\lambda-t}\right)^\alpha \left(\frac{\lambda}{\lambda-t}\right)^\beta = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha+\beta}$$

$$\sim \Gamma(\alpha+\beta, \lambda).$$

Ex $X \sim P(\lambda)$.

$$p(X=k) = p(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k=0,1,2,\cdots$$

$$E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Ex $Y \sim P(\mu)$

$X+Y \sim P(\lambda+\mu)$. $M_{X+Y}(t) = e^{(\lambda+\mu)(e^t-1)}$

Ex $X \sim$ Bernoulli $(p, 1-p)$

$$M_X(t) = pe^t + (1-p)$$

$X_1, X_2, \cdots X_n$ are n ind. Bernoulli w/. Parameter p.

$$Y_i = \frac{X_i - p}{\sqrt{p(1-p)}}$$

$$E[Y_i] = \frac{1}{\sqrt{p(1-p)}} E[X_i] - \frac{p}{\sqrt{p(1-p)}} = 0$$

$$Var(Y_i) = 1.$$

$$M_{Y_i}(t) = E[e^{tY_i}] = E\left[e^{t\frac{X_i - p}{\sqrt{p(1-p)}}}\right] = e^{\frac{-tp}{\sqrt{p(1-p)}}}\left(p e^{\frac{t}{\sqrt{p(1-p)}}} + (1-p)\right)$$

$$= p \cdot e^{\frac{1-p}{\sqrt{p(1-p)}}t} + (1-p) \cdot e^{-\frac{p}{\sqrt{p(1-p)}}t} \qquad p = \frac{1}{2}$$

$$= \cancel{pe^{t} + (1-p)} = \frac{e^{t} + e^{-t}}{2} = \cos t$$

$$\boxed{e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots}$$

$$M_{\frac{Y_1 + Y_2 + \cdots + Y_2}{\sqrt{n}}}(t) \approx e^{\frac{t^2}{2}} \quad \sim N(0,1)$$

☆ $(\ln M_X(t))'' = (\ln E[e^{tX}])''$ take derivative twice and evaluate at $t=0$

$$= \left(\frac{E[Xe^{tX}]}{E[e^{tX}]}\right)' = \frac{E[X^2 e^{tX}] E[e^{tX}] - E[Xe^{tX}] E[Xe^{tX}]}{E[e^{tX}]^2}$$

$t=0$ $\quad E[X^2] - E[X]^2 = Var(X)$

Ex $\quad M_X(t) = e^{\mu t + \frac{\sigma^2}{2}t^2} \quad = (\mu + \sigma^2 t)' = \sigma^2$

$$(\ln M_X(t))'' = \left(\mu t + \frac{\sigma^2}{2}t^2\right)''\Big|_{t=0} = \sigma^2$$

Ex. $\quad X \sim P(\lambda) \quad M_X(t) = e^{\lambda(e^t - 1)}$

$$\ln M_X(t)'' = (\lambda(e^t - 1))'' = \lambda e^t \Big|_{t=0} = \lambda.$$

$$cov(X,Y) \doteq E[(X - E[X])(Y - E[Y])]$$

if X & Y are indepedent $\quad cov(X,Y) = 0$

$$Var(X_1+X_2+\cdots+X_n) = E\left[\left(\Sigma X_i - \Sigma E(X_i)\right)^2\right] = E\left[\left(\Sigma(X_i - E(X_i))\right)^2\right]$$
$$= E\left[\sum_{i=1}^{n}(X_i - E(X_i))^2\right] + 2E\left[\sum_{i<j}(X_i-E(X_i))(X_j-E(X_j))\right]$$
$$= \sum_{i=1}^{n} E(X_i - E(X_i))^2 + 2\sum_{i<j} E\left[(X_i-E(X_i))(X_j-E(X_j))\right]$$
$$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} cov(X_i, X_j).$$

computations with joint normal distribution function

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_x\sigma_y} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\frac{\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_x^2}} \exp\left\{-\frac{\left(x-(\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y))\right)^2}{2(1-\rho^2)\sigma_x^2}\right\}$$

$$E[X|Y=y] = \begin{cases} \sum_x x\, P_{X|Y}(x|y) \\ \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx \end{cases}$$

<u>Ex.</u> Let T_1, T_2 be ind. $\mathcal{E}(\lambda)$.

$U = T_1$, $V = T_1 + T_2$. find $E(U|V=v)$.

$E(X|Y) = G(Y)$

$$E[G(Y)] = \sum_y G(y)\cdot P_Y(y) = \sum_y E(X|Y=y)\cdot P_Y(y)$$
$$= \sum_x\sum_y x\cdot P_{X|Y}(x|y)\,P_Y(y) = \sum_x x\sum_y P_{XY}(x|y)\,P_Y(y) = \sum_x x\cdot P_X(x)$$
$$= E[X]$$

$$E[E[X|Y]] = E[X]$$

Ex. $T_1, T_2, \dots$ are ind. $\mathcal{E}(\lambda)$ and independent of all those,

N is geometric w/ parameter p, compute.

$$E\left[\sum_{i=1}^{N} T_i\right]$$

$$G(n) = E\left[\sum_{i=1}^{N} T_i \mid N=n\right] = \sum_{i=1}^{n} E[T_i] = n\lambda.$$

$$E\left[\sum_{i=1}^{N} T_i\right] = E(G(N)) = E[N\lambda] = \frac{\lambda}{p}$$

— Chebyshev's inequality

$$P(|X-\mu| > a) \le \frac{\sigma^2}{a^2}.$$

proof: $P(|X-\mu|>a) = \int_A f_X(x)\,dx \le \int_A \frac{(x-\mu)^2}{a^2} f_X(x)\,dx \le \frac{1}{a^2}\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$

$$= \frac{\sigma^2}{a^2}.$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$E\left[\frac{S_n}{n}\right] = \mu \qquad Var\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \ge a\right) \le \frac{\sigma^2}{na^2}$$

LLN

$\overline{X}_n = \frac{S_n}{n} \longrightarrow \mu$ in probability. (weak law of large numbers)

— Central limit theorem.

Let $X_1, X_2, \dots$ be iid r.v.s. $E[X_1] = \mu$, $Var(X_1) = \sigma^2$

$$S_n = \sum_{j=1}^{n} X_j$$

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

— Continuity theorem

If Z_n is a sequence of random variables and Z is another random variable for which

$\lim_{n\to\infty} M_{Z_n}(t) = M_Z(t)$ for all t in some interval containing 0, then

$\lim_{n\to\infty} F_{Z_n}(z) = F_Z(z)$. at all points of continuity of $F_Z(z)$

EX. $X_1, X_2, \cdots$ Bernoulli r.v.s w/ parameter p of success

$\frac{S_n - np}{\sqrt{np(1-p)}}$ is approximately $N(0,1)$ distributed

~~$X \sim P(81)$ approximate $p(X \le 72)$~~

EX. toss a coin 100 times and lands heads up 60 times. Is the coin fair?

$1 - \Phi(2) = 0.0228$

— $U \sim N(0,1)$ $U^2 \sim \chi_1^2$ chi-squared with 1 degree of freedom

$$E[e^{tU^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tu^2} e^{-\frac{u^2}{2}} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2(1-2t)}{2}} du = \frac{1}{(1-2t)^{1/2}}$$

$U_1, U_2, \cdots U_n \overset{iid}{\sim} N(0,1)$

$$U_1^2 + U_2^2 + \cdots + U_n^2 \sim \chi_n^2$$

$$E[e^{t(U_1^2 + \cdots + U_n^2)}] = \frac{(\frac{1}{2})^{n/2}}{(\frac{1}{2} - t)^{n/2}} \sim \Gamma(\frac{n}{2}, \frac{1}{2})$$

— if $U \sim \chi_m^2$ ind $V \sim \chi_n^2$ m, n are non-negative

$$\frac{U/m}{V/n} \sim F_{m,n}.$$

— $Z \sim N(0,1) \overset{ind}{;} U \sim \chi_n^2$ then $Z / \sqrt{U/n}$ student t distribution.

go to Examples

$X_1, X_2, \cdots, X_n$ are r.v.s

Sample mean $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Sample Variance $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$.

if $X_1, X_2, \cdots X_n$ are iid ~~independent~~ $Var(X_1) = \sigma^2$ then $E[S^2] = \sigma^2$.
unbiased

$\overline{X}$ is independent of $(X_1 - \overline{X}, X_2 - \overline{X}, \cdots X_n - \overline{X})$.

Ex 11

Probability and stats

X, Y ind. $N(0,1)$ r.v.s , Find dist'n of $\frac{X}{Y}$

① $$P\left(\frac{X}{Y} \le x\right) = \iint_{\{(u,v):\, \frac{u}{v} \le x\}} f_{XY}(u,v)\,du\,dv$$

$$= f_X(u) f_Y(v) = \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}$$

②

$$U = \frac{X}{Y} = g_1(X,Y), \quad V = Y = g_2(X,Y)$$

$$f_{UV}(u,v) = f_{XY}(h_1(u,v), h_2(u,v))\,|J^{-1}(h_1(u,v), h_2(u,v))|$$

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} \quad \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}$$

$X = UV$
$Y = V$
$h_1(u,v) = uv$
$h_2(u,v) = v$

i) U and V independent? (NO).

$$= \frac{1}{2\pi} e^{-\frac{u^2v^2+v^2}{2}} |v|.$$

$$f_U(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{u^2v^2+v^2}{2}} |v|\,dv = \frac{1}{\pi}\int_0^{\infty} e^{-\frac{u^2v^2+v^2}{2}} v\,dv$$

$$= \frac{1}{\pi(1+u^2)}$$ Cauchy density

density factorized?

$$\frac{1}{2\pi} e^{-\frac{(u^2+1)v^2}{2}} |v|.$$

ii) $$f_{U|V}(u,v) = \frac{f_{UV}(u,v)}{f_V(v)} = \frac{\frac{|v|}{2\pi} e^{\frac{-(1+u^2)v^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}} = \frac{|v|}{\sqrt{2\pi}} e^{-\frac{u^2v^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi\frac{1}{v^2}}} e^{-\frac{u^2}{2\cdot\frac{1}{v^2}}} \sim N\left(0, \frac{1}{v^2}\right)$$

$$f_{V|U}(u,v) = \frac{f_{UV}(u,v)}{f_U(u)} = \frac{\frac{|v|}{2\pi} e^{-\frac{(1+u^2)v^2}{2}}}{\frac{1}{\pi(1+u^2)}} = \frac{1}{2}|v|(1+u^2)\,e^{-(1+u^2)v^2/2}$$

Moment generating functions.

$X \sim \mathcal{E}(\lambda)$ exponential w/ parameter λ.

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \lambda \int_0^\infty e^{tx} e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-(\lambda-t)x}\,dx$$
$$= \frac{\lambda}{\lambda - t} \quad (t < \lambda)$$

Variation 1. If $X_1, X_2, \cdots X_n$ are iid $\mathcal{E}(\lambda)$.
what is dist'n of $X_1 + X_2 + \cdots + X_n$?

$$M_{X_1+X_2+\cdots+X_n}(t) = E\left[e^{t(X_1+X_2+\cdots+X_n)}\right] = E\left[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}\right] = E[e^{tX_1}]^n$$
$$= \left(\frac{\lambda}{\lambda-t}\right)^n$$

If $Y \sim \Gamma(\alpha, \lambda)$

$$f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot y^{\alpha-1} \cdot e^{-\lambda y} \quad y \geq 0$$

$$M_Y(t) = E[e^{tY}] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty e^{ty} y^{\alpha-1} e^{-\lambda y}\,dy = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-(\lambda-t)y}\,dy$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$$

$$X_1 + X_2 + \cdots + X_n \sim \Gamma(n, \lambda)$$

Variation 2. If $X_1, X_2, X_3 \cdots$ are ind $\mathcal{E}(\lambda)$
and N is ind. of $X_1, X_2, X_3 \cdots$ geometric dist'n
w/ parameter p, what is dist'n of $\sum_{i=1}^{N} X_i$?

$$M_Z(t) = E[e^{tZ}] = E\left[E[e^{tZ} \mid N]\right] = E[g(N)]$$

$$g(n) = E\left[e^{tZ} \mid N=n\right] = \int_0^\infty e^{tz} f_{Z|N}(z|n)\,dz = \frac{\lambda^n}{\Gamma(n)} \int_0^\infty z^{n-1} e^{-(\lambda-t)z}\,dz$$

$$f_{Z|N}(z|n) = \frac{\lambda^n}{\Gamma(n)} \cdot z^{n-1} e^{-\lambda z} \quad z \geq 0 \qquad = \frac{\lambda^n}{\Gamma(n)} \cdot \frac{\Gamma(n)}{(\lambda-t)^n} = \left(\frac{\lambda}{\lambda-t}\right)^n$$

$$E[g(N)] = \sum_{k=1}^{\infty} g(k) \cdot (1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda-t}\right)^k (1-p)^{k-1} \cdot p = p \cdot \frac{\lambda}{\lambda-t} \sum_{k=1}^{\infty} \left(\frac{\lambda(1-p)}{\lambda-t}\right)^{k-1}$$
$$= \frac{\lambda p}{\lambda - t} \cdot \sum_{j=0}^{\infty} \left(\frac{\lambda(1-p)}{\lambda-t}\right)^j = \frac{\lambda p}{\lambda-t} \cdot \frac{1}{1 - \frac{\lambda(1-p)}{\lambda-t}} = \frac{\lambda p}{\lambda p - t}$$

Geometric series. exponential w/ parameter λp.

— $X_1\ X_2 \cdots X_n$ Bernoulli $p<\frac{1}{2}$

what is $P(X_1+X_2+\dots+X_n > \frac{n}{2})$ (approximately) ?

$\frac{X_1+X_2+\dots+X_n-np}{\sqrt{p(1-p)n}}$ has an approximately $N(0,1)$ dist'n

\$1/2 to play

\$1 win

$$P(X_1+X_2+\dots+X_n > \frac{n}{2}) = P\left(\frac{X_1+X_2+\dots+X_n-np}{\sqrt{p(1-p)n}} > \frac{(\frac{1}{2}-p)n}{\sqrt{p(1-p)n}}\right)$$

$$\approx P\left(Z > \frac{\frac{1}{2}-p}{\sqrt{p(1-p)}}\sqrt{n}\right) \qquad Z\sim N(0,1)$$

$$\le \frac{1}{\sqrt{2\pi}}\int_{\frac{\frac{1}{2}-p}{\sqrt{p(1-p)}}\sqrt{n}}^{\infty} \frac{y}{\frac{\frac{1}{2}-p}{\sqrt{p(1-p)}}\sqrt{n}} e^{-\frac{y^2}{2}}dy \le \frac{1}{\sqrt{2\pi}}\frac{\sqrt{p(1-p)}}{(\frac{1}{2}-p)\sqrt{n}}\int ye^{-\frac{y^2}{2}}dy$$

— X_1 = # of α-particles emitted in 1 hour.

$X_1 \sim P(\lambda)$. where $E(X_1)=\lambda=10$

$$P(X_1=k)=\frac{\lambda^k}{k!}e^{-\lambda} \quad k=0,1,2,\cdots$$

X_i = # emitted in i^{th} hr. $X_1, X_2, \cdots, X_{10}$ are ind.

$$Y = X_1+X_2+\dots+X_{10} \sim P(10\lambda)$$

$$E[e^{tX_1}] = \sum_{k=0}^{\infty} e^{tk}P(X_1=k) = \sum_{k=0}^{\infty} e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

[Central limit Theorem]

$$\frac{Y-100}{\sqrt{100}} \sim N(0,1)$$

$$P(Y>95) = P\left(\frac{Y-100}{10} > -0.5\right) = P(Z<\tfrac{1}{2})$$

$= 0.6915$

EX 4

If $X_1, X_2, \cdots X_n$ are iid r.v.s with $Var(X_1)=\sigma^2$

What is an unbiased estimator of σ^2 ?

$$\bar{X} = \frac{1}{n}(X_1+X_2+\cdots+X_n) \quad E(\bar{X}) = E(X_1) = \mu$$

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n}X_i^2 - \frac{n}{n-1}\bar{X}^2$$

$$E\left[\frac{1}{n-1}\sum_{i=1}^{n}X_i^2 - \frac{n}{n-1}\bar{X}^2\right] = \frac{n}{n-1}(\sigma^2+\mu^2) - \frac{n}{n-1}\left(\mu^2+\frac{1}{n}\sigma^2\right)$$

$$= \sigma^2$$

$$E[\bar{X}^2] = \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right]$$

$$= \frac{1}{n^2}E\left[\sum_{i=1}^{n}X_i^2 + \sum_{i\neq j}X_iX_j\right]$$

$$= \frac{1}{n^2}\left[n(\mu^2+\sigma^2) + (n^2-n)\mu^2\right]$$

$$= \mu^2 + \frac{1}{n}\sigma^2$$

$$\frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$$

$$\frac{n-1}{\sigma^2} S^2 = \frac{1}{\sigma^2} \Sigma (X_i - \bar{X})^2 = \Sigma \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{n-1}$$

Simple random Sampling

population. $X_1, X_2 \cdots X_N$.

true mean $\mu = \frac{1}{N} \sum_{i=1}^{N} X_i$.

true variance $\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (X_i - \mu)^2$

$X_1, X_2 \cdots X_n$. — select n individuals randomly

random Sampling

to estimate the population parameters μ, σ^2.

take a random sample of size n. w/out replacement.

$X_1, X_2, \cdots, X_n$

$E[\bar{X}] = \mu$

to tell how good $\bar{X}$ is an estimator of μ, we should get info about ~~the~~ $Var(\bar{X})$.

$$\boxed{Cov(X_i, X_j) = -\frac{\sigma^2}{N-1}}$$

$$Var(\bar{X}) = -\frac{n^2 - n}{n^2} \frac{\sigma^2}{N-1} + \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i).$$

$$= \frac{1}{n^2} \sum_{i,j=1}^{n} Cov(X_i, X_j)$$

$$\boxed{= \frac{\sigma^2}{n} \underbrace{\left[1 - \frac{n-1}{N-1}\right]}_{\text{finite population correction}}}$$

Standard error $= \sqrt{Var(\text{estimator})}$

when estimating μ. we give a confidence interval for that parameter. This means we select an $\alpha \in (0, 1)$, close to 1, and find randomly located interval I ~~set~~ s.t. $P(\mu \in I) = \alpha$ ~~100(1-α)%~~ $100\alpha\%$ confidence interval

$N = 8{,}000$ $\quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = 1.6$ $\quad X_i$ # cars i^{th} resident has

$n = 100$

$\cancel{S} = 0.8$ $\quad S_{\overline{X}} = \frac{0.8}{\sqrt{100}}\sqrt{1-\frac{n-1}{N-1}} = 0.08$ $\quad \boxed{S_{\overline{X}}^2 = \frac{S^2}{n}\left(1-\frac{n-1}{N-1}\right)}$

σ^2 $\quad \cancel{\frac{S}{\sqrt{n}}\sqrt{1-\frac{n-1}{N-1}} \rightarrow Var(\overline{X})}$

$$\frac{\overline{X}-\mu}{\sqrt{S_{\overline{X}}^2}} = \frac{\overline{X}-E(\overline{X})}{Var(\overline{X})} = \frac{\overline{X}-1.6}{0.08}$$ approximatly $N(0,1)$

We want to find a 95% C.I for μ.

$$P\left(-Z(\alpha) \le \frac{\overline{X}-1.6}{0.08} \le Z(\alpha)\right) = \alpha = 0.95$$

$$P(1.6-0.08Z(\alpha) \le \overline{X} \le 1.6+0.08Z(\alpha)) = 0.95$$

C.I: $[1.6-0.08Z(\alpha),\ 1.6+0.08Z(\alpha)]$

$$= 2P(Z \le Z(\alpha)) - 1 = 0.95$$

$$P(Z \le Z(\alpha)) = 0.975 \qquad Z(\alpha) = 1.96$$

Unbiased estimators

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ to estimate μ.

$\frac{S^2}{n}\left(1-\frac{n}{N}\right)$ to estimate $\cancel{\sigma^2}$ $Var(\overline{X})$ $\quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$