TOTAL $\frac{18}{20}$

Excellent!

1) a. $\sup a = 5$ $\inf a = 1$

5 and 1 are in the set, as they are natural numbers

b. $\sup b = \sqrt{29}$ $\inf b = -\sqrt{29}$

the supremum and infimum are not in the given set, as they are irrational.

c. $\sup c = \sqrt{29}$ $\inf c = -\sqrt{29}$

the supremum and infimum are in the set as $\sqrt{29} \in \mathbb{R}$ and $-\sqrt{29} \in \mathbb{R}$

Good! $\frac{3}{3}$

2) a. Claim $\left(-\frac{1}{\sqrt{2n}}\right)_{n\geq 1} \to 0$

Proof let $\varepsilon > 0$.

$$\left|-\frac{1}{\sqrt{2n}} - 0\right| = \left|-\frac{1}{\sqrt{2n}}\right| = \frac{1}{\sqrt{2n}} < \varepsilon$$

$$\frac{1}{\varepsilon} < \sqrt{2n}$$

$$n > \frac{1}{2\varepsilon^2}$$

set $N = \left\lceil \frac{1}{2\varepsilon^2} \right\rceil$

hence, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N, \left|-\frac{1}{\sqrt{2n}} - 0\right| < \varepsilon$

so, $\left(-\frac{1}{\sqrt{2n}}\right)_{n\geq 1} \to 0$

Good.

b. claim $\left(\frac{1-e^{-n}}{2}\right)_{n\geq 1} \to \frac{1}{2}$

Proof : let $\varepsilon > 0$

For $n \in \mathbb{N}$, $\left|\frac{1-e^{-n}}{2} - \frac{1}{2}\right| = \left|-\frac{e^{-n}}{2}\right| = \frac{e^{-n}}{2} < \varepsilon$

$\Leftrightarrow e^{-n} < 2\varepsilon$

$\Leftrightarrow -n \ln e < \ln 2\varepsilon$

$\iff -n\ln e < \ln 2\varepsilon$ Remember the $\iff$ signs & explanation to fully form your argument.

$\iff -n < \ln 2\varepsilon$

$\iff n > -\ln 2\varepsilon$

So set $N = \max(\lceil -\ln 2\varepsilon \rceil, 0)$ $\because N > 0$ $\lceil \ln 2\varepsilon \rceil \leq 0$ if $\varepsilon > \frac{1}{2}$

hence, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\left| \frac{1-e^{-n}}{2} - \frac{1}{2} \right| < \varepsilon$

so, $\left(\frac{1-e^{-n}}{2}\right)_{n\geq 1} \to \frac{1}{2}$

5/6 Good! Just note the comments above.

c. claim $\left(\frac{n-1}{n}\right)_{n\geq 1} \to 1$

proof: let $\varepsilon > 0$

$\left|\frac{n-1}{n} - 1\right| = \left|\frac{n-1-n}{n}\right| = \frac{1}{n} < \varepsilon$

set $N = \lceil \frac{1}{\varepsilon} \rceil$

hence, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\left|\frac{n-1}{n} - 1\right| < \varepsilon$

so, $\left(\frac{n-1}{n}\right)_{n\geq 1} \to 1$ Good!

d. claim $(c)_{n\geq 1} \to c$

proof let $\varepsilon > 0$

$|c-c| = |0| = 0 < \varepsilon$

set $N = 0$

hence, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|c-c| < \varepsilon$

so, $(c)_{n\geq 1} \to c$ Nice.

3) $\lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right) = 1$

proof Take $l_n = 1 - \frac{1}{n}$ and $u_n = 1$, given $a_n = 1 - \frac{1}{n^2}$

Then, we know Why? Justify briefly $l_n \leq a_n \leq u_n$, $l_n = \frac{n-1}{n} \to 1$ and $u_n \to 1$

Hence, by the sandwich theorem, $a_n \to 1$ as $n \to \infty$. 5/6

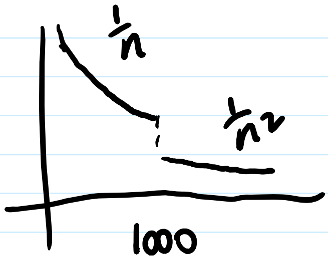
Good, just the one small gap.

4) a. claim $(a_n)_{n\geq 1} \to 0$

4) a. <u>claim</u> $(a_n)_{n \geq 1} \to 0$

<u>proof</u> let $\varepsilon > 0$

choose $n \geq 1000$, as
$1 \leq n < 1000$ has a finite number of terms.



$\Rightarrow a_n = \frac{1}{n^2}$

$\left| \frac{1}{n^2} - 0 \right| = \left| \frac{1}{n^2} \right| = \frac{1}{n^2} < \varepsilon$

$\Rightarrow n > \frac{1}{\sqrt{\varepsilon}}$ for $n \geq 1000$

set $N = \max\left(\left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil, 1000\right)$

Hence, $\forall \varepsilon > 0, \exists N, n > N$ s.t. $|a_n - 0| = a_n = \frac{1}{n^2} < \varepsilon$

so, $a_n \to 0$

b. <u>claim</u>: $a_n \to 0$

$a_n = \begin{cases} 1/n & \text{for even } n \\ -1/n & \text{for odd } n \end{cases}$

$\Rightarrow a_n = \frac{(-1)^n}{n}$

<u>proof</u> let $\varepsilon > 0$

$|a_n - 0| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} < \varepsilon$

$\Rightarrow n > \frac{1}{\varepsilon} \Rightarrow$ set $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$

Hence, $\forall \varepsilon > 0, \exists N, n > N$ s.t. $\left| \frac{(-1)^n}{n} - 0 \right| < \varepsilon$

so, $a_n \to 0$

Nice.

5. <u>claim</u> $a_n = 2^{-n} \to 1/8$

let $\varepsilon > 0$, $\left| a_n - \frac{1}{8} \right| = \left| 2^{-n} - \frac{1}{8} \right| < \varepsilon$

for $n \geq 4$,

$\left| 2^{-n} - \frac{1}{8} \right| = \frac{1}{8} - 2^{-n} < \varepsilon$

$\Rightarrow 2^{-n} > \frac{1}{8} - \varepsilon$

$\Rightarrow -n \ln 2 > \ln\left(\frac{1}{8} - \varepsilon\right)$

$\Rightarrow n < \frac{\ln\left(\frac{1}{8} - \varepsilon\right)}{-\ln 2} \Rightarrow$ There is no N for $n > N$ s.t. $\left| 2^{-n} - \frac{1}{8} \right| < \varepsilon$

$\Rightarrow a_n$ does not tend to $1/8$.

Good.

$\Rightarrow$ a_n does not tend to $1/8$.

6) a. $a_n = n^{-\alpha}$ $(\alpha > 0)$

<u>claim</u> $a_n \to 0$

<u>proof</u> let $\varepsilon > 0$

$$|a_n - 0| = |n^{-\alpha} - 0| = n^{-\alpha} < \varepsilon$$

$$\Leftrightarrow \frac{1}{n^{\alpha}} < \varepsilon$$

$$\Leftrightarrow n^{\alpha} > \frac{1}{\varepsilon} \quad \alpha \neq 0$$

$$\Leftrightarrow n > \varepsilon^{-\frac{1}{\alpha}} \Rightarrow \text{set } N = \lceil \varepsilon^{-1/\alpha} \rceil$$

Hence, $\forall \varepsilon > 0, \exists N, n > N$, s.t. $|a_n - 0| < \varepsilon$
so, $a_n \to 0$

b. <u>claim</u> $\frac{n!}{n^n} \to 0$

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots(1)}{n \cdot n \cdot n \cdot n \cdots n}$$

$$= 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n}, \quad \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)(\ldots) < 1$$

$$\Rightarrow \frac{n!}{n^n} \leq \frac{1}{n}$$

$\Rightarrow$ since $n > 0$, $\frac{n!}{n^n} > 0$

$$\Rightarrow 0 < \frac{n!}{n^n} \leq \frac{1}{n}$$

<u>proof</u> take $l_n = 0$, $u_n = \frac{1}{n}$

Then, we know $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$, so $l_n \leq a_n \leq u_n$.

since $l_n = 0 \to 0$, $u_n = \frac{1}{n} = n^{-\alpha}$ for $\alpha = 1$

$= n^{-1} \to 0$

since $l_n = 0 \to 0$, $u_n = \frac{1}{n} = n^{-\alpha}$ for $\alpha = 1$

$= n^{-1} \to 0$

$a_n = \frac{n!}{n^n} \to 0$ by the sandwich theorem.

Great!

c. $\frac{n!}{n^p} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \underbrace{\boxed{(n-p)\cdot(n-p-1)\cdots 2 \cdot 1}}_{=(n-p)!}$

let $k_n = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-p+1}{n}$

$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-p+1}{n}$

$= \lim_{n\to\infty} \frac{n}{n} \cdot \lim_{n\to\infty} \frac{n-1}{n} \cdots \lim_{n\to\infty} \frac{n-p+1}{n}$

since each term converges to 1, & is a finite product $k_n \to 1$

$\Rightarrow \lim_{n\to\infty} k_n \cdot (n-p)! = \lim_{n\to\infty} (n-p)! \to \infty$

when diverging to $+\infty$, best not to write "lim", only when the limit is finite

$\frac{5}{5}$

7) claim $a_n = \frac{\sin n\theta}{2^n} \to 0$

proof let $\varepsilon > 0$.

$\left|\frac{\sin n\theta}{2^n} - 0\right| = \left|\frac{\sin n\theta}{2^n}\right|$ $\quad -1 \leq \sin n\theta \leq 1$

$\Rightarrow |\sin n\theta| \leq 1$

$\leq \frac{1}{2^n} < \varepsilon$

$\Rightarrow \frac{1}{\varepsilon} < 2^n$

$\Rightarrow \ln\frac{1}{\varepsilon} < n \ln 2$

$\Rightarrow n > \frac{\ln\frac{1}{\varepsilon}}{\ln 2}$

$\Rightarrow$ set $N = \left\lceil \frac{\ln\frac{1}{\varepsilon}}{\ln 2} \right\rceil$

Hence, $\forall \varepsilon > 0, \exists N, n > N$ s.t. $|a_n - 0| = \left|\frac{\sin n\theta}{2^n} - 0\right| < \varepsilon$

so, $a_n \to 0$.

8) a. $a_n = \left(1 + \frac{1}{n}\right)^n$, $n \geq 1$

if a_n is increasing, $\forall n, a_{n+1} \geq a_n$

if a_n is increasing, $\forall n, a_{n+1} \geq a_n$

proof we need to show $\left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^n$

Great work!