Calculus: *Continuity, Integration, Series*

Autumn Term 2024

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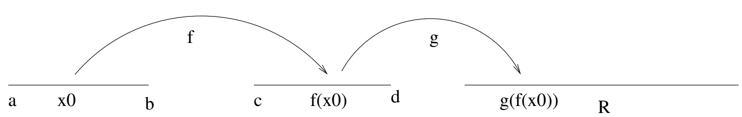
Second Part of Assessed Coursework: Questions **3b, 4c, 5d, 9** are assessed.

1. Suppose $f : (a,b) \to \mathbb{R}$ and $g : (c,d) \to \mathbb{R}$ with $x_0 \in (a,b)$ and $f(x) \in (c,d)$ for all $x \in (a,b)$. Assuming that f is continuous at x_0 and g is continuous at $f(x_0)$, prove that the composition function $g \circ f : (a,b) \to \mathbb{R}$ is coontinuous at x_0 in two different ways as follows:

   (a) Directly, using the $\epsilon - \delta$ definition of continuity.

   (b) Indirectly, reducing the continuity of the two functions to convergence of sequences.



2. **Exam 2024** Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = x^3$. Consider the partition P_n of $[0,1]$ into n equal intervals of length $1/n$.

   - Using the monotonicity of x^3 in $[0,1]$, find the lower sum $L(f, P_n)$ and the upper sum $U(f, P_n)$.
   - Hence, show that f is integrable and find $\int_0^1 x^3\, dx$.

   (You can use the formula $\sum_{i=1}^{n} i^3 = (n(n+1))^2/4$.)

3. Use the Comparison Test to establish whether each of the series below converges or diverges. You may assume common series convergence results.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n+6}$ **Exam standard.**

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{5n^2-4}$

(c) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n-4}$ **Exam standard.**

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$

4. Use the Limit Comparison Test to investigate the convergence or otherwise of the following series. You may assume common series convergence and divergence results.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3+9}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+4n-2}$ **Exam standard.**

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^n}$ for $\alpha > 2$ **Exam standard.**

5. Use d'Alembert's *Limit* Ratio Test to determine the convergence or divergence of the following series. You may assume common series convergence and divergence results.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ **Exam standard.**

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$ **Exam standard.**

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ **Exam standard.**

   You may use the fact that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

6. Using the Integral Test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges.

7. **Exam standard.** Using a suitable convergence technique, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for all $\alpha > 1$.

8. **Exam standard.** Using any technique, investigate the convergence or divergence properties (for different values of the parameter x, if present) of each of the following series.

(a) $\sum_{n=1}^{\infty} \frac{1}{3n+2}$

(b) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

(c) $\sum_{n=1}^{\infty} n!x^n$

(d) $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$

(e) $\sum_{n=1}^{\infty} \sin \frac{\pi}{n}$ using the fact that $2x/\pi < \sin x < x$ for $0 < x < \pi/2$

(f) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$

9. Use an appropriate test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$$

converges.