Calculus: *Limits and Sequences*

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First Assessed Coursework: Questions **1, 2(b), 3, 6(c)** are assessed.

1. What is the *least upper bound* or supremum and *greatest lower bound* or infimum in the reals $\mathbb{R}$ of the following sets of numbers:

   (a) $\{x \in \mathbb{N} \mid 1 \leq x^2 \leq 29\}$

   (b) $\{x \in \mathbb{Q} \mid 1 \leq x^2 \leq 29\}$

   (c) $\{x \in \mathbb{R} \mid 1 \leq x^2 \leq 29\}$

   In each case, state whether the infimum and supremum are in the given set.

2. For each of the sequences below $(a_n)_{n\geq 1}$, guess the limit and prove directly[1] that the sequence tends to that limit.

   (a) $a_n = -\dfrac{1}{\sqrt{2n}}$ for all $n \geq 1$

   (b) $a_n = \dfrac{1-e^{-n}}{2}$ for all $n \geq 1$

   (c) $a_n = \dfrac{n-1}{n}$ for all $n \geq 1$

   (d) $a_n = C$ for all $n \geq 1$, where C is a real constant.

3. Use the Sandwich theorem and any previous result to prove that:

$$\lim_{n\mapsto\infty} \left(1 - \frac{1}{n^2}\right) = 1$$

4. **Exam standard.** You are given that both of the sequences $(a_n)_{n\geq 1}$, with a_n defined for $n \geq 1$ below, tend to 0. Prove that they do so directly or using the Sandwich theorem.

   (a) $a_n = \begin{cases} 1/n & : \text{if } n < 1000 \\ 1/n^2 & : \text{if } n \geq 1000 \end{cases}$

[1] Direct proof of sequence convergence requires that you use the ϵ-N method.

(b) $a_n = \begin{cases} 1/n & : \text{if } n \text{ even} \\ -1/n & : \text{if } n \text{ odd} \end{cases}$

5. The sequence $(a_n = 2^{-n})_{n\geq 1}$ converges to 0. Attempt a direct proof that $\lim_{n\mapsto\infty} a_n = 1/8$ instead and explain where it fails.

6. **Exam standard.**

   (a) Let α be a positive, real constant. Use a direct proof to show that the sequence $(a_n)_{n\geq 1} = (n^{-\alpha})_{n\geq 1}$ converges to 0 as n tends to infinity.

   (b) Use the Sandwich theorem to show that

   $$\lim_{n\mapsto\infty} \frac{n!}{n^n} = 0$$

   (c) Let p be a fixed natural number. What happens to the sequence

   $$\left(\frac{n!}{n^p}\right)_{n\geq 1}$$

   as $n \to \infty$?

7. **Exam standard.** Let θ be any constant in $\mathbb{R}$. Using any techniques of your choice, prove that the following sequence converges to 0 as $n \to \infty$:

   $$(a_n)_{n\geq 1} = \left(\frac{\sin n\theta}{2^n}\right)_{n\geq 1}$$

8. Let $a_n = \left(1 + \frac{1}{n}\right)^n$ for all $n \geq 1$. Show that

   (a) $(a_n)_{n\geq 1}$ is an increasing sequence.

   (b) $(a_n)_{n\geq 1}$ is upper bounded by e.

   Hint: You may use the fact that $\ln x \leq x - 1$.

9. Let $a_1 = 1$ and define $a_{n+1} = \sqrt{1 + 2a_n}$ for all $n \geq 1$.

   (a) Show that the sequence $(a_n)_{n\geq 1}$ is increasing.

   (b) Show that the sequence $(a_n)_{n\geq 1}$ is upper bounded by 3.

   (c) Explain why $\lim_{n\to\infty} a_n$ exists and determine its value.