Modular SMT-Based Analysis of Nonlinear Hybrid Systems

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Abstract—**We present SMT-based techniques for analyzing networks of nonlinear hybrid systems, which interact with each other in both discrete and continuous ways. We propose a modular encoding method to reduce reachability problems of hybrid components, involving *continuous I/O* as well as usual discrete I/O, into the satisfiability of first-order logic formulas over the real numbers. We identify a generic class of logical formulas to modularly encode networks of hybrid systems, and present an SMT algorithm for checking the satisfiability of such logical formulas. The experimental results show that our techniques significantly increase the performance of SMT-based analysis for networks of nonlinear hybrid components.**

I. INTRODUCTION

Formal analysis of hybrid systems can be reduced to the satisfiability of SMT formulas over the real numbers. This approach can combine state-of-the-art SMT techniques with numerical methods to analyze continuous dynamics, governed by ordinary differential equations (ODEs). The satisfiability of these formulas are in general undecidable for nonlinear hybrid systems, but important advances have been made by various approaches using approximation, including [1]–[8].

In principle, these methods can deal with networks of nonlinear hybrid systems by combining the SMT encodings of all components. Following SMT-based approaches for digital systems, *discrete communication* between components, such as synchronization or message passing, can be encoded using first-order variables that commonly occur in the encodings of several components. In order for this technique to work, all interactions between components must be discrete.

However, many networks of hybrid systems also include *continuous interaction* as well as discrete communication. For example, consider the problem of controlling the temperature of several adjacent rooms. The temperature of one room, besides reacting to its own heater being on or off, can continuously affect the temperature of all adjacent rooms. If we model this system as a network of several thermostat systems, it is quite clear that such continuous interactions cannot be captured only using discrete communication.

This kind of *continuous I/O* is common in control systems that are composed of mechanically connected components; e.g, cars [9], airplanes [10], plants [11], [12], etc. Indeed, formal models of hybrid systems, such as hybrid automata [13] and hybrid I/O automata [14], can precisely specify continuous interactions (either implicitly or explicitly).

The analysis of networks of nonlinear hybrid systems has not been studied much in existing SMT-based approaches for hybrid systems. To apply existing SMT algorithms, one may define a sound discrete approximation of continuous I/O. But this is very difficult for nonlinear systems because continuous I/O can involve nonlinear real functions over time, not just single values. Or one can build a single hybrid component that is equivalent to the entire network of hybrid systems, but at the cost of the state explosion problem. Therefore, neither can provide a satisfactory solution at present.

The goal of this paper is to provide an SMT technique to analyze networks of nonlinear hybrid systems involving continuous I/O as well as discrete I/O. The contribution of this paper is twofold: (1) to directly provide a new modular SMT encoding for networks of nonlinear hybrid systems with continuous I/O, and (2) to develop an SMT solving algorithm to check the satisfiability of such logical formulas.

The basic idea is to encode continuous interaction between different components by means of *uninterpreted real functions*, not first-order variables. We use invariant conditions specified by *universally quantified equalities over time* to encode a continuous synchronization of uninterpreted real functions. To logically decompose continuous I/O expressed as systems of ODEs, we also make use of *parameterized integration operators*. We identify a syntactic subclass of formulas that is expressive enough to modularly encode continuous I/O yet that allows an efficient decision procedure.

We present a novel SMT algorithm to check the satisfiability of the proposed class of logical formulas. Clearly, existing SMT algorithms cannot directly deal with uninterpreted real functions, universally quantified equalities, and parameterized integration operators at the same time. Our algorithm is based on an equisatisfiable process that removes uninterpreted real functions and parameterized integration operators. This process can be considered as a wrapper of a theory solver for the real numbers and ODEs, and thus can easily be combined with existing DPLL($\mathcal{T}$)-based algorithms with minimal cost.

We have implemented our algorithm in the dReal solver [15], and performed experiments on a range of nontrivial networks of nonlinear hybrid systems. These case studies include both discrete and continuous interactions between hybrid components. The experimental results show that our techniques can greatly increase the performance of SMT-based analysis for networks of general nonlinear hybrid systems.
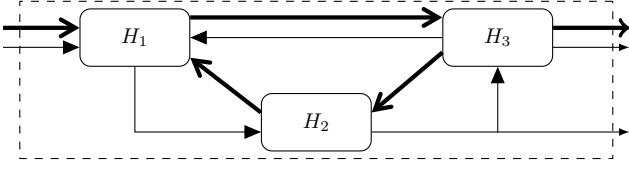
Fig. 1. Hybrid System Components

The rest of the paper is organized as follows. Section II explains networks of hybrid systems. Section III proposes a modular encoding for networks of nonlinear hybrid systems. Section IV formally identifies a class of logical formulas for the encoding, and presents an SMT solving algorithm. Section V provides an overview of the case studies. Section VI presents the experimental results. Section VII discusses related work, and Section VIII presents some concluding remarks.

II. NETWORK OF HYBRID SYSTEM COMPONENTS

We consider a network of hybrid systems components that interact with each other in discrete or continuous ways. As shown in Fig. 1, each component has two types of input and output; continuous I/O (denoted by bold lines) and discrete I/O (denoted by thin lines). A hybrid system can be specified by *hybrid automata* [13], finite state machines with continuous variables. This paper uses an extended version of hybrid automata with explicit I/O, similar to one presented in [14].

A. Hybrid I/O Automata

In *hybrid I/O automata*, discrete states are given by a finite set of *modes* Q, and continuous states are specified by using a finite set of real-valued *variables* $X = \{x_1, \ldots, x_l\}$. The domain of X's values $\boldsymbol{v} = (v_1, \ldots, v_l)$ is denoted by $val(X) = \mathbb{R}^l$. A combined state is then a pair $\langle q, \boldsymbol{v} \rangle$ of a mode $q \in Q$ and a vector $\boldsymbol{v} = (v_1, \ldots, v_l) \in \mathbb{R}^l$ of real numbers. Given a (possibly infinite) set of actions Σ, a discrete transition between two states $\langle q, \boldsymbol{v} \rangle \xrightarrow{a} \langle q', \boldsymbol{v'} \rangle$, identified with an *action* $a \in \Sigma$, is specified by a *jump condition* $jump_{q,q'}(\boldsymbol{v}, a, \boldsymbol{v'})$.

Each mode $q \in Q$ of a hybrid I/O automaton defines extra conditions to specify the continuous behavior of the variables X in the mode q. An *invariant condition* inv_q defines all possible values of the variables X in mode q. A *flow condition* $flow_q$ defines *trajectories* of the variables X—describing continuous changes of X's values over time—in mode q, typically using ordinary differential equations (ODEs). An *initial condition* $init_q$ defines a set of initial states.

Discrete input and output of a component are identified by using disjoint sets of *input actions* $\Sigma_I \subseteq \Sigma$ and *output actions* $\Sigma_O \subseteq \Sigma$. Other "local" actions in the set $\Sigma \setminus (\Sigma_I \cup \Sigma_O)$ of a component are called *internal actions*. Likewise, continuous input and output of a component are identified by using *continuous trajectories* for disjoint sets of *input variables* $X_I \subseteq X$ and *output variables* $X_O \subseteq X$, and other "local" variables in $X \setminus (X_I \cup X_O)$ are called *internal variables*.

Definition 1. *A hybrid I/O automaton (HIOA) is defined as a tuple* $H = (Q, X, \Sigma, \{inv_q\}_{q \in Q}, \{flow_q\}_{q \in Q}, \{init_q\}_{q \in Q}, \{jump_{q,q'}\}_{q,q' \in Q}, (X_I, X_Y), (\Sigma_I, \Sigma_O))$.

The *init*, *inv*, and *jump* conditions are often written as predicates $init_q(\boldsymbol{x})$, $inv_q(\boldsymbol{x})$, and $jump_{q,q'}(\boldsymbol{x}, a, \boldsymbol{x'})$ over the variables X. The *flow* condition is written as a system of ODEs of the form $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\iota})(t)$, where the input variables $\boldsymbol{\iota}$ in X_I appear as *free variables* in the ODEs. That is, the trajectories of the variables $\boldsymbol{x}$ in $X \setminus X_I$ are defined for any (valid) trajectories of the input variables $\boldsymbol{\iota}$ in X_I.

B. Composition

A network of hybrid system components is specified by a parallel composition of hybrid I/O automata:

$$\mathcal{H} = H_1 \parallel H_2 \parallel \cdots \parallel H_n$$

A communication between components is specified using I/O actions and variables. We assume that: (i) output actions and variables of one component are not output of any other components, and (ii) internal actions and variables of one component are not parts of any other components.

There are two types of communications in a network $\mathcal{H}$. Consider a source component H_o and target components $H_{i_1}, \ldots, H_{i_m}$. A *discrete communication* from one component H_o to other components $H_{i_1}, \ldots, H_{i_m}$ in $\mathcal{H}$ is modeled by *joint synchronous actions* in $\Sigma_O^o \cap \Sigma_I^{i_1} \cap \cdots \cap \Sigma_I^{i_m}$ that are output actions of the source H_o and input actions of the targets $H_{i_1}, \ldots, H_{i_m}$ at the same time. A *continuous interaction* from one component H_o to other components $H_{i_1}, \ldots, H_{i_m}$ is modeled by *shared variables* in $X_O^o \cap X_I^{i_1} \cap \cdots \cap X_I^{i_m}$ that are output variables of the source H_o and input variables of the targets $H_{i_1}, \ldots, H_{i_m}$ at the same time.

A discrete state of a network $\mathcal{H}$ is defined as a vector $(q_1, \ldots, q_n) \in Q_1 \times \cdots \times Q_n$ of modes of its components $H_1, \ldots, H_n$. Components in the network $\mathcal{H}$ synchronize their discrete transitions labeled with a joint action. When one component performs a discrete transition labeled with an action a, every component that includes the action a must perform a discrete transition labeled with the same action a. Notice that a component can individually perform a discrete transition with an internal action.

A continuous state of a network $\mathcal{H}$ is defined as trajectories of the variables $X_1 \cup \cdots \cup X_n$ of its components $H_1, \ldots, H_n$. The values of the network $\mathcal{H}$'s variables $X_1 \cup \cdots \cup X_n$ evolve simultaneously over time according to their flow and invariant conditions for the current modes. Flow conditions of an input variable are given by those of its connected output variable. That is, an output variable and its corresponding input variables must have the same value at all times.

Consequently, a network of HIOA is semantically equivalent to a single HIOA (see [14] for a formal definition). The output action of the composition of the network $\mathcal{H}$ becomes the union $\Sigma_O = \Sigma_O^1 \cup \cdots \cup \Sigma_O^n$, and the input action becomes the set $(\Sigma_I^1 \cup \cdots \cup \Sigma_I^n) \setminus \Sigma_O$. Similarly, the output variable is the union $X_O = X_O^1 \cup \cdots \cup X_O^n$, and the input variable is the set $(X_I^1 \cup \cdots \cup X_I^n) \setminus X_O$. For example, in Fig. 1, the composition of the hybrid automata H_1, H_2, and H_3 can be seen as the single hybrid automaton enclosed by the outer box.
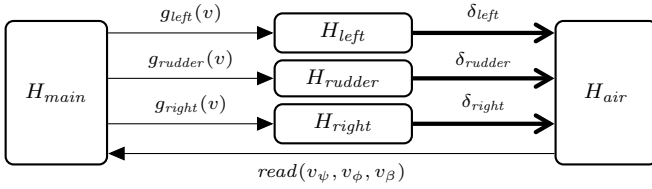
Fig. 2. Controllers for turning an airplane

C. Example: Turning an Airplane

We consider a networked controller for turning an airplane, adapted from [10], [16]. An aircraft makes a turn by controlling three control surfaces, namely, two *ailerons* (surfaces attached to the end of the wings) and a *rudder* (a surface attached to the vertical tail).[1] As depicted in Fig. 2, the main controller orchestrates the three subcontrollers for the ailerons and the rudder to achieve a coordinated turn. The entire system is given by the composition

$$H_{main} \parallel H_{left} \parallel H_{rudder} \parallel H_{right} \parallel H_{air}.$$

A subcontroller H_M for $M \in \{left, right, rudder\}$ has three modes acc, dec, and con. Its angle δ_M and the moving rate r_M of the surface changes according to the ODEs with some positive constant $c_M > 0$:

$$\dot{\delta}_M = r_M,$$
$$\dot{r}_M = c_M \ (acc), \quad \dot{r}_M = -c_M \ (dec), \quad \dot{r}_M = 0 \ (con).$$

The subcontroller H_M performs a jump transition with an input action $g_M(v)$ to determine a next mode based on the current angle and the goal angle v. The sets of I/O variables and I/O actions are defined by:

$$X_O^M = \{\delta_M\}, \qquad \Sigma_I^M = \{g_M(v) \mid v \in \mathbb{R}\},$$
$$X_I^M = \emptyset, \qquad \Sigma_O^M = \emptyset.$$

The lateral dynamics of the aircraft is modeled by H_{air} that has a single mode with the nonlinear ODEs:

$$\dot{\beta} = Y(\beta, \boldsymbol{\delta})/mV - r + V \cos\beta \sin\phi/g,$$
$$\dot{\psi} = g/V \cdot \tan\phi,$$
$$\dot{p} = (c_1 r + c_2 p)r \tan\phi + c_3 L(\beta, \boldsymbol{\delta}) + c_4 N(\beta, \boldsymbol{\delta}),$$
$$\dot{\phi} = p,$$
$$\dot{r} = (c_8 p - c_2 r)r \tan\phi + c_4 L(\beta, \boldsymbol{\delta}) + c_9 N(\beta, \boldsymbol{\delta})$$

with β the yaw angle, ψ the direction, p the rolling moment, ϕ the roll angle, and r the yawing moment. Y, L, and N are fixed linear functions of β and angles $\boldsymbol{\delta} = (\delta_{left}, \delta_{right}, \delta_{rudder})$. The sets of I/O variables and I/O actions are defined by:

$$X_I^{air} = \{\boldsymbol{\delta}\}, \qquad \Sigma_O^{air} = \Sigma_I^{main},$$
$$X_O^{air} = \emptyset, \qquad \Sigma_I^{air} = \emptyset.$$

[1] An aircraft rolls in the direction of a turn by moving its *ailerons* (surfaces attached to the end of the wings). Since the tilted lift of the wings then acts as the centripetal force, the aircraft moves in a circular motion. But the rolling makes the aircraft yaw in a wrong direction. The *rudder* (a surface attached to the vertical tail) is used to counter this undesirable *adverse yaw*.

Finally, the main controller H_{main} provides a goal angle v to a subcontroller M using a discrete transition with an output action $g_M(v)$, and monitors the current position of the aircraft from the component H_{air} using a discrete transition with an input action $read(v_\psi, v_\phi, v_\beta)$, for v_ψ the direction, v_ϕ the roll angle, and v_β the yaw angle. The component H_{main} has no I/O variables, but has the sets of I/O actions:

$$\Sigma_I^{main} = \{read(v_\psi, v_\phi, v_\beta) \mid v_\psi, v_\phi, v_\beta \in \mathbb{R}\}$$
$$\Sigma_O^{main} = \Sigma_I^{left} \cup \Sigma_I^{right} \cup \Sigma_I^{rudder}$$

We are interested in a reachability problem to find outputs of H_{main} to reach a goal direction in a reasonable time, while keeping the yaw angle β close to 0 during the turn. Notice that the input variables $\boldsymbol{\delta}$ are free variables in H_{air}'s ODEs whose interpretation is defined by the other components.

The continuous interaction between $(H_{left}, H_{rudder}, H_{right})$ and H_{air} cannot be specified using discrete synchronization without I/O variables. We can eliminate I/O variables by building a single HIOA that is equivalent to the composition of the network $H_{air} \parallel H_{left} \parallel H_{rudder} \parallel H_{right}$, but at the cost of the state explosion problem. For example, a single hybrid automaton for $H_1 \parallel \ldots \parallel H_n$ has total $\prod_{i=1}^{n} m_i$ modes, if each component H_i has m_i modes, for $i = 1, \ldots, n$.

III. SMT Encoding of Hybrid Components

In this section we propose a modular method to encode a network of hybrid system components that involve *both discrete and continuous I/O* at the same time. Our technique generalizes previous SMT-based approaches that only focus on discrete communication using synchronous actions. An SMT algorithm for new encoding is shown in the following section.

A. Encoding of Discrete Transitions

We use first-order logic formulas over the real numbers, called $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formulas, along with a collection $\mathcal{F}$ of Type 2 computable real functions [3]. A Type 2 computable real function can be numerically evaluated up to an arbitrary precision, such as polynomials, exponentiation, trigonometric functions, and solutions of Lipschitz-continuous ODEs. The syntax is defined in the standard way, with c real number constants, y first-order variables, and f real functions in $\mathcal{F}$:

$$t ::= c \mid y \mid f(t_1, \ldots, t_n)$$
$$\varphi ::= t > 0 \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \exists y.\varphi \mid \forall y.\varphi.$$

Consider a HIOA H with I/O variables (X_I, X_Y) and I/O actions (Σ_I, Σ_O). As usual, a discrete transition of H with an action can simply be encoded by using a first-order variable w that denotes the action. Discrete synchronization by the action is encoded by using the same variable w in the formula for each corresponding component.

Definition 2. An $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-encoding for a (synchronized) discrete transition of H from state $\langle q, \boldsymbol{y} \rangle$ to $\langle q', \boldsymbol{y}' \rangle$ by an action w is:

$$dt(q, \boldsymbol{y}, w, q', \boldsymbol{y}') \equiv (w \in \Sigma \rightarrow jump_{q,q'}^H(\boldsymbol{y}, w, \boldsymbol{y}'))$$
$$\wedge (w \notin \Sigma \rightarrow q = q' \wedge \boldsymbol{y} = \boldsymbol{y}').$$

B. Encoding of Continuous Flows

The continuous behavior of one component is parameterized by the trajectories of its input variables, which are composed of the trajectories of the corresponding output variables in other components. Unlike actions, these (potentially nonlinear) trajectories cannot be encoded as first-order variables in $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$, because they are actually real-valued functions.

We use an *uninterpreted real function symbol* $\mathbb{R} \to \mathbb{R}$ in $\mathcal{F}$, instead of a first-order variable, to represent a trajectory for each input variable of H. Consider a flow condition for mode q, expressed as a system of Lipschitz-continuous ODEs:

$$\left[\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f_q}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{\iota})(t), \quad \frac{d\boldsymbol{o}}{dt} = \boldsymbol{g_q}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{\iota})(t) \right]$$

with $\boldsymbol{x}$ the internal variables, $\boldsymbol{o}$ the output variables, and $\boldsymbol{\iota}$ the input variables. Mathematically, these ODE variables in the terms $\boldsymbol{f_q}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{\iota})(t)$ and $\boldsymbol{g_q}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{\iota})(t)$ denote unary real functions $\mathbb{R} \to \mathbb{R}$ over time t.

Given function symbols $\dot{o}_1, \ldots, \dot{o}_l : \mathbb{R} \to \mathbb{R}$ to denote the derivatives of the output variables $\boldsymbol{o} = (o_1, \ldots, o_l)$, and function symbols $\dot{x}_1, \ldots, \dot{x}_n : \mathbb{R} \to \mathbb{R}$ to denote the derivatives of the internal variables $\boldsymbol{x} = (x_1, \ldots, x_n)$, we can express the trajectories of the internal variables $\boldsymbol{x}$ and the output variables $\boldsymbol{o}$ as the $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formula

$$\forall u \in [0, t]. \ [\dot{\boldsymbol{x}}, \dot{\boldsymbol{o}}](u) = [\boldsymbol{f_q}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{\iota})(u), \ \boldsymbol{g_q}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{\iota})(u)].$$

The trajectories of the input variables $\boldsymbol{\iota}$ (that appear as free variables in $\boldsymbol{f_q}$ and $\boldsymbol{g_q}$) are then defined by using *output derivatives of other components*.

Using these I/O derivative symbols, we can express the continuous change of hybrid system component H's states from initial values $\boldsymbol{y^0}$ to new values $\boldsymbol{y^t}$ in mode q for duration t according to the ODE system as the $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formula:

$$\boldsymbol{y^t} = \boldsymbol{y^0} + \int_0^t [\dot{\boldsymbol{x}}, \dot{\boldsymbol{o}}, \dot{\boldsymbol{\iota}}] \, ds$$

This $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formula is parameterized by the internal derivatives $\dot{\boldsymbol{x}} = (\dot{x}_1, \ldots, \dot{x}_n)$, the output derivatives $\dot{\boldsymbol{o}} = (\dot{o}_1, \ldots, \dot{o}_l)$, and the input derivatives $\dot{\boldsymbol{\iota}} = (\dot{\iota}_1, \ldots, \dot{\iota}_m)$.

The ODE solution term $\boldsymbol{y^0} + \int_0^t [\dot{\boldsymbol{x}}, \dot{\boldsymbol{o}}, \dot{\boldsymbol{\iota}}] \, ds$ can be considered as a computable real function $F(\boldsymbol{y^0}, t)$ (with parameters $\boldsymbol{y^0}$ and t) in the collection $\mathcal{F}$, provided that $\boldsymbol{f_q}$, $\boldsymbol{g_q}$, and the ODEs for the input variables (given by connected output variables) are all Lipschitz-continuous [17], [18]. The extra derivative function symbols $\dot{o}_1, \ldots, \dot{o}_l, \dot{\iota}_1, \ldots, \dot{\iota}_m$ are also in the collection $\mathcal{F}$, but must be bounded by a specific set of computable real functions (see Sec. IV-A below). In sum, an encoding of continuous flows is defined as follows:

Definition 3. *An $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-encoding for a continuous flow of H from one values $\boldsymbol{y^0}$ to new values $\boldsymbol{y^t}$ in mode q for duration t, that involves I/O variables $(\boldsymbol{\iota}, \boldsymbol{o})$ and satisfies the invariant condition inv_q^H, where $F(\boldsymbol{y^0}, t)$ denotes the ODE solution term $\boldsymbol{y^0} + \int_0^t [\dot{\boldsymbol{x}}, \dot{\boldsymbol{o}}, \dot{\boldsymbol{\iota}}] \, ds$, is the $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formula:*

$$\begin{aligned}
ct(q, \boldsymbol{y^0}, \boldsymbol{y^t}, t \mid \boldsymbol{\iota}, \boldsymbol{o}, \boldsymbol{x}) \equiv \ & \boldsymbol{y^t} = F(\boldsymbol{y^0}, t) \\
& \wedge \ \forall u \in [0, t]. \ [\dot{\boldsymbol{x}}, \dot{\boldsymbol{o}}](u) = [\boldsymbol{f_q}, \boldsymbol{g_q}](u) \\
& \wedge \ \forall u \in [0, t]. \ inv_q^H(F(\boldsymbol{y^0}, u)).
\end{aligned}$$

We use different uninterpreted functions for variables in different components, even if they are "shared" variables in a network of HIOA. The use of the same uninterpreted functions for different components may cause unintended semantic effects, as explained in Sec. IV-A below. This is not a strict restriction, because we can always use "syntactic copies" to enforce that input variables $\boldsymbol{\iota}$ and output variables $\boldsymbol{o}$ follow the isomorphic system of ODEs $\boldsymbol{f}$ up to renaming, by means of *connection formulas* between $\boldsymbol{\iota}$ and $\boldsymbol{o}$ as follows.

Definition 4. *An $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-encoding of connections between input variables $\boldsymbol{\iota}$ and output variables $\boldsymbol{o}$ is as follows (it includes internal variables $\boldsymbol{x}$ and $\boldsymbol{x'}$, since $\boldsymbol{o}$ may depend on $\boldsymbol{x}$):*

$$\begin{aligned}
conn(\boldsymbol{\iota}, \boldsymbol{o}, \boldsymbol{x}, \boldsymbol{x'}) \ \equiv \ & (\forall u \in [0, t]. \ [\dot{\boldsymbol{o}}, \dot{\boldsymbol{x}}](u) = \boldsymbol{f}(\dot{\boldsymbol{o}}, \boldsymbol{x})(u)) \\
& \leftrightarrow (\forall u \in [0, t]. \ [\dot{\boldsymbol{\iota}}, \dot{\boldsymbol{x'}}](u) = \boldsymbol{f}(\dot{\boldsymbol{\iota}}, \boldsymbol{x'})(u)).
\end{aligned}$$

C. Encoding of Bounded Reachability

For a network of HIOA $H_1 \parallel \cdots \parallel H_N$, the reachability up to the k-th discrete step—that involves continuous I/O as well as discrete synchronization—can be encoded in $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$ as follows. This formula is defined as just a conjunction of N subformulas, each of which encodes the reachability of each individual component H_j up to the k-th discrete step.

Definition 5. *An $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-encoding for the k-step reachability of a network of HIOA $H_1 \parallel \cdots \parallel H_N$ is the $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formula of size $O(\sum_{j=1}^N k \cdot |Q_j|^2)$, where $\Sigma = \bigcup_{j=1}^N \Sigma_j$:*

$$\exists \{t_i \in [0, t_{\max}], \ w_i \in \Sigma\}_{i=0}^k.$$

$$\exists \{\{m_i^j \in Q_j, \ \boldsymbol{y_{j}}_i^0, \boldsymbol{y_{j}}_i^t \in val(X_j)\}_{j=1}^N\}_{i=0}^k.$$

$$\bigwedge_{j=1}^N \left[\begin{array}{l} \left[\begin{array}{l} init_{m_0^j}^j(\boldsymbol{y_{j}}_0^0) \\ \wedge \ ct(m_0^j, \boldsymbol{y_{j}}_0^0, \boldsymbol{y_{j}}_0^t, t_0 \mid \boldsymbol{\iota}_0^j, \boldsymbol{o}_0^j, \boldsymbol{x}_0^j) \end{array} \right] \\ \wedge \ \bigwedge_{i=1}^k \left[\begin{array}{l} dt(m_{i-1}^j, \boldsymbol{y_{j}}_{i-1}^t, w_i, m_i^j, \boldsymbol{y_{j}}_i^0) \\ \wedge \ ct(m_i^j, \boldsymbol{y_{j}}_i^0, \boldsymbol{y_{j}}_i^t, t_i \mid \boldsymbol{\iota}_i^j, \boldsymbol{o}_i^j, \boldsymbol{x}_i^j) \end{array} \right] \end{array} \right]$$

$$\wedge \ goal(m_k^1, \ldots, m_k^N, \boldsymbol{y_1}_k^t, \ldots, \boldsymbol{y_N}_k^t)$$

$$\wedge \ \bigwedge_{i=0}^k conn(\boldsymbol{\iota}_i^1, \ldots, \boldsymbol{\iota}_i^N, \boldsymbol{o}_i^1, \ldots, \boldsymbol{o}_i^N, \boldsymbol{x}_i^1, \ldots, \boldsymbol{x}_i^N)$$

The encoding can be explained as follows. For each i-th discrete step of duration t_i (at most $t_{\max} > 0$), component H_j is in mode m_i^j, and the values of H_j's variables X_j begin with $\boldsymbol{y_{j}}_i^0$ and end with $\boldsymbol{y_{j}}_i^t$. At the first step ($i = 0$), the initial values $\boldsymbol{y_{j}}_0^0$ of H_j's variables satisfy the initial condition $init_{m_0^j}^j$. To begin the $(i+1)$-th step, every component synchronizes its transition with the same action w_i. Each input derivative $\dot{\boldsymbol{\iota}}_i^j$ of H_j is defined by means of universally quantified $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-formulas for output derivatives $\dot{\boldsymbol{o}}_i^n$ of another component H_n, as explained in Def. 3. For the k-th step, the goal formula holds for H_j's final state. The connection formulas $conn$ link input and output variables.

D. Example

Consider the airplane controller example in Sec. II-C. For a subcontroller M, an $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-encoding of a continuous flow from initial values (δ_M^0, r_M^0) to new values (δ_M^t, r_M^t) with the invariant condition $-45 < \delta_M < 45$ is the formula below.

$$(\delta_M^t, r_M^t) = (\delta_M^0, r_M^0) + \int_0^t [\dot{\delta}_M, \dot{r}_M]\, \mathrm{d}s$$

$$\wedge \begin{bmatrix} (q_M = acc \rightarrow \forall u \in [0,t].\ [\dot{\delta}_M, \dot{r}_M](u) = [r_M, c_M]) \wedge \\ (q_M = dec \rightarrow \forall u \in [0,t].\ [\dot{\delta}_M, \dot{r}_M](u) = [r_M, -c_M]) \wedge \\ (q_M = con \rightarrow \forall u \in [0,t].\ [\dot{\delta}_M, \dot{r}_M](u) = [r_M, 0]) \end{bmatrix}$$

$$\wedge\ \forall u \in [0,t].\ -45 < \pi_1\big((\delta_M^0, r_M^0) + \int_0^u [\dot{\delta}_M, \dot{r}_M]\, \mathrm{d}s\big) < 45.$$

The second conjunct defines the derivatives of the variables (δ_M, r_M) for three modes acc, dec, and con. The last line expresses the invariant condition $-45 < \delta_M < 45$ using the solution term $(\delta_M^0, r_M^0) + \int_0^u [\dot{\delta}_M, \dot{r}_M]\, \mathrm{d}s$ and the projection function $\pi_1(a,b) = a$ for any $(a,b) \in \mathbb{R}^2$.

For the airplane component H_{air}, an $\mathcal{L}_{\mathbb{R}_\mathcal{F}}$-encoding of its continuous flow is simply the single ODE solution term:

$$(\beta^t, \psi^t, \phi^t, p^t, r^t, \boldsymbol{\delta}^t) = (\beta^0, \psi^0, \phi^0, p^0, r^0, \boldsymbol{\delta}^0) + \int_0^t \boldsymbol{F}(s)\, \mathrm{d}s,$$

that is parameterized by the component H_{air}'s input variables $\boldsymbol{\delta}^{air} = (\delta_{left}^{air}, \delta_{right}^{air}, \delta_{rudder}^{air})$, where $\boldsymbol{F}(s)$ is given by:

$$\begin{bmatrix} Y(\beta, \boldsymbol{\delta}^{air})/mV - r + V/g \cdot \cos\beta \sin\phi \\ g/V \cdot \tan\phi \\ p \\ (c_1 r + c_2 p) r \tan\phi + c_3 L(\beta, \boldsymbol{\delta}^{air}) + c_4 N(\beta, \boldsymbol{\delta}^{air}) \\ (c_8 p - c_2 r) r \tan\phi + c_4 L(\beta, \boldsymbol{\delta}^{air}) + c_9 N(\beta, \boldsymbol{\delta}^{air}) \\ \dot{\boldsymbol{\delta}}^{air} \end{bmatrix}$$

The behavior of the input variables $\boldsymbol{\delta}^{air}$ is given by connection formulas such as:

$$(\forall u \in [0,t].\ [\dot{\delta}_M, \dot{r}_M](u) = [r_M, c_M])$$
$$\leftrightarrow\ (\forall u \in [0,t].\ [\dot{\delta}_M^{air}, \dot{r}_M^{air}](u) = [r_M^{air}, c_M])$$
$$(\forall u \in [0,t].\ [\dot{\delta}_M, \dot{r}_M](u) = [r_M, -c_M])$$
$$\leftrightarrow\ (\forall u \in [0,t].\ [\dot{\delta}_M^{air}, \dot{r}_M^{air}](u) = [r_M^{air}, -c_M])$$
$$(\forall u \in [0,t].\ [\dot{\delta}_M, \dot{r}_M](u) = [r_M, 0])$$
$$\leftrightarrow\ (\forall u \in [0,t].\ [\dot{\delta}_M^{air}, \dot{r}_M^{air}](u) = [r_M^{air}, 0])$$

where variable r_M^{air} is a "copy" of the internal variable r_M of M. We also require that the initial values of such copies are identical, which can be defined as equalities of first-order variables. The behavior of $\boldsymbol{\delta}^{air}$ is provided by the encodings of three subcontrollers using uninterpreted derivative functions $\dot{\boldsymbol{\delta}} = (\dot{\delta}_{left}, \dot{\delta}_{right}, \dot{\delta}_{rudder})$ and universal quantification.

E. The Correctness of the Encoding

Our modular encoding is correct in the sense that it is equisatisfiable to the encoding of the composition, which is a single hybrid automaton.

Theorem 1. *Given a network of HIOA $\mathcal{H} = H_1 \parallel \cdots \parallel H_n$, its modular encoding (of size $O(k \sum_{j=1}^N |Q_j|^2)$) and the encoding of its composition (of size $O(k \prod_{j=1}^N |Q_j|)$) are equisatisfiable.*

Proof Sketch. The modular encoding of the k-step reachability of the network $\mathcal{H}$ is as follows, where the modes of the components in $\mathcal{H}$ are *explicitly* written by using disjunction, instead of using existential quantification as Def 5:

$$\bigwedge_{j=1}^N \left[\begin{matrix} \bigvee_{m_0^j \in Q_j} \begin{bmatrix} init_{m_0^j}^j(\boldsymbol{y_{j0}^0}) \wedge \\ ct(m_0^j, \boldsymbol{y_{j0}^0}, \boldsymbol{y_{j0}^t}, t_0 \mid \boldsymbol{\iota_0^j}, \boldsymbol{o_0^j}, \boldsymbol{x_0^j}) \end{bmatrix} \wedge \\ \bigwedge_{i=1}^k \left[\bigvee_{m_{i-1}^j, m_i^j \in Q_j} \begin{matrix} dt(m_{i-1}^j, \boldsymbol{y_{ji-1}^t}, w_i, m_i^j, \boldsymbol{y_{ji}^0}) \wedge \\ ct(m_i^j, \boldsymbol{y_{ji}^0}, \boldsymbol{y_{ji}^t}, t_i \mid \boldsymbol{\iota_i^j}, \boldsymbol{o_i^j}, \boldsymbol{x_i^j}) \end{matrix} \right] \end{matrix} \right]$$

$$\wedge\ goal(m_k^1, \ldots, m_k^N, \boldsymbol{y_{1k}^t}, \ldots, \boldsymbol{y_{Nk}^t})$$
$$\wedge\ \bigwedge_{i=0}^k conn(\boldsymbol{\iota_i^1}, \ldots, \boldsymbol{\iota_i^N}, \boldsymbol{o_i^1}, \ldots, \boldsymbol{o_i^N}, \boldsymbol{x_i^1}, \ldots, \boldsymbol{x_i^N})$$

We can apply the distributive laws to rearrange the formula according to vectors $\boldsymbol{m_i} = (m_i^0, \ldots, m_i^N)$ of modes that denote discrete states of the composition of the network $\mathcal{H}$:

$$\bigvee_{\boldsymbol{m_0} \in Q_0 \times \cdots \times Q_N} \begin{bmatrix} \bigwedge_{j=1}^N init_{m_0^j}^j(\boldsymbol{y_{j0}^0}) \wedge \\ \bigwedge_{j=1}^N ct(m_0^j, \boldsymbol{y_{j0}^0}, \boldsymbol{y_{j0}^t}, t_0 \mid \boldsymbol{\iota_0^j}, \boldsymbol{o_0^j}, \boldsymbol{x_0^j}) \wedge \\ \wedge\ conn(\boldsymbol{\iota_0^1}, \ldots, \boldsymbol{x_0^N}) \end{bmatrix} \wedge$$

$$\bigwedge_{i=1}^k \left[\bigvee_{\substack{\boldsymbol{m_{i-1}}, \boldsymbol{m_i} \in \\ Q_0 \times \cdots \times Q_N}} \begin{bmatrix} \bigwedge_{j=1}^N dt(m_{i-1}^j, \boldsymbol{y_{ji-1}^t}, w_i, m_i^j, \boldsymbol{y_{ji}^0}) \wedge \\ \bigwedge_{j=1}^N ct(m_i^j, \boldsymbol{y_{ji}^0}, \boldsymbol{y_{ji}^t}, t_i \mid \boldsymbol{\iota_i^j}, \boldsymbol{o_i^j}, \boldsymbol{x_i^j}) \\ \wedge\ conn(\boldsymbol{\iota_i^1}, \ldots, \boldsymbol{x_i^N}) \end{bmatrix} \right]$$

$$\wedge\ goal(m_k^1, \ldots, m_k^N, \boldsymbol{y_{1k}^t}, \ldots, \boldsymbol{y_{Nk}^t})$$

Now we can apply Lemma 1 (see Sec. IV-A) to remove uninterpreted function symbols and parameterized integration operators in the subformulas of the form ($i = 0, \ldots, k$):

$$\bigwedge_{j=1}^N ct(m_i^j, \boldsymbol{y_{ji}^0}, \boldsymbol{y_{ji}^t}, t_i \mid \boldsymbol{\iota_i^j}, \boldsymbol{o_i^j}, \boldsymbol{x_i^j})\ \wedge\ conn(\boldsymbol{\iota_i^1}, \ldots, \boldsymbol{x_i^N})$$

which are only part of the formula that cannot be expressed in $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$. As will be discussed in Sec. IV-D, because only one universally quantified $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literal is chosen for each mode, formulas for extra consistency checking are not needed. Suppose that the resulting $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-subformula is written as:

$$ct(\boldsymbol{m_i}, \boldsymbol{y0_i^0}, \ldots, \boldsymbol{yN_i^0}, \boldsymbol{y0_i^t}, \ldots, \boldsymbol{yN_i^t}, t_i)$$

Finally, we obtain the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula of size $O(k \prod_{j=1}^N |Q_j|)$ that is equisatisfiable to the original modular encoding:

$$\bigvee_{\boldsymbol{m_0} \in Q_0 \times \cdots \times Q_N} \begin{bmatrix} \bigwedge_{j=1}^N init_{m_0^j}^j(\boldsymbol{y_{j0}^0}) \wedge \\ ct(\boldsymbol{m_0}, \boldsymbol{y0_0^0}, \ldots, \boldsymbol{yN_0^0}, \boldsymbol{y0_0^t}, \ldots, \boldsymbol{yN_0^t}, t_0) \end{bmatrix} \wedge$$

$$\bigwedge_{i=1}^k \left[\bigvee_{\substack{\boldsymbol{m_{i-1}}, \boldsymbol{m_i} \in \\ Q_0 \times \cdots \times Q_N}} \begin{bmatrix} \bigwedge_{j=1}^N dt(m_{i-1}^j, \boldsymbol{y_{ji-1}^t}, w_i, m_i^j, \boldsymbol{y_{ji}^0}) \wedge \\ ct(\boldsymbol{m_i}, \boldsymbol{y0_i^0}, \ldots, \boldsymbol{yN_i^0}, \boldsymbol{y0_i^t}, \ldots, \boldsymbol{yN_i^t}, t_i) \end{bmatrix} \right]$$

$$\wedge\ goal(m_k^1, \ldots, m_k^N, \boldsymbol{y_{1k}^t}, \ldots, \boldsymbol{y_{Nk}^t})$$

This $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula extactly describes the concrete behavior of the composition of the network $H_1 \parallel \cdots \parallel H_n$ for the combined mode $(q_1, \ldots, q_n) \in Q_1 \times \cdots \times Q_n$. Thus, it can be considered as an encoding of a single hybrid automaton. Consequently, the modular encoding and the encoding of the composition are equisatisfiable. $\square$

IV. SMT Algorithm for Hybrid Components

We syntactically identify a generic class of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}}}$-formulas that involve universal quantification for uninterpreted real functions. This class includes $\mathcal{L}_{\mathbb{R}_{\mathcal{F}}}$-formulas for networks of hybrid I/O automata in Def. 5. We present an SMT procedure for checking the satisfiability of these $\mathcal{L}_{\mathbb{R}_{\mathcal{F}}}$-formulas.

A. Syntactic Classification

We explicitly decompose a collection $\mathcal{F}$ as $\mathcal{F}' \cup \mathcal{G}$. A collection $\mathcal{G}$ is composed of uninterpreted unary differentiable functions $\mathbb{R} \rightarrow \mathbb{R}$ to denote variables $x_1, \ldots, x_l$ in ODEs and their derivatives $\dot{x}_1, \ldots, \dot{x}_l$ (including I/O variables and I/O derivatives), whereas $\mathcal{F}'$ includes only interpreted real functions. For each pair $(x, \dot{x})$ in $\mathcal{G}$, the function $\dot{x} \in \mathcal{G}$ is a derivative of the function $x \in \mathcal{G}$ (in a model of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$).

We explicitly take into account *parameterized integration operators* over ODE variables $\boldsymbol{x} \in \mathcal{G}$ and derivatives $\boldsymbol{\iota} \in \mathcal{G}$:

$$\boldsymbol{y^0} + \int_0^t [\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\iota})(s), \boldsymbol{\iota}(s)] \, \mathrm{d}s,$$

specifying solution functions of parameterized ODE systems of the form $\frac{\mathrm{d}}{\mathrm{d}s}[\boldsymbol{x}, \boldsymbol{\iota}] = [\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\iota})(s), \boldsymbol{\iota}(s)]$ with initial values $\boldsymbol{y^0}$ over time t. As mentioned, we can define the behavior of each input derivative ι by universally quantified $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas over time of the form

$$\forall u \in [0, t]. \, i(u) = g(\boldsymbol{z})(u),$$

where $\boldsymbol{z} \in \mathcal{G}$ denote unary function symbols for ODE variables. A negated form

$$\exists u \in [0, t]. \, i(u) \neq g(\boldsymbol{z})(u)$$

is also considered. We assume that "interpreted" real functions $\boldsymbol{f}$ and g are Lipschitz-continuous functions in $\mathcal{F}'$.

We identify a subclass of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas that allows a reduction to $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$ without extra uninterpreted real functions in $\mathcal{G}$. We can apply any (existing) algorithms for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas after the reduction. As a matter of fact, the entire class of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas is too expressive to have any kinds of efficient decision procedures. For example, control problems of nonlinear systems are undecidable in general [19], but can be written as $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas.[2] Our goal is to identify a syntactic subclass of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$ that allows a reduction to $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$ without uninterpreted real functions.

We require that each uninterpreted function symbol in $\mathcal{G}$ only occurs in a single integration term (however, the same integration term can appear in a formula many times). If not, a "shared" function symbol in different terms leads to an unintended semantic restriction that all the integration terms have the same solution function. The decomposition using parameterized integration in Sec. III-B is therefore no longer equisatisfiable to the original formula. This is why we encode variables in different components as different functions.

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[2]Consider a nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ with initial values $\boldsymbol{x_0}$ and controls $\boldsymbol{u}$. The controllability to $\boldsymbol{x_t}$ can be expressed as the satisfiability of $\boldsymbol{x_t} = \boldsymbol{x_0} + \int_0^t \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \, \mathrm{d}s$ with uninterpreted real functions $\boldsymbol{u}$.

For example, consider the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formula that involves the ODE $\frac{\mathrm{d}x}{\mathrm{d}s} = x$ with different initial values 1 and 2:

$$x_1 = 1 + \int_0^t x(s) \, \mathrm{d}s \ \wedge \ x_2 = 2 + \int_0^t x(s) \, \mathrm{d}s \ \wedge \ t = 1,$$

which is satisfiable with $x_1 = e$ and $x_2 = 2e$. Now let us replace the term $x(s)$ with an uninterpreted function $\dot{x}(s)$:

$$x_1 = 1 + \int_0^t \dot{x}(s) \, \mathrm{d}s \ \wedge \ x_2 = 2 + \int_0^t \dot{x}(s) \, \mathrm{d}s \ \wedge \ t = 1$$
$$\wedge \ \forall u \in [0, t]. \, \dot{x}(u) = x(u).$$

This formula is *not* satisfiable, because there are no single interpretations of $\dot{x}$ and x that denote solution functions of $\frac{\mathrm{d}x}{\mathrm{d}s} = x$ for both initial values 1 and 2 at the same time.

We restrict our attention to $\mathcal{G}$ with the following constraints. Every uninterpreted function $x \in \mathcal{G}$ that occurs in the same integration term has the same domain $[0, t_x]$. Each derivative $\dot{x} \in \mathcal{G}$ is bounded by a finite set of Lipschitz-continuous functions $G_x = \{g_1(\boldsymbol{z}), g_2(\boldsymbol{z}), \ldots, g_m(\boldsymbol{z})\}$. This *boundedness constraint* for $\dot{x} \in \mathcal{G}$ can be expressed as the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formula:

$$bound_{G_x}(\dot{x}) \equiv \bigvee_{g(\boldsymbol{z}) \in G_x} \forall u \in [0, t_x]. \, \dot{x}(u) = g(\boldsymbol{z})(u).$$

Definition 6. *An $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formula for networks of hybrid I/O automata has the form:* $\exists \boldsymbol{y}. \, \varphi \wedge \bigwedge_{\dot{x} \in \mathcal{G}} bound_{G_x}(\dot{x})$, *where:*
- *each subformula of φ is quantifier-free or containing only universally quantified subformulas over time;*
- *each uninterpreted function in $\mathcal{G}$ appears in integration terms, universally quantified formulas over time of the form $\forall u \in [0, t_x]. \, \dot{x}(u) = g(\boldsymbol{z})(u)$, or formulas of the negated form $\exists u \in [0, t_x]. \, \dot{x}(u) \neq g(\boldsymbol{z})(u)$;*
- *each uninterpreted function in $\mathcal{G}$ appears in at most one integration term (but the same integration term can appear in a formula many times).*

The formula in Def. 5 is in the syntactic class of Def. 6, if every input variable corresponds to some output variable.[3] The syntax of Definition 6 allows ODE fragments to occur under arbitrary Boolean connectives, which often enables further simplification of the encoding (e.g., Sec III-D).

B. $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-Reduction of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-Formulas

A key part of our algorithm is *the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction* that removes uninterpreted functions in $\mathcal{G}$ from $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas, summarized in Alg. 1. For a conjunction μ of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals that *satisfy the boundedness constraint and the syntactic restriction in Def. 6*, the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction procedure builds an $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula that is equisatisfiable to the conjunction μ.

Definition 7. *An $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literal is an atomic $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula, a universally quantified formula $\forall u \in [0, t_x]. \, \dot{x}(u) = g(\boldsymbol{z})(u)$, or a negation of these $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-atoms.*

Without loss of generality, we assume that for each derivative $\dot{x} \in \mathcal{G}$, the conjunction μ includes a universally quantified $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literal of the form $\forall u \in [0, t_x]. \, \dot{x}(u) = g(\boldsymbol{z})(u)$ from the boundedness constraint $bound_{G_x}(\dot{x})$.

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[3]Each uninterpreted unary real function encodes an input variable of one component, and its behavior is bounded by a connected output variable.

Algorithm 1: $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas.

Input: A conjunction $\mu = l_1 \wedge \cdots \wedge l_m$ of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals
Output: An equisatisfiable $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula

1 **for** *each $\dot{x} \in \mathcal{G}$ inside μ's integration terms* **do**
2   pick a literal $l = \forall u \in [0, t_x].\, \dot{x}(u) = g(\boldsymbol{z})(u)$ in μ;
3   replace each $\dot{x}$ by $g(\boldsymbol{z})$ in μ's integration terms;
4   **for** *each literal l' in μ including $\dot{x}$ other than l* **do**
5   **if** $l' = \forall u \in [0, t_x].\, \dot{x}(u) = g'(\boldsymbol{z})(u)$ **then**
6   combine $\forall u \in [0, t_x].\, g(\boldsymbol{z})(u) = g'(\boldsymbol{z})(u)$ with μ;
7   **else**
8   combine $\exists u \in [0, t_x].\, g(\boldsymbol{z})(u) \neq g'(\boldsymbol{z})(u)$ with μ;
9 **return** μ;

The $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction begins with choosing a universally quantified $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literal $\forall u \in [0, t_x].\; \dot{x}(u) = g(\boldsymbol{z})(u)$ for each derivative $\dot{x} \in \mathcal{G}$ (line 2). We replace every occurrence of derivative $\dot{x}$ in parameterized integration terms by the term $g(\boldsymbol{z})(u)$ (line 3). This procedure preserves the satisfiability of the formula as stated in Lemma 1, since each uninterpreted function in $\mathcal{G}$ appears in at most one integration term.

Lemma 1. *Given a conjunction μ of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals including $\forall u \in [0, t_x].\; \dot{x}(u) = g(\boldsymbol{z})(u)$, the formula μ' obtained from μ by replacing each occurrence of $\dot{x}$ in integration terms by $g(\boldsymbol{z})$ is equisatisfiable to the conjunction μ.*

Proof. By the assumptions in Def. 6, one integration term $[\boldsymbol{z}_0, x_0] + \int_0^{t_x} [\boldsymbol{f}(\boldsymbol{z}, x)(s), \dot{x}(s)]\, \mathrm{d}s$ can include $\dot{x}$ in μ (we are done if no such integration term exists). Suppose that the conjunction μ is satisfiable. There exist real numbers $\boldsymbol{z}_0^{\mathbb{R}}$, $x_0^{\mathbb{R}}$, $t_x^{\mathbb{R}}$ and real functions $\dot{x}^{\mathbb{R}}$, $x^{\mathbb{R}}$, $\boldsymbol{z}^{\mathbb{R}}$ of domain $[0, t_x^{\mathbb{R}}]$, where

- $\dot{x}^{\mathbb{R}}(u) = g(\boldsymbol{z}^{\mathbb{R}})(u)$ for $u \in [0, t_x^{\mathbb{R}}]$,
- $\dot{x}^{\mathbb{R}}$ is a derivative of $x^{\mathbb{R}}$,
- $\boldsymbol{z}^{\mathbb{R}}$ satisfies the ODEs with $\boldsymbol{z}^{\mathbb{R}}(0) = \boldsymbol{z}(0)$:

$$\frac{\mathrm{d}}{\mathrm{d}s}[\boldsymbol{z}, x] = [\boldsymbol{f}(\boldsymbol{z}, x), g(\boldsymbol{z}^{\mathbb{R}})](s)$$

Now let us consider the initial value problem of the ODEs

$$\frac{\mathrm{d}}{\mathrm{d}s}[\boldsymbol{z}^{\mathbb{R}}, x^{\mathbb{R}}] = [\boldsymbol{f}(\boldsymbol{z}, x), g(\boldsymbol{z})](s)$$

with $[\boldsymbol{z}, x](0) = [\boldsymbol{z}_0^{\mathbb{R}}, x_0^{\mathbb{R}}]$. By the Picard-Lindelöf theorem, this problem has a unique solution, but notice that $[\boldsymbol{z}^{\mathbb{R}}, x^{\mathbb{R}}]$ is also a solution of it. This means that the replacement result μ' including the term

$$[\boldsymbol{z}_0, x_0] + \int_0^{t_x} [\boldsymbol{f}(\boldsymbol{z}, x)(s), g(\boldsymbol{z})(s)]\, \mathrm{d}s$$

is satisfiable by the same real numbers $\boldsymbol{z}_0^{\mathbb{R}}$, $x_0^{\mathbb{R}}$, $t_x^{\mathbb{R}}$ and real functions $\dot{x}^{\mathbb{R}}$, $x^{\mathbb{R}}$, $\boldsymbol{z}^{\mathbb{R}}$. The other direction is similar; the same real numbers and real functions in a satisfiable model of μ' also satisfy the original formula μ. $\square$

As a result, every parameterized integration term in the conjunction μ becomes *concrete solution $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-term* without free ODE variables. That is, each integration term is now considered as a computable real function in $\mathcal{F}'$.

We add extra constraints to ensure the consistency between the chosen $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals and the other $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals in μ (line 4). For each pair of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals in the conjunction μ, we must ensure that they are actually satisfiable at the same time. We need two types of extra constraints:

- $\forall u \in [0, t_x].\, g(\boldsymbol{z})(u) = g'(\boldsymbol{z})(u)$ for each universally quantified literal $\forall u \in [0, t_x].\, \dot{x}(u) = g'(\boldsymbol{z})(u)$ in μ; and
- $\exists u \in [0, t_x].\, g(\boldsymbol{z})(s) \neq g'(\boldsymbol{z})(s)$ for each negated literal of the form $\exists u \in [0, t_x].\, \dot{x}(u) \neq g'(\boldsymbol{z})(u)$ in μ.

These new constraints are often *not* $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas, since they can still include ODE variables (e.g., $\boldsymbol{z}$).

We express these constraints in $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$ by adding extra ODEs to corresponding integration terms. For a constraint $\forall u \in [0, t_x].\, g(\boldsymbol{z})(u) = g'(\boldsymbol{z})(u)$, we define a new ODE $\frac{\mathrm{d}w}{\mathrm{d}s} = g'(\boldsymbol{z})(s) - g(\boldsymbol{z})(s)$ with a fresh variable w. If the constraint is true, the value of the variable w is always 0; that is, the invariant condition $\forall u \in [0, t_x].\, w(u) = 0$ must hold. We formally define this process as follows.

Definition 8. *For constraints $\forall u \in [0, t_x].\, g(\boldsymbol{z})(u) = g'(\boldsymbol{z})(u)$ or $\exists u \in [0, t_x].\, g(\boldsymbol{z})(s) \neq g'(\boldsymbol{z})(s)$, the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction of integration $F(\boldsymbol{y^0}, t) \equiv \boldsymbol{y^0} + \int_0^t \boldsymbol{f}(\boldsymbol{z})(s)\, \mathrm{d}s$ is the term:*

$$\hat{F}(\boldsymbol{y^0}, t) \equiv [0, \boldsymbol{y^0}] + \int_0^t [g'(\boldsymbol{z})(s) - g(\boldsymbol{z})(s),\ \boldsymbol{f}(\boldsymbol{z})(s)]\, \mathrm{d}s,$$

and the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reductions of the constraints are respectively (with the projection function $\pi_1(a, b) = a$):

$$\forall u \in [0, t_x].\, \pi_1(\hat{F}(\boldsymbol{y^0}, u)) = 0, \ \exists u \in [0, t_x].\, \pi_1(\hat{F}(\boldsymbol{y^0}, u)) \neq 0.$$

This process also preserves the satisfiability of the formula as stated in Lemma 2, because every corresponding integration term of a constraint is identical by assumption.

Lemma 2. *A conjunction μ with an extra constraint is equisatisfiable to the formula μ' obtained from μ by replacing corresponding integration terms by their $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reductions and replacing the constraint by its $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction.*

Proof. Suppose that μ is satisfiable. There exists a solution function $\boldsymbol{z}^{\mathbb{R}}$ of the integration term $\boldsymbol{y^0} + \int_0^t \boldsymbol{f}(\boldsymbol{z})(s)\, \mathrm{d}s$ that satisfies the extra constraint; that is, either:

$$\forall u \in [0, t_x].\, g(\boldsymbol{z})(u) = g'(\boldsymbol{z})(u), \ \text{or}$$
$$\exists u \in [0, t_x].\, g(\boldsymbol{z})(s) \neq g'(\boldsymbol{z})(s)$$

In the formula μ', since w is a fresh ODE variable, $[\boldsymbol{z}^{\mathbb{R}}, 0]$ is the solution function of the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction of the integration term (where 0 is the zero function). Similarly, if the formula μ' is satisfiable, there exists a solution function $[0, \boldsymbol{z}^{\mathbb{R}}]$ of the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction of the integration term, which is the solution function of the original term and satisfies the constraint. $\square$

The time complexity of the entire $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction process is $O(n^2)$, where n denotes the number of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-literals in the conjunction μ.[4] The correctness of our $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction procedure follows from the fact that each $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction step in Alg. 1 preserves the satisfiability.

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[4]There can be many identical integration terms for each step, but the replacements can be performed in constant time using subformula sharing.

Theorem 2. *Given a conjunction μ that meet the boundedness constraint and the syntactic restriction in Def. 6, Algorithm 1 generates an $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula that is equisatisfiable to μ.*

Proof. At line 3, one uninterpreted function in parameterized integration terms is removed from μ, while preserving the satisfiability by Lemma 1. After executing the loop of line 4, the chosen uninterpreted function is entirely removed from μ, and the result is equisatisfiable to μ by Lemma 2. Therefore, for each iteration, one uninterpreted function is removed (and all in the end), while preserving the satisfiability. $\qquad\square$

C. Checking Satisfiability Using $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-Reduction

We consider the satisfiability of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-formulas of the syntactic class of Def. 6. We apply the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction process to have equisatisfiable $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas without extra function symbols in $\mathcal{G}$, and then employ an existing algorithm to check $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-satisfiability as a subroutine. Therefore, our algorithm terminates if the underlying $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-satisfiability subroutine is terminated. This provides a theory solver for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$, by means of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction and an $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-solver.

Algorithm 2: SMT procedure for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-formulas.

Input: An $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-formula $\exists \boldsymbol{y}.\, \varphi \wedge \bigwedge_{\dot{x}\in\mathcal{G}} bound_{G_x}(\dot{x})$
Output: Unsat, or Sat with a satisfiable assignment

1 **while** $\exists$ *a propositionally satisfiable set L of literals* **do**
2 $\quad$ $\phi \leftarrow \mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction($\bigwedge_{l\in L} l$);
3 $\quad$ **if** ϕ *is* Sat *by* $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-*satisfiability solving* **then**
4 $\quad\quad$ **return** Sat *and a satisfiable assignment of* $\boldsymbol{y}$;
   $\quad\quad$ // learn the conflicts in case of unsat
5 $\quad$ learn the conflicts between $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals;
6 **return** Unsat;

Algorithm 2 summarizes our algorithm. We employ the DPLL($\mathcal{T}$) framework to obtain a propositionally satisfiable set of literals by using Boolean satisfiability solving (line 1). The formula is unsatisfiable if no propositionally satisfiable set exists (line 6). Otherwise, each set imposes a conjunction of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals. Since we only consider formulas in the class of Def. 6, the conjunction satisfies the boundedness constraint.

We apply the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction procedure in Alg. 1 to obtain an *equisatisfiable $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula* ϕ that has no uninterpreted functions in the collection $\mathcal{G}$ (line 2). We use an SMT algorithm for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formula as a subroutine (line 3). If the resulting formula ϕ is satisfiable, then the original formula is also satisfiable (line 4), and a satisfiable assignment is obtained.[5] If the resulting formula ϕ is not satisfiable, conflict clauses can be used by SAT solving for the next iteration based on conflict-driven clause learning (line 5).

The combined procedure is briefly shown in Figure 3. The two boxes in the right-hand side denote Alg. 1, and the two boxes in the left-hand side denotes Alg. 2.

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[5] Since the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction does not alter first-order variables, satisfiable assignments for μ and ϕ have the same values for the first-order variables.
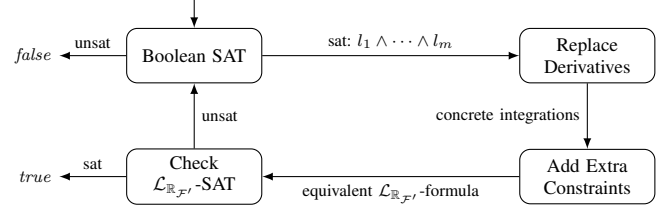

Fig. 3. SMT procedure for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$ formulas

D. δ-Complete SMT for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-Formulas

Our algorithm is best suited for nonlinear hybrid systems where continuous I/O cannot be encoded as discrete actions. For example, δ-complete SMT can be applied to check the satisfiability of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas up to a given precision $\delta > 0$, called δ-*satisfiability* [3]. The satisfiability of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas is in general undecidable for nonlinear real functions [20], but the δ-satisfiability of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas is decidable.

Finding conflict $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals is very important for the performance of DPLL($\mathcal{T}$) and conflict-driven clause learning, but this process is nontrivial for δ-complete SMT. The reason is that δ-consistency actually depends on the value of δ. For example, two universally quantified $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals

$$\forall u \in [0, 0.5].\, \dot{x}(u) = 1 + 0.5u,$$
$$\forall u \in [0, 0.5].\, \dot{x}(u) = \sqrt{u+1}$$

are inconsistent up to precision $\delta = 0.01$, but consistent up to different precision $\delta = 0.1$.

To facilitate the process of finding conflict $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals, we use *uniqueness lemmas* for incompatible $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals. Consider two $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals $\forall u \in [0, t_x].\, \dot{x}(u) = g(\boldsymbol{z})(u)$ and $\forall u \in [0, t_x].\, \dot{x}(u) = g'(\boldsymbol{z})(u)$. If we know in advance that two functions g and g' are not equal, we add the formula

$$\neg[\forall u \in [0, t_x].\, \dot{x}(u) = g(\boldsymbol{z})(u) \wedge \forall u \in [0, t_x].\, \dot{x}(u) = g'(\boldsymbol{z})(u)]$$

asserting that two $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literals cannot be true at the same time. This can greatly improve the performance of DPLL($\mathcal{T}$), since the consistency checking can be performed at line 1, only using Boolean satisfiability solving.

For the reachability of networks of hybrid I/O automata, we apply a heuristic specialized for the encoding in Def. 5. Since each mode q can correspond to only one flow condition, if a formula $\forall u \in [0, t].\, [\dot{\boldsymbol{x}}, \dot{\boldsymbol{o}}](u) = [\boldsymbol{f_q}, \boldsymbol{g_q}](u)$ for one continuous output is true, the truths of other continuous output formulas are not relevant. In Alg. 1, we choose one universally quantified $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'\cup\mathcal{G}}}$-literal for each mode, and disregard extra consistency checking in line 4 in this case.

V. Case Studies

This section shows a number of examples of networks of hybrid system components, besides the airplane example. They include discrete and continuous interactions between components, and involve nontrivial nonlinear ODEs.

A. Driving Simple Cars.

A number of cars are running in sequence, while each car follows the behavior of the car in front (the first car moves according to its own scenario). The position (x_i, y_i) and the direction θ_i of each car i of length L_i depends on its speed v_i and steering angle ϕ_i, given by the nonlinear ODEs [21]:

$$\dot{x}_i = v_i \cos\theta_i, \qquad \dot{\theta}_i = v_i/L_i \cdot \tan\phi_i,$$
$$\dot{y}_i = v_i \sin\theta_i, \qquad \dot{\phi}_i = -k_i(\phi_i - \phi_{i-1}),$$

To keep a safe distance, each car has three modes acc, dec, and $keep$ for acceleration, deceleration, and following the speed of the front car, respectively:

$$\dot{v}_i = \begin{cases} -K_i(v_i - v_{i-1}) & \text{if} \quad q_i = keep \\ C & \text{if} \quad q_i = acc \\ -C & \text{if} \quad q_i = dec \end{cases}$$

for $C > 0$. A car controller performs a discrete transition when the distance is either too close or too far. The goal is to find a mode change schedule for driving cars safely.

B. Network of Thermostat Controllers.

A number of rooms are interconnected by open doors (Fig. 4). The temperature x_i of each room i is separately controlled by each thermostat, depending on both the heater's mode $q_i \in \{m_{\text{on}}, m_{\text{off}}\}$ and the temperatures of the adjacent rooms. The value of x_i changes according to the ODEs:

$$\frac{dx_i}{dt} = \begin{cases} K_i\big(h_i - (c_i x_i - d_i \sum_{j \in A_i} x_j)\big) & \text{if} \quad q_i = m_{\text{on}} \\ -K_i\big(c_i x_i - d_i \sum_{j \in A_i} x_j\big) & \text{if} \quad q_i = m_{\text{off}}, \end{cases}$$

where $A_i \subseteq \mathbb{N}$ is the set of the adjacent rooms, and K_i, h_i, c_i, d_i depend on the size of the room, the heater's power, and the size of the open doors, respectively. Notice that the ODEs for room i include variables x_j for adjacent rooms $j \in A_i$. A thermostat controller H_i performs a discrete transition by using the input action on^i or off^i to turn on or off the heater. The goal is to keep each temperature in a desired range, while the outside temperatures change.

C. Network of Water Tanks.

A number of water tanks are connected by pipes (Fig. 5), adapted from [11], [12]. The water level x_i of each tank i is separately controlled by its pump in the tank, depending on the pump's mode $m_i \in \{m_{\text{on}}, m_{\text{off}}\}$ and the levels of the adjacent tanks. The value of x_i changes according to the nonlinear ODEs:

$$A_i \dot{x}_i = (q_i + a\sqrt{2g}\sqrt{x_{i-1}}) - a\sqrt{2g}\sqrt{x_i} \quad \text{if} \quad m_i = m_{\text{on}}$$
$$A_i \dot{x}_i = a\sqrt{2g}\sqrt{x_{i-1}} - a\sqrt{2g}\sqrt{x_i} \quad \text{if} \quad m_i = m_{\text{off}}$$

($x_0 = 0$ for the leftmost tank 1), where A_i, q_i, a depend on the size of the tank, the power of the pump, and the width of the pipe, respectively, and g is the standard gravity constant. Similarly, a controller H_i performs a discrete transition using the input action on^i or off^i to turn on or off the pump. The goal is to keep each water level in a desired range.
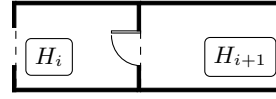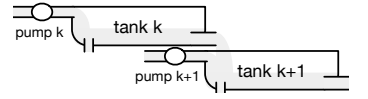

Fig. 4. Connected rooms

Fig. 5. Connected water tanks

D. Multiple battery Usage.

Given a number of fully charged batteries, a controller switches load between them to achieve longer lifetime out of the batteries, adapted from [22]. Each battery i has three modes $switchedOn$, $switchedOff$, and $dead$. The battery charge dynamics is expressed by the ODEs:

$$\text{(on)} \begin{array}{l} \dot{d}_i = L/c - kd_i \\ \dot{g}_i = -L, \end{array} \quad \text{(off)} \begin{array}{l} \dot{d}_i = -kd_i \\ \dot{g}_i = 0, \end{array} \quad \text{(dead)} \begin{array}{l} \dot{d}_i = 0 \\ \dot{g}_i = 0, \end{array}$$

with d_i its kinetic energy difference, g_i its total charge, L its load, and $c \in [0, 1]$ its threshold. If $g_i > (1-c)d_i$, battery i is dead. Otherwise, it can be either on or off. When k batteries are on, load to each battery is divided by k. A goal is to find a switching schedule to achieve a desired lifetime.

VI. EXPERIMENTAL RESULTS

We have implemented our algorithm for $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-formulas in version 2 of the dReal SMT solver [15]. It can decide the δ-satisfiability of a wide range of $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-formulas (also containing universal quantification over time).[6] We base our implementation on dReal, since the tool is open-source and supports nontrivial *nonlinear ODEs* in our case studies.[7] However, in principle our techniques can be combined with any other SMT-based approaches for hybrid systems to support continuous I/O between hybrid components.

We have compared the performance of our algorithm for the $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-encoding with one by a non-modular $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-encoding. Because our examples involve continuous I/O that cannot be encoded by discrete actions, no modular $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-encoding is possible. Therefore, we use an $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-encoding of a single hybrid automaton that is equivalent to the composition, which has been studied in many approaches [1]–[5], [17], [23].

The experimental results are summarized in Fig. 6. (see Table I for more details). The case studies and the experimental results are available in the supplementary material. We have performed reachability analysis for the five case studies up to bound $k = 5$. We consider two variants for each example, with double or triple components (for the airplane example, nonlinear ODEs or approximate linear ODEs). For example, we consider a network of two water tanks and a network of three water tanks. We consider both sat (reachable) and $unsat$ (unreachable) cases using different goals. All the experiments in this section were conducted on Intel Xeon 2.6 GHz with 512 GB memory. We set a timeout of 12 hours.

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[6]dReal uses interval constraint propagation (ICP) to numerically evaluate ODE integration functions up to a precision $\delta > 0$ [17].

[7]For example, many analysis tools for nonlinear hybrid systems only support polynomials or linear ODEs, not nonlinear ODEs.
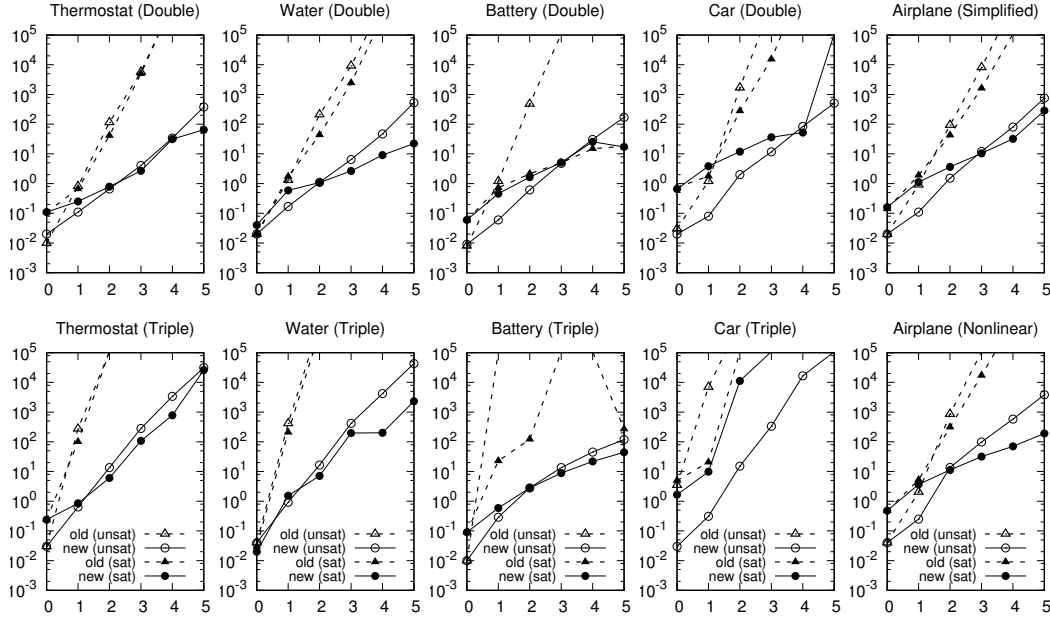
Fig. 6. Running time (in seconds) of k-step reachability analysis (solid lines for the new approach and dashed lines for the non-modular approach)

The results show that our approach with the new modular $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-encoding *significantly outperforms* the old approach with the non-modular encoding. For example, the analysis with the non-modular $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-encoding for three interconnected thermostats did not terminate within 12 hours for bound $k = 2$, whereas the same analysis by our algorithm with the modular $\mathcal{L}_{\mathbb{R}_{\mathcal{F}' \cup \mathcal{G}}}$-encoding gave the result less than 15 seconds.

According to the results, the performance improvement tends to be more apparent for bigger models and for the *unsat* cases. For the *sat* cases, it is possible that a satisfiable assignment can be found in an early stage of DPLL($\mathcal{T}$), e.g., the *sat* case of two batteries where the difference is small. In a bigger model the size of the generated formula becomes larger, and thus the benefit of using the modular encoding is clearly more explicit (e.g., the cases of three batteries).

This performance improvement is due to the fact that the modular encoding allows a much compact size of the formulas, and a more efficient ODE solving by decomposition of complex systems of ODEs. Also, conflict-driven clause learning can be fully exploited for continuous connections in this way, because conflicts caused by continuous I/O can be detected and then learned in our algorithm.

VII. RELATED WORK

One of the early studies on SMT-based analysis of nonlinear hybrid systems is [24], which proposes constraint solving algorithms for nonlinear reachability problems. There are several approaches that explicitly formulate analysis problems of nonlinear hybrid systems as SMT formulas over the real numbers, such as MathSAT/HyCOMP [5], [6], hydlogic [4], iSAT/HySAT [1], [2], and dReal/dReach [3], [15], [17], [23] and HybridSAL [8], [25]. In particular, hydlogic, iSAT-ODE, and dReal/dReach can deal natively with ODEs.

But a modular SMT encoding of networks of nonlinear hybrid systems has not been much investigated for continuous I/O in existing SMT-based approaches, which is what we aim to improve in this paper. All of these techniques assume that interactions between components can be specified through discrete synchronization (e.g., using joint actions). Hence, continuous interactions between components, which occur frequently in many networks of hybrid systems as shown in Sec. V, cannot be properly dealt with in a modular way without discrete approximation of continuous I/O.

In iSAT-ODE [1], the syntax allows ODE fragments to occur positively in formulas, and thus the encoding of Def. 6 can be expressed in principle. But a modular encoding has not been studied for iSAT-ODE, and the tool does not support it either. Our work formally identifies a generic class of formulas that *strictly* includes one supported by iSAT-ODE. Our algorithm is based on equisatisfiable $\mathcal{L}_{\mathbb{R}_{\mathcal{F}'}}$-reduction, whereas iSAT-ODE uses a specialized algorithm that extends DPLL.

The dReach tool [23] encodes reachability analysis problems of nonlinear hybrid systems as SMT problems for dReal. In order to deal with the formula explosion problem, dReach *explicitly* enumerates all mode paths of a hybrid automaton by generating many small SMT formulas for these paths. However, the power of SMT techniques (such as DPLL($\mathcal{T}$) and CDCL) cannot be fully exploited in this way.

In addition to SMT-based approaches, there are many other approaches to analyze nonlinear hybrid systems by calculating a set of reachable states, e.g., [26]–[29]. Our technique and such reachable-set computation techniques focus on different aspects of hybrid system analysis. Reachable-set computation can be used as an ODE solver in Alg. 2, while our technique addresses higher-level composition and modular analysis.

TABLE I
RUNNING TIME OF k-STEP REACHABILITY ANALYSIS (- MEANS TIMEOUT)

Benchmark			Time (s)					
			$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
Thermo (double)	unsat	new	0.02	0.11	0.66	4.07	34.23	377.22
		old	0.01	0.81	114.33	5723.48	-	-
	sat	new	0.11	0.25	0.79	2.69	31.55	64.73
		old	0.11	0.66	41.17	5015.66	-	-
Thermo (triple)	unsat	new	0.03	0.64	13.55	280.08	3390.64	31470.83
		old	0.03	272.50	-	-	-	-
	sat	new	0.24	0.85	6.07	107.87	782.46	25854.26
		old	0.24	100.13	-	-	-	-
Water (double)	unsat	new	0.02	0.17	1.10	6.42	46.79	528.18
		old	0.02	1.31	210.07	9139.09	-	-
	sat	new	0.04	0.59	1.08	2.65	9.06	22.30
		old	0.02	1.72	43.72	2448.90	-	-
Water (triple)	unsat	new	0.04	0.92	16.57	418.35	4225.10	43093.32
		old	0.04	413.66	-	-	-	-
	sat	new	0.02	1.53	7.10	196.84	201.38	2319.91
		old	0.03	208.29	-	-	-	-
Battery (double)	unsat	new	0.01	0.06	0.61	4.77	30.47	171.11
		old	0.01	1.20	479.19	-	-	-
	sat	new	0.06	0.46	1.65	5.16	25.41	17.17
		old	0.06	0.72	2.13	4.99	14.96	18.02
Battery (triple)	unsat	new	0.01	0.29	2.80	13.55	45.09	118.55
		old	0.01	-	-	-	-	-
	sat	new	0.09	0.59	2.77	8.86	21.92	44.31
		old	0.09	22.61	120.96	-	-	269.07
Car (double)	unsat	new	0.02	0.08	2.01	11.73	83.42	509.29
		old	0.03	1.20	1681.85	-	-	-
	sat	new	0.66	3.86	11.96	36.46	51.95	-
		old	0.74	1.78	277.90	15297.14	-	-
Car (triple)	unsat	new	0.03	0.31	15.06	334.30	16690.72	-
		old	3.47	6972.87	-	-	-	-
	sat	new	1.66	9.85	11218.88	-	-	-
		old	4.94	20.54	-	-	-	-
Simplified Airplane	unsat	new	0.02	0.11	1.51	12.03	79.72	755.13
		old	0.02	0.91	93.15	8143.61	-	-
	sat	new	0.16	1.12	3.63	10.45	32.05	285.17
		old	0.14	1.90	42.90	1615.98	-	-
Nonlinear Airplane	unsat	new	0.04	0.25	13.61	98.32	577.95	3829.00
		old	0.04	2.05	852.17	-	-	-
	sat	new	0.48	3.67	11.27	31.57	70.24	190.94
		old	0.48	5.25	310.39	17130.38	-	-

VIII. CONCLUSIONS

We have presented new SMT-based techniques for analyzing networks of nonlinear hybrid systems. We have shown that continuous interactions between hybrid components, which cannot be captured by discrete communication in general, can be decomposed and modularly encoded as SMT formulas. Since existing SMT algorithms cannot deal with the modular encoding, we have presented a new SMT solving algorithm, which can greatly increase the performance of SMT-based analysis for networks of nonlinear hybrid systems.

Future work will include investigating approximation and decomposition methods to achieve further scalability when analyzing networks of nonlinear hybrid systems. We also plan to implement our techniques in version 3 of the dReal SMT solver [15] (which is a new implementation of dReal) and to integrate it with the dReach reachability analysis tool. This also requires to develop a suitable modeling language for dReach to specify a network of nonlinear hybrid systems, because the dReach language currently only allow to specify a single hybrid system component.

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