Hybrid PALS for Distributed Hybrid Systems and Their SMT-Based Analysis

Kyungmin Bae[1,2], Soonho Kong[1], Sicun Gao[3], Peter Csaba Ölveczky[4], and Edmund M. Clarke[1]

[1] Carnegie Mellon University
[2] SRI International
[3] Massachusetts Institute of Technology
[4] University of Oslo

Abstract. The PALS methodology reduces the problem of designing and verifying a *virtually synchronous* distributed real-time system to the much simpler task of designing and verifying its underlying *synchronous* design. This paper presents *Hybrid PALS*, which extends PALS to the important class of virtually synchronous cyber-physical systems (CPSs) where distributed controllers communicate with each other and interact with physical entities having *continuous dynamics*. We prove a bisimulation equivalence between the synchronous and the distributed models. We then explain how a number of verification problems for synchronous Hybrid PALS models, such as bounded reachability and inductive analysis, can be reduced to SMT solving over real numbers with nonlinear ordinary differential equations with arbitrary precision. Since such SMT-based analysis typically becomes unfeasible due to the *formula explosion problem*, we propose a new SMT framework to effectively encode distributed hybrid systems in a modular way.

1 Introduction

Virtually synchronous distributed real-time systems consist of a number of *distributed* controllers that should logically behave in a synchronous way. Designing and analyzing such systems is very hard because of race conditions, clock skews, network delays, execution times, and the state space explosion caused by the interleavings. To overcome these problems, the *PALS* (physically asynchronous, logically synchronous) methodology [1,2,5,19] has been developed to reduce the design and verification of a virtually synchronous distributed real-time system to the much simpler task of designing and verifying the underlying *synchronous* models, provided that the infrastructure can guarantee bounds on computation times, network delays, and imprecision of the local clocks.

In this paper we present *Hybrid PALS* that extends PALS to the important class of virtually synchronous *cyber-physical systems* (CPSs), which includes avionics, automotive, robotics, and medical systems. The controllers interact with their local *physical environments*, and the *continuous dynamics* can be specified by ordinary differential equations (ODEs). We prove a bisimulation equivalence relating distributed hybrid systems and the underlying synchronous models. This means that the design and analysis of distributed hybrid systems can be reduced to those of the synchronous models.

Verifying the synchronous Hybrid PALS models is still challenging. Different components read their physical values and give actuator commands at different times because of local clock skews, and these timings cannot be abstracted away. The continuous dynamics often involves *nonlinear* ODEs which in general may not have exact solutions. This paper presents SMT solving techniques to address the challenges of analyzing Hybrid PALS models of CPSs. The verification of a distributed CPS involving ODEs and clock skews is reduced to checking the satisfiability of SMT formulas over real numbers and ODEs, which is decidable up to any user-given precision $\delta > 0$ [13, 15]. The number δ, provided by the user, is the bound on numerical errors that is tolerable in the analysis.

One problem with SMT-based methods is that they do not scale well for distributed hybrid systems with nonlinear ODEs. We cope with this scalability problem as follows. First, the discrete part and the continuous part of the system are encoded separately, so that ODEs are considered *only after* the discrete part of the system has been fully analyzed. Second, in addition to bounded reachability analysis, *inductive* SMT analysis is used to verify safety properties, without any time bound. Third, compositional SMT analysis is used to analyze each part of a distributed CPS by a divide-and-conquer approach.

In distributed CPS, the states of different components can be physically correlated to each other. Consider, for example, two adjacent rooms: then the temperature of one room immediately affects the temperature of the other room. Their behaviors are specified as *coupled* ODEs in which variables evolve over time simultaneously. Existing SMT approaches use non-modular encodings for such systems. The size of the SMT formula can be huge, which leads to the *formula explosion problem* that makes SMT-based analysis practically infeasible.

To overcome this problem, we present a new SMT framework to effectively encode formal analysis problems of distributed hybrid systems with coupled ODEs *in a modular way*. We show that a satisfaction problem in the new theory can be reduced to one in the standard theory of the real numbers *at no cost*. For a composition $H_1 \parallel \cdots \parallel H_n$ of n hybrid systems, where each component H_i has m_i control modes, the size of the SMT formula by the new modular encoding is $O(\sum_{i=1}^{n} k_i)$, whereas by the previous encoding the size is $O(\prod_i k_i)$.

We have implemented our SMT techniques within the dReal SMT solver [14]. We have applied our techniques on a range of advanced CPSs, including: a system for turning an airplane, networked water tank controllers, and networked thermostat controllers. The experimental results show that it can dramatically improve the performance of SMT-based verification of distributed CPSs.

The rest of the paper is organized as follows. Section 2 gives a background on PALS. Section 3 introduces Hybrid PALS. Section 4 shows how synchronous Hybrid PALS models and their verification problems can be encoded as logical formulas, and presents a new SMT framework supporting the modular encoding of distributed hybrid systems. Section 5 gives an overview of the Hybrid PALS

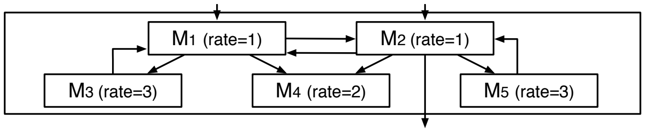


Fig. 1. A multirate ensemble $\mathcal{E}$, with M_1 and M_2 slow machines.

verification case studies. Finally, Section 6 discusses related work, and Section 7 gives some concluding remarks.

2 Preliminaries on PALS

PALS transforms a *synchronous design SD* into a distributed real-time system $\mathcal{MA}(SD, T, \Gamma)$ with period T, satisfying the same temporal logic properties, provided that the underlying infrastructure guarantees bounds Γ on network delays, execution times, and clock skews. This section overviews the synchronous models SD, the distributed models $\mathcal{MA}(SD, T, \Gamma)$, and the relationship between SD and $\mathcal{MA}(SD, T, \Gamma)$ (we refer to [5, 19] for details).

2.1 Discrete Synchronous Models.

The synchronous model SD is specified as an *ensemble* $\mathcal{E}$ of state machines with input and output ports. In each iteration, a machine performs a transition based on its inputs, proceeds to the next state, and generates new outputs.

Definition 1. *A* typed machine *M is a tuple $M = (D_i, S, D_o, \delta_M)$, where (i) $D_i = D_{i_1} \times \cdots \times D_{i_n}$ an input set (a value to the k-th* input port *is an element of D_{i_k}), (ii) S a set of states, (iii) $D_o = D_{o_1} \times \cdots \times D_{o_m}$ an output set, and (iv) $\delta_M \subseteq (D_i \times S) \times (S \times D_o)$ a total transition relation.*

A collection $\{M_j\}_{j \in J_S \cup J_F}$ of state machines with different periods can be composed into a *multirate ensemble* $\mathcal{E}$, as illustrated in Fig. 1. The period of a slow machine $s \in J_S$ (with $\mathit{rate}(s) = 1$) is a multiple of the period of a fast machine $f \in J_F$ (with $\mathit{rate}(f) > 1$). A *wiring diagram* connects the input and output ports, where there are no connections between two fast machines.

In each iteration, all components in $\mathcal{E}$ perform a transition each *in lockstep.* A fast machine f is *slowed down* and performs $k = \mathit{rate}(f)$ *internal* transitions in one global synchronous step. Since a fast machine produces k-tuples of outputs in one step, *input adapters* are used to generate single values (e.g., the last value, or the average of the k values) for a slow machine. Likewise, a single output from a slow machine is adapted to a k-tuple of inputs for a fast machine.

Definition 2. *A tuple $\mathcal{E} = (J_S, J_F, e, \{M_j\}_{j \in J_s \cup J_F}, E, \mathit{src}, \mathit{rate}, \mathit{adap})$ denotes a* multirate ensemble*, where: (i) J_S is a set of "slow machine" indices; (ii) J_F is a set of "fast machine" indices ($J_S \cap J_F = \emptyset$); (iii) $e \notin J_S \cup J_F$ is the* *"ensemble interface" index; (iv)* $\{M_j\}_{j \in J_S \cup J_F}$ *is a family of typed machines; (v)* $E = (D_i^e, D_o^e)$ *is the ensemble's* interface *with* D_i^e *the* input set *and* D_o^e *its* output set*; (vi) src is a wiring diagram assigning to each input port* (j, n) *(input port n of machine j) its "source" output port; (vii) rate assigns to each fast machine its rate; and (viii) adap assigns an input adaptor to each machine.*

To formally define a synchronous composition, we first recall some notations. The k-step deceleration of a machine M performs k transitions in one step, where each input and output port handles a k-tuple of values. The adaptor closure of M with an *input adaptor* $\alpha = \{\alpha_j : D'_j \to D_{i_j}\}_{j \in \{1,\dots,n\}}$ is the machine in which each input port j is enclosed by each input adaptor function α_j.

Definition 3. *For a machine* $M = (D_{i_1} \times \cdots \times D_{i_n}, S, D_{o_1} \times \cdots \times D_{o_m}, \delta_M)$*, its k-step* deceleration *is* $M^{\times k} = (D_{i_1}^k \times \cdots \times D_{i_n}^k,\ S,\ D_{o_1}^k \times \cdots \times D_{o_m}^k,\ \delta_{M^{\times k}})$*, where:* $(((\boldsymbol{i}_1, \dots, \boldsymbol{i}_n), s_0), (s_k, (\boldsymbol{o}_1, \dots, \boldsymbol{o}_m))) \in \delta_{M^{\times k}}$ *iff* $\exists s_1, \dots, s_{k-1} \in S$ *such that* $\bigwedge_{j=0}^{k-1}(((\pi_j(\boldsymbol{i}_1), \dots, \pi_j(\boldsymbol{i}_n)), s_j), (s_{j+1}, (\pi_j(\boldsymbol{o}_1), \dots, \pi_j(\boldsymbol{o}_m)))) \in \delta_M$*, where* $\pi_j(\boldsymbol{d})$ *denotes the j-th component of a k-tuple* $\boldsymbol{d}$*.*

Definition 4. *Given a machine* $M = (D_i, S, D_o, \delta_M)$ *and an input adaptor* $\alpha = \{\alpha_j : D'_j \to D_{i_j}\}_{j \in \{1,\dots,n\}}$*, where* $\alpha(d_1, \dots, d_n) = (\alpha_1(d_1), \dots, \alpha_n(d_n))$*, the* adaptor closure *is the machine* $M_\alpha = ((D'_1 \times \cdots \times D'_n), S, D_o, \delta_{M_\alpha})$ *such that* $((\boldsymbol{i}, s), (s', \boldsymbol{o})) \in \delta_{M_\alpha} \iff ((\alpha(\boldsymbol{i}), s), (s', \boldsymbol{o})) \in \delta_M$*.*

The *synchronous composition* of a multirate ensemble $\mathcal{E}$ is equivalent to a single machine $M_\mathcal{E} = (D_i^\mathcal{E}, S^\mathcal{E}, D_o^\mathcal{E}, \delta_{M_\mathcal{E}})$. If a machine in $\mathcal{E}$ has a feedback wire connected to itself or to another component, then the output becomes an input of the destination in the next iteration. That is, $M_\mathcal{E}$'s states $S^\mathcal{E}$ consist of the states S_j of its subcomponents M_j and the "feedback" outputs D_{OF}^j (i.e., the outputs from M_j to some machine in $\mathcal{E}$). For example, the synchronous composition $M_\mathcal{E}$ of the ensemble $\mathcal{E}$ in Fig. 1 is the machine given by the outer box.

Definition 5. *For an ensemble* $\mathcal{E} = (J_S, J_F, e, \{M_j\}_{j \in J_s \cup J_F}, E, src, rate, adap)$*. let* $\overline{M}_j = (M_j^{\times rate(j)})_{adap(j)}$ *for* $j \in J_S \cup J_F$*. The* synchronous composition *of* $\mathcal{E}$ *is the machine* $M_\mathcal{E} = (D_i^\mathcal{E}, S^\mathcal{E}, D_o^\mathcal{E}, \delta_\mathcal{E})$*, where: (i)* $D_i^\mathcal{E} = D_o^e$ *and* $D_o^\mathcal{E} = D_i^e$*; (ii)* $S^\mathcal{E} = \prod_{j \in J_S \cup J_F}(S_j \times D_{OF}^j)$*; and (iii)* $\delta_\mathcal{E} \subseteq (D_i^\mathcal{E} \times S^\mathcal{E}) \times (S^\mathcal{E} \times D_o^\mathcal{E})$ *that "combines" the transitions of* $\{\overline{M}_j\}_{j \in J_S \cup J_F}$ *into a synchronous step:*

$$\begin{aligned} &\big((\boldsymbol{i}, \{s_j, \boldsymbol{f}_j\}_{j \in J_S \cup J_F}), (\{s'_j, fout_l(\boldsymbol{o}'_l)\}_{j \in J_S \cup J_F}, in_e(\{\boldsymbol{o}'_j\}_{j \in J_S \cup J_F}))\big) \in \delta_\mathcal{E} \\ \iff &\text{ for each } l \in J_S \cup J_F,\ \big((in_l(\boldsymbol{i}, \{\boldsymbol{f}_j\}_{j \in J_S \cup J_F}), s_l), (s'_l, \boldsymbol{o}'_l)\big) \in \delta_{\overline{M}_l}. \end{aligned}$$

where $fout_j(\boldsymbol{o}_j)$ *is the feedback part of* $\overline{M}_j$*'s output* $\boldsymbol{o}_j$*,* $in_e(\{\boldsymbol{o}'_j\}_{j \in J_S \cup J_F})$ *is the output to the interface e from outputs* $\{\boldsymbol{o}'_j\}_{j \in J_S \cup J_F}$ *of the subcomponents, and* $in_l(\boldsymbol{i}, \{\boldsymbol{f}_j\}_{j \in J_S \cup J_F})$ *is the input to* $\overline{M}_l$ *from interface input* $\boldsymbol{i}$ *and feedback outputs* $\{\boldsymbol{f}_j\}_{j \in J_S \cup J_F}$*, all of which are given by the wiring diagram src.*

The transition system for $M_\mathcal{E}$ is a tuple $ts(M_\mathcal{E}) = (S^\mathcal{E} \times D_i^\mathcal{E}, \longrightarrow_\mathcal{E})$, where $(\boldsymbol{s}_1, \boldsymbol{i}_1) \longrightarrow_\mathcal{E} (\boldsymbol{s}_2, \boldsymbol{i}_2)$ iff an ensemble in state $\boldsymbol{s}_1$ with input $\boldsymbol{i}_1$ from the interface

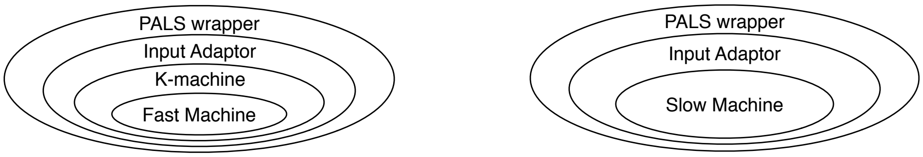


Fig. 2. The wrapper hierarchies in PALS distributed real-time models.

has a transition to state $\boldsymbol{s}_2$ (i.e., $\exists \boldsymbol{o}\ ((\boldsymbol{i}_1, \boldsymbol{s}_1), (\boldsymbol{s}_2, \boldsymbol{o})) \in \delta_{M_\mathcal{E}}$). Notice that this model is an untimed model (apart from the period T): the result of applying a transition is independent of *when* the transition is applied in a round.

2.2 PALS Distributed Real-Time Models

PALS assumes that the underlying infrastructure provides performance bounds $\Gamma = (\epsilon, \alpha_{\min}, \alpha_{\max}, \mu_{\min}, \mu_{\max})$, with (i) ϵ a maximal clock skew with respect to the global clock, (ii) $[\alpha_{\min}, \alpha_{\max}]$ bounds for executing a transition, and (iii) $[\mu_{\min}, \mu_{\max}]$ bounds for the network transmission delay.

Each component in the distributed model $\mathcal{MA}(\mathcal{E}, T, \Gamma)$ is composed of a machine in $\mathcal{E}$ and *wrappers* around it, as illustrated in Fig. 2. The outermost wrapper is the PALS wrapper, which encloses an input adaptor wrapper, which encloses either a (slow) machine or a k-machine wrapper, which encloses a (fast) machine. Each machine in $\mathcal{MA}(\mathcal{E}, T, \Gamma)$ performs at its own rate according to its local clock. At the beginning of its periods, it reads its input from the layer above, performs a transition, and then generates the outputs.

A wrapper has I/O buffers, timers, and access to the machine's local clock. Each PALS wrapper has the same *global period* T and stores received inputs in its input buffer. When the i-th round begins according to its local clock (at time $u_0 \in (iT - \epsilon, iT + \epsilon)$), it delivers the contents of its input buffer to the inner input adaptor wrapper, and sets its *backoff timer* to $2\epsilon - \mu_{min}$. When the execution of the inner components is finished *and* the backoff timer expires, the contents of the output buffer are sent out into the network. All inputs are read in a round-consistent way if $T \geq 2\epsilon + \mu_{max} + \max(2\epsilon - \mu_{min}, \alpha_{max})$ [19].[5]

An input adaptor wrapper reads the inputs from the PALS wrapper and applies input adaptor functions for each global period T. A k-machine wrapper (i) extracts each value from the k-tuple input and delivers it to the enclosed fast machine at each fast period T/k, and (ii) delivers the *k-tuples* from the outputs of the fast machine to its outer layer at each global period T.

As depicted in Fig. 3, a fast machine M_f may *not* be able to finish all of its k internal transitions in a global round *before* the outputs must be sent to arrive before the next round. The number of transitions that M_f can perform before

[5] The outputs are sent out before $u_0 + \max(2\epsilon - \mu_{min}, \alpha_{max})$, and delivered before $u = \mu_{max} + u_0 + \max(2\epsilon - \mu_{min}, \alpha_{max})$. All inputs are read in a round-consistent way if $T \geq 2\epsilon + \mu_{max} + \max(2\epsilon - \mu_{min}, \alpha_{max})$, since $u < (i + 1)T - \epsilon$.

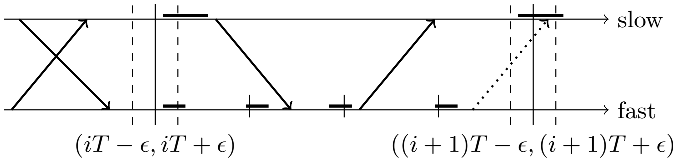

slow
fast
$(iT - \epsilon, iT + \epsilon)$
$((i+1)T - \epsilon, (i+1)T + \epsilon)$

Fig. 3. Timeline for $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})$ (with $k = 4$ and $k' = 3$). Diagonal arrows denote network transmission and short horizontal lines denote the execution. The dotted arrow illustrates a deadline miss caused when the fast machine finishes all its transitions.

the deadline is $k' = 1 + \lfloor \max(T - (2\epsilon + \mu_{max} + \alpha_{max_f}), 0) \cdot (k/T) \rfloor$, where $\alpha_{\max_f}$ is the maximal execution time for M_f. If $k' < k$, then M_f's k-machine wrapper only sends the first k' values (followed by $k - k'$ "don't care" values $\bot$). The input adaptor of each input port whose source is M_f must be $(k'+1)$-*oblivious*, i.e., it ignore the last $k - k'$ values $v_{k'+1}, \ldots, v_k$ in a k-tuple $(v_1, \ldots, v_k)$.

2.3 Relating the Synchronous and Distributed Models

Stable state of the distributed model $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})$ are snapshots of the system at times $iT - \epsilon$, just before the components in $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})$ start performing local machine transitions [19]. As depicted in Fig. 3, network transmission in $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})$ can happen only in the time interval $(iT + \epsilon, (i+1)T - \epsilon)$. In stable states at times $iT - \epsilon$, all the input buffers of the PALS wrappers are full, and all the other input and output buffers are empty. Therefore, there exists the function $\mathit{sync} : \mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})) \to S^{\mathcal{E}} \times D_i^{\mathcal{E}}$ that maps stable states to the corresponding states of the synchronous composition $M_{\mathcal{E}}$ in a natural way.

Definition 6. *Given a stable state $C \in \mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}))$, $\mathit{sync}(C)$ is a pair $(\{s_j, \boldsymbol{f}_j\}_{j \in J_S \cup J_F}, \boldsymbol{i}) \in S^{\mathcal{E}} \times D_i^{\mathcal{E}}$ such that: (i) the states of the typed machines in C give the states $\{s_j\}_{j \in J_S \cup J_F}$ of the machines; and (ii) the values in the input buffers of the PALS wrappers in C give the values $\{\boldsymbol{f}_j\}_{j \in J_S \cup J_F}$ in the feedback wires and the input $\boldsymbol{i}$ from the ensemble interface.*

Assuming $(k'+1)$-oblivious of the input adaptors, we can relate two stable states by $C_1 \sim_{obi} C_2$ iff their machine states are identical and their corresponding input buffer contents *cannot* be distinguished by input adaptors.

Definition 7. *For two stable states $C_1, C_2 \in \mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}))$, $C_1 \sim_{obi} C_2$ iff (i) each machine M_j has the same machine state in C_1 and C_2; and (ii) for each machine M_j, if $k' \leq \mathit{rate}(j)$ is its cutoff number, then each input buffer of M_j's PALS wrapper has the same first k' values in C_1 and C_2.*

Big-step transitions $\longrightarrow_{st}$ are defined between two stable states, and those in the transition system $\mathit{ts}(\mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}))) = (\mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})), \longrightarrow_{st})$ are related to single synchronous steps of the transition system $\mathit{ts}(M_{\mathcal{E}})$.

Theorem 1. *[5] The binary relation* $(\sim_{obi}\,;\, sync)$ *is a* bisimulation *between the transition systems* $ts(M_{\mathcal{E}})$ *and* $ts(\mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \Gamma)))$.

Also, if the state labeling function $\mathcal{L} : S^{\mathcal{E}} \times D_i^{\mathcal{E}} \to 2^{AP}$ for an ensemble $\mathcal{E}$ and it cannot distinguish between $\sim_{obi}$-equivalent states, then the transition systems $ts(M_{\mathcal{E}})$ and $ts(\mathit{Stable}(\mathcal{MA}(\mathcal{E}, T, \Gamma)))$ (more precisely, their corresponding Kripke structures) satisfy the same CTL^* formulas [5].

3 Hybrid PALS

This section presents *Hybrid PALS*, which extends PALS to distributed hybrid systems. One main difference of Hybrid PALS from PALS is that the precise times at which "physical" events happen are also included in the *synchronous* Hybrid PALS models. In PALS, the time at which an event takes place does not matter as long as it happens within a certain time interval, and this allows us to relate a distributed real-time system to an essentially untimed synchronous model. But for hybrid systems, we cannot abstract from the time at which a continuous value is read or an actuator command is given (both of which depend on a component's imprecise local clock).

In Hybrid PALS, the standard PALS models $\mathcal{E}$ and $\mathcal{MA}(\mathcal{E}, T, \Gamma)$ are both nondeterministic models defined for *any possible* environment behaviors. The *environment restrictions* $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright E$ and $\mathcal{E} \upharpoonright E$ define the behavior of the models constrained by the physical environment E. To have any control over when values are read from, and sent to, physical environments, *sampling and response timing policies* are added to the Hybrid PALS models. We then prove a bisimulation equivalence relating $\mathcal{E} \upharpoonright E$ and $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright E$.

3.1 Controlled Physical Environments

A state of a physical environment is a tuple $\boldsymbol{v} = (v_1, \ldots, v_l) \in \mathbb{R}^l$ of its physical parameters $\boldsymbol{x} = (x_1, \ldots, x_l)$, and the behavior of $\boldsymbol{x}$ can be modeled by ODEs that specify *trajectories* $\tau_1, \ldots, \tau_l$ of the parameters $\boldsymbol{x}$ over time. A trajectory of duration T is a function $\tau : [0, T] \to \mathbb{R}$. Let $\mathcal{T}$ denote the set of all trajectories, and $\boldsymbol{\tau}(t) = (\tau_1(t), \ldots, \tau_l(t))$ for an l-tuple of trajectories $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_l)$.

Definition 8. *Given a trajectory* $\tau : [0, T] \to \mathbb{R}$ *and* $u \in [0, T]$*,* τ*'s* prefix *is denoted by* $\tau \trianglelefteq u$ *(that is,* $\tau \trianglelefteq u(t) = \tau(t)$ *for* $t \in [0, u]$*), and* τ*'s* suffix *is denoted by* $\tau \trianglerighteq u$ *(that is,* $\tau \trianglerighteq u(t) = \tau(t + u)$ *for* $t \in [0, T - u]$*).*

A physical environment E_M of machine M is specified as a *controlled physical environment* that defines every possible trajectory of its physical parameters $\boldsymbol{x}$ for the control commands from M. For a state $\boldsymbol{v} \in \mathbb{R}^l$, a control command a, and a duration $t \in \mathbb{R}$, a physical environment E_M gives a trajectory $\boldsymbol{\tau} \in \mathcal{T}^l$ of its parameters $\boldsymbol{x}$ of duration t, as illustrated in Fig. 4 (e.g., state v_1, command a, duration $t_2 - t_1$ yields trajectory τ_1).

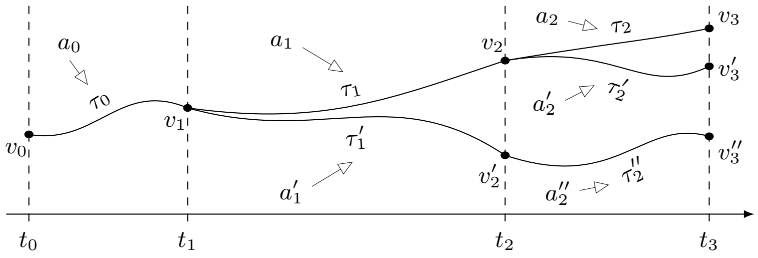


Fig. 4. A controlled physical environment E_M; for example, $((a_0, v_0, t_1 - t_0), \tau_0) \in \Lambda$, $((a_1, v_1, t_2 - t_1), \tau_1) \in \Lambda$, $((a_1', v_1, t_2 - t_1), \tau_1') \in \Lambda$, etc.

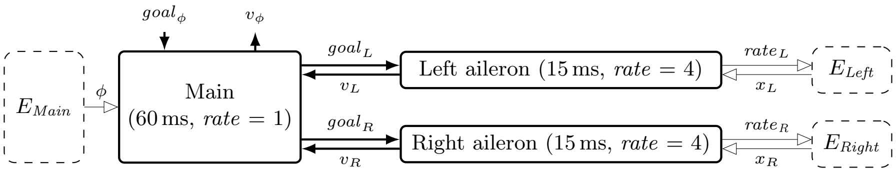


Fig. 5. The simple CPS controller to roll an airplane. The period of the main controller (rate 1) is four times longer than the periods of the subcontrollers (rate 4).

Definition 9. *For $l \in \mathbb{N}$, an l-dimensional* controlled physical environment *is a tuple $E_M = (C, \boldsymbol{x}, \Lambda)$, where:*

- *C is a set of* control commands *from the controller M;*
- *$\boldsymbol{x} = (x_1, \ldots, x_l)$ is a vector of real number variables; and*
- *$\Lambda \subseteq (C \times \mathbb{R}^l \times \mathbb{R}_{\geq 0}) \times \mathcal{T}^l$ is a* physical transition relation*: $((a, \boldsymbol{v}, t), \boldsymbol{\tau}) \in \Lambda$ iff for a control command $a \in C$ that lasts for duration t, E_M's physical state $\boldsymbol{x}$ follows the trajectory $\boldsymbol{\tau} \in \mathcal{T}^l$ from state $\boldsymbol{\tau}(0) = \boldsymbol{v} \in \mathbb{R}^l$.*[6]

Several physical environments may be physically correlated, and one local environment may immediately affect another environment. Such correlations are naturally expressed as *time-invariant constraints* ($\forall t.\, \psi$) of physical parameters over time t. For example, if parameter x_1 of E_{M_1} must be equal to parameter x_2 of another environment E_{M_2}, then the time-invariant constraint is the formula $\forall t.\ x_1(t) = x_2(t)$ with variable t for time. Without loss of generality, we assume that parameter names of different components are all different.

Example 1. Consider a simple distributed CPS controller to roll an airplane by moving its *ailerons*, illustrated in Fig. 5. An aileron is a hinged surface attached to the end of the left or the right wing. The subcontrollers for the ailerons move their surfaces towards the goal angles specified by the main controller. The subcontrollers and the main controller operate at different rates.

[6] If E_M defines a trajectory τ, then E_M should also define every prefix of τ. That is, we also require that if $((a, \boldsymbol{v}, t), \boldsymbol{\tau}) \in \Lambda$, then $((a, \boldsymbol{v}, t'), \boldsymbol{\tau} \trianglelefteq t') \in \Lambda$ for any $t' < t$.

The aileron angle x_M for subcontroller M changes according to the moving rate rate_M (the control command from M). The continuous dynamics of x_M is specified by the ODE $\dot{x}_M = \mathit{rate}_M$. The controlled physical environment of M is given by $E_M = (\mathbb{R}, x_M, \Lambda_M)$, where: (i) $\mathbb{R}$ is the domain of the control command rate_M; (ii) x_M is the physical parameter for the aileron angle; and (iii) $\Lambda_M \subseteq (\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{\geq 0}) \times \mathcal{T}$ is the physical transition relation such that

$$\big((\mathit{rate}_M, v_M, 15\,\mathrm{ms}), \tau\big) \in \Lambda_M \iff \forall t \in [0, 15\,\mathrm{ms}].\ \tau(t) = v_M + \int_0^t \mathit{rate}_M \,\mathrm{d}t.$$

Disregarding the yawing effect caused by the rolling, the physical dynamics of an aircraft is specified as the ODEs $\dot{\phi} = p$ and $\dot{p} = c(\delta_R - \delta_L)$, where ϕ is the roll angle, p is the rolling moment, δ_R is the angle of the right aileron, δ_L is the angle of the left aileron, and c is a constant determined by the geometry of the aircraft. The controlled physical environment of the main controller is given by $E_{Main} = (\{*\}, (\phi, p, \delta_L, \delta_R), \Lambda_{Main})$, where: (i) $\{*\}$ is the singleton set to indicate that E_{Main} has no control command; (ii) $(\phi, p, \delta_L, \delta_R)$ are the physical parameters of E_{Main}; and (iii) $\Lambda_{Main} \subseteq (\{*\} \times \mathbb{R}^4 \times \mathbb{R}_{\geq 0}) \times \mathcal{T}^4$ is the physical transition relation such that

$$\begin{aligned} &\big((*, (v_\phi, v_p, v_{\delta_L}, v_{\delta_R}), 60\,\mathrm{ms}), (\tau_\phi, \tau_p, \tau_{\delta_L}, \tau_{\delta_R})\big) \in \Lambda_{Main} \\ \iff \forall t \in [0, 60\,\mathrm{ms}].\ &\begin{bmatrix} \tau_\phi \\ \tau_p \end{bmatrix}(t) = \begin{bmatrix} v_\phi \\ v_p \end{bmatrix} + \int_0^t \begin{bmatrix} p(t) \\ c(\tau_{\delta_R}(t) - \tau_{\delta_L}(t)) \end{bmatrix} \mathrm{d}t. \end{aligned}$$

The behavior of E_{Main} depends on the trajectories of the control angles (δ_L, δ_R), determined by the subcontrollers's physical environments. The control angles δ_L and δ_R must be the same as the respective aileron angles x_L and x_R of the subcontrollers. However, because the main controller and the subcontrollers have *different periods with local clock skews*, the ODEs of the subcontrollers cannot be directly "plugged" into E_{Main}. Instead, their physical correlations can be specified as the time-invariant constraint

$$\forall t.\ (\delta_L(t) = x_L(t)) \wedge (\delta_R(t) = x_R(t)).$$

3.2 Sampling and Response Timing

A controller M interacts with its physical environment E_M according to its local clock, which may differ from global time by up to the maximal clock skew ϵ. Let $c_M : \mathbb{N} \to \mathbb{R}_{>0}$ denote a a *periodic local clock* of M that gives the *global time* at the beginning of the $(i+1)$-th period according to M's local clock. That is, $c_M(0) = 0$ and $c_M(n) \in (nT - \epsilon, nT + \epsilon)$ for each $n > 0$.

Fig. 6 depicts the behavior of the controller M with respect to its physical environment E_M and a periodic local clock c_M in a PALS distributed model for a time interval $[iT - \epsilon, (i+1)T - \epsilon]$ of duration T:

1. The $(i+1)$-th period begins at time $c_M(i) \in (iT - \epsilon, iT + \epsilon)$. Because of PALS bounds, M has already received all the inputs $\boldsymbol{i}$ before time $iT - \epsilon$.

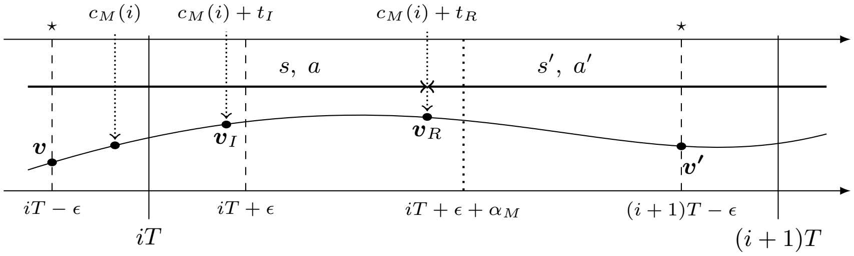


Fig. 6. Timeline for an environment-restricted controller with a local clock c_M, an environment sampling time t_I, and an response time t_R.

2. M samples the appropriate values of its environment E_M. The sampling takes time t_I and the physical state $\boldsymbol{v}_I$ of E_M is read at time $c_M(i)+t_I$.
3. M executes its transition based on the inputs $\boldsymbol{i}$, the sampled physical state $\boldsymbol{v}_I$, and the current machine state s.
4. After the execution, the machine state changes to s', and the new controller command a is sent to E_M at time $c_M(i)+t_R$, where t_R is the response time for the transition execution and the actuator processing.
5. The new outputs $\boldsymbol{o}$ from M are delivered to their destinations for the next round before time $(i+1)T-\epsilon$.

We assume that a control command from M to its physical environment E_M only depends on M's current state s. In Fig. 6, a current control command a remains effective until the execution of M ends at time $u_R = c_M(i)+t_R$, and then a new control command a' takes effect. That is, E_M defines trajectories $\boldsymbol{\tau}$ in an interval $[iT-\epsilon,(i+1)T-\epsilon]$ of duration T with respect to (a,a',u_R).

Definition 10. *For a physical environment $E_M=(C,\boldsymbol{x},\mathit{\Lambda})$, trajectories $\boldsymbol{\tau}\in\mathcal{T}^l$ of duration T are* realizable *with respect to control commands $a,a'\in C$ and a response duration $u_R\in\mathbb{R}$, denoted by $\boldsymbol{\tau}\in\mathcal{R}^T_{E_M}(a,u_R,a')$, iff:*

$$\big((a,\boldsymbol{\tau}(0),u_R),\boldsymbol{\tau}\trianglelefteq u_R\big)\in\mathit{\Lambda}\quad\wedge\quad\big((a',\boldsymbol{\tau}(u_R),T-u_R),\boldsymbol{\tau}\trianglerighteq u_R\big)\in\mathit{\Lambda}.$$

The sampling times of a controller M depends on its state s, and the response times of M depend on its state s and input $\boldsymbol{i}$. In order to compose M with its physical environment E_M, we define the "interface" between the controller M and its environment E_M, given by the following projection functions:

Definition 11. *The* interface *of machine $M=(D_i,S,D_o,\delta_M)$ is given by the projection functions $\pi=(\pi_T,\pi_R,\pi_I,\pi_C)$, for $s\in S$ and $\boldsymbol{i}\in D_i$:*

- $\pi_T(s)\in\mathbb{N}$ *a round number (i.e., state s is in the $(\pi_T(s)+1)$-th iteration);*
- $\pi_R(s,\boldsymbol{i})\in\mathbb{R}$ *an environment response time;*
- $\pi_I(s)\in\mathbb{R}$ *an environment sampling time, where $\pi_I(s)\leq\pi_R(s,\boldsymbol{i})$; and*
- $\pi_C(s)\in C$ *the control command of M to E_M.*

3.3 Environment-Restricted Controllers

A controller M is a *nondeterministic* machine parameterized by any behavior of its physical environment E_M. Such a controller M needs to observe "some" physical parameters of E_M (not necessary all parameters of E_M). Therefore, M has a state space of the form $S \times \mathbb{R}^m$, where $\mathbb{R}^m$ is the state space of the m physical parameters that M can observe. Likewise, E_M has a state space of the form $\mathbb{R}^l = \mathbb{R}^m \times \mathbb{R}^n$, where $\mathbb{R}^m$ is the *observable* part of its state space $\mathbb{R}^l$. For state $\boldsymbol{v} \in \mathbb{R}^l$ of E_M, let $\pi_O(\boldsymbol{v}) \in \mathbb{R}^m$ denote the observable part of $\boldsymbol{v}$.

The *environment restriction* $M \upharpoonright E_M$ is then defined as a normal typed machine with a combined state space $S \times \mathbb{R}^l$, which records "snapshots" of E_M's physical states at times $iT - \epsilon$. A transition of $M \upharpoonright E_M$ from state $(s, \boldsymbol{v})$ to $(s', \boldsymbol{v}')$ corresponds to realizable trajectories $\boldsymbol{\tau}$ for an interval $[iT - \epsilon, (i+1)T - \epsilon]$ with respect to the control commands $\pi_C(s)$ and $\pi_C(s')$ and the response duration $u_R = (c_M(i) + \pi_R(s)) - (iT - \epsilon)$. The controller M performs a transition based on the observable physical state $\pi_O(\boldsymbol{\tau}(u_I))$ of E_M sampled after the sampling duration $u_I = (c_M(i) + \pi_I(s)) - (iT - \epsilon)$.

Definition 12. *Given a controller $M = (D_i, S \times \mathbb{R}^m, D_o, \delta_M)$, an l-dimensional physical environment $E_M = (C, \boldsymbol{x}, \Lambda)$, an interface π, and a period $T \in \mathbb{R}$, the* environment restriction *of M by E_M is the typed machine*

$$M \upharpoonright_\pi E_M = (D_i, S \times \mathbb{R}^l, D_o, \delta_{M \upharpoonright_\pi E_M}),$$

where $((\boldsymbol{i}, (s, \boldsymbol{v})), ((s', \boldsymbol{v}'), \boldsymbol{o})) \in \delta_{M \upharpoonright_\pi E_M}$ iff for the round number $i = \pi_T(s)$, $u_I = (c_M(i) + \pi_I(s)) - (iT - \epsilon)$, and $u_R = (c_M(i) + \pi_R(s)) - (iT - \epsilon)$:

- *$\boldsymbol{\tau}(0) = \boldsymbol{v}$ and $\boldsymbol{\tau}(T) = \boldsymbol{v}'$ for some $\boldsymbol{\tau} \in \mathcal{R}^T_{E_M}(\pi_C(s), u_R, \pi_C(s'))$.*
- *$\big((\boldsymbol{i}, (s, \pi_O(\boldsymbol{\tau}(u_I)))), ((s', \pi_O(\boldsymbol{v}')), \boldsymbol{o})\big) \in \delta_M$, and $\pi_T(s') = i + 1$.*

Example 2. Consider a subcontroller M to move an aileron in Example 1. The period of a subcontroller M is 15 ms, and the $(i+1)$-th round of M begins at time $c_M(i) \in (i \cdot 15\,\mathrm{ms} - \epsilon, i \cdot 15\,\mathrm{ms} + \epsilon)$ with the maximal clock skew $\epsilon > 0$. In each round, M receives a goal angle g_M, sets a new moving rate rate'_M based on g_M and the observed angle v_M, and sends back v_M. Such a subcontroller M can be specified as the typed machine $M = (\mathbb{R}, \mathbb{N} \times \mathbb{R}^2, \mathbb{R}, \delta_M)$, where

$$\begin{aligned} &\big((g_M, (i, \mathit{rate}_M, v_M)), ((i', \mathit{rate}'_M, v'_M), o)\big) \in \delta_M \\ \iff\ & \mathit{rate}'_M = (g_M - v_M)/15\,\mathrm{ms} \quad \wedge \quad o = v_m \quad \wedge \quad i' = i + 1. \end{aligned}$$

The interface π_M of M can be defined by projection functions as follows: (i) $\pi_C(i, \mathit{rate}_M, v_M) = \mathit{rate}_M$ (the controller command); (ii) $\pi_T(i, \mathit{rate}_M, v_M) = i$ (the round number); (iii) $\pi_I(i, \mathit{rate}_M, v_M) = 1\,\mathrm{ms}$ (the environment sampling time); and (iv) $\pi_R((i, \mathit{rate}_M, v_M), g_M) = 2\,\mathrm{ms}$ (the environment response time). For the environment $E_M = (\mathbb{R}, x_M, \Lambda_M)$ in Example 1, its observable state is given by the function $\pi_O(x_M) = x_M$, since M observes the aileron angle x_M.

The environment restriction is defined as a typed machine $M \upharpoonright_{\pi_M} E_M$. Its transitions are associated with the interval $[i \cdot 15\,\mathrm{ms} - \epsilon, (i+1) \cdot 15\,\mathrm{ms} - \epsilon]$. The angle x_M follows a "realizable" trajectory, by the current moving rate rate_M up to the response time $c_M(i) + 2\,\mathrm{ms}$ and then by the new rate rate'_M. The controller M performs a transition using the angle observed at the sampling time $c_M(i) + 1\,\mathrm{ms}$. That is, $M \upharpoonright_{\pi_M} E_M = (\mathbb{R}, \mathbb{N} \times \mathbb{R}^2, \mathbb{R}, \delta_{M \upharpoonright_{\pi_M} E_M})$, where for $u_I = (c_M(i) + 1\,\mathrm{ms}) - (i \cdot 15\,\mathrm{ms} - \epsilon)$ and $u_R = (c_M(i) + 2\,\mathrm{ms}) - (i \cdot 15\,\mathrm{ms} - \epsilon)$:

$$\begin{aligned} & ((g_M, (i, \mathit{rate}_M, v_M)), ((i', \mathit{rate}'_M, v'_M), v_M)) \in \delta_{M \upharpoonright_{\pi_M} E_M} \\ \iff \quad & \exists \tau \in \mathcal{R}^{15\,\mathrm{ms}}_{E_M}(\mathit{rate}_M, u_R, \mathit{rate}'_M).\ \ v_M = \tau(0) \ \wedge \ v'_M = \tau(15\,\mathrm{ms}) \wedge \\ & \qquad\qquad ((g_M, (i, \mathit{rate}_M, \tau(u_I))), ((i', \mathit{rate}'_M, v'_M), v_M)) \in \delta_M. \end{aligned}$$

Example 3. Now consider the main controller M_{Main} in Example 1. Its period is 60 ms, and its $(i+1)$-th round begins at $c_{Main}(i) \in (i \cdot 60\,\mathrm{ms} - \epsilon, i \cdot 60\,\mathrm{ms} + \epsilon)$. In each round, M_{Main} receives a desired roll angle g_ϕ and the aileron angles (v_L, v_R), and sends the new goal angles (g_L, g_R). The main controller can be specified as $M_{Main} = (\mathbb{R}^3, \mathbb{N} \times \mathbb{R}^2, \mathbb{R}^3, \delta_{Main})$, where

$$\begin{aligned} & \big(((g_\phi, v_L, v_R), (i, v_\phi)), ((i', v'_\phi), (o, g_L, g_R))\big) \in \delta_{Main} \\ \iff & \ g_R = 0.3 \cdot (g_\phi - v_\phi) \ \ \wedge \ \ g_L = -g_R \ \wedge \ o = v_\phi \ \ \wedge \ \ i' = i + 1. \end{aligned}$$

The interface π_{Main} can be defined as follows (since there is no controller command, the value of the response time is not important): (i) $\pi_C(i, v_\phi) = *$; (ii) $\pi_I(i, v_\phi) = 3\,\mathrm{ms}$; (iii) $\pi_T(i, v_\phi) = i$; and (iv) $\pi_R((i, v_\phi), (g_\phi, v_L, v_R)) = 10\,\mathrm{ms}$. For $E_{Main} = (\{*\}, (\phi, p, \delta_L, \delta_R), \mathit{\Lambda}_{Main})$ in Example 1, its observable state is given by $\pi_O(\phi, p, \delta_L, \delta_R) = \phi$, since M_{Main} only observes the roll angle ϕ.

The environment restriction of M_{Main} by E_{Main} is then the typed machine $M_{Main} \upharpoonright_{\pi_{Main}} E_{Main} = (\mathbb{R}^3, \mathbb{N} \times \mathbb{R}^4, \mathbb{R}^3, \delta_{M_{Main} \upharpoonright_{\pi_{Main}} E_{Main}})$. Each transition of $M_{Main} \upharpoonright_{\pi_{Main}} E_{Main}$ is associated with the interval $[i \cdot 60\,\mathrm{ms} - \epsilon, (i+1) \cdot 60\,\mathrm{ms} - \epsilon]$. The controller M_{Main} performs a transition using the roll angle observed at the sampling time $c_{Main}(i) + 3\,\mathrm{ms}$. For $u_I = (c_{Main}(i) + 3\,\mathrm{ms}) - (i \cdot 60\,\mathrm{ms} - \epsilon)$, $\boldsymbol{v} = (v_\phi, v_p, v_{\delta_L}, v_{\delta_R})$, $\boldsymbol{v}' = (v'_\phi, v'_p, v'_{\delta_L}, v'_{\delta_R})$, and $\boldsymbol{\tau} = (\tau_\phi, \tau_p, \tau_{\delta_L}, \tau_{\delta_R})$:

$$\begin{aligned} & \big(((g_\phi, v_L, v_R), (i, \boldsymbol{v})), ((i', \boldsymbol{v}'), (v_\phi, g_L, g_R))\big) \in \delta_{M_{Main} \upharpoonright_{\pi_{Main}} E_{Main}} \\ \iff \quad & \exists \boldsymbol{\tau} \in \mathcal{R}^{60\,\mathrm{ms}}_{E_{Main}}(*, 10\,\mathrm{ms}, *).\ \ \boldsymbol{v} = \boldsymbol{\tau}(0) \ \ \wedge \ \ \boldsymbol{v}' = \boldsymbol{\tau}(60\,\mathrm{ms}) \ \ \wedge \\ & \quad \big(((g_\phi, v_L, v_R), (i, \tau_\phi(u_I))), ((i', v'_\phi), (v_\phi, g_L, g_R))\big) \in \delta_{Main}. \end{aligned}$$

The k-step deceleration $(M \upharpoonright E_M)^{\times k}$ of the environment restriction $M \upharpoonright E_M$ can also be defined in a straightforward way. A transition of $(M \upharpoonright E_M)^{\times k}$ consists of k successive transitions of $M \upharpoonright E_M$, and its realizable trajectories are just a concatenation of the k realizable trajectories for the k successive transitions. The definition of "one-step" realizable trajectories can easily be extended for such k-steps of realizable trajectories as follows:

Definition 13. *Trajectories $\boldsymbol{\tau} \in \mathcal{T}^l$ of duration T are* k-step realizable *for control commands $a_0, a_1, \ldots, a_k \in C$ and response durations $u^1_R, \ldots, u^k_R \in \mathbb{R}$, denoted by $\boldsymbol{\tau} \in \mathcal{R}^T_{E_M}(a_0, u^1_R, a_1, \ldots, u^k_R, a_k)$, iff for $u_{R_0} = 0$ and $u_{R_{k+1}} = T$: $\bigwedge_{j=0}^{k} \big((a_j, \boldsymbol{\tau}(u_{R_j}), u_{R_{j+1}} - u_{R_j}), \boldsymbol{\tau} \trianglerighteq u_{R_j} \trianglelefteq u_{R_{j+1}}\big) \in \mathit{\Lambda}$.*

3.4 Hybrid PALS Synchronous Models

The synchronous model in Hybrid PALS is specified as a multirate ensemble $\mathcal{E}$ and the physical environments of subcomponents, where physical correlations between those environments are specified as time-invariant constraints.

Definition 14. *A* hybrid multirate ensemble *is a triple* $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$*, composed of: (i) a multirate ensemble* $\mathcal{E}$*, (ii) a family of interface functions* $\Pi = \{\pi_j\}_{j \in J_S \cup J_F}$ *(with* J_S *an index set of slow machines and* J_F *an index set of fast machines), and (iii) a family of local physical environments* $E_{\mathcal{E}} = \langle \{E_{M_j}\}_{j \in J_S \cup J_F}, (\forall t)\, \psi \rangle$ *with* $(\forall t)\, \psi$ *the time-invariant constraints.*

Example 4. For the simple CPS controller to roll an airplane in Example 1, the multirate ensemble $\mathcal{E} = (J_S, J_F, e, \{M_j\}_{j \in J_s \cup J_F}, E, \mathit{src}, \mathit{rate}, \mathit{adap})$ is defined by: (i) $J_S = \{\mathit{Main}\}$ and $J_F = \{L, R\}$; (ii) M_{Main} in Example 3, and M_L and M_R in Example 2; (iii) $E = (\mathbb{R}, \mathbb{R})$ for $(v_\phi, \mathit{goal}_\phi)$; (iv) src given as Fig. 5; (v) $\mathit{rate}(\mathit{Main}) = 1$ and $\mathit{rate}(L) = \mathit{rate}(R) = 4$; and (vi) $\mathit{adap}(\mathit{Main})$ taking the last value from a 4-tuple, and $\mathit{adap}(L)$ and $\mathit{adap}(R)$ giving (v, v, v, v) from a single value v. The interface functions $\Pi = \{\pi_{\mathit{Main}}, \pi_L, \pi_R\}$ and the physical environments $\{E_{\mathit{Main}}, E_L, E_R\}$ are defined in Examples 2–3. The time-invariant constraint $(\forall t)\, \psi \equiv (\forall t)\, (\delta_L(t) = x_L(t)) \wedge (\delta_R(t) = x_R(t))$ is given in Example 1. The resulting hybrid ensemble is then $\mathcal{E} \upharpoonright_{\Pi} \langle \{E_{\mathit{Main}}, E_L, E_R\}, (\forall t)\, \psi \rangle$.

A hybrid multirate ensemble $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ induces a normal multirate ensemble $\overline{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}}$ composed of the environment restrictions $\{M_j \upharpoonright_{\pi} E_{M_j}\}_{j \in J_S \cup J_F}$, which disregards the time-invariant constraints $(\forall t)\, \psi$. The behaviors of $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ are a *subset* of the behaviors of $\overline{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}}$, namely, the behaviors restricted by $(\forall t)\, \psi$. As mentioned, a transition of a (decelerated) environment-restricted machine $M_j \upharpoonright_{\pi} E_{M_j}$ corresponds to realizable trajectories $\boldsymbol{\tau}$ for $[iT - \epsilon, (i+1)T - \epsilon]$ for a global period T. Hence, a lockstep composition of such environment-restricted transitions *whose realizable trajectories* $\boldsymbol{\tau}$ *satisfy the time-invariant constraints* $(\forall t)\, \psi$ gives a transition of the synchronous composition $M_{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}}$ from one state at time $iT - \epsilon$ to another state at time $(i+1)T - \epsilon$.

Definition 15. *Given a hybrid multirate ensemble* $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$*, the* synchronous composition *of* $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ *is the typed machine* $M_{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}} = (D_i^{\mathcal{E}}, S^{\mathcal{E}}, D_o^{\mathcal{E}}, \delta_{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}})$*, where (i)* $D_i^{\mathcal{E}} = D_o^e$ *and* $D_o^{\mathcal{E}} = D_i^e$*, (ii)* $S^{\mathcal{E}} = \prod_{j \in J}((S_j \times \mathbb{R}^{l_j}) \times D_{OF}^j)$*, and (iii)* $\delta_{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}} \subseteq (D_i^{\mathcal{E}} \times S^{\mathcal{E}}) \times (S^{\mathcal{E}} \times D_o^{\mathcal{E}})$ *such that, for* $J = J_S \cup J_F$*:*

$$\big((\boldsymbol{i}, \{(s_j, \boldsymbol{v}_j), \boldsymbol{f}_j\}_{j \in J}), (\{(s_j', \boldsymbol{v}_j'), \mathit{fout}_l(\boldsymbol{o}_l')\}_{j \in J}, \mathit{in}_e(\{\boldsymbol{o}_j'\}_{j \in J}))\big) \in \delta_{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}}$$

iff, given $\overline{M_j \upharpoonright_{\pi} E_{M_j}} = ((M_j \upharpoonright_{\pi} E_{M_j})^{\times \mathit{rate}(j)})_{\mathit{adap}(j)}$ *for* $j \in J$*:*

1. *for each machine* $l \in J$ *with* $k_l = \mathit{rate}(l)$*:*
   (a) $\big((\mathit{in}_l(\boldsymbol{i}, \{\boldsymbol{f}_j\}_{j \in J}), (s_l, \boldsymbol{v}_l)), ((s_l', \boldsymbol{v}_l'), \boldsymbol{o}_l')\big) \in \delta_{\overline{M_j \upharpoonright_{\pi} E_{M_j}}}$

(b) for some realizable trajectories $\boldsymbol{\tau}_l$ with $\boldsymbol{\tau}_l(0) = \boldsymbol{v}_l$ and $\boldsymbol{\tau}_l(T) = \boldsymbol{v}'_l$;[7] *and*

2. the collection $\{\boldsymbol{\tau}_l\}_{j \in J}$ satisfies the time-invariant constraints $(\forall t)\, \psi$ in $E_{\mathcal{E}}$;

where $fout_j(\boldsymbol{o}_j)$ is the feedback part of output $\boldsymbol{o}_j$, $in_e(\{\boldsymbol{o}'_j\}_{j \in J})$ is the output to the interface e from outputs $\{\boldsymbol{o}'_j\}_{j \in J}$ of the machines, and $in_l(\boldsymbol{i}, \{\boldsymbol{f}_j\}_{j \in J})$ is the input to $\overline{M_j \restriction_{\pi} E_{M_j}}$ from interface input $\boldsymbol{i}$ and feedback outputs $\{\boldsymbol{f}_j\}_{j \in J}$.

Notice that for a multirate ensemble $\overline{\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}}$ of $\{M_j \restriction_{\pi} E_{M_j}\}_{j \in J_S \cup J_F}$, only the condition (1a) is necessary to perform synchronous transitions, and (1b) and (2) further restrict transitions for a hybrid ensemble $\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$.

3.5 Hybrid PALS Distributed Models

Hybrid PALS maps a hybrid multirate ensemble $\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ with a global period T, together with PALS bounds $\mathit{\Gamma}$, to the hybrid system $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$. The distributed components in $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ are exactly the same as $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})$ given by the wrapper hierarchies in Fig. 2, but the controllers in $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ are also interact with their physical environments in $E_{\mathcal{E}}$, according to the *sampling and response timing policy $\mathit{\Pi}$* that determine sensor sampling timing t_I and actuator response timing t_R for each controller.

The behaviors of the Hybrid PALS distributed system $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ are the subset of those of the PALS distributed system $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma})$ that can be realized by the physical environments and the time-invariant constraints in $E_{\mathcal{E}}$. The continuous dynamics of $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ is "completely" decided by the controllers, their physical environments, the timing policies, and the local clocks of the controllers; that is, it abstracts from asynchronous communication, network delays, message buffering, backoff timers, etc.

More precisely, recall that a hybrid ensemble $\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ induces an ensemble $\overline{\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}}$ composed of the environment restrictions $\{M_j \restriction_{\pi} E_{M_j}\}_{j \in J_S \cup J_F}$. The behaviors of $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ are the behaviors of $\mathcal{MA}(\overline{\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}}, T, \mathit{\Gamma})$ that are restricted by the time-invariant constraints. As explained in Section 2.3, for $\mathcal{MA}(\overline{\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}}, T, \mathit{\Gamma})$, *big-step transitions* are defined from one stable state at time $iT - \epsilon$ to another stable state at $(i+1)T - \epsilon$. For such time intervals $[iT - \epsilon, (i+1)T - \epsilon]$, each environment-restricted controller $M_j \restriction_{\pi} E_{M_j}$ provides (k-step) realizable trajectories $\boldsymbol{\tau}_j$, based on control commands, sampling timings, response timings, and round numbers. Therefore, big-step stable transitions of $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ are exactly those of $\mathcal{MA}(\overline{\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}}, T, \mathit{\Gamma})$ whose associated trajectories $\boldsymbol{\tau}_j$ satisfy the time-invariant constraints $(\forall t)\, \psi$.

The correctness of Hybrid PALS immediately follows from the fact that all physical measurements and physical activation *happen at the same time* in both $\mathcal{E} \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ and $\mathcal{MA}(\mathcal{E}, T, \mathit{\Gamma}) \restriction_{\mathit{\Pi}} E_{\mathcal{E}}$ with the *same timing policies $\mathit{\Pi}$*.

[7] $\boldsymbol{\tau}_l \in \mathcal{R}^T_{E_{M_l}}(\pi_C(s_0), u_{R_1}, \pi_C(s_1), \ldots, u_{R_{k_l}}, \pi_C(s_{k_l}))$ are k_l-step realizable trajectories with $s_0 = s_l$, $s_{k_l} = s'_l$, some intermediate states $s_1, \ldots, s_{k_l - 1}$ for k-step deceleration, and $u_{R_i} = (c_{M_l}(\pi_T(s_{i-1})) + \pi_{R_l}(s_{i-1})) - (\pi_T(s_{i-1})(T/k_l) - \epsilon)$ for $1 \leq i \leq k_l$.

Theorem 2. $(\sim_{obi}\, ; \mathit{sync})$ *is a bisimulation between the transition system by* $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ *and the big-step transition system induced by* $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright_{\Pi} E_{\mathcal{E}}$*, exhibiting the exactly same set of realizable trajectories.*

Proof. Suppose that there exists a transition $s \longrightarrow_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}} s'$ by $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ and $s\ (\sim_{obi}\, ; \mathit{sync})\ C$ for a stable state C of $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright_{\Pi} E_{\mathcal{E}}$. By definition, there exists a transition $s \longrightarrow_{\overline{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}} s'$ by $M_{\overline{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}}$ with some realizable trajectories $\boldsymbol{\tau}$ that satisfy the time-invariant constraints $(\forall t)\, \psi$. By Theorem 1, $(\sim_{obi}\, ; \mathit{sync})$ is a bisimulation between $ts(M_{\overline{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}})$ and $ts(\mathit{Stable}(\mathcal{MA}(\overline{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}}, T, \Gamma)))$, and thus for some stable state C', there exists a big-step transition $C \longrightarrow_{st} C'$ by $\mathcal{MA}(\overline{\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}}, T, \Gamma)$ such that $s'\ (\sim_{obi}\, ; \mathit{sync})\ C'$, where both $s \longrightarrow_{\overline{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}} s'$ and $C \longrightarrow_{st} C'$ involve exactly the same machine states and inputs. By construction, control commands, sampling timings, response timings, and round numbers all depend on machine states and inputs. Therefore, $\boldsymbol{\tau}$, which satisfy $(\forall t)\, \psi$, are also realizable for $C \longrightarrow_{st} C'$. Consequently, there exists a big-step transition $C \longrightarrow_{st} C'$ by $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright_{\Pi} E_{\mathcal{E}}$. The other direction is similar. □

Given a set of atomic propositions AP, consider a state labeling function $\mathcal{L} : S^{\mathcal{E}} \times D_i^{\mathcal{E}} \rightarrow 2^{AP}$ for $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$. If $\mathcal{L}$ cannot distinguish $\sim_{obi}$-equivalent states (i.e., $s \sim_{obi} s' \implies \mathcal{L}(s, \boldsymbol{i}) = \mathcal{L}(s', \boldsymbol{i})$), then Theorem 2 implies that $(\sim_{obi}\, ; \mathit{sync})$ is also a bisimulation between Kripke structures for $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ and $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright_{\Pi} E_{\mathcal{E}}$. Therefore, they satisfy the same CTL^* formulas.

Corollary 1. *For a hybrid ensemble* $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ *and its state labeling function* $\mathcal{L} : S^{\mathcal{E}} \times D_i^{\mathcal{E}} \rightarrow 2^{AP}$ *that cannot distinguish* $\sim_{obi}$*-equivalent states, the Kripke structures for* $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ *and* $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright_{\Pi} E_{\mathcal{E}}$ *satisfy the same set of* CTL^* *formulas (for* $(\sim_{obi}\, ; \mathit{sync})$*-related initial states and the same initial input).*

4 SMT-Based Verification of Hybrid PALS Models

This section first show how analysis problems for a hybrid multirate ensemble $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$, for *all possible local clocks*, can be encoded as logical formulas over the real numbers and ODEs. Theorem 2 implies that a distributed hybrid model $\mathcal{MA}(\mathcal{E}, T, \Gamma) \upharpoonright_{\Pi} E_{\mathcal{E}}$ can then be verified by checking the satisfiability of these formulas for the much simpler synchronous model $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$. It is worth nothing that hybrid automata can also be used (see Appendix A), but it is quite complex since Hybrid PALS models are time-triggered and consider clock skews.

The satisfiability of logical formulas over the real numbers is undecidable for nonlinear hybrid systems, but becomes *decidable* up to any given precision $\delta > 0$ [13,15]. But formulas encoding Hybrid PALS models may contain *uninterpreted functions on reals*, which are not supported by existing SMT solving techniques. For this purpose, we present a new SMT framework to provide an efficient SMT decision procedure for formulas encoding Hybrid PALS models.

4.1 Hybrid PALS Models as Logical Formulas

Transition of the synchronous composition $M_{\mathcal{E}\restriction_{\Pi} E_{\mathcal{E}}}$ of an ensemble $\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}$ are expressed as formulas of the form $\phi^{T}_{\mathcal{E}\restriction_{\Pi} E_{\mathcal{E}}}(\boldsymbol{i}, \boldsymbol{y} \mid \boldsymbol{y'}, \boldsymbol{o} \mid \boldsymbol{\tau})$, where: (i) $\boldsymbol{i}$ denotes the interface inputs, (ii) $\boldsymbol{y}$ denotes the states at the beginning of the round at time $iT - \epsilon$, (iii) $\boldsymbol{y'}$ denotes the states at the end of the round at time $(i+1)T - \epsilon$, (iv) $\boldsymbol{o}$ denotes the interface outputs, and (v) $\boldsymbol{\tau}$ *unary function symbols over time* denoting the trajectories of duration T for the physical parameters in $E_{\mathcal{E}}$.

Encoding Typed Machines. A controller M can be expressed as a logic formula of the form $\phi_M(\boldsymbol{i}, \boldsymbol{y} \mid \boldsymbol{y'}, \boldsymbol{o})$, with variables $\boldsymbol{i}$, $\boldsymbol{y}$, $\boldsymbol{y'}$, and $\boldsymbol{o}$ denoting, respectively, the input, the current state, the next state, and the output. That is, $\phi_M(\boldsymbol{i}, s \mid s', \boldsymbol{o}) \iff ((\boldsymbol{i}, s), (s', \boldsymbol{o})) \in \delta_M$.

Example 5. For an aileron subcontroller M in Example 2, the logical formula $\phi_M(y_{g_M}, y_i, y_{rate_M}, y_{v_M} \mid y'_i, y'_{rate_M}, y'_{v_M}, y_o)$ can be defined as the conjunction: $y'_{rate_M} = (y_{g_M} - y_{v_M})/15 \ \wedge\ y_o = y_{v_M} \ \wedge\ y'_i = y_i + 1$.

For the main controller M_{Main} to govern the subcontrollers in Example 3, the logical formula $\phi_{Main}(y_{g_\phi}, y_{v_L}, y_{v_R}, y_i, y_{v_\phi} \mid y'_i, y'_{v_\phi}, y_o, y_{g_L}, y_{g_R})$ is defined by: $y_{g_R} = 0.3 \cdot (y_{g_\phi} - y_{v_\phi}) \ \wedge\ y_{g_L} = -y_{g_R} \ \wedge\ y_o = y_{v_\phi} \ \wedge\ y'_i = y_i + 1$.

Encoding Physical Environments. A controlled physical environment E_M is encoded as a formula of the form $\phi_{E_M}(\boldsymbol{a}, \boldsymbol{v}, u_0, u_t \mid \boldsymbol{\tau})$, with: *unary function symbols* $\boldsymbol{\tau}$ denoting the trajectories of E_M's physical parameters $\boldsymbol{x}$, and *variables* $\boldsymbol{a}$, $\boldsymbol{v}$, u_0, and u_t denoting, respectively, control commands, the initial values of $\boldsymbol{\tau}$ at time u_0, the times at the beginning and the end of the trajectory duration. That is, $\phi_{E_M}(\boldsymbol{a}, \boldsymbol{v}, u_0, u_t \mid \boldsymbol{\tau})$ iff $((\boldsymbol{a}, \boldsymbol{v}, u_t - u_0), \boldsymbol{\tau} \trianglerighteq u_0) \in \Lambda$.

If the continuous dynamics of $\boldsymbol{x}$ is specified as ODEs $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = F_{\boldsymbol{a}}(\boldsymbol{x}, t)$ for a control command $\boldsymbol{a}$ and a time interval $[u_0, u_t]$, then ϕ_{E_M} includes universal quantification over time along with solutions of the ODEs, such as a formula of the form: $\bigvee \big(\mathit{guard}(\boldsymbol{a}) \to \forall t \in [u_0, u_t].\ \boldsymbol{\tau}(t) = \boldsymbol{v} + \int_0^{t-u_0} F_{\boldsymbol{a}}(\boldsymbol{x}, t)\,\mathrm{d}t\big)$.

Example 6. For the physical environment E_M of a subcontroller M in Example 1, the formula $\phi_{E_M}(y_{rate_M}, y_{v_M}, y_{u_0}, y_{u_t} \mid \tau_M)$ is defined by:

$$\forall t \in [y_{u_0}, y_{u_t}].\ \tau_M(t) = y_{v_M} + \int_0^{t-y_{u_0}} y_{rate_M}\,\mathrm{d}t.$$

For the physical environment E_{Main} of the main controller M_{Main}, the formula $\phi_{E_{Main}}(y_{v_\phi}, y_{v_p}, y_{v_{\delta_L}}, y_{v_{\delta_R}}, y_{u_0}, y_{u_t} \mid \tau_\phi, \tau_p, \tau_{\delta_L}, \tau_{\delta_R})$ is defined by:

$$\forall t \in [y_{u_0}, y_{u_t}].\ \begin{bmatrix} \tau_\phi \\ \tau_p \end{bmatrix}(t) = \begin{bmatrix} y_{v_\phi} \\ y_{v_p} \end{bmatrix} + \int_0^{t-y_{u_0}} \begin{bmatrix} \tau_p(t) \\ c(\tau_{\delta_R}(t) - \tau_{\delta_L}(t)) \end{bmatrix} \mathrm{d}t.$$

Encoding Environment Restrictions. A environment restriction $M \upharpoonright_\pi E_M$ is encoded as a formula of the form $\phi^{T,i}_{M \upharpoonright_\pi E_M}(\boldsymbol{i}, \boldsymbol{y}, \boldsymbol{v} \mid \boldsymbol{y'}, \boldsymbol{v'}, \boldsymbol{o} \mid \boldsymbol{\tau})$, with unary function symbols $\boldsymbol{\tau}$ denoting E_M's trajectories, and variables: (i) $\boldsymbol{i}$ denoting input, (ii) $(\boldsymbol{y}, \boldsymbol{v})$ denoting a state at the beginning of the round, (iii) $(\boldsymbol{y'}, \boldsymbol{v'})$ denoting a state at the end of the round, and (iv) $\boldsymbol{o}$ denoting output, given a period $T \in \mathbb{R}$ and a round number $i \in \mathbb{N}$.

The exact values of sampling duration $u_I = (c_M(i)+t_I)-(iT-\epsilon)$ and response duration $u_R = (c_M(i)+t_R)-(iT-\epsilon)$ are unknown, since the concrete values of the imprecise local clock $c_M(i)$ are determined on-the-fly by *clock synchronization* mechanisms. Since $iT-\epsilon < c_M(i) < iT+\epsilon$, we represent those times as formulas $t_I < u_I < t_I + 2\epsilon$ and $t_R < u_R < t_R + 2\epsilon$, so that $\phi_{M \upharpoonright_\pi E_M}(\boldsymbol{i}, s, \boldsymbol{v} \mid s', \boldsymbol{v'}, \boldsymbol{o})$ iff $((\boldsymbol{i}, (s, \boldsymbol{v})), ((s', \boldsymbol{v'}), \boldsymbol{o})) \in \delta_{M \upharpoonright_\pi E_M}$ for some c_M.

Definition 16. *For an environment restriction $M \upharpoonright_\pi E_M$ with interface π, the formula $\phi^{T,i}_{M \upharpoonright_\pi E_M}(\boldsymbol{i}, \boldsymbol{y}, \boldsymbol{v} \mid \boldsymbol{y'}, \boldsymbol{v'}, \boldsymbol{o} \mid \boldsymbol{\tau})$ is defined by:*[8]

$$\begin{aligned}
(\exists \boldsymbol{a}, \boldsymbol{a'}, u_I, u_R, \boldsymbol{v}_I, \boldsymbol{v}_R)\ \ \boldsymbol{a} = \pi_C(\boldsymbol{y}) \ &\wedge\ \pi_I(\boldsymbol{y}) < u_I < \pi_I(\boldsymbol{y}) + 2\epsilon \ \wedge \\
\boldsymbol{a'} = \pi_C(\boldsymbol{y'}) \ &\wedge\ \pi_R(\boldsymbol{y}, \boldsymbol{i}) < u_R < \pi_R(\boldsymbol{y}, \boldsymbol{i}) + 2\epsilon \ \wedge \\
\boldsymbol{v}_I = \boldsymbol{\tau}(u_I) \ \wedge\ \boldsymbol{v}_R = \boldsymbol{\tau}(u_R) \ &\wedge\ \boldsymbol{v} = \boldsymbol{\tau}(0) \ \wedge\ \boldsymbol{v'} = \boldsymbol{\tau}(T) \ \wedge \\
\phi_{E_M}(\boldsymbol{a}, \boldsymbol{v}, iT, iT + u_R \mid \boldsymbol{\tau}) \ &\wedge\ \phi_{E_M}(\boldsymbol{a'}, \boldsymbol{v}_R, iT + u_R, (i+1)T, \mid \boldsymbol{\tau}) \\
&\wedge\ \phi_M(\boldsymbol{i}, \langle \boldsymbol{y}, \pi_O(\boldsymbol{\tau}(u_I)) \rangle \mid \langle \boldsymbol{y'}, \pi_O(\boldsymbol{v'}) \rangle, \boldsymbol{o}).
\end{aligned}$$

Example 7. For an environment-restricted subcontroller $M \upharpoonright_\pi E_M$ in Example 2, the formula $\phi^{15,0}_{M \upharpoonright_\pi E_M}(y_{g_M}, y_i, y_{rate_M}, y_{v_M} \mid y'_i, y'_{rate_M}, y'_{v_M}, y_o \mid \tau_M)$ is defined by:

$$\begin{aligned}
&(\exists u_I, u_R, v_I, v_R)\ \ 1 < u_I < 1 + 2\epsilon \ \wedge\ 2 < u_R < 2 + 2\epsilon \ \wedge \\
&\quad v_I = \tau_M(u_I) \ \wedge\ v_R = \tau_M(u_R) \ \wedge\ y_{v_M} = \tau_M(0) \ \wedge\ y'_{v_M} = \tau_M(15) \ \wedge \\
&\quad\quad \phi_{E_M}(y_{rate_M}, y_{v_M}, 0, u_R \mid \tau_M) \ \wedge\ \phi_{E_M}(y'_{rate_M}, v_R, u_R, 15, \mid \tau_M) \ \wedge \\
&\qquad\qquad\qquad\qquad \phi_M(y_{g_M}, y_i, y_{rate_M}, v_I \mid y'_i, y'_{rate_M}, y'_{v_M}, y_o).
\end{aligned}$$

For the environment restriction $M_{Main} \upharpoonright_{\pi_{Main}} E_{Main}$ in Example 3, the formula $\phi^{60,0}_{M_{Main} \upharpoonright_{\pi_{Main}} E_{Main}}(y_{g_\phi}, y_{v_L}, y_{v_R}, y_i, \boldsymbol{y} \mid y'_i, \boldsymbol{y'}, y_o, y_{g_L}, y_{g_R} \mid \boldsymbol{\tau})$ is defined by:

$$\begin{aligned}
&(\exists u_I, v_I^\phi)\ \ 3 < u_I < 3 + 2\epsilon \ \wedge\ v_I^\phi = \tau_\phi(u_I) \ \wedge\ \boldsymbol{y} = \boldsymbol{\tau}(0) \ \wedge\ \boldsymbol{y'} = \boldsymbol{\tau}(60) \ \wedge \\
&\qquad \phi_{E_{Main}}(\boldsymbol{y}, 0, 60 \mid \boldsymbol{\tau}) \ \wedge\ \phi_{Main}(y_{g_\phi}, y_{v_L}, y_{v_R}, y_i, v_I^\phi \mid y'_i, y'_{v_\phi}, y_o, y_{g_L}, y_{g_R}).
\end{aligned}$$

for $\boldsymbol{y} = (y_{v_\phi}, y_{v_p}, y_{v_{\delta_L}}, y_{v_{\delta_R}})$, $\boldsymbol{y'} = (y'_{v_\phi}, y'_{v_p}, y'_{v_{\delta_L}}, y'_{v_{\delta_R}})$, and $\boldsymbol{\tau} = (\tau_\phi, \tau_p, \tau_{\delta_L}, \tau_{\delta_R})$ Notice that there are no formulas for response behaviors in $\phi^{60,0}_{M_{Main} \upharpoonright_{\pi_{Main}} E_{Main}}$, since E_{Main} does not have control commands.

Encoding *k*-Step Decelerations. The encoding $\phi^{T,i}_{(M \upharpoonright_\pi E_M)^{\times k}}$ of the k-step *deceleration* of an environment restriction $M \upharpoonright_\pi E_M$ is defined by sequentially composing the k formulas $\phi^{T/k, ik}_{M \upharpoonright_\pi E_M}, \ldots, \phi^{T/k, (ik+k-1)}_{M \upharpoonright_\pi E_M}$ corresponding to the k *subintervals* $[(ik+n-1)T/k - \epsilon, (ik+n)T/k - \epsilon]$ for $n = 1, \ldots, k$.

[8] Instead of $iT - \epsilon$ and $(i+1)T - \epsilon$, we use the "ϵ-shifted" time axis for $[iT, (i+1)T]$.

Definition 17. *Given a environment restriction $M \restriction_\pi E_M$ and $k \in \mathbb{N}$, the formula $\phi^{T,i}_{(M\restriction_\pi E_M)^{\times k}}(\langle \boldsymbol{i_1}, \dots, \boldsymbol{i_k}\rangle, \boldsymbol{y_0}, \boldsymbol{v_0} \mid \boldsymbol{y}_k, \boldsymbol{v}_k, \langle \boldsymbol{o_1}, \dots, \boldsymbol{o_k}\rangle \mid \boldsymbol{\tau})$ is:*

$$\exists \{\boldsymbol{y_n}, \boldsymbol{v_n}\}_{n=1}^{k-1}.\ \bigwedge_{n=1}^{k} \phi^{T/k,(ik+n-1)}_{M\restriction_\pi E_M}(\boldsymbol{i_n}, \boldsymbol{y}_{n-1}, \boldsymbol{v}_{n-1} \mid \boldsymbol{y_n}, \boldsymbol{v_n}, \boldsymbol{o}_n \mid \boldsymbol{\tau})$$

Example 8. The 4-step deceleration of $M \restriction_\pi E_M$ in Example 2 is encoded as the logical formula $\phi^{60,i}_{(M\restriction_\pi E_M)^{\times 4}}(\langle y^1_{g_M}, \dots, y^4_{g_M}\rangle, \boldsymbol{y}_0, y^0_{v_M} \mid \boldsymbol{y}_0, y^4_{v_M}, \langle y^1_o, \dots, y^4_o\rangle \mid \tau_M)$ using the formula $\phi^{15,i}_{M\restriction_\pi E_M}$ in Example 7, where $\boldsymbol{y}_n = (y^n_i, y^n_{rate_M})$, such that:

$$\exists \{\boldsymbol{y_n}, y^n_{v_M}\}_{n=1}^{3}.\ \bigwedge_{n=1}^{4} \phi^{15,(4i+n-1)}_{M\restriction_\pi E_M}(y^n_{g_M}, \boldsymbol{y}_{n-1}, y^{n-1}_{v_M} \mid \boldsymbol{y}_n, y^n_{v_M}, y^n_o \mid \tau_M)$$

Encoding Wiring Diagrams. The wiring diagram of an ensemble $\mathcal{E}$ is encoded as a conjunction of appropriate equalities between variables denoting input and output ports. Each equality corresponds to a connection in $\mathcal{E}$ (together with an input adaptor for typed machines with different rates). Since feedback outputs becomes input of their destinations in the next step, we use a separate set of variables for such output ports connected to machines in $\mathcal{E}$.

Definition 18. *The formula ϕ_{wire} is the conjunction of the equalities, composed of: (i) $\alpha^l_j(i^l_j) = f^n_k$ and $f^{n\prime}_k = o^n_k$ for connection from M_k's n-th output port to M_j's l-th input port with its adaptor α^l_j, where f^l_k denotes a feedback output from the previous step, and $f^{l\prime}_k$ denotes one for the next step; (ii) $\alpha^l_j(i^l_j) = i^n_e$ for connection from $\mathcal{E}$'s n-th input port to M_j's l-th input port with adaptor α^l_j; (iii) $o^n_e = o^l_j$ for connection from M_j's l-th output port to $\mathcal{E}$'s n-th output port.*

Example 9. For the ensemble $\mathcal{E}$ in Example 4 for the simple CPS controller in Example 1, the formula ϕ_{wire} is defined as follows (where different variables with the same names are appropriately distinguished using superscripts):

$$\begin{aligned} &(y^{Main}_{v_L} = f^4_{o_L} \ \wedge\ f^{4\prime}_{o_L} = y^4_{o_L}) \ \wedge\ (\textstyle\bigwedge_{i=1}^4 y^i_{g_L} = f^{Main}_{g_L} \ \wedge\ f^{Main\prime}_{g_L} = y^{Main}_{g_L}) \\ \wedge\ &(y^{Main}_{v_R} = f^4_{o_R} \ \wedge\ f^{4\prime}_{o_R} = y^4_{o_R}) \ \wedge\ (\textstyle\bigwedge_{i=1}^4 y^i_{g_R} = f^{Main}_{g_R} \ \wedge\ f^{Main\prime}_{g_R} = y^{Main}_{g_R}) \\ &\qquad\qquad \wedge\ (y^{Main}_{g_\phi} = y^{\mathcal{E}}_{g_\phi} \ \wedge\ y^{\mathcal{E}}_{v_\phi} = y^{Main}_{v_\phi}). \end{aligned}$$

Encoding Hybrid Ensembles. A hybrid ensemble $\mathcal{E} \restriction_\Pi E_{\mathcal{E}}$ is encoded as a formula $\phi^T_{\mathcal{E}\restriction_\Pi E_{\mathcal{E}}}(\boldsymbol{i}, \{\boldsymbol{z_j}, \boldsymbol{f_j}\}_{j\in J_S\cup J_F} \mid \{\boldsymbol{z'_j}, \boldsymbol{f'_j}\}_{j\in J_S\cup J_F}, \boldsymbol{o} \mid \{\boldsymbol{\tau}_j\}_{j\in J_S\cup J_F})$, with unary function symbols $\boldsymbol{\tau}_j$ denoting trajectories for physical environments, and variables $\boldsymbol{i}$, $\boldsymbol{z_j}$, $\boldsymbol{f_j}$, $\boldsymbol{z'_j}$, $\boldsymbol{f'_j}$, and $\boldsymbol{o}$ denoting, respectively, inputs, the state of $M_j \restriction_{\pi_j} E_{M_j}$ at the beginning of the round, feedback outputs from the previous round, the state of $M_j \restriction_{\pi_j} E_{M_j}$ at the end of the round, the feedback outputs for the next round, and the output.

Definition 19. *For a hybrid ensemble* $\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}$ *of machines* $\{M_j\}_{j \in J_S \cup J_F}$*, their physical environments* $\{E_{M_j}\}_{j \in J_S \cup J_F}$ *and the time-invariant constraint* $(\forall t.\ \psi)$*,* $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}(\boldsymbol{i}, \{\boldsymbol{y_j}, \boldsymbol{v_j}, \boldsymbol{f_j}\}_{j \in J_S \cup J_F} \mid \{\boldsymbol{y'_j}, \boldsymbol{v'_j}, \boldsymbol{f'_j}\}_{j \in J_S \cup J_F}, \boldsymbol{o} \mid \{\boldsymbol{\tau}_j\}_{j \in J_S \cup J_F})$ *is:*

$$\begin{aligned} \exists \{\boldsymbol{i_j}, \boldsymbol{o_j}\}_{j \in J_S \cup J_F}.\ & \textstyle\bigwedge_{s \in J_S} \big(\phi^{T,0}_{M_s \restriction_{\pi_s} E_{M_s}}(\boldsymbol{i_s}, \boldsymbol{y_s}, \boldsymbol{v_s} \mid \boldsymbol{y'_s}, \boldsymbol{v'_s}, \boldsymbol{o}_s \mid \boldsymbol{\tau}_s)\big) \\ & \wedge\ \textstyle\bigwedge_{f \in J_F} \big(\phi^{T,0}_{(M_f \restriction_{\pi_f} E_{M_f})^{\times rate(f)}}(\boldsymbol{i_f}, \boldsymbol{y_f}, \boldsymbol{v_f} \mid \boldsymbol{y'_f}, \boldsymbol{v'_f}, \boldsymbol{o}_f \mid \boldsymbol{\tau}_f)\big) \\ & \wedge \phi_{wire}(\boldsymbol{i}, \boldsymbol{o}, \{\boldsymbol{i_j}, \boldsymbol{o_j}, \boldsymbol{f_j}, \boldsymbol{f'_j}\}_{j \in J_S \cup J_F})\ \wedge\ (\forall t.\ \psi). \end{aligned}$$

Example 10. Consider the hybrid ensemble $\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}$ in Example 4. For variables $\boldsymbol{y} = (y_{v_\phi}, y_{v_p}, y_{v_{\delta_L}}, y_{v_{\delta_R}})$, $\boldsymbol{y'} = (y'_{v_\phi}, y'_{v_p}, y'_{v_{\delta_L}}, y'_{v_{\delta_R}})$, $\boldsymbol{\tau} = (\tau_\phi, \tau_p, \tau_{\delta_L}, \tau_{\delta_R})$, and $\boldsymbol{y}^m_n = (y^n_{i_m}, y^n_{rate_m})$ for $m \in \{L, R\}$ and $1 \leq n \leq 4$:

$$\begin{aligned} & \phi^{60}_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}(y^{\mathcal{E}}_{g_\phi}, (y_i, \boldsymbol{y}, f^{Main}_{gL}, f^{Main}_{gR}), (\boldsymbol{y}^L_0, y^0_{v_L}, f^4_{o_L}), (\boldsymbol{y}^R_0, y^0_{v_R}, f^4_{o_R}) \mid \\ & \quad (y'_i, \boldsymbol{y'}, f^{Main'}_{gL}, f^{Main'}_{gR}), (\boldsymbol{y}^L_4, y^4_{v_L}, f^{4\prime}_{o_L}), (\boldsymbol{y}^R_4, y^4_{v_R}, f^{4\prime}_{o_R}), y^{\mathcal{E}}_{v_\phi} \mid \boldsymbol{\tau}, \tau_L, \tau_R) \\ \Updownarrow & \\ & \phi^{60,0}_{M_{Main} \restriction_{\pi_{Main}} E_{Main}}(y^{Main}_{g_\phi}, y^{Main}_{v_L}, y^{Main}_{v_R}, y_i, \boldsymbol{y} \mid y'_i, \boldsymbol{y'}, y^{Main}_o, y^{Main}_{gL}, y^{Main}_{gR} \mid \boldsymbol{\tau}) \\ & \qquad \wedge \phi^{60,0}_{(M_L \restriction_\pi E_L)^{\times 4}}(\langle y^1_{gL}, \ldots, y^4_{gL} \rangle, \boldsymbol{y}^L_0, y^0_{v_L} \mid \boldsymbol{y}^L_4, y^4_{v_L}, \langle y^1_{o_L}, \ldots, y^4_{o_L} \rangle \mid \tau_L) \\ & \qquad \wedge \phi^{60,0}_{(M_R \restriction_\pi E_R)^{\times 4}}(\langle y^1_{gR}, \ldots, y^4_{gR} \rangle, \boldsymbol{y}^R_0, y^0_{v_R} \mid \boldsymbol{y}^R_4, y^4_{v_R}, \langle y^1_{o_R}, \ldots, y^4_{o_R} \rangle \mid \tau_R) \\ & \qquad \wedge \phi_{wire}\ \wedge\ (\forall t.\ \tau_{\delta_L}(t) = \tau_L(t)\ \wedge\ \tau_{\delta_R}(t) = \tau_R(t)). \end{aligned}$$

By construction, a formula $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ is satisfiable iff there is a corresponding transition of the synchronous composition $M_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ for *some* local clocks. Hence, by the bisimulation equivalence (Theorem 2):

Theorem 3. $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ *is satisfiable iff there is a corresponding* stable transition *for some local clocks in* $\mathcal{MA}(\mathcal{E}, T, \Gamma) \restriction_{\Pi} E_{\mathcal{E}}$*.*

4.2 Encoding Verification Problems

Our goal is to verify safety properties of a distributed CPS $\mathcal{MA}(\mathcal{E}, T, \Gamma) \restriction_{\Pi} E_{\mathcal{E}}$. We exploit the bisimulation equivalence $M_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}} \approx \mathcal{MA}(\mathcal{E}, T, \Gamma) \restriction_{\Pi} E_{\mathcal{E}}$ (by Theorem 2) to verify the distributed hybrid system $\mathcal{MA}(\mathcal{E}, T, \Gamma) \restriction_{\Pi} E_{\mathcal{E}}$ by using the simpler hybrid system $M_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$. A safety property is expressed as a formula of the form $\mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t))$ for *state variables* $\boldsymbol{y}$, *trajectories* $\boldsymbol{\tau}$, and time variable t. A transition of $M_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ for a global period $[iT - \epsilon, (i+1)T - \epsilon]$ is encoded as a formula $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}(\boldsymbol{i}, \boldsymbol{y} \mid \boldsymbol{y'}, \boldsymbol{o} \mid \boldsymbol{\tau})$. Using these formulas *safe* and $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ as building blocks, This section shows how standard verification problems for hybrid systems, such as bounded reachability, inductive reasoning, and compositional assume-guarantee reasoning, can be encoded as formulas.

Bounded Reachability. To verify a safety property up to a given bound $n \in \mathbb{N}$ (i.e., for the time interval $[-\epsilon, nT - \epsilon]$), we encode its *bounded counterexamples* for the synchronous hybrid model $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ as an SMT formula. If the formula is unsatisfiable (i.e., no counterexample exists), then, by Theorem 3, the system satisfies the safety property in $[-\epsilon, nT-\epsilon]$ *for any local clocks.*

Definition 20. *A bounded reachability problem of $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ for a safety property $\mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t))$* up to $n \in \mathbb{N}$ rounds *with an initial condition $\mathit{init}(\boldsymbol{y})$ and an input constraint $\mathit{in}(\boldsymbol{i})$ is encoded by:*

$$\exists \boldsymbol{y}_0, \{\boldsymbol{y}_k, \boldsymbol{i}_k, \boldsymbol{o}_k, t_k\}_{k=1}^{n}.\ \mathit{init}(\boldsymbol{y}_0) \ \wedge \ \bigwedge_{k=1}^{n} \big(\phi^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}(\boldsymbol{i}_k, \boldsymbol{y}_{k-1} \shortmid \boldsymbol{y}_k, \boldsymbol{o}_k \shortmid \boldsymbol{\tau}_k) \wedge \mathit{in}(\boldsymbol{i}_k)\big)$$
$$\wedge \ \bigvee_{k=1}^{n} \neg \mathit{safe}(\boldsymbol{y}_k, \boldsymbol{\tau}_k(t_k)).$$

Some initial state $\boldsymbol{y}_0$ satisfies the *init* condition, and $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ then performs n steps of synchronous transitions, each of which is from some state $\boldsymbol{y}_{k-1}$ to $\boldsymbol{y}_k$ using trajectories $\boldsymbol{\tau}_k$ of duration T with some input $\boldsymbol{i}_k$ satisfying the input constraint *in*. The system $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ has bounded counterexamples if some state $\boldsymbol{y}_k$ and physical state $\boldsymbol{\tau}_k(t_k)$ at some time t_k violates safety *safe*.

Example 11. For the simple CPS controller in Example 1, consider the safety property that the roll angle ϕ is always in a given bound $[-\gamma, \gamma]$. As a formula, $\mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t)) \equiv (\mathrm{abs}(v_\phi) \le \gamma \wedge \mathrm{abs}(\tau_\phi(t)) \le \gamma)$. Let $\mathit{init}(\boldsymbol{y}) \equiv (\boldsymbol{y} = \boldsymbol{0})$, and $\mathit{in}(y^{\mathcal{E}}_{g_\phi}) \equiv (y^{\mathcal{E}}_{g_\phi} = 0)$. Using the formula $\phi^{60}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ in Example 10, the n-step bounded reachability problem can be encoded according to Definition 20.

Inductive Reasoning. For *unbounded* time verification of a safety property $\mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t))$, we encode its inductive proof as a logical formula by using an inductive condition for $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$'s synchronous transitions, which implies the safety property. Such an inductive invariant consists of two formulas: (i) $\mathit{ind}_d(\boldsymbol{y})$ for state variables $\boldsymbol{y}$, and (ii) $\forall t \in [0, T].\ \mathit{ind}_c(\boldsymbol{\tau}(t))$ for trajectories $\boldsymbol{\tau}$.[9]

Definition 21. *An inductive analysis problem of $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ for a safety property $\mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t))$, an initial condition $\mathit{init}(\boldsymbol{y})$, and an input constraint $\mathit{in}(\boldsymbol{i})$, using an inductive invariant $(\forall t.\ \mathit{ind}(\boldsymbol{y}, \boldsymbol{\tau}(t))) \equiv (\mathit{ind}_d(\boldsymbol{y}) \ \wedge \ \forall t \in [0, T].\ \mathit{ind}_c(\boldsymbol{\tau}(t)))$, where time variable t does not occur in $\mathit{ind}_d(\boldsymbol{y})$, is encoded by:*

- $\forall \boldsymbol{y}.\ \mathit{init}(\boldsymbol{y}) \implies \mathit{ind}_d(\boldsymbol{y})$
- $\forall \boldsymbol{y}, \boldsymbol{y'}, \boldsymbol{i}, \boldsymbol{o}.\ (\mathit{ind}_d(\boldsymbol{y}) \ \wedge \ \mathit{in}(\boldsymbol{i}) \ \wedge \ \phi^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}(\boldsymbol{i}, \boldsymbol{y} \shortmid \boldsymbol{y'}, \boldsymbol{o} \shortmid \boldsymbol{\tau})) \implies \forall t.\ \mathit{ind}(\boldsymbol{y}, \boldsymbol{\tau}(t))$
- $\forall \boldsymbol{y}.\ (\forall t \in [0, T].\ \mathit{ind}(\boldsymbol{y}, \boldsymbol{\tau}(t))) \implies \forall t \in [0, T].\ \mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t))$

These formula encode the classical inductive analysis approach. The *init* condition implies the ind_d condition for any state variables $\boldsymbol{y}$. If a transition of $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ is taken from state $\boldsymbol{y}$ satisfying the ind_d condition, then: (i) the

[9] One of an important problem is to find such an inductive invariant for $M_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$, but providing general solutions for this problem is beyond the scope of this paper.

ind_d condition again holds for any next state $\boldsymbol{y'}$, and (ii) in the meantime the ind_c condition holds for trajectories $\boldsymbol{\tau}$. The inductive invariant $\forall t.\, \mathit{ind}(\boldsymbol{y}, \boldsymbol{\tau}(t))$ implies the safety property $\forall t \in [0, T].\, \mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t))$ for one round. By proving the these conditions, we show that the safety property holds for unbounded time with unbounded number of transitions.The formulas can be proved by checking the unsatisfiability of their *negated versions* using SMT solvers.[10]

Example 12. For the simple CPS controller in Example 1, consider the safety property $\mathit{safe}(\boldsymbol{y}, \boldsymbol{\tau}(t)) \equiv (\mathrm{abs}(v_\phi) \leq \gamma \wedge \mathrm{abs}(\tau_\phi(t)) \leq \gamma)$, the initial condition $\mathit{init}(\boldsymbol{y}) \equiv (v_\phi = 0) \wedge (v_p = 0) \wedge (v_L = 0) \wedge (\mathit{rate}_L = 0) \wedge (v_R = 0) \wedge (\mathit{rate}_R = 0)$, and the input constraint $\mathit{in}(y^{\mathcal{E}}_{g_\phi}) \equiv (y^{\mathcal{E}}_{g_\phi} = 0)$ in Example 11. Using simple inductive formulas $\mathit{ind}_d(\boldsymbol{y}) \equiv (\boldsymbol{y} = \boldsymbol{0})$ and $\mathit{ind}_c(\boldsymbol{\tau}(t)) \equiv (\boldsymbol{\tau}(t) = 0)$, the inductive analysis problem can be encoded according to Definition 21.

Compositional Reasoning. We encode a divide-and-conquer proof to verify a safety property based on standard assume-guarantee reasoning. In $M_{\mathcal{E} \upharpoonright_\Pi E_\mathcal{E}}$, conceptually, each subcomponent $M_j \upharpoonright_{\pi_j} E_{M_j}$ performs a transition based on its input $\boldsymbol{i}_j$ and trajectories $\boldsymbol{\tau}_j$ that are restricted by time-invariant constraints. Therefore, we consider an input condition $c^j_{\mathit{in}}(\boldsymbol{i}_j, \boldsymbol{\tau}_j(t))$ and an output condition $c^j_{\mathit{out}}(\boldsymbol{o}_j, \boldsymbol{\tau}_j(t))$ for each subcomponent $M_j \upharpoonright_{\pi_j} E_{M_j}$ during a global round such that a collection of output conditions $\{c^j_{\mathit{out}}(\boldsymbol{o}, \boldsymbol{\tau}_j(t))\}_{j \in J_S \cup J_F}$ implies each input condition $c^j_{\mathit{in}}(\boldsymbol{i}_j, \boldsymbol{\tau}_j(t))$, which again implies the safety property $\mathit{safe}_j(\boldsymbol{y}_j, \boldsymbol{\tau}_j(t))$ of the subcomponent $M_j \upharpoonright_{\pi_j} E_{M_j}$ for that round.

Definition 22. *For each subcomponent $j \in J_S \cup J_F$ in $M_{\mathcal{E} \upharpoonright_\Pi E_\mathcal{E}}$, consider a safety property $\mathit{safe}_j(\boldsymbol{y}_j, \boldsymbol{\tau}_j(t))$, an input constraint $\mathit{in}_j(\boldsymbol{i}_j)$ for input from $\mathcal{E}$'s interface, an initial condition $\mathit{init}_j(\boldsymbol{y}_j)$, and I/O conditions*

$$\begin{aligned} \forall t.\ c^j_{\mathit{in}}(\boldsymbol{i}_j, \boldsymbol{\tau}_j(t)) \ &\equiv\ (c^j_{\mathit{in},d}(\boldsymbol{i}_j) \ \wedge\ \forall t \in [0, T].\ c^j_{\mathit{in},c}(\boldsymbol{\tau}_j(t))) \\ \forall t.\ c^j_{\mathit{out}}(\boldsymbol{o}_j, \boldsymbol{\tau}_j(t)) \ &\equiv\ (c^j_{\mathit{out},d}(\boldsymbol{i}_j) \ \wedge\ \forall t \in [0, T].\ c^j_{\mathit{out},c}(\boldsymbol{\tau}_j(t))) \end{aligned}$$

Let $\overline{M}_j$ denote its decelerated version $(M_j \upharpoonright_{\pi_j} E_{M_j})^{\times \mathit{rate}(j)}$ for a fast component $j \in J_F$, and $M_j \upharpoonright_{\pi_j} E_{M_j}$ for a slow component $j \in J_S$. Necessary constraints on I/O conditions for a compositional analysis of $M_{\mathcal{E} \upharpoonright_\Pi E_\mathcal{E}}$ are encoded by:

- *I/O correspondence: for each component $j \in J_S$, $\forall \boldsymbol{i}_j, \boldsymbol{o}_j, \boldsymbol{y}_j, \boldsymbol{y}'_j$:*

$$\big(\forall t.\ c^j_{\mathit{in}}(\boldsymbol{i}_j, \boldsymbol{\tau}_j(t)) \ \wedge\ \phi^{T,0}_{\overline{M}_j}(\boldsymbol{i}_j, \boldsymbol{y}_j \shortmid \boldsymbol{y}'_j, \boldsymbol{o}_j \shortmid \boldsymbol{\tau}_j)\big) \implies \forall t.\ c^j_{\mathit{out}}(\boldsymbol{o}_j, \boldsymbol{\tau}_j(t))$$

- *Intercomponent correspondence:*

$$\Big(\bigwedge_{j \in J_S \cup J_F} \forall t.\ c^j_{\mathit{out}}(\boldsymbol{o}, \boldsymbol{\tau}_j(t))\Big) \implies \bigwedge_{j \in J_S \cup J_F} \forall t.\ c^j_{\mathit{in}}(\boldsymbol{i}_j, \boldsymbol{\tau}_j(t))$$

[10] This may involve universally quantified time variable t for the last condition. For example, the dReal SMT solver can deal with such constraints.

We can consider both bounded reachability and inductive analysis problems based on compositional reasoning. A compositional bounded reachability problem up to $n \in \mathbb{N}$ rounds for component j is encoded by:

$$\exists \boldsymbol{y}_0^j, \{\boldsymbol{y}_k^j, \boldsymbol{i}_k^j, \boldsymbol{o}_k^j, t_k\}_{k=1}^n.\ \mathit{init}_j(\boldsymbol{y}_0^j) \ \wedge \ \bigwedge_{k=1}^n \phi^{T,0}_{\overline{M}_j}(\boldsymbol{i}_k^j, \boldsymbol{y}_{k-1}^j \shortmid \boldsymbol{y}_k^j, \boldsymbol{o}_k^j \shortmid \boldsymbol{\tau}_k^j)$$
$$\wedge \ \bigwedge_{k=1}^n \big(\mathit{in}_j(\boldsymbol{i}_k^j) \ \wedge \ \forall t.\ c_{in}^j(\boldsymbol{i}_k^j, \boldsymbol{\tau}_k^j(t))\big) \ \wedge \ \bigvee_{k=1}^n \neg \mathit{safe}_j(\boldsymbol{y}_k^j, \boldsymbol{\tau}_k^j(t_k)).$$

A compositional inductive analysis problem for component j with an inductive invariant $\forall t.\ \mathit{ind}^j(\boldsymbol{y}_j, \boldsymbol{\tau}_j(t)) \equiv \mathit{ind}_d^j(\boldsymbol{y}_j) \wedge \forall t \in [0,T].\ \mathit{ind}_c^j(\boldsymbol{\tau}_j(t))$ is encoded by:

- $\forall \boldsymbol{y}_j.\ \mathit{init}_j(\boldsymbol{y}_j) \implies \mathit{ind}_d^j(\boldsymbol{y}_j)$
- $\forall \boldsymbol{y}_j, \boldsymbol{y}'_j, \boldsymbol{i}_j, \boldsymbol{o}_j.\ (\mathit{ind}_d^j(\boldsymbol{y}_j) \wedge \mathit{in}_j(\boldsymbol{i}_j) \wedge \forall t.\ c_{in}^j(\boldsymbol{i}_j, \boldsymbol{\tau}_j(t)) \wedge \phi^{T,0}_{\overline{M}_j}(\boldsymbol{i}, \boldsymbol{y} \shortmid \boldsymbol{y'}, \boldsymbol{o} \shortmid \boldsymbol{\tau}))$ $\implies \forall t.\ \mathit{ind}_j(\boldsymbol{y}_j, \boldsymbol{\tau}_j(t))$
- $\forall \boldsymbol{y}_j.\ (\forall t \in [0,T].\ \mathit{ind}_j(\boldsymbol{y}_j, \boldsymbol{\tau}_j(t))) \implies \forall t \in [0,T].\ \mathit{safe}_j(\boldsymbol{y}_j, \boldsymbol{\tau}_j(t))$

Two necessary constraints in this definition state that given I/O conditions are correct assumptions for compositional reasoning: (i) for each component, assuming its input condition, if a transition is taken, then its output condition holds, and (ii) output conditions for subcomponents implies input conditions for their connected components, either by a wiring diagram or by time-invariant constraints. Bounded reachability and inductive analysis can then be separately performed for each component in a compositional way, by additionally assuming input conditions for individual components. The validity of such formulas for compositional analysis can also be automatically checked using SMT solving by checking the unsatisfiability of their *negations* (see Section 5 for examples).

4.3 SMT Encoding of Hybrid PALS Models

SMT-based techniques for hybrid systems normally use the SMT theory $\mathcal{L}_\mathcal{F}$ of the real numbers and computable (nonlinear) real functions, such as polynomials, trigonometric functions, and solutions of Lipschitz-continuous ODEs. Notice that solutions of ODEs are considered *atomic functions* in $\mathcal{L}_\mathcal{F}$.

Definition 23. *For a finite set $\mathcal{F}$ of computable real functions (with real number constants seen as 0-ary functions), $\mathcal{L}_\mathcal{F} = (\mathcal{F}, >)$ denotes the first-order signature over the real numbers with the functions in $\mathcal{F}$, and $\mathbb{R}_\mathcal{F} = (\mathbb{R}, \mathcal{F}^\mathbb{R}, >^\mathbb{R})$ is the standard structure of the theory of the real numbers.*

However, formulas encoding Hybrid PALS models may contain ODEs and universal quantification over *uninterpreted functions on the real numbers*, such as a formula $\forall t \in [u_0, u_t].\ \boldsymbol{x}(t) = \boldsymbol{v} + \int_0^{t-u_0} F_{\boldsymbol{a}}(\boldsymbol{x}, t)\,\mathrm{d}t$, or time-invariant constraints $(\forall t.\ \psi)$, which are not supported by current state-of-the-art SMT techniques. This section shows how formulas for hybrid PALS models can be equivalently encoded *without* using uninterpreted real functions and universal quantification.

The restrictive syntax of $\mathcal{L}_{\mathcal{F}}$ makes it difficult to effectively encode Hybrid PALS models as $\mathcal{L}_{\mathcal{F}}$-formulas, since the structure of ODEs cannot be used for SMT algorithms in $\mathcal{L}_{\mathcal{F}}$. We therefore present a new SMT theory, by extending $\mathcal{L}_{\mathcal{F}}$, that allows to express solutions of ODEs in a modular way, so that the size of the formula can be significantly reduced.

Time Segments for ODEs. We first restrict our attention to time-invariant constraints with only equality terms, since physical correlations for CPS can be typically expressed using only equalities (for example, the simple CPS controller for an airplane). Equality constraints, such as $x_1(t) = x_2(t)$, can be removed from the formula by replacing one side with the other, for example, by replacing each function symbol x_1 with x_2. From now on we assume that time-invariant equality constraints have been removed from the formula in this way.

The basic idea is to assign a *complete system* of ODEs, which combine all partial ODE systems for physical environments in $\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}$, to every time point in a global round. Suppose that a partial ODE system $\frac{\mathrm{d}\boldsymbol{x}_j}{\mathrm{d}t} = F^j_{\boldsymbol{a}_j}(\boldsymbol{x}_j, t)$ is assigned to an interval $[u_j, u'_j]$ in a global round by each environment-restricted controller $M_j \restriction_{\pi_j} E_{M_j}$. A complete ODE system at time z is then given by the ODEs $\{\frac{\mathrm{d}\boldsymbol{x}_j}{\mathrm{d}t} = F^j_{\boldsymbol{a}_j}(\boldsymbol{x}_j, t)\}_{j \in J_S \cup j_F}$, where time z is included in every interval $[u_j, u'_j]$ for $M_j \restriction_{\pi_j} E_{M_j}$ (that is, $z \in [u_j, u'_j]$), provided that variable are accordingly renamed by the equality time-invariant constraints.

Recall that each occurrence of (partial) ODEs in formulas for Hybrid PALS models has the form $\forall t \in [u_0, u_t].\ \boldsymbol{x}(t) = \boldsymbol{v} + \int_0^{t-u_0} F_{\boldsymbol{a}}(\boldsymbol{x}, t)\,\mathrm{d}t$, which means exactly that the (partial) ODE system $F_{\boldsymbol{a}}(\boldsymbol{x}, t)$ is assigned to the interval $[u_0, u_t]$. Since a formula $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ for one global round of a hybrid ensemble $\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}$ includes a finite number of such subformulas, there exists only a finite number of combinations of complete ODE systems in one global round. Therefore, it is possible to construct an equivalent formula that only includes a complete system of ODEs (e.g., transforming $\phi^T_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ to its disjunctive normal form, and then performing case analysis on any possible arrangements of time intervals).

Example 13. For the hybrid ensemble $\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}$ in Example 4, the formula $\phi^{60}_{\mathcal{E} \restriction_{\Pi} E_{\mathcal{E}}}$ in Example 10 involves the following time segments: (i) for the main controller, the ODEs $\dot{\tau_\phi} = \tau_p$ and $\dot{\tau_\phi} = c(\tau_{\delta_R} - \tau_{\delta_L})$ to the interval $[0, 60]$, and (ii) for each subcontroller M, the ODE $\dot{\tau_M} = y^I_{rate^i_M}$ to $[15i, 15i + u^M_{R,i}]$, and $\dot{\tau_M} = y^R_{rate^i_M}$ to $[15i + u^M_{R,i}, 15(i+1)]$ for $i \in \{0, 1, 2, 3\}$. Given an ordering $u^L_{R,i} \leq u^R_{R,i}$ for each i (i.e., the left aileron controller always responds earlier than the right one), then the complete ODE system assigned to the interval $[15i + u^L_{R,i}, 15i + u^R_{R,i}]$ is

$$\dot{\tau_\phi} = \tau_p, \quad \dot{\tau_\phi} = c(\tau_R - \tau_L), \quad \dot{\tau_L} = y^R_{rate^i_L}, \quad \dot{\tau_R} = y^I_{rate^i_R},$$

where τ_{δ_R} and τ_{δ_L} are renamed by $(\forall t.\ \tau_{\delta_L}(t) = \tau_L(t) \ \wedge\ \tau_{\delta_R}(t) = \tau_R(t))$. This ODE system indicates a period that the left subcontroller has responded (with the new rate $y^R_{rate^i_L}$) but the right subcontroller has not responded yet.

SMT for ODE Assignments. The restrictive syntax of the standard logic $\mathcal{L}_{\mathcal{F}}$ makes it difficult to efficiently encode logical formulas for Hybrid PALS. For example, since there is no way in $\mathcal{L}_{\mathcal{F}}$ to "assign" a partial ODE system to a time segment, it is generally not possible to construct such $\mathcal{L}_{\mathcal{F}}$-formulas in a modular way: complete ODE systems need to be explicitly constructed by "expanding" the formula (e.g., using disjunctive normal forms), which may significantly increase the size of the resulting formulas. We therefore present a new SMT theory, by extending $\mathcal{L}_{\mathcal{F}}$, that allows to express solutions of ODEs in a modular way, so that the size of the formula can be significantly reduced.

Our theory extends $\mathcal{L}_{\mathcal{F}}$ by adding *function names* to denote time segments of a certain partial ODE. We consider a *two-sorted* first-order logic with sorts *Real* and *Name*, where *Real* denotes the real numbers, and *Name* denotes *name constants* for unary functions composed of the functions in $\mathcal{F}$. For example, given $\{1, +, \times, \sin, x, y\} \subset \mathcal{F}$ and two name constants $\mathtt{m_1}$ and $\mathtt{m_2}$ of sort *Name*, we can have two *named* unary real functions:

$$(\mathtt{m_1})\ \ \sin(x(t)) : \mathit{Real} \to \mathit{Real}, \qquad (\mathtt{m_2})\ \ [y(t)+1, z(t)^2] : \mathit{Real} \to \mathit{Real}^2.$$

A collection of *application operators* $\mathit{app}^n : \mathit{Name} \times \mathit{Real} \to \mathit{Real}^n$ connects a name constant to its underlying function with range Real^n, e.g.,

$$\mathit{app}^1(\mathtt{m_1}, 3) \equiv \sin(x(3)), \quad \mathit{app}^2(\mathtt{m_2}, u) \equiv [y(u)+1, z(u)^2].$$

Unlike the previous approach, we explicitly take into account a collection of *integral operators* $\mathit{int}^{k_1,\dots,k_n} : \mathit{Real} \times \mathit{Name}^n \to \mathit{Real}^{\sum_{i=1}^{n} k_i}$. An integral term $\mathit{int}^{k_1,\dots,k_n}(u, \nu_1, \dots, \nu_n)$ takes time value u and a list of name constants $\nu_1, \dots, \nu_n$, to respectively denote unary real functions $f_1, \dots, f_n$ with ranges $\mathit{Real}^{k_1}, \dots, \mathit{Real}^{k_n}$, and returns the value $\int_0^u [f_1(t), \dots, f_n(t)]\, \mathrm{d}t$. For example:

$$\mathit{int}^2(1, \mathtt{m_2}) \equiv \int_0^1 [y+1, z^2]\mathrm{d}t,$$
$$\mathit{int}^{1,2}(T, \mathtt{m_1}, \mathtt{m_2}) \equiv \int_0^T [\sin(x), y+1, z^2]\mathrm{d}t.$$

Notice that solutions of ODEs are *no longer* atomic functions in the new logic, but can be constructed using integral operators and function names.

Definition 24 (Theory of the Real Numbers with Function Names). *For a set $\mathcal{N}$ of name constants and a set $\mathcal{O}$ of operators app^n and $\mathit{int}^{k_1,\dots,k_n}$, the first order signature is given by $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}} = (\mathcal{F} \cup \mathcal{N} \cup \mathcal{O}, >)$. The first order structure is given by $\mathbb{R}_{\mathcal{F}\cup\mathcal{N}} = (\mathbb{R} \cup N, \mathcal{F}^{\mathbb{R}} \cup \mathcal{N}^N \cup \mathcal{O}^{N,\mathbb{R}}, >^{\mathbb{R}})$ for a fixed finite set N of* name objects*, where the interpretation $\mathcal{N}^N$ of name constants and the interpretation $\mathcal{O}^{N,\mathbb{R}}$ of application and integral operators are given as explained above. The syntax and semantics of $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formulas is defined by means of $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$ and $\mathbb{R}_{\mathcal{F}\cup\mathcal{N}}$ in the standard way.*[11]

[11] The signature $\mathcal{L}_{\mathcal{F}}$ is a subsignature of $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$, and the structure $\mathbb{R}_{\mathcal{F}}$ for the real numbers is a substructure of $\mathbb{R}_{\mathcal{F}\cup\mathcal{N}}$.

The satisfiability problems of $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formulas in the new theory $\mathbb{R}_{\mathcal{F}\cup\mathcal{N}}$ can be reduced to ones in the standard theory $\mathbb{R}_{\mathcal{F}}$ at minimal cost, provided that all *Name* variables in the formulas are only existentially quantified. Consider an $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$ formula $\exists \boldsymbol{n}.\ \psi(\boldsymbol{n})$, where $\boldsymbol{n}$ are only variables of sort *Name* and $\psi(\boldsymbol{n})$ contains no quantifier for $\boldsymbol{n}$. Since there are only a finite number of name objects in $\mathbb{R}_{\mathcal{F}\cup\mathcal{N}}$, by using the standard SMT solving for equalities, we can enumerate all *consistent* assignments $\boldsymbol{n} = \overrightarrow{name}_1, \ldots, \boldsymbol{n} = \overrightarrow{name}_N$ for $\boldsymbol{n}$. Let $\hat{\psi}_i$ be the formula obtained from $\psi(\boldsymbol{n})$ by replacing each name by its related function according to the i-th assignment $\boldsymbol{n} = \overrightarrow{name}_i$. Then, $\vee_{i=1}^{N}\hat{\psi}_i$ is an ordinary $\mathcal{L}_{\mathcal{F}}$ formula, whose satisfiability can be determined by existing SMT techniques.

The satisfiability of $\mathcal{L}_{\mathcal{F}}$ formulas is undecidable for nonlinear hybrid systems, but is *decidable* up to a given precision $\delta > 0$ [13, 15]. A *δ-complete decision procedure* for a formula ϕ returns false if ϕ is unsatisfiable, and returns true if its *syntactic numerical perturbation* of ϕ by bound δ is satisfiable.[12] This is very useful in practice, since sampling exact values of physical parameters is not possible in reality. As mentioned above, the satisfiability of an $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formula of the form $\exists \boldsymbol{n}.\ \psi(\boldsymbol{n})$ is reduced to the satisfiability of an $\mathcal{L}_{\mathcal{F}}$ formula $\vee_{i=1}^{N}\hat{\psi}_i$, which is now decidable using δ-complete decision procedures.

Theorem 4. *The satisfiability of $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$ formulas of the form $\exists \boldsymbol{n}.\ \psi(\boldsymbol{n})$, where $\boldsymbol{n}$ are only name variables in ψ, is decidable by δ-complete SMT solving.*

Encoding Hybrid PALS Models in $\boldsymbol{\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}}$. We now show how Hybrid PALS models can be encoded in the new SMT logic $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$. Basically, we follow the above-mentioned approach to transform the logical formula $\phi^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ for a hybrid ensemble $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ into an $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formula *without* using uninterpreted real functions and universal quantification, using time segments by building *symbolic time segments* using name variables and assignments.

First, a global period $[0, T]$ for one synchronous round of $\mathcal{E} \upharpoonright_{\Pi} E_{\mathcal{E}}$ is dividid into N contiguous subintervals $[0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_{N-2}, t_{N-1}], [t_{N-1}, T]$ such that $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{N-1} \leq T$. Each interval denotes a single time segment to which a complete system of ODEs is assigned. The number N is determined by the total number of interval assignments in one round, namely, a number of ODE subformulas of the form $\forall t \in [u_0, u_t].\ \boldsymbol{x}(t) = \boldsymbol{v} + \int_0^{t-u_0} \boldsymbol{F_a}(\boldsymbol{x}, t)\,\mathrm{d}t$ in the formula $\phi^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$. Because each ODE subformula involves two time points u_0 and u_t, if there are k such ODE subformulas in $\phi^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$, then the total number of possible time segments with different ODE assignments is $N = 2 \cdot k$.

Second, for each physical environment E_{M_j}, N *name variables* $f_j^1, f_j^2, \ldots, f_j^N$ are declared, which denote the behavior of E_{M_j}'s physical parameters $\boldsymbol{x}_j$ for the N subintervals . Each variable f_j^i denotes a (partial) system of ODEs for corresponding to the interval $[t_{i-1}, t_i]$ (with $t_0 = 0$ and $t_N = T$). As illustrated in Figure 7, an ODE system $\boldsymbol{F_a}(\boldsymbol{x_j}, t)$ with name $\mathtt{m}_{F_a(\boldsymbol{x_j},t)}$ is assigned to *some*

[12] E,g,, if $\psi \equiv (x > 3) \wedge (y = z)$, then its syntactic numerical perturbation by $\delta > 0$ is $(x - 3 > -\delta) \wedge (y - z \geq -\delta) \wedge (z - y \geq -\delta)$.

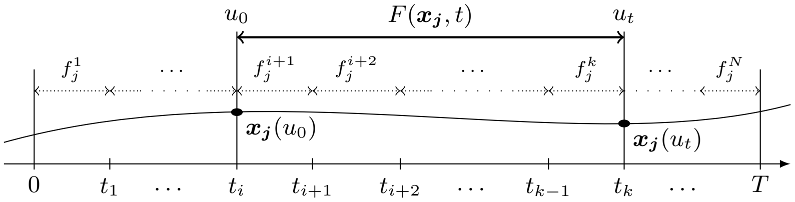


Fig. 7. $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-transformation for ODEs

interval $[t_i, t_k]$ by declaring $f_j^l = \mathtt{m}_{F_a(\boldsymbol{x_j},t)}$ for $i < l \leq k$. That is, for an ODE formula $\forall t \in [u_0, u_t].\ \boldsymbol{x}_j(t) = \boldsymbol{v}_j + \int_0^{t-u_0} F_{\boldsymbol{a}}(\boldsymbol{x}_j, t)\,\mathrm{d}t$, we declare:

$$(t_i = u_0)\ \wedge\ (t_k = u_t)\ \wedge\ (\boldsymbol{x_j}(u_0) = \boldsymbol{v_j})\ \wedge\ \textstyle\bigwedge_{l=i+1}^{k} f_j^l = \mathtt{m}_{F_a(\boldsymbol{x_j},t)}.$$

Therefore, we can *transform* each ODE subformula in $\mathcal{L}_{\mathcal{F}}$ into an $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formula (where existential quantifiers for intervals are expressed using disjunctions). Let $tr(\phi^T_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}})$ be the $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formula obtained from the logical formula $\phi^T_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}$ by repeatedly performing the $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-transformation for every ODE subformula.

Definition 25. *For an ODE formula* $\forall t \in [u_0, u_t].\ \boldsymbol{x_j}(t) = \boldsymbol{v_j} + \int_0^{t-u_0} F(\boldsymbol{x_j}, t)\,\mathrm{d}t$ *for a subcomponent* $M_j \upharpoonright_{\pi_j} E_{M_j}$*, if name constant* $\mathtt{m}_{F(\boldsymbol{x_j},t)}$ *denotes* $F(\boldsymbol{x_j}, t)$*, then its* $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$*-transform is the* $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$*-formula:*

$$\bigvee_{0\leq i\leq k\leq n} \left[(t_i = u_0) \wedge (t_k = u_t)\ \wedge\ (\boldsymbol{x_j}(u_0) = \boldsymbol{v_j})\ \wedge\ \bigwedge_{l=i+1}^{k} f_j^l \equiv \mathtt{m}_{F(\boldsymbol{x_j},t)}\right].$$

Example 14. For the formula $\phi^{60}_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}$ in Example 10 for the simple airplane controller, as explained in Example 13, the beginning and end times for interval assignments are given by the set $\{15i, 15i+u^L_{R,1}, 15i+u^R_{R,1} \mid i = 0, 1, 2, 3\} \cup \{60\}$, and therefore $N = 12$. We define time variables $t_1, t_2, \ldots, t_{12}$, and name variables $f^1_{Main}, \ldots, f^{12}_{Main}, f^1_L, \ldots, f^{12}_L, f^1_R, \ldots, f^{12}_R$. For the main controller, the formula $\forall t \in [0, 60].\ [\tau_\phi, \tau_p](t) = [y_{v_\phi}, y_{v_p}] + \int_0^{t-y_{u_0}} [\tau_p(t), c(\tau_{\delta_R}(t) - \tau_{\delta_L}(t))]\mathrm{d}t$ becomes

$$[\tau_\phi, \tau_p](0) = [y_{v_\phi}, y_{v_p}]\ \wedge\ \bigwedge_{l=1}^{12} f^l_{Main} \equiv \mathtt{m}_{[\tau_p(t), c(\tau_{\delta_R}(t)-\tau_{\delta_L}(t))]}.$$

For each subcontroller M and its intermediate step $i = 0, 1, 2, 3$, the formulas $(\forall t \in [15i, 15i + u^M_R].\ \tau_M(t) = y^i_{v_M} + \int_0^{u^M_R} y^I_{rate^i_M}\ \mathrm{d}t)$ and $(\forall t \in [15i + u^M_R, 15(i + 1)].\ \tau_M(t) = y^i_{v_M} + \int_0^{15-u^M_R} y^R_{rate^i_M}\ \mathrm{d}t)$, respectively, become:

$$\textstyle\bigvee_{k=3i+1}^{3i+2} \left(\tau_M(15i) = y^i_{v_M}\ \wedge\ \bigwedge_{l=3i+1}^{k} f^l_M = \mathtt{m}_{y^I_{rate^i_M}}\right)$$

$$\textstyle\bigvee_{k=3i+2}^{3(i+1)} \left(\tau_M(15i + u^M_R) = y^i_{v_M}\ \wedge\ \bigwedge_{l=k}^{3(i+1)} f^l_M = \mathtt{m}_{y^R_{rate^i_M}}\right)$$

Notice that we have already discarded unsatisfiable time segment assignments from these formulas (e.g., $t_0 = 30$ and $t_{12} = 30 + u_R^L$).

Finally, we build complete systems of ODEs *parameterized by name variables*, using time segments of a global period $[0, T]$. Consider a subinterval $[t_{i-1}, t_i]$ and its corresponding name variables $\{f_j^i\}_{j\in J_S\cup J_F}$. Suppose that $\{\boldsymbol{g}_{i-1}^j\}_{j\in J_S\cup J_F}$ denote the values of the physical parameters at time t_{i-1}. Then, the relationship between those variables are expressed by:

$$[\boldsymbol{g}_i^j]_{j\in J_S\cup J_F} = [\boldsymbol{g}_{i-1}^j]_{j\in J_S\cup J_F} + \int_0^{t_i - t_{i-1}} [f_j^i]_{j\in J_S\cup J_F}\,\mathrm{d}t,$$

written as a term $[\boldsymbol{g}_i^j]_{j\in J_S\cup J_F} = [\boldsymbol{g}_{i-1}^j]_{j\in J_S\cup J_F} + \mathit{int}^{\boldsymbol{l}}(t_i - t_{i-1}, [f_j^i]_{j\in J_S\cup J_F})$ in $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$, where $\boldsymbol{l}$ denotes the dimensions of the physical environments.

Definition 26. *Given a number N, a* global ODE formula $\phi_{ODE}(\boldsymbol{f}, \boldsymbol{t}, \boldsymbol{g})$ *is a conjunction of N ODE formulas, parameterized by name variables, in $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$:*

$$\boldsymbol{x}(t_0) = \boldsymbol{g}^0 \;\wedge \bigwedge_{1\le i\le N} (\boldsymbol{g}^i = \boldsymbol{g}^{i-1} + \mathit{int}^{\boldsymbol{l}}(t_i - t_{i-1}, \boldsymbol{f}^i) \;\wedge\; \boldsymbol{x}(t_i) = \boldsymbol{g}^i),$$

where: (i) $\boldsymbol{t} = \{t_0, t_1, \dots, t_N\}$ are time variables *denoting N time segments with $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_N = T$; (ii) $\boldsymbol{f} = \{f_j^1, \dots, f_j^N\}_{j\in J_S\cup J_F}$ are* name variables *denoting (partial) ODE systems of each E_{M_j} for each subinterval $[t_{i-1}, t_i]$; (iii) $\boldsymbol{g} = \{\boldsymbol{g}_j^0, \dots, \boldsymbol{g}_j^N\}_{j\in J_S\cup J_F}$ are* value variables *denoting the values of all physical parameters of $\mathcal{E} \upharpoonright_\Pi E_\mathcal{E}$ at each time t_i; and (iv) $\boldsymbol{l} = \{l_j\}_{j\in J_S\cup J_F}$ are the* dimensions *of E_{M_j}'s parameters $\boldsymbol{x_j} = (x_j^1, \dots, x_j^{l_j})$.*

Example 15. Consider the time segments in Examples 13 and 14 for the simple airplane controller. The global ODE formula ϕ_{ODE} is given by:

$$[\tau_\phi(t_0), \tau_p(t_0), \tau_L(t_0), \tau_R(t_0)] = [g_\phi^0, g_p^0, g_L^0, g_R^0] \;\wedge$$

$$\bigwedge_{i=1}^{12} \left(\begin{bmatrix} g_\phi^i \\ g_p^i \\ g_L^i \\ g_R^i \end{bmatrix} = \begin{bmatrix} g_\phi^{i-1} \\ g_p^{i-1} \\ g_L^{i-1} \\ g_R^{i-1} \end{bmatrix} + \mathit{int}^{2,1,1}(t_i - t_{i-1}, \begin{bmatrix} f_{Main}^i \\ f_L^i \\ f_R^i \end{bmatrix}) \;\wedge\; \begin{bmatrix} \tau_\phi(t_i) \\ \tau_p(t_i) \\ \tau_L(t_i) \\ \tau_R(t_i) \end{bmatrix} = \begin{bmatrix} g_\phi^i \\ g_p^i \\ g_L^i \\ g_R^i \end{bmatrix} \right)$$

Using $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-transformations and ϕ_{ODE}, we obtain the $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-formula that includes no uninterpreted real functions and universal quantification:

$$\tilde{\phi}_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}^T \equiv (\exists \boldsymbol{f}, \boldsymbol{t}, \boldsymbol{g}).\; \phi_{ODE}(\boldsymbol{f}, \boldsymbol{t}, \boldsymbol{g}) \wedge tr(\phi_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}^T),$$

and the satisfiability of the resulting formula $\tilde{\phi}_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}^T$ can be determined by using δ-complete SMT solving (by Theorem 4).

Theorem 5. *$\phi_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}^T$ is satisfiable iff $\tilde{\phi}_{\mathcal{E}\upharpoonright_\Pi E_\mathcal{E}}^T$ is satisfiable.*

Proof (Proof Sketch). ($\Rightarrow$) If the logical formula $\phi^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ is satisfiable, then there exists a collection $\{\boldsymbol{x_j}\}_{j \in J_S \cup J_F}$ of trajectories for a global round, and we can find the concrete values of $(\boldsymbol{f}, \boldsymbol{t}, \boldsymbol{g})$ from the trajectories. ($\Leftarrow$) If $\tilde{\phi}^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ is satisfiable, then there exist resulting values of $(\boldsymbol{f}, \boldsymbol{t}, \boldsymbol{g})$, and a collection $\{\boldsymbol{x_j}\}_{j \in J_S \cup J_F}$ of trajectories can be constructed. Then, by construction, any ODE subformula $\forall t \in [u_0, u_t].\, \boldsymbol{x_j}(t) = \boldsymbol{v_j} + \int_0^{t-u_0} F(\boldsymbol{x_j}, t)\, \mathrm{d}t$ is true iff its transformed formula by Definition 25 is true, where $u_0 = t_i$ and $u_t = t_k$ for some $0 \leq i \leq k \leq n$. □

Notice that in the new encoding $\tilde{\phi}^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ ODE literals (i.e., terms including integral operators) only appear in the global ODE subformula ϕ_{ODE}. Although it would be possible to represent Hybrid PALS models in the standard SMT logic $\mathcal{L}_{\mathcal{F}}$ using the time segment approach, the size of such a formula in $\mathcal{L}_{\mathcal{F}}$ is much bigger than the size of $\tilde{\phi}^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$ in $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$. For each time segment, an $\mathcal{L}_{\mathcal{F}}$-encoding can include many ODE literals to represent different combinations of complete ODE systems, but there is only one ODE literal in the $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-encoding $\tilde{\phi}^{T}_{\mathcal{E}\upharpoonright_{\Pi} E_{\mathcal{E}}}$, parameterized by name variables. Because the most computationally expensive operation is a decision procedure for ODEs, minimizing the number of ODE literals is a very effective way to enhance the performance of analysis.

The use of name variables in $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$ speeds up the DPLL($\mathcal{T}$) procedure for δ-complete SMT. In DPLL($\mathcal{T}$), identifying inconsistent literals is important to to prune the Boolean search space, but it is computationally expensive for ODE literals in δ-complete SMT. For example, consider two functions $f_1(t) = \sqrt{t+1}$ and $f_2(t) = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \frac{1}{16}t^3$, where $f_2(t)$ is the third order taylor expansion of $\sqrt{t+1}$. The formulas $\forall t \in [0, T].\, g(t) = f_1(t)$ and $\forall t \in [0, T].\, g(t) = f_2(t)$ are consistent up to precision $\delta = 0.1$ for $T = 0.8$, but not consistent if $\delta = 0.01$. But for hybrid systems, we need not to check whether two continuous functions are identical. In this case, we can statically add *uniqueness lemmas* of the form $\neg(w = \mathtt{m}_{f_1} \ \wedge \ w = \mathtt{m}_{f_2})$ in $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$, and this can significantly reduce the Boolean search space (see Section 5.4 for some comparison and experimental results).

5 Case Studies and Experimental Results

This section gives an overview of some case studies using Hybrid PALS and SMT-based analysis to verify distributed CPSs. These case studies involve nontrivial (nonlinear) ODEs, due to the continuous physical interaction between distributed components. We have verified safety properties using inductive and compositional SMT encodings for *any possible* set of local clocks with maximal clock skew ϵ. All experiments were conducted on an Intel Xeon 2.0 GHz with 64 GB memory. The case studies and the experimental evaluation are available at `http://dreal.github.io/benchmarks/networks`.

5.1 Turning an Airplane

We consider a multirate distributed CPS to turn an airplane (adapted from [4]). To make a turn, an aircraft rolls towards the direction of the turn by moving its *ailerons*. The rolling causes a yawing moment in the opposite direction, called *adverse yaw*, which is countered by using its *rudder* (a surface attached to the vertical stabilizer). The subcontrollers for the ailerons and the rudder operate at different rates, and the main controller orchestrates them to achieve a smooth turn. Fig. 8 illustrates the multirate ensemble $\mathcal{E}$ of our system.

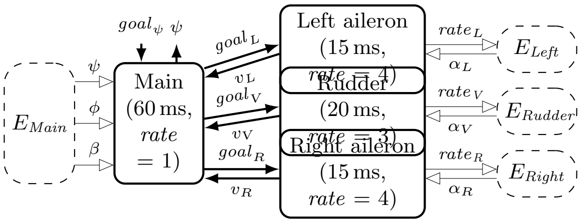


Fig. 8. The distributed controllers for turning an airplane.

Each subcontroller M gradually moves its surface towards the goal angle $goal_M$ specified by the main controller M_{Main}. In each round, M receives $goal_M$ from M_{Main}, determines the moving rate $rate_M$ based on $goal_M$ and the current sampled value v_M of the surface angle α_M, and sends back v_M to M_{Main} (e.g., the new value of $rate_M$ can be given by $\mathrm{sign}(goal_M - v_M) \cdot \min(\mathrm{abs}(goal_M - v_M)/T, max_M))$. The continuous dynamics of α_M is specified by the local physical environment E_M as the ODE $\frac{\mathrm{d}\alpha_M}{\mathrm{d}t} = rate_M$,

The main controller M_{Main} determines the goal angles for the subcontrollers to make a coordinated turn. In each round, M_{Main} receives a desired direction $goal_\psi$ (from the pilot) and the surface angles (v_L, v_V, v_R) from the subcontrollers, and then sends back the new goal angles $(goal_L, goal_V, goal_R)$, based on the current sampled *position* values $(v_\psi, v_\phi, v_\beta)$ of the direction angle ψ, the roll angle ϕ, and the yaw angle β.

We consider a simple control logic to decide the new goal angles $(goal_L, goal_V, goal_R)$, by three control functions [4]: (i) $f_\phi(v_\phi, v_\psi, goal_\psi)$ returns the goal roll angle $goal_\phi$, (ii) $f_R(v_\phi, goal_\phi)$ returns the goal angle $goal_R$ for the right aileron (where the goal angle $goal_L$ for the left aileron is the opposite angle $-goal_R$); and (iii) $f_V(v_\beta)$ returns the goal rudder angle $goal_V$. For example, for $h = 0.32(g_\psi - v_\psi)$, $f_\phi(v_\phi, v_\psi, g_\psi) = v_\phi + \mathrm{sign}(h - v_\phi) \cdot \min(\mathrm{abs}(h - v_\phi), 1.5)$, $f_R(v_\phi, g_\phi) = \mathrm{sign}(g_\phi - v_\phi) \cdot \min(0.3\,\mathrm{abs}(g_\phi - v_\phi), 15))$, and $f_V(v_\beta) = \mathrm{sign}(-v_\beta) \cdot \min(0.3\,\mathrm{abs}(v_\beta), 10)$.

In the physical environment E_{Main}, the lateral dynamics of an aircraft can be specified as the following nonlinear ODEs, where p is the rolling moment, r is the yawing moment, and $Y_{\delta_L,\delta_V,\delta_R,\beta}$, $L_{\delta_L,\delta_V,\delta_R,\beta}$, and $N_{\delta_L,\delta_V,\delta_R,\beta}$ are (linear) functions of the control angles $(\delta_L, \delta_V, \delta_R)$ and β [22]:

$$\begin{aligned}
\dot{\beta} &= Y_{\delta_L,\delta_V,\delta_R,\beta}/mV - r + (g/V)\cos\beta\sin\phi, \\
\dot{\phi} &= p, \qquad\qquad\qquad\quad \dot{\psi} = (g/V)\tan\phi, \\
\dot{p} &= (c_1 r + c_2 p)\cdot r\tan\phi + c_3 L_{\delta_L,\delta_V,\delta_R,\beta} + c_4 N_{\delta_L,\delta_V,\delta_R,\beta}, \\
\dot{r} &= (c_8 p - c_2 r)\cdot r\tan\phi + c_4 L_{\delta_L,\delta_V,\delta_R,\beta} + c_9 N_{\delta_L,\delta_V,\delta_R,\beta}.
\end{aligned}$$

The physical environment E_{Main} clearly depends on the subcontrollers's physical environments. Each control angle δ_M in E_{Main} must be the same as the corresponding surface angle α_M. Such immediate physical correlations between the local environments are specified by the time-invariant constraint

$$\forall t.\, (\delta_L(t) = \alpha_L(t)) \wedge (\delta_V(t) = \alpha_V(t)) \wedge (\delta_R(t) = \alpha_R(t)).$$

Notice that since the main controller and the subcontrollers have *different periods with local clock skews*, the ODEs of the subcontrollers cannot be directly “plugged” into E_{Main}.

The safety property is that the yaw angle β is always close to 0 (e.g., $\forall t.\ \beta(t) < 0.2$). In the analysis, we assume that the maximal clock skew is $\epsilon = 0.2\,\text{ms}$, the sampling time t_I is $0\,\text{ms}$ for every controller, and the response time t_R is $3\,\text{ms}$ for the subcontrollers. To verify the safety property $\forall t.\ \beta(t) < 0.2$, we have performed a bounded reachability analysis up to bound $k = 20$ from the initial condition $(\beta, \phi, \psi, p, r) = \mathbf{0}$. To avoid the formula explosion problem, we have also used compositional analysis by (i) showing that each subcontroller gradually moves its surface towards its goal direction, and (ii) then performing a bounded reachability analysis, using dReal with precision $\delta = 0.0001$, only for the main controller, assuming (i) (the analysis took 2 minutes).

5.2 Networked Water Tank Controllers

In this case study, adapted from [17,20], a number of water tanks are connected by pipes as shown in Figure 9. The water level in each tank is controlled by a pump in the tank, and depends on the pump's mode $m \in \{m_{\texttt{on}}, m_{\texttt{off}}\}$ and the water levels of the adjacent tanks. The water level x_i of tank i changes according to the ODEs:[13]

$$\begin{aligned}
A_i\dot{x}_i &= (q_i + a\sqrt{2g}\sqrt{x_{i-1}}) - b\sqrt{2g}\sqrt{x_i} && \text{if } m_i = m_{\texttt{on}}, \\
A_i\dot{x}_i &= a\sqrt{2g}\sqrt{x_{i-1}} - b\sqrt{2g}\sqrt{x_i} && \text{if } m_i = m_{\texttt{off}},
\end{aligned}$$

We set $x_0 = 0$ for the leftmost tank 1. Every pipe controller performs its transitions according to its local clock and sets the pump to on if $x_i \leq L_m$ and to off if $x_i > L_M$.

[13] A_i, q_i, a, b are constants determined by the size of the tank, the power of the pump, and the widths of the I/O pipes, and g is the gravity constant.

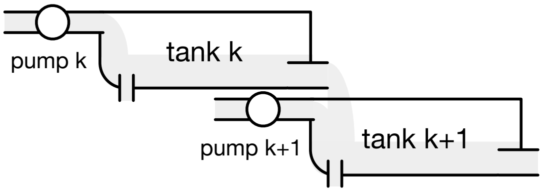


Fig. 9. Connected water tanks.

The safety property is that the water level of each tank is in a certain range $I = [L_m - \eta, L_M + \eta]$. We have verified this safety property for *any number* of connected water tanks using inductive and compositional analysis.[14] For maximal clock skew $\epsilon = 30\,\text{ms}$, sampling time $t_I = 20\,\text{ms}$, and response time $t_R = 100\,\text{ms}$, we have proved the compositional safety property using dReal with precision $\delta = 0.001$ (the analysis took 4.3 seconds).[15] However, if $\epsilon = 150\,\text{ms}$, then the compositional inductive condition is violated by the trajectory in Fig. 10 (the analysis took 1.46 seconds).

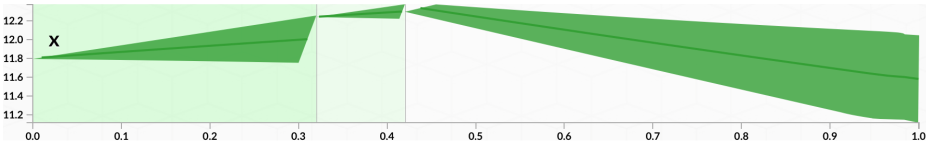


Fig. 10. The counterexample trajectory when $\epsilon = 150\,\text{ms}$, where the water level increases for extra 300 ms to violate the inductive condition.

5.3 Networked Thermostat Controllers

A number of rooms are interconnected by open doors, as shown in Fig. 18. The temperature of each room is separately controlled by its own thermostat controller that turns the heater on and off. That is, it depends on the heater's mode $m \in \{m_{\texttt{on}}, m_{\texttt{off}}\}$ and the temperatures of the other rooms. The temperature x_i

[14] For a tank k and $I' = [L_m - \eta', L_M + \eta'] \subseteq I$ with $\eta' < \eta$, provided that $x_{k-1} \in I$ always holds, we show that $x_k \in I'$ is an inductive condition, and $x_k \in I$ always holds if $x_k \in I'$ at the beginning of each round (i.e., $(x_k(0) \in I' \wedge \phi^{T,0}_{\mathcal{E}\restriction_\Pi E_\mathcal{E}}) \to (x_k(T) \in I') \wedge (\forall t \in [0,T].\, x_k(t) \in I)$).

[15] In the analysis, we use the parameters $a = 0.5$, $b = 0.6$, $g = 9.8$, $A = 2$, $q = 4$, $L_m = 8$, $L_M = 10$, $\eta = 3$, $\eta' = 2$, and $T = 1\,\text{s}$.

of room i changes according to the ODEs:[16]

$$\begin{aligned} \dot{x}_i &= K_i(h_i - ((1-2c)x_i + cx_{i-1} + cx_{i+1})) && \text{if } m_i = m_{\mathtt{on}} \\ \dot{x}_i &= -K_i((1-2c)x_i + cx_{i-1} + cx_{i+1}) && \text{if } m_i = m_{\mathtt{off}} \end{aligned}$$

In each transition, a controller turns on the heater if $x_i \leq T_m$, and turns it off if $x_i > T_M$.

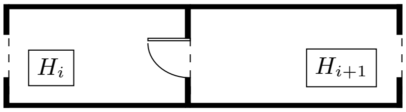


Fig. 11. Interconnected rooms.

The safety property is that the temperature of each room is in the range $I = [T_m - \eta, T_M + \eta]$. We have verified the safety property $\forall t.\ x_i \in I$ for *any number* of interconnected thermostat controllers by inductive and compositional analysis.[17] For $\epsilon = 2\,\text{ms}$, $t_I = 10\,\text{ms}$, and $t_R = 200\,\text{ms}$, we have proved (in 2.6 seconds) the compositional safety property using dReal with precision $\delta = 0.001$.[18] However, if $\epsilon = 20\,\text{ms}$, then the compositional inductive condition is violated by the trajectory in Fig. 12 (the analysis took 0.56 seconds).

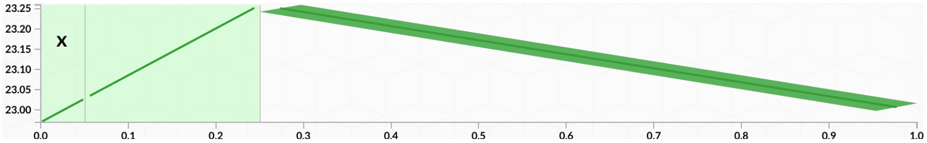


Fig. 12. The counterexample trajectory when $\epsilon = 20\,\text{ms}$.

5.4 Bounded Reachability Comparison

We have implemented the SMT algorithm for our new logic $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$ in the dReal SMT solver [14], and have compared the performance of the new $\mathcal{L}_{\mathcal{F}\cup\mathcal{N}}$-encoding with one of the previous *non-modular* $\mathcal{L}_{\mathcal{F}}$-encoding for Hybrid PALS models.

[16] $K_i, h_i \in \mathbb{R}$ are constants depending on the size of room i and the heater's power, respectively, and $c \in \mathbb{R}$ is determined by the size of the open door.

[17] For $I' = [T_m - \eta', T_M + \eta'] \subseteq I$, provided that both $x_{i-1} \in I$ and $x_{i+1} \in I$ always hold, we show that $x_i \in I'$ is an inductive condition, and $x_i \in I$ always holds if $x_i \in I'$ at the beginning of each round.

[18] In the analysis, we use the parameters $K = 0.015$, $h = 100$, $c = 0.01$, $T_M = 21$, $T_m = 19$, $\eta = 3$, $\eta' = 2$, and $T = 1\,\text{s}$.

In this comparison, we only consider special cases with no clock skews ($\epsilon = 0$) due to the lack of expressiveness of $\mathcal{L}_{\mathcal{F}}$, and only bounded reachability analysis, since it needs to generate bigger formulas than other types of analysis.

We have performed bounded reachability analysis up to $k = 5$ for the safety properties for the "concrete" instances of the case studies (using dReal without the use of other SMT heuristics).[19] The results for k-step bounded reachability analysis are summarized in Fig. 13 and show that the new encoding significantly outperforms the non-modular encoding. For example, the SMT analysis using the non-modular encoding for three interconnected thermostats did not terminate in 30 hours even for $k = 2$.

6 Related Work.

PALS [1,2,5,19] targets distributed real-time systems, whose absence of continuous behaviors means that continuous behaviors and local clocks do not need to be taken into account in the synchronous models, which can therefore be verified by any explicit-state model checker. In contrast, Hybrid PALS synchronous models must take both clock skews and continuous behaviors into account and hence cannot be analyzed by explicit-state techniques. The initial steps towards a hybrid extension of PALS were taken in [6]. However, that work imposes strong conditions on sampling and response times of sensors and actuators, and, furthermore, does not give a bisimulation equivalence. Our work presents a more general model, where the relative sampling and actuating times are system parameters, and also provides a crucial bisimulation result between the synchronous and the distributed model.

More clear differences should be explained

However, the main difference is in the verification part. Previous PALS work [4] used explicit-state model checking, based on numerical simulation for the continuous dynamics. Obviously, this cannot guarantee the correctness of hybrid systems. Furthermore, there was no hope of verifying a system for all possible local clocks. The paper [6] shows that two interconnected thermostats can be verified using dReal, but does not present any general SMT techniques for Hybrid PALS. Also, the notion of stable states are quite different in [6], and so sampling and actuating times are quite restricted and guarantees only the trace equivalence, not bisimulation. Our work presents SMT encodings for bounded reachability, inductive, and compositional analysis, besides a new SMT framework and more case studies.

Because of the difficulty of handling SMT formulas over the reals with nonlinear functions, SMT-solving-based verification is a fairly new direction for nonlinear hybrid systems. The research direction is initiated in [21], which uses constraint solving algorithms for handling nonlinear reachability problems. Two main lines of work that explicitly formulate problems as SMT formulas are based on the HySAT/iSAT solver [10,11] and the MathSAT solver [8,9]. But neither

[19] In the airplane case study, we also use a single-rate system in which every controller has period 0.5 s. We consider both `unsat` (verified) and `sat` (counterexample found) cases by using different safety properties.

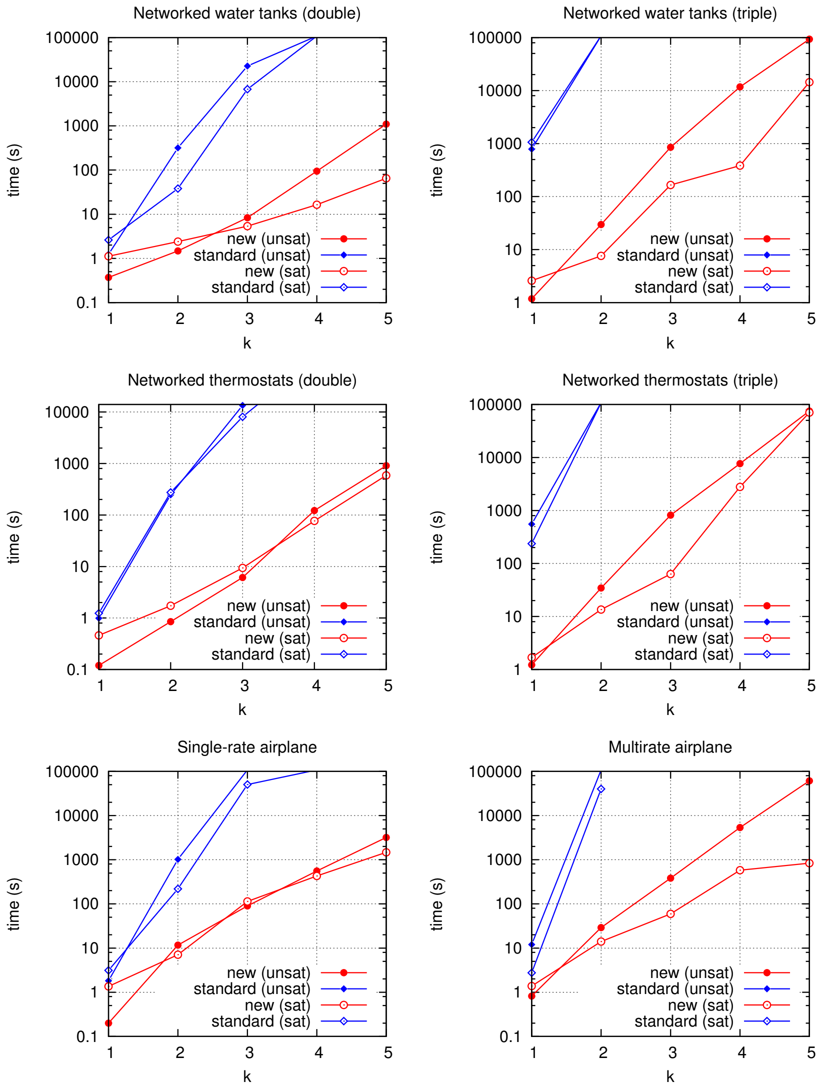


Fig. 13. Running time of k-step bounded reachability analysis, where red lines denote the new encoding, and blue lines denote the non-modular encoding.

efficient encodings of distributed hybrid systems nor inductive and compositional analysis has been investigated in existing work along these lines. It is also worth noting that our methods are orthogonal to reachable set computation tools like SpaceEX [12] and Flow* [7], since they can also be used as ODE solvers for δ-compete SMT (e.g., in dReal).

7 Concluding Remarks

We have presented and proved the correctness of the Hybrid PALS methodology that greatly simplifies the design and verification of virtually synchronous cyber-physical systems whose components have environments with continuous behaviors. Although Hybrid PALS does not allow us to abstract from (imprecise) local clocks and the times transitions are performed, it allows us to abstract from asynchronous communication (and the resulting interleavings), message buffering, network delays, backoff timers, and so on.

We have shown that verification problems for Hybrid PALS models can be expressed as SMT formulas and have developed efficient SMT-solving-based verification methods for Hybrid PALS. We have implemented these techniques in the dReal SMT solver and have applied our methodology on a number of non-trivial CPSs. Our experiments have shown that our techniques dramatically increase the performance of SMT analysis for distributed hybrid systems with multiple control modes and nonlinear ODEs up to precision δ.

References

1. Al-Nayeem, A., Sun, M., Qiu, X., Sha, L., Miller, S.P., Cofer, D.D.: A formal architecture pattern for real-time distributed systems. In: RTSS'09. IEEE (2009)
2. Al-Nayeem, A., Sha, L., Cofer, D.D., Miller, S.P.: Pattern-based composition and analysis of virtually synchronized real-time distributed systems. In: ICCPS'12. IEEE (2012)
3. Alur, R., Courcoubetis, C., Halbwachs, N., Henzinger, T.A., Ho, P.H., Nicollin, X., Olivero, A., Sifakis, J., Yovine, S.: The algorithmic analysis of hybrid systems. Theoretical Computer Science 138, 3–34 (1995)
4. Bae, K., Krisiloff, J., Meseguer, J., Ölveczky, P.C.: Designing and verifying distributed cyber-physical systems using Multirate PALS: An airplane turning control system case study. Science of Computer Programming 103, 13–50 (2015)
5. Bae, K., Meseguer, J., Ölveczky, P.C.: Formal patterns for multirate distributed real-time systems. Science of Computer Programming 91, Part A, 3 – 44 (2014)
6. Bae, K., Ölveczky, P.C.: Hybrid multirate PALS. In: Logic, Rewriting, and Concurrency. LNCS, Springer (2015), to appear
7. Chen, X., Ábrahám, E., Sankaranarayanan, S.: Flow*: An analyzer for non-linear hybrid systems. In: CAV'13. LNCS, vol. 8044. Springer (2013)
8. Cimatti, A., Mover, S., Tonetta, S.: A quantifier-free SMT encoding of non-linear hybrid automata. In: FMCAD'12. IEEE (2012)
9. Cimatti, A., Mover, S., Tonetta, S.: SMT-based verification of hybrid systems. In: AAAI. AAAI Press (2012)

10. Eggers, A., Fränzle, M., Herde, C.: SAT modulo ODE: A direct SAT approach to hybrid systems. In: ATVA'08, LNCS, vol. 5311. Springer (2008)
11. Fränzle, M., Herde, C.: HySAT: An efficient proof engine for bounded model checking of hybrid systems. Formal Methods in System Design 30(3), 179–198 (2007)
12. Frehse, G., Le Guernic, C., Donzé, A., Cotton, S., Ray, R., Lebeltel, O., Ripado, R., Girard, A., Dang, T., Maler, O.: SpaceEx: Scalable verification of hybrid systems. In: CAV'11. LNCS, vol. 6806. Springer (2011)
13. Gao, S., Avigad, J., Clarke, E.M.: δ-complete decision procedures for satisfiability over the reals. In: IJCAR'12. LNCS, vol. 7364. Springer (2012)
14. Gao, S., Kong, S., Clarke, E.M.: dReal: An SMT solver for nonlinear theories over the reals. In: CADE'13. LNCS, vol. 7898. Springer (2013)
15. Gao, S., Kong, S., Clarke, E.M.: Satisfiability modulo ODEs. In: FMCAD'13. IEEE (2013)
16. Henzinger, T.A.: The theory of hybrid automata. Springer (2000)
17. Kowalewski, S., Stursberg, O., Fritz, M., Graf, H., Hoffmann, I., Preußig, J., Remelhe, M., Simon, S., Treseler, H.: A case study in tool-aided analysis of discretely controlled continuous systems: The two tanks problem. In: Hybrid Systems V, LNCS, vol. 1567. Springer (1999)
18. Lynch, N., Segala, R., Vaandrager, F.: Hybrid I/O automata. Information and Computation 185(1), 105–157 (2003)
19. Meseguer, J., Ölveczky, P.C.: Formalization and correctness of the PALS architectural pattern for distributed real-time systems. Theoretical Computer Science 451, 1–37 (2012)
20. Raisch, J., Klein, E., O'Young, S., Meder, C., Itigin, A.: Approximating automata and discrete control for continuous systems — two examples from process control. In: Hybrid Systems V, LNCS, vol. 1567. Springer (1999)
21. Ratschan, S., She, Z.: Safety verification of hybrid systems by constraint propagation-based abstraction refinement. ACM Transactions on Embedded Computing Systems (TECS) 6(1) (2007)
22. Stevens, B.L., Lewis, F.L.: Aircraft control and simulation. John Wiley & Sons (2003)

A Hybrid Automata Semantics

A hybrid automaton [16] is one of the most widely used formalism for modeling hybrid systems. This section explains the relationship between our Hybrid PALS formalism and hybrid automata. The hybrid automata representation of a Hybrid PALS synchronous model is quite complex due to extra timer variables and communication jump conditions as shown in this section.

A.1 Preliminaries on Hybrid Automata

A state of a hybrid automaton is a pair $(m, \boldsymbol{v})$ of a discrete mode m and a vector $\boldsymbol{v} \in \mathbb{R}^l$ of real numbers. The behavior of a hybrid automaton is specified by using both discrete *jump* conditions and continuous *flow* conditions. Let $\mathfrak{T}$ be a set of trajectories over a closed time interval beginning at 0, e.g., $f : [0, \infty) \to \mathbb{R}^l$ or $f : [0, t] \to \mathbb{R}^l$ for $t \geq 0$ (see [16, 18] for general conditions of trajectories).

Definition 27. *A* hybrid automaton $H = (X, Q, \Sigma, \mathit{init}, \mathit{inv}, \mathit{flow}, \mathit{jump})$ *has:*

- $X = \{x_1, \ldots, x_l\}$ *a finite set of real-numbered variables* ($\mathit{val}(X) = \mathbb{R}^l$)*;*
- Q *a set of control modes, and* Σ *a set of actions;*
- $\mathit{init} \subseteq Q \times \mathit{val}(X)$ *a set of initial states;*
- $\mathit{inv} : Q \to 2^{\mathit{val}(X)}$ *assigning to each mode* q *its invariant condition (i.e., a set of all possible values of* X *in mode* q*);*
- $\mathit{flow} : Q \to 2^{\mathfrak{T}}$ *assigning to each mode* q *its flow condition (i.e., a set of all trajectories of* X *in mode* q*); and*
- $\mathit{jump} : (Q \times \mathit{val}(X)) \times \Sigma \times (Q \times \mathit{val}(X))$ *a jump condition to define a discrete transition* $(q, \boldsymbol{v}) \xrightarrow{a} (q', \boldsymbol{v}')$ *with action* a*.*

Note that *init*, *inv*, and *flow* are often defined as predicates over the variables X: e.g., for each mode $q \in Q$, predicates: $\mathit{init}_q(x_1, \ldots, x_l)$, $\mathit{inv}_q(x_1, \ldots, x_l)$, and $\mathit{flow}_q(x_1, \ldots, x_l, \dot{x}_1, \ldots, \dot{x}_l)$, where $\dot{x}_i$ denotes the first derivative $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ of x_i.

Now consider two hybrid automata H_1 and H_2. A (discrete) communication between H_1 and H_2 is modeled by joint synchronous actions in $\Sigma_1 \cap \Sigma_2$, and their (continuous) interaction is modeled by using shared variables in $X_1 \cap X_2$. Let $\boldsymbol{v}\restriction_{X_i} \in \mathit{val}(X_i)$, $i = 1, 2$, denote the restriction of $\boldsymbol{v} \in \mathit{val}(X_1 \cup X_2)$ by X_i in which both $\boldsymbol{v}$ and $\boldsymbol{v}\restriction_{X_i}$ give the same value for each common variable $x \in X_i$.

Definition 28. *Given two automata* $H_i = (X_i, Q_i, \Sigma_i, \mathit{init}_i, \mathit{inv}_i, \mathit{flow}_i, \mathit{jump}_i)$ *for* $i = 1, 2$*, their parallel composition* $H_1 \parallel H_2$ *is defined by the hybrid automata* $H_1 \parallel H_2 = (X_1 \cup X_2,\ Q_1 \times Q_2,\ \Sigma_1 \cup \Sigma_2,\ \mathit{init},\ \mathit{inv},\ \mathit{flow},\ \mathit{jump})$*, where:*

- $\mathit{init} = \{((q_1, q_2), \boldsymbol{v}) \mid (q_1, \boldsymbol{v}\restriction_{X_1}) \in \mathit{init}_1, (q_2, \boldsymbol{v}\restriction_{X_2}) \in \mathit{init}_2\}$*;*
- $\mathit{inv}(q_1, q_2) = \{\boldsymbol{v} \in \mathit{val}(X_1 \cup X_2) \mid \boldsymbol{v}\restriction_{X_1} \in \mathit{inv}_1(q_1), \boldsymbol{v}\restriction_{X_2} \in \mathit{inv}_2(q_2)\}$*;*
- $f \in \mathit{flow}$ *iff there exist compatible trajectories* $f_1 \in \mathit{flow}_i$, $i \in 1, 2$, *such that* $\mathit{dom}(f) = \mathit{dom}(f_i)$ *and* $(\forall t \in \mathit{dom}(f))\ f(t)\restriction_{X_i} = f_i(t)$*; and*
- $((q_1, q_2), \boldsymbol{v}) \xrightarrow{a} ((q_1', q_2'), \boldsymbol{v}') \in \mathit{jump}$ *iff* $a \in \Sigma_1 \cup \Sigma_2$ *and for each* $i = 1, 2$*:*

$$\begin{cases} (q_i, \boldsymbol{v}\restriction_{X_i}) \xrightarrow{a} (q_i', \boldsymbol{v}'\restriction_{X_i}) \in \mathit{jump}_i & \text{if } a \in \Sigma_i \\ (q_i, \boldsymbol{v}\restriction_{X_i}) = (q_i', \boldsymbol{v}'\restriction_{X_i}) & \text{if } a \notin \Sigma_i. \end{cases}$$

A.2 Hybrid Automata for Hybrid PALS Synchronous Models

We can define a hybrid automaton for a Hybrid PALS synchronous model $\mathcal{E}$. For each typed machine M and its physical environment $\mathcal{E}_M$ in a multirate ensemble $\mathcal{E}$, there exists a corresponding hybrid automaton $H_{M \upharpoonright E_M}$ in which the discrete jumps are given by the transitions of M and the continuous flows are given by the physical transitions of $\mathcal{E}_M$. Recall that a communication between components in $\mathcal{E}$ happens at the beginning of their hyper-period. Therefore, we extend each $H_{M \upharpoonright E_M}$ to the communication hybrid automaton $\bar{H}_{M_j}$ by adding discrete jumps for such synchronous communications, and then their composition $||_{M_j \in \mathcal{E}} \bar{H}_{M_j}$ becomes the hybrid automaton for $\mathcal{E}$. Finally, the time-invariant constraints for $\mathcal{E}$ are encoded in $||_{M_j \in \mathcal{E}} \bar{H}_{M_j}$ using shared variables.

The environment restriction $M \upharpoonright E_M$ of a controller $M = (D_i, S, D_o, \delta_M)$ by its environment $E_M = (C, \boldsymbol{x}, U, \mathit{\Lambda})$ corresponds to a hybrid automaton $H_{M \upharpoonright E_M}$. The variables of $H_{M \upharpoonright E_M}$ include the physical parameters $\boldsymbol{x}$ of E_M, and an extra *timer* variable τ_M to keep track of its periods. A discrete state of $H_{M \upharpoonright E_M}$ consists of a state of M, input buffers $D_{i\perp}$, and output buffers $D_{o\perp}$, where

$$D_{i\perp} = (D_{i_1} \cup \{\perp\}) \times \cdots \times (D_{i_n} \cup \{\perp\}),$$
$$D_{o\perp} = (D_{o_1} \cup \{\perp\}) \times \cdots \times (D_{o_m} \cup \{\perp\}),$$

with $\perp$ an empty buffer. As $M \upharpoonright E_M$, the observed behavior $\pi_{\boldsymbol{x}}(s)$ by M in a state $s \in S$ should be the same as the observable behavior $\pi_S(\boldsymbol{v})$ of E_M. Initially, the timer τ_M is set to 0. During each period when $0 \leq \tau_M \leq c_M(\pi_t(s')) - c_M(\pi_t(s))$, the value of the parameter $\boldsymbol{x}$ changes according to E_M. At the end of each period (when $\tau_M = c_M(\pi_t(s')) - c_M(\pi_t(s))$), a discrete transition $((\boldsymbol{i}, s), (s', \boldsymbol{o})) \in \delta_M$ is taken using the value $\boldsymbol{i}$ in its input buffer, the output $\boldsymbol{o}$ is stored in its output buffer, and the timer τ_M is reset to 0. The value in an input buffer is updated in the ensemble level (see below).

Definition 29. *Given a machine $M = (D_i, S, D_o, \delta_M)$, its physical environment $E_M = (C, \boldsymbol{x}, U, \mathit{\Lambda})$ with $\boldsymbol{x} = (x_1, \ldots, x_l)$, and a local clock $c_M : \mathbb{N} \to \mathbb{R}_{>0}$, the corresponding automaton is $H_{M \upharpoonright E_M} = (X, Q, \mathit{\Sigma}, \mathit{init}, \mathit{inv}, \mathit{flow}, \mathit{jump})$, where:*

- *$X = \{x_1, \ldots, x_l, \tau_M\}$ with τ_M a δ_M-timer variable;*
- *$Q = S \times D_{i\perp} \times D_{o\perp}$ with $D_{i\perp}$ input buffers and $D_{o\perp}$ output buffers;*
- *$\mathit{\Sigma} = \{\ni_M\}$ with $\ni_M$ an* internal *action;*
- *$\mathit{init} = \{((s, \boldsymbol{i}, \perp), (\boldsymbol{v}, 0)) \in Q \times \mathbb{R}^{l+1} \mid \pi_t(s) = 0, \pi_{\boldsymbol{x}}(s) = \pi_S(\boldsymbol{v})\}$;*
- *$\mathit{inv}(s, \boldsymbol{i}, \boldsymbol{o}) = \{(\boldsymbol{v}, t) \in \mathbb{R}^{l+1} \mid 0 \leq t \leq c_M(\pi_t(s) + 1) - c_M(\pi_t(s))\}$;*
- *$(\boldsymbol{f}, f_{\tau_M}) \in \mathit{flow}(s, \boldsymbol{i}, \boldsymbol{o})$ iff $\dot{f}_{\tau_M} = 1$ and $(\exists \boldsymbol{v} \in \mathbb{R}^l)$ $((\pi_C(s), \boldsymbol{v}), \boldsymbol{f}) \in \mathit{\Lambda}$; and*
- *$((s, \boldsymbol{i}, \perp), (\boldsymbol{v}', T)) \xrightarrow{\ni_M} ((s', \perp, \boldsymbol{o}), (\boldsymbol{v}', 0)) \in \mathit{jump}$ iff:*

$$((\boldsymbol{i}, s), (s', \boldsymbol{o})) \in \delta_M, \quad \pi_{\boldsymbol{x}}(s') = \pi_S(\boldsymbol{v}'), \quad T = c_M(\pi_t(s')) - c_M(\pi_t(s)).$$

The following lemma states that the behaviors of $M \upharpoonright E_M$ and $H_{M \upharpoonright E_M}$ are equivalent with respect to their periods.

Lemma 1. *Given an environment restriction $M \restriction E_M$ and its hybrid automaton $H_{M \restriction E_M}$, an $M \restriction E_M$'s transition $((\boldsymbol{i}, (s, \boldsymbol{v})), ((s', \boldsymbol{v}'), \boldsymbol{o})) \in \delta_{M \restriction E_M}$ holds iff:*

- $((s, \boldsymbol{i}, \bot), (\boldsymbol{v}, 0)) \xrightarrow{T} ((s, \boldsymbol{i}, \bot), (\boldsymbol{v}', T))$ *holds by flow of $H_{M \restriction E_M}$ during time $T = c_M(\pi_t(s) + 1) - c_M(\pi_t(s))$ for $\pi_{\boldsymbol{x}}(s) = \pi_S(\boldsymbol{v})$; and*
- $((s, \boldsymbol{i}, \bot), (\boldsymbol{v}', T)) \xrightarrow{\ni_M} ((s', \bot, \boldsymbol{o}), (\boldsymbol{v}', 0))$ *holds by jump of $H_{M \restriction E_M}$.*

Proof. If $((\boldsymbol{i}, (s, \boldsymbol{v})), ((s', \boldsymbol{v}'), \boldsymbol{o})) \in \delta_{M \restriction E_M}$, then we have: $((\boldsymbol{i}, s), (s', \boldsymbol{o})) \in \delta_M$, $((\pi_C(s), \boldsymbol{v}), \boldsymbol{f}) \in \mathit{\Lambda}$, $\boldsymbol{f}(0) = \boldsymbol{v}$, $\boldsymbol{f}(c_M(\pi_t(s')) - c_M(\pi_t(s))) = \boldsymbol{v}'$ $\pi_t(s') = \pi_t(s) + 1$, $\pi_{\boldsymbol{x}}(s) = \pi_S(\boldsymbol{v})$, and $\pi_{\boldsymbol{x}}(s') = \pi_S(\boldsymbol{v}')$. First, by *jump* of $H_{M \restriction E_M}$, for the timer value $T = c_M(\pi_t(s) + 1) - c_M(\pi_t(s)) = c_M(\pi_t(s')) - c_M(\pi_t(s))$, the transition $((s, \boldsymbol{i}, \bot), (\boldsymbol{v}', T)) \xrightarrow{\ni_M} ((s', \bot, \boldsymbol{o}), (\boldsymbol{v}', 0))$ clearly holds. Second, during time T, by *flow* and *inv* of $H_{M \restriction E_M}$, the value of $\boldsymbol{x}$ can follow the trajectory $\boldsymbol{f}$ from $\boldsymbol{v}$ and the value of the timer τ_M follows from 0 the differential equation $\dot{f}_{\tau_M} = 1$. Therefore, $((s, \boldsymbol{i}, \bot), (\boldsymbol{v}, 0)) \xrightarrow{T} ((s, \boldsymbol{i}, \bot), (\boldsymbol{v}', T))$ holds. The other direction is also similarly straightforward by the definitions of $M \restriction E_M$ and $H_{M \restriction E_M}$. □

Now consider an ensemble $\mathcal{E} = (J_S \cup J_F, e, \{M_j\}_{j \in J_S \cup J_F}, E, \mathit{src}, \mathit{rate}, \mathit{adap})$, where each M_j is given as an environment restriction. For each M_j, we have the corresponding hybrid automaton H_{M_j}. Each automaton H_{M_j} is extended into its communication hybrid automaton $\bar{H}_{M_j}$ with extra output buffers and communication jump conditions. If M_j is a slow machine with $\mathit{rate}(j) = 1$, then the output buffers are just $D_{o\bot}$. If M_j is a fast machine with $\mathit{rate}(j) > 1$, then M_j should perform $\mathit{rate}(j)$ internal transitions in a single step. Therefore, the extended automaton $\bar{H}_{M_j}$ should have $\mathit{rate}(j)$ output buffers for each output port: that is, $D_{o\bot}^{\mathit{rate}(j)} = (D_{o_1} \cup \{\bot\})^{\mathit{rate}(j)} \times \cdots \times (D_{o_m} \cup \{\bot\})^{\mathit{rate}(j)}$. For output buffers $(\boldsymbol{o}_1, \ldots, \boldsymbol{o}_m) \in D_{o\bot}^{k}$ with k items, the append operation is given by $(\boldsymbol{o}_1, \ldots, \boldsymbol{o}_m) \,;\, (d_1, \ldots, d_m) = ((\boldsymbol{o_1}, d_1), \ldots, (\boldsymbol{o}_m, d_m)) \in D_{o\bot}^{k+1}$.

When the h-th input port (j, h) of M_j is connected to the l-th output port (k, l) of M_k in $\mathcal{E}$ (i.e., $\mathit{src}(j, h) = (k, l)$), their synchronous communication is expressed in hybrid automata using discrete jumps with action $\mathit{out}_{(k,l)}(d)$. There are two kinds of communication jumps: the "output" automaton $\bar{H}_{M_k}$ performs a discrete jump with action $\mathit{out}_{(k,l)}(d)$ when the value in its l-th output buffer is d, while all the "input" automata for each (j, h) such that $\mathit{src}(j, h) = (k, l)$ (recall that an output port can be connected to many input ports) perform jumps with action $\mathit{out}_{(k,l)}(d)$ that update the values in their input buffers by the new value d, together with their input adaptors $\mathit{adap}(j)$.

Such communication jumps must happen after every machine jump is taken, since machine jumps of H_{M_j} with period T_j happen at the *end* of its period, at each time $c_{M_j}(i)$ with $c_{M_j} = 0$ for $i = 0$ and $iT_j - \epsilon < c_{M_j}(i) < iT_j + \epsilon$ for $i > 0$, where $\epsilon \geq 0$ denotes a maximal clock skew, Therefore, each $\bar{H}_{M_j}$ has an extra timer variable τ_{src}, initially set to $T_C = T_{\mathcal{E}} + \epsilon$ with the hyper-period $T_{\mathcal{E}}$ (which is always equal to $T_j \cdot \mathit{rate}(j)$ for any $M_j \in \mathcal{E}$). The communication jumps are taken when $\tau_{\mathit{src}} = T_C$. Then, for the first synchronous step, τ_{src} is reset to 0 (so that next communication jumps happen after $T_{\mathcal{E}} + \epsilon$ elapsed), and for any other step, τ_{src} is reset to ϵ (so that next jumps happen after $T_{\mathcal{E}}$ elapsed).

Definition 30. *Consider a controller M_j with period T_j, its rate $rate(j)$, an input adaptor $adap(j) = \{\alpha_i\}_{l \in \{1,\dots,n\}}$, a wiring diagram src, and a maximal clock skew $\epsilon \geq 0$. Let $T_C = T_j \cdot rate(j) + \epsilon$. Given the machine hybrid automata $H_{M_j} = (X, Q, \Sigma, init, inv, flow, jump)$, the communication hybrid automata is defined by $\bar{H}_{M_j} = (\bar{X}, \bar{Q}, \bar{\Sigma}, \overline{init}, \overline{inv}, \overline{flow}, \overline{jump})$, where:*

- $\bar{X} = X \cup \{\tau_{src}\}$ *with τ_{src} a communication timer variable;*
- $\bar{Q} = S \times D_{i\perp} \times D_{o\perp}^{rate(j)}$, *if $Q = S \times D_{i\perp} \times D_{o\perp}$;*
- $\bar{\Sigma}$ *contains an out action for each connection in src involving M, that is:*

$$\begin{aligned}\bar{\Sigma} = \Sigma &\cup \{out_{(k,l)}(d) \mid d \in D_{i_h}, src(j,h) = (k,l)\} \\ &\cup \{out_{(j,h)}(d) \mid d \in D_{o_h}, src(k,l) = (j,h)\};\end{aligned}$$

- $\overline{init} = \{((s, \perp, \boldsymbol{o}), ((\boldsymbol{v}, 0), T_C)) \mid \pi_t(s) = 0, \pi_{\boldsymbol{x}}(s) = \pi_S(\boldsymbol{v})\}$ *(unlike init, input buffers are empty and output buffers are full);*
- $\overline{inv}(q) = \{(\boldsymbol{v}, t) \mid \boldsymbol{v} \in inv, 0 \leq t \leq T_C\}$ *(i.e., $0 \leq \tau_{src} \leq T_C$);*
- $(\boldsymbol{f}, f_{\tau_{src}}) \in \overline{flow}(q)$ *iff $\boldsymbol{f} \in flow(q)$ and $\dot{f}_{\tau_{src}} = 1$; and*
- $\overline{jump}$ *extends jump by adding extra communication jumps that, when τ_{src} is T_C: (i) receives inputs if input buffers are not full, (ii) sends outputs if output buffers are not empty, and (iii) resets the timer τ_{src} if input buffers are full and output buffers are empty. When $\tau_{src} < T_C$, $\overline{jump}$ performs the original jump and appends the output to the output buffer. That is:*
  - $(q, (\boldsymbol{v}, T_C)) \xrightarrow{out_{(k,l)}(d)} (rec_h(d,q), (\boldsymbol{v}, T_C)) \in \overline{jump}$, *if $src(j,h) = (k,l)$ and the h-th input buffer is empty, where $q = (s, (d_1, \dots, \perp, \dots, d_n), \boldsymbol{o})$ and $rec_h(d,q) = (s, (d_1, \dots, \alpha_h(d), \dots, d_n), \boldsymbol{o})$;*
  - $(q, (\boldsymbol{v}, T_C)) \xrightarrow{out_{(j,h)}(d)} (snd_h(d,q), (\boldsymbol{v}, T_C)) \in \overline{jump}$, *if $src(k,l) = (j,h)$ and the h-th output buffer is d, where $q = (s, \boldsymbol{i}, (d_1, \dots, d, \dots, d_m)))$ and $snd_h(d,q) = (s, \boldsymbol{i}, (d_1, \dots, \perp, \dots, d_m))$;*
  - $(q, (\boldsymbol{v}, T_C)) \xrightarrow{\ni M} (q, (\boldsymbol{v}, T_0)) \in \overline{jump}$, *if all the input buffers are full and all the output buffers are empty (i.e., $q = (s, \boldsymbol{i}, \perp)$ for $\boldsymbol{i} \in D_i$), where $T_0 = 0$ for the first synchronous step, and $T_0 = \epsilon$ for the other steps.*
  - $((s, \boldsymbol{i}, \boldsymbol{o}), (\boldsymbol{v}, t)) \xrightarrow{\ni M} ((s', \perp, \boldsymbol{o}; \boldsymbol{d}), (\boldsymbol{v}', t)) \in \overline{jump}$, *if $t < T_C$ holds and $((s, \boldsymbol{i}, \perp), \boldsymbol{v}) \xrightarrow{\ni M} ((s', \perp, \boldsymbol{d}), \boldsymbol{v}') \in jump$ (i.e., performing an original jump when $\tau_{src} < T_C$, and adding the output $\boldsymbol{d}$ to the output buffer $\boldsymbol{o}$).*

The composition $||_{j \in J_S \cup J_F} \bar{H}_{M_j}$ becomes the desired hybrid automaton $H_{\mathcal{E}}$ for $\mathcal{E}$. Notice that an "output" action $out_{(j,h)}(d)$ of $\bar{H}_{M_j}$ can be taken only if there exists the same output action by other automata receiving d.[20] The time-invariant constraints for $\mathcal{E}$ can be easily encoded in each $\bar{H}_{M_j}$ by using shared variables and the invariant condition *inv* of hybrid automata (e.g., see Section A.3).

[20] This also implies that a self-loop connection for a machine M cannot be expressed by the above transformation using actions, because joint synchronous jumps for a single automaton is not allowed in hybrid automata. However, it is always possible to remove such self-loop connections using state variables.

Recall that each state of $\mathcal{E}$'s synchronous composition consists of the states $\{s_j, \boldsymbol{v}_j\}_{j\in J_S\cup J_F}$ of the subcomponents and the feedback outputs $\{\boldsymbol{d}_j\}_{j\in J_S\cup J_F}$ of the subcomponents to be transferred in the next step. For an ensemble state $(\{s_j, \boldsymbol{v}_j\}_{j\in J_S\cup J_F}, \{\boldsymbol{d}_j\}_{j\in J_S\cup J_F})$ of $\mathcal{E}$, its corresponding hybrid automaton state has the form $||_{j\in J_S\cup J_F}((s_j, \boldsymbol{i}_j, \boldsymbol{o}_j), ((\boldsymbol{v}_j, t_j), T_C))$ for $T_C = T_\mathcal{E} + \epsilon$ with $T_\mathcal{E}$ the hyper-period, where each machine timer t_j (of τ_{M_j}) must be *consistent* with the communication timer t_{src} (of τ_{src}). The communication timer τ_{src} is initially set to T_C, and then immediately set to 0 after applying communication jumps. Whenever $\tau_{src} = T_C$ again, communication jumps are taken and τ_{src} is set to ϵ. That is, when $\tau_{src} = T_C$, the global time is either 0 or $nT_\mathcal{E} + \epsilon$ for a certain $n \in \mathbb{N}$. If the global time is 0, then $t_j = 0$. If the global time is $nT_\mathcal{E} + \epsilon$, which is in the $(n \cdot rate(j))$-th round for M_j, then $t_j = nT_\mathcal{E} + \epsilon - c_{M_j}(n \cdot rate(j))$.

Definition 31. *Given a hybrid state* $||_{j\in J_S\cup J_F}((s_j, \boldsymbol{i}_j, \boldsymbol{o}_j), ((\boldsymbol{v}_j, t_j), T_C))$ *with* $T_C = T_\mathcal{E} + \epsilon$*, a machine timer* t_j *is* consistent *with* T_C *iff for the global time* T_g*:*

$$t_j = \begin{cases} 0 & \text{if } \ T_g = 0 \\ nT_\mathcal{E} + \epsilon - c_{M_j}(n \cdot rate(j)) & \text{if } \ (\exists n \in \mathbb{N}) \ \ T_g = nT_\mathcal{E} + \epsilon. \end{cases}$$

An ensemble $\mathcal{E}$ may also have external interface ports. Let us consider a *closed* ensemble with no interface ports The following lemmas show that the multirate synchronous composition of a closed ensemble $\mathcal{E}$ is equivalent to the composed hybrid automaton $||_{j\in J_S\cup J_F} \bar{H}_{M_j}$ (the case of an *open* ensemble $\mathcal{E}$ with interface ports is similar, plus more complexity due to interface connections).

Lemma 2. *Given a closed ensemble* $\mathcal{E}$ *and its corresponding hybrid automaton* $H_\mathcal{E} = ||_{j\in J} \bar{H}_{M_j}$ *for* $J = J_S \cup J_F$*, assuming* $\epsilon < T_j$ *for any period* T_j *in* $\mathcal{E}$*, a transition* $((*, (\{s_j, \boldsymbol{v}_j\}_{j\in J}, \{\boldsymbol{d}_j\}_{j\in J})), ((\{s'_j, \boldsymbol{v}'_j\}_{j\in J}, \{\boldsymbol{d}'_j\}_{j\in J}), *)) \in \delta_\mathcal{E}$ *holds iff:*

- $||_{j\in J}((s_j, \bot, \boldsymbol{d}_j), ((\boldsymbol{v}_j, t_j), T_C)) \longrightarrow^* ||_{j\in J}((s_j, \boldsymbol{i}_j, \bot), ((\boldsymbol{v}_j, t_j), T_0))$ *by only communication jumps, where* $T_0 = 0$ *for the first synchronous step,* $T_0 = \epsilon$ *for the other steps, and each timer* t_j *is consistent with* $T_\mathcal{E}$*, and*
- $||_{j\in J}((s_j, \boldsymbol{i}_j, \bot), ((\boldsymbol{v}_j, t_j), T_0)) \longrightarrow^* ||_{j\in J}((s'_j, \bot, \boldsymbol{d}'_j), ((\boldsymbol{v}'_j, t'_j), T_C))$ *by using only machine jumps and flows with consistent timers* t_j*.*

Proof. Because $\mathcal{E}$ is closed, every output port in $\mathcal{E}$ is a feedback output port, and each ensemble state $(\{s_j, \boldsymbol{v}_j\}_{j\in J}, \{\boldsymbol{d}_j\}_{j\in J})$ corresponds to a hybrid automaton state $||_{j\in J}((s_j, \bot, \boldsymbol{d}_j), ((\boldsymbol{v}_j, t_j), T_C))$. Because $\tau_{src} = T_C$, only communication jumps can be applied until all the input buffers are full and all the output buffers are empty. Then, τ_{src} is reset to T_0, either 0 (for the first step) or ϵ (for the other steps). In the resulting state $||_{j\in J}((s_j, \boldsymbol{i}_j, \bot), ((\boldsymbol{v}_j, t_j), T_0))$, because of the timer consistency, all the machine transitions in the previous synchronous step must be taken, and any machine transition in the next step has *not* been taken yet, provided that $\epsilon < T_j$ for any period T_j. Now because $\tau_{src} < T_C$, only flows and machine jumps can be applied until $\tau_{src} = T_C$ and all the input buffers are empty again, resulting in the state $||_{j\in J}((s'_j, \bot, \boldsymbol{d}'_j), ((\boldsymbol{v}'_j, t'_j), T_C))$. The other direction is straightforward by construction. □

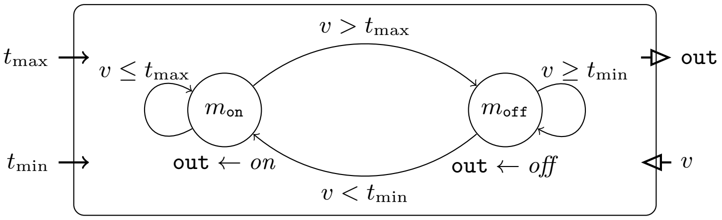


Fig. 14. A digital thermostat controller [4].

A.3 Example

Consider the single thermostat controller M of Figure 15 for the open room environment E_M.

Figure 15 shows a thermostat controller with a local clock, adapted from [3,4], operating at a certain rate to control a temperature, specified by the typed machine $M = (\mathbb{R}^2, S, \{*\}, \delta_M)$, where

- $\mathbb{R}^2$ is the input set to denote M's two inputs $(t_{\max}, t_{\min}) \in \mathbb{R}^2$, with $t_{\max}$ the desired maximum temperature and $t_{\min}$ for desired minimum temperature;
- $S = \{m_{\texttt{on}}, m_{\texttt{off}}\} \times \mathbb{N} \times \mathbb{R}$ is the state set to denote a triple (m, n, v) of:
  - $m \in \{m_{\texttt{on}}, m_{\texttt{off}}\}$ a heater's mode,
  - $n \in \mathbb{N}$ a counter for time $c_M(n)$ at the beginning of the period, and
  - $v \in \mathbb{R}$ an observed temperature at time $c_M(n)$;
- $\{*\}$ is the singleton output set to indicate that M has no output port; and
- $\delta_M \subseteq (\mathbb{R}^2 \times S) \times (S \times \{*\})$ the transition relation to define the next state (m', n', v') from input $(t_{\max}, t_{\min})$ and a state (m, n, v), where:

$$\big(((t_{\max}, t_{\min}), (m_{\texttt{on}}, n, v)), ((\textbf{if } v \leq t_{\max} \textbf{ then } m_{\texttt{on}} \textbf{ else } m_{\texttt{off}} \textbf{ fi}, n+1, v'), *)\big) \in \delta_M$$
$$\big(((t_{\max}, t_{\min}), (m_{\texttt{off}}, n, v)), ((\textbf{if } v \geq t_{\min} \textbf{ then } m_{\texttt{off}} \textbf{ else } m_{\texttt{on}} \textbf{ fi}, n+1, v'), *)\big) \in \delta_M.$$

For each synchronous step, the controller M receives two inputs $(t_{\max}, t_{\min}) \in \mathbb{R}^2$ from other controllers. The *next* observed temperature $v' \in \mathbb{R}$ at time $c_M(n+1)$ (i.e., the beginning of the next period) can be any value, but is determined by its physical environment E_M below. During each period (from $c_M(n)$ to $c_M(n+1)$), the controller M gives a control command `out` through its actuator to turn the heater *on*/*off* according to the mode $m \in \{m_{\texttt{on}}, m_{\texttt{off}}\}$.

For the nondeterministic controller M, its local physical environment E_M models the open room environment shown in Figure 16, where the temperature x of the room changes based on the mode of M and the outside temperature x_o. This environment is specified by $E_M = (\{\mathit{on}, \mathit{off}\}, (x, x_o), T + 2\epsilon, \Lambda)$ with

- $\{\mathit{on}, \mathit{off}\}$ a set of control commands from the controller M;
- (x, x_o) the physical parameters of M for the room's temperature x and the outside temperature x_o, denoting a physical state $(v, v_o) \in \mathbb{R}^2$;
- $T + 2\epsilon \in \mathbb{R}_{>0}$ the period of E_M; and

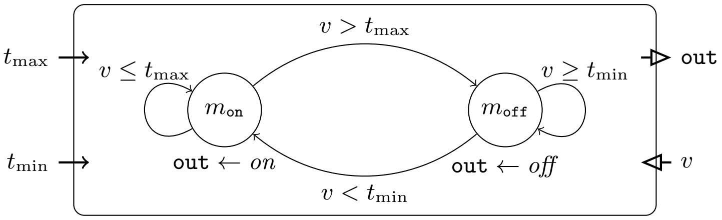


Fig. 15. A digital thermostat controller [4].

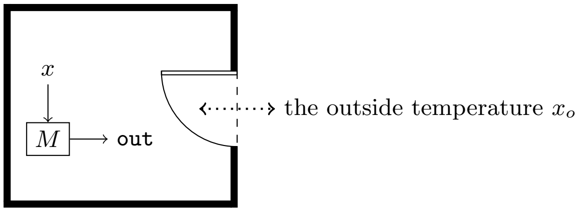


Fig. 16. Open room environment.

- $\mathit{\Lambda} \subseteq (\{on, off\} \times \mathbb{R}^2) \times \mathcal{T}^2_{T+2\epsilon}$ the physical transition relation such that a transition $\big((\texttt{out}, (v, v_o)), (\tau, \tau_o)\big) \in \mathit{\Lambda}$ holds iff there exist two continuous trajectories $\tau, \tau_o : [0, T + 2\epsilon] \to \mathbb{R}$, where $\tau(0) = v$, $\tau_o(0) = v_o$, and:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{cases} K(h - ((1-k)x + kx_o)) & \text{if } \texttt{out} = on \\ -K((1-k)x + kx_o) & \text{if } \texttt{out} = off. \end{cases}$$

The values of the constants $K, h, k \in \mathbb{R}$ depend on the size of the room, the power of the heater, and the size of the door, respectively. That is, given a *nondeterministically chosen* trajectory τ_o of the outside temperature x_o, if the heater is on, then the temperature x rises according to the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = K(h - ((1-k)x + kx_o))$, and if the heater is off, then the temperature x falls according to the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = -K((1-k)x + kx_o)$.

The environment restriction is $M \upharpoonright E_M = (\mathbb{R}^2, S \times \mathbb{R}^2, \{*\}, \delta_{M \upharpoonright E_M})$, with M's state space $S = \{m_{\texttt{on}}, m_{\texttt{off}}\} \times \mathbb{N} \times \mathbb{R}$ and E_M's physical state space $\mathbb{R}^2$. The combined transition $(((t_M, t_m), (m, n, v, v, v_o)), ((m', n', v', v', v'_o), *)) \in \delta_{M \upharpoonright E_M}$ of the machine $M \upharpoonright E_M$ holds iff:

- $(((t_M, t_m), (m, n, v)), ((m', n', v'), *)) \in \delta_M$ (i.e., the M's transition holds) with $n' = n + 1$, where M's periodic local time is given by

$$\pi_t(m, n, v) = n,$$

- the observed/observable physical parameters of E_M are given by:

$$\pi_{\boldsymbol{x}}(m, n, v) = v, \qquad \pi_S(v, v_o) = v,$$

- $((\pi_C(m,n,v),(v,v_o)),(\tau,\tau_o)) \in \Lambda$ (i.e., the E_M's transition holds) for some trajectories $(\tau,\tau_o) \in \mathcal{T}^2_{T+2\epsilon}$ such that $(\tau,\tau_o)(c_M(n+1)-c_M(n)) = (v',v'_o)$ and $(\tau,\tau_o)(0) = (v,v_o)$, where the control commands to E_M are given by:

$$\pi_C(m_{\mathtt{on}},n,v) = \mathit{on}, \qquad \pi_C(m_{\mathtt{off}},n,v) = \mathit{off}.$$

For example, given a *constant* outside temperature (i.e., $\frac{\mathrm{d}x_o}{\mathrm{d}t}=0$), the combined behavior of the environment restriction $M \upharpoonright E_M$ is shown in Figure 17 in which from a physical state $\boldsymbol{v}_i$ at time $c_M(i)$, the next *observed* physical state $\boldsymbol{v}_{i+1}$ (at time $c_M(i+1)$) is given according to the differential equations:

$$\frac{\mathrm{d}x_o}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \begin{cases} K(h-((1-k)x+kx_o)) & \text{if } \mathtt{out} = \mathit{on} \\ -K((1-k)x+kx_o) & \text{if } \mathtt{out} = \mathit{off}. \end{cases}$$

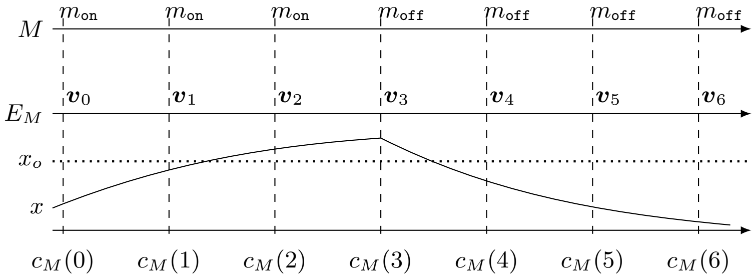


Fig. 17. The behavior of $M \upharpoonright E_M$, when $\frac{\mathrm{d}x_o}{\mathrm{d}t} = 0$.

We consider digital thermostat controllers M_1 and M_2 that are respectively installed in two *adjacent* rooms connected by an open door, as illustrated in Figure 18. The temperature of each room is separately controlled by the normal thermostat controller in Figure 15 (for $i = 1,2$):

$$M_i = (\mathbb{R}^2, \{m_{\mathtt{on}}, m_{\mathtt{off}}\} \times \mathbb{N} \times \mathbb{R}, \{*\}, \delta_{M_i})$$

which turns on/off the switch of the heater in the room. The local physical environment E_{M_i} of each controller M_i can be just considered as the open room environment in Figure 16 explained above (for $i = 1,2$):

$$E_{M_i} = (\{\mathit{on}, \mathit{off}\}, (x_i, x_{o_i}), T_i + 2\epsilon, \Lambda_i).$$

Each physical transition relation Λ_i states that the room's temperature x_i and the outside temperature x_{o_i} are governed by the differential equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \begin{cases} K_i(h_i-((1-k)x_i+kx_{o_i})) & \text{if } \mathtt{out}_i = \mathit{on} \\ -K_i((1-k)x_i+kx_{o_i}) & \text{if } \mathtt{out}_i = \mathit{off}. \end{cases}$$

where $K_i, h_i \in \mathbb{R}$ are constants given by the size of each room and the heater's power, respectively, and $k \in \mathbb{R}$ is a constant based on the size of the open door. Recall that the behavior of each E_{M_i} depends on *nondeterministic chosen* trajectories of the outside temperature x_{o_i}.

Because those two rooms 1 and 2 are physically connected to each other by the open door, the outside temperature x_{o_1} of the room 1 must be equal to the temperature x_2 of the room 2, and similarly, the outside temperature x_{o_2} of the room 2 must be equal to the temperature x_1 of the room 1. This requirement can be precisely specified by the time-invariant constraint between E_{M_1} and E_{M_2}:

$$(\forall t)\ x_{o_1}(t) = x_2(t)\ \wedge\ x_{o_2}(t) = x_1(t).$$

The composed behavior of the nondeterministic physical environments E_{M_1} and E_{M_2} is now *deterministic* by this constraint, since the outside temperature x_{o_i} of each E_{M_i} is determined by the temperature of the other room. That is, in the combined system, x_1 and x_2 follows the differential equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \begin{cases} K_1(h_1 - ((1-k)x_1 + kx_2)) & \text{if } \mathtt{out}_1 = \mathit{on} \\ -K_1((1-k)x_1 + kx_2) & \text{if } \mathtt{out}_1 = \mathit{off}, \end{cases}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \begin{cases} K_2(h_2 - ((1-k)x_2 + kx_1)) & \text{if } \mathtt{out}_2 = \mathit{on} \\ -K_2((1-k)x_2 + kx_1) & \text{if } \mathtt{out}_2 = \mathit{off}. \end{cases}$$

Figure 19 shows a multirate ensemble $\mathcal{E}$ for controlling the temperatures of the two adjacent rooms. The ensemble $\mathcal{E}$ consists of three discrete components. The main controller `Main` sets a maximum temperature $t_{\max}$ and a minimum temperature $t_{\min}$ for both of the rooms, and each thermostat controller M_i ($i = 1, 2$) then controls the room's heather according to given $t_{\max}$ and $t_{\min}$. The thermostat controllers M_1 and M_2 have different rates

$$\mathit{rate}(1) = 2 \quad \text{and} \quad \mathit{rate}(2) = 3.$$

That is, $2T_1 = 3T_2$ holds for the periods T_1 and T_2 of M_1 and M_2, respectively. Each thermostat controller in $\mathcal{E}$ is the environment restriction $M_i \upharpoonright E_{M_i}$, which also satisfies the also time-invariant constraint. Notice that the behavior of each $M_i \upharpoonright E_{M_i}$ is also parameterized by the exact local clock c_{M_i} of each controller M_i, since M_i "observes" the current temperature based on c_{M_i}.

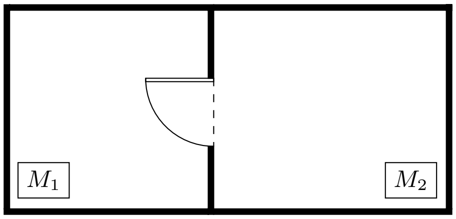
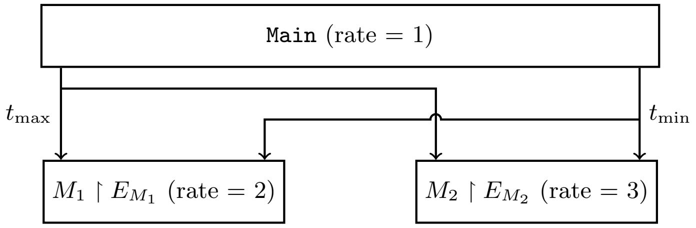


Fig. 18. Two rooms connected by an open door.


Main (rate = 1)
$t_{\max}$
$t_{\min}$
$M_1 \upharpoonright E_{M_1}$ (rate = 2)
$M_2 \upharpoonright E_{M_2}$ (rate = 3)

$$(\forall t)\ x_{o_1}(t) = x_2(t)\ \wedge\ x_{o_2}(t) = x_1(t).$$

Fig. 19. A multirate ensemble $\mathcal{E}$.

This system can be expressed as the hybrid automaton $H_{M\upharpoonright E_M} = (X, Q, \{\ni_M$ $\}, \mathit{init}, \mathit{inv}, \mathit{flow}, \mathit{jump})$, where[21]

- $X = \{x, x_o, \tau_M\}$ for the room temperature x, the outside temperature x_o, and the timer τ_M;
- $Q = \{m_{\texttt{on}}, m_{\texttt{off}}\} \times \mathbb{N} \times \mathbb{R}^2_\perp$, denoting a tuple $(m, n, t_{\max}, t_{\min})$ of mode m, local clock n, and input buffers $t_{\max}$ and $t_{\min}$;
- $\mathit{init} = \{((m, 0, t_M, t_m), (v, v_o, 0)) \mid t_M, t_m, v, v_o \in \mathbb{R}, m = \{m_{\texttt{on}}, m_{\texttt{off}}\}\}$;
- $\mathit{inv}(m, n, t_M, t_m) = \{(v, v_o, t) \in \mathbb{R}^3 \mid 0 \le t \le c_M(n+1) - c_M(n)\}$;
- $(f, f_o, f_{\tau_M}) \in \mathit{flow}(m, n, t_M, t_m)$ iff

$$\frac{\mathrm{d}f_{\tau_M}}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}f}{\mathrm{d}t} = \begin{cases} K(h - ((1-k)f + kf_o)) & \text{if } m = m_{\texttt{on}} \\ -K((1-k)f + kf_o) & \text{if } m = m_{\texttt{off}}, \end{cases}$$

  where the constants $K, h, k \in \mathbb{R}$ depend on the size of the room, the power of the heater, and the size of the door, respectively;
- $((m, n, t_M, t_m), (v, v_0, T)) \xrightarrow{\ni_M} ((m', n+1, \perp, \perp), (v, v_0, 0)) \in \mathit{jump}$ holds iff $T = c_M(n{+}1) - c_M(n)$, $m' = \textbf{if } v \le t_{\max} \textbf{ then } m_{\texttt{on}} \textbf{ else } m_{\texttt{off}} \textbf{ fi}$ for $m = m_{\texttt{on}}$, and $m' = \textbf{if } v \ge t_{\min} \textbf{ then } m_{\texttt{off}} \textbf{ else } m_{\texttt{on}} \textbf{ fi}$ for $m = m_{\texttt{off}}$.

Now consider the multirate ensemble $\mathcal{E}$ in Figure 18 that consists of one main controller `Main` and two thermostat controllers M_1 and M_2 for two adjacent rooms, where $\mathit{rate}(\texttt{Main}) = 1$, $\mathit{rate}(1) = 2$, and $\mathit{rate}(2) = 3$. For T_i a period of M_i, $i = 1, 2$, let $T_C = 2 \cdot T_1 + \epsilon = 2 \cdot T_3 + \epsilon$. The communication automaton for M_1 is $H_{M_1\upharpoonright E_{M_1}} = (\bar{X}_1, \bar{Q}_1, \bar{\Sigma}_1, \overline{\mathit{init}}_1, \overline{\mathit{inv}}_1, \overline{\mathit{flow}}_1, \overline{\mathit{jump}}_1)$, where:

- $\bar{X}_1 = \{x_1, x_{o_1}, \tau_{M_1}, \tau_{src}\}$, and $\bar{Q}_1 = \{m_{\texttt{on}}, m_{\texttt{off}}\} \times \mathbb{N} \times \mathbb{R}^2_\perp$ (no output port);
- $\bar{\Sigma}_1 = \{\ni_{M_1}\} \cup \{\mathit{out}_{(\texttt{Main}, t_{\max})}(d) \mid d \in \mathbb{R}\} \cup \{\mathit{out}_{(\texttt{Main}, t_{\min})}(d) \mid d \in \mathbb{R}\}$;
- $\overline{\mathit{init}}_1 = \{(q, (v_1, v_{o_1}, 0, T_C)) \mid (q, (v_1, v_{o_1}, 0)) \in \mathit{init}_1\}$;
- $\overline{\mathit{inv}}_1(q) = \{(v_1, v_{o_1}, t, u) \mid (v_1, v_{o_1}, t) \in \mathit{inv}_1(q), 0 \le u \le T_C\}$;

[21] According to the original definition, $Q = \{m_{\texttt{on}}, m_{\texttt{off}}\} \times \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3_\perp$ with extra observed parameters, which is omitted here for simplicity reasons, since observed parameters are always equal to observable parameter in $\mathit{val}(X)$.

- $(f_1, f_{o_1}, f_{\tau_M}, f_{\tau_{src}}) \in \overline{\mathit{flow}}(q)$ iff $(f_1, f_{o_1}, f_{\tau_M}) \in \mathit{flow}(q)$ and $\dot{f}_{\tau_{src}} = 1$; and
- When $\tau_{src} = T_C$, the communication jumps:

  $$\begin{aligned}((m, n, \bot, t_m), (\boldsymbol{v}, T_C)) &\xrightarrow{out_{(\mathtt{Main}, t_{\max})}(d_1, d_2)} ((m, n, d_2, t_m), (\boldsymbol{v}, T_C)) \in \overline{jump}_1,\\ ((m, n, t_M, \bot), (\boldsymbol{v}, T_C)) &\xrightarrow{out_{(\mathtt{Main}, t_{\min})}(d_1, d_2)} ((m, n, t_M, d_2), (\boldsymbol{v}, T_C)) \in \overline{jump}_1,\\ ((m, n, t_M, t_m), (\boldsymbol{v}, T_C)) &\xrightarrow{\ni M} ((m, n, t_M, t_m), (\boldsymbol{v}, T_0)) \in \overline{jump}_1 \ \ (\text{if } t_M, t_m \in \mathbb{R}),\end{aligned}$$

  happen, where $T_0 = 0$ for the first step, and $T_0 = \epsilon$ for the other steps, and when $\tau_{src} < T_C$, the machine jumps happen (where $u < T_C$):

  $$\begin{aligned}&((m, n, t_M, t_m), (\boldsymbol{v}, u)) \xrightarrow{\ni M} ((m', n', \bot, \bot), (\boldsymbol{v}', u)) \in \overline{jump}_1\\ \iff\ &((m, n, t_M, t_m), \boldsymbol{v}) \xrightarrow{\ni M} ((m', n', \bot, \bot), \boldsymbol{v}') \in jump_1.\end{aligned}$$

The communication automaton $H_{M_2 \restriction E_{M_2}}$ for M_2 is similar. The communication automaton for `Main` is $\bar{H}_{\mathtt{Main}} = (\bar{X}, \bar{Q}, \bar{\Sigma}, \overline{init}, \overline{inv}, \overline{\mathit{flow}}, \overline{jump})$—which contains only timer variables, since the main controller `Main` is a discrete controller with no physical environments—where:

- $\bar{X} = \{\tau_{\mathtt{Main}}, \tau_{src}\}$;
- $Q = \mathbb{N} \times \mathbb{R}^2_\bot$ for a tuple $(n, t_{\max}, t_{\min})$ of local clock n and output buffers;
- $\bar{\Sigma} = \{\ni_{\mathtt{Main}}\} \cup \{out_{(\mathtt{Main}, t_{\max})}(d) \mid d \in \mathbb{R}\} \cup \{out_{(\mathtt{Main}, t_{\min})}(d) \mid d \in \mathbb{R}\}$;
- $\overline{init} = \{((0, t_M, t_m), (0, T_C)) \mid t_M, t_m \in \mathbb{R}\}$;
- $\overline{inv}(n, t_M, t_m) = \{(t, u) \in \mathbb{R}^2 \mid 0 \leq t \leq c_{\mathtt{Main}}(n + 1) - c_{\mathtt{Main}}(n), 0 \leq u \leq T_C\}$;
- $(f_{\tau_{\mathtt{Main}}}, f_{\tau_{src}}) \in \overline{\mathit{flow}}(n, t_M, t_m)$ iff $\dot{f}_{\tau_{\mathtt{Main}}} = 1$ and $\dot{f}_{\tau_{src}} = 1$;
- When $\tau_{src} = T_C$, the communication jumps:

  $$\begin{aligned}((n, t_M, t_m), (\boldsymbol{v}, T_C)) &\xrightarrow{out_{(\mathtt{Main}, t_{\max})}(t_M)} ((n, \bot, t_m), (\boldsymbol{v}, T_C)) \in \overline{jump} \ \ (\text{if } t_M \in \mathbb{R}),\\ ((n, t_M, t_m), (\boldsymbol{v}, T_C)) &\xrightarrow{out_{(\mathtt{Main}, t_{\min})}(t_m)} ((n, t_M, \bot), (\boldsymbol{v}, T_C)) \in \overline{jump} \ \ (\text{if } t_m \in \mathbb{R}),\\ ((n, \bot, \bot), (\boldsymbol{v}, T_C)) &\xrightarrow{\ni M} ((n, \bot, \bot), (\boldsymbol{v}, T_0)) \in \overline{jump}_1,\end{aligned}$$

  happen, where $T_0 = 0$ for the first step, and $T_0 = \epsilon$ for the other steps, and when $\tau_{src} < T_C$, the machine jumps (that generates any pair $(t_M, t_m) \in \mathbb{R}$ for the output) happen, where $u < T_C$:

  $$\begin{aligned}&((n, \bot, \bot), (T, u)) \xrightarrow{\ni M} ((n + 1, t_M, t_m), (0, u)) \in jump\\ \iff\ &t_M, t_m \in \mathbb{R} \ \wedge\ T = c_{\mathtt{Main}}(n + 1) - c_{\mathtt{Main}}(n)\end{aligned}$$

The time-invariant constraint $(\forall t)\ x_{o_1}(t) = x_2(t)\ \wedge\ x_{o_2}(t) = x_1(t)$ for $\mathcal{E}$ can be encoded in $\bar{H}_{M_1 \restriction E_{M_1}}$ and $\bar{H}_{M_2 \restriction E_{M_2}}$ by renaming the variable x_{o_1} to x_2, and renaming the variable x_{o_2} to x_1.[22] Then, the ensemble hybrid automaton $H_{\mathcal{E}}$ is given by the composition $\bar{H}_{M_1 \restriction E_{M_1}} \parallel \bar{H}_{M_2 \restriction E_{M_2}} \parallel \bar{H}_{\mathtt{Main}}$.

[22] In general, we can add all the variables in time-invariant constraints into the variable set of each hybrid automata, and make the constraints as their invariants.

To illustrate, when $T_1 = 3, T_2 = 2$, and $T_{\mathcal{E}} = T_{\mathtt{Main}} = 6$, given a maximal clock skew $0 \leq \epsilon < 2$, an initial state of $H_{\mathcal{E}}$ can be $((m_{\mathtt{on}}, 0, \bot, \bot), (v_1, v_2, 0, 6+\epsilon)) \parallel ((m_{\mathtt{on}}, 0, \bot, \bot), (v_2, v_1, 0, 6+\epsilon)) \parallel ((0, t_M, t_m), (0, 6+\epsilon))$. Since $\tau_{src} = T_{\mathcal{E}} + \epsilon$ for every automaton, communication jumps can happen until all the input buffers are full and all the output fullers are empty. For example, there exist:

$$((m_{\mathtt{on}}, 0, \bot, \bot), (v_1, v_2, 0, 6+\epsilon)) \xrightarrow{out_{(\mathtt{Main}, t_{\max})}(t_M)} ((m_{\mathtt{on}}, 0, t_M, \bot), (v_1, v_2, 0, 6+\epsilon)) \in \overline{jump}_1$$

$$((m_{\mathtt{on}}, 0, \bot, \bot), (v_2, v_1, 0, 6+\epsilon)) \xrightarrow{out_{(\mathtt{Main}, t_{\max})}(t_M)} ((m_{\mathtt{on}}, 0, t_M, \bot), (v_1, v_2, 0, 6+\epsilon)) \in \overline{jump}_2$$

$$((0, t_M, t_m), (0, 6+\epsilon)) \xrightarrow{out_{(\mathtt{Main}, t_{\max})}(t_M)} ((0, \bot, t_m), (0, 6+\epsilon)) \in \overline{jump},$$

and therefore we have the next composed state $((m_{\mathtt{on}}, 0, t_M, \bot), (v_1, v_2, 0, 6+\epsilon)) \parallel ((m_{\mathtt{on}}, 0, t_M, \bot), (v_2, v_1, 0, 6+\epsilon)) \parallel ((0, \bot, t_m), (0, 6+\epsilon))$.