Solutions for *Category Theory in Context*

A collection of proofs and worked exercises.

Kyle Mickelson

Last updated: February 3rd, 2025

Contents

Categories, Functors, Natural Transformations **2**
- 1.1 Abstract and concrete categories . . . 2
- 1.2 Duality . . . 3
- 1.3 Functoriality . . . 4
- 1.4 Naturality . . . 7
- 1.5 Equivalence of categories . . . 7
- 1.6 The art of the diagram chase . . . 7
- 1.7 The 2-category of categories . . . 10

Universal Properties, Representability, and the Yoneda Lemma **13**
- 2.1 Representable functors . . . 13
- 2.2 The Yoneda Lemma . . . 13
- 2.3 Universal properties and universal elements . . . 13
- 2.4 The category of elements . . . 13

Limits and Colimits **14**
- 3.1 Limits and colimits as universal cones . . . 14
- 3.2 Limits in the category of sets . . . 19
- 3.3 Preservation, reflection, and creation of limits and colimits . . . 19
- 3.4 The representable nature of limits and colimits . . . 20
- 3.5 Complete and cocomplete categories . . . 21
- 3.6 Functoriality of limits and colimits . . . 21
- 3.7 Size matters . . . 21
- 3.8 Interactions between limits and colimits . . . 22

Adjunctions **23**
- 4.1 Adjoint functors . . . 23
- 4.2 The unit and counit as universal arrows . . . 26
- 4.3 Contravariant and multivariable adjoint functors . . . 27
- 4.4 The calculus of adjunctions . . . 27
- 4.5 Adjunctions, limits, and colimits . . . 27
- 4.6 Existence of adjoint functors . . . 29

Monads and their Algebras **29**

All Categories are Kan Extensions **29**

Theorems in Category Theory **29**

1 Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Exercise 1.1.1. (i) Show that a morphism can have at most one inverse isomorphism.
(ii) Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

Proof. (i) If $f : x \to y$ is a morphism with two inverses $g, h : y \to x$, then we know in particular that $fh = 1_y$ and $gf = 1_x$. Using these two equations, we find that

$$g = g1_y = g(fh) = (gf)h = 1_x h = h,$$

to which these inverses must coincide; hence there exists at most one inverse for a morphism.

(ii) If we assume the existence of a pair of morphisms $g, h : y \rightrightarrows x$ such that $gf = 1_x$ and $fh = 1_y$, then

$$g = g1_y = g(fh) = (gf)h = 1_x h = h$$

must hold once more. Setting $f^{-1} := g = h$ we find that $ff^{-1} = 1_y$ and $f^{-1}f = 1_x$, hence f^{-1} is an inverse morphism for f and so f is an isomorphism. ■

Exercise 1.1.2. Let $\mathbf{C}$ be a category. Show that the collection of isomorphisms in $\mathbf{C}$ defines a subcategory, the **maximal groupoid** inside $\mathbf{C}$.

Proof. Let $\mathbf{C}_0$ denote the maximal groupoid inside $\mathbf{C}$. Take the objects of $\mathbf{C}$ to be the objects of $\mathbf{C}_0$ and for morphisms in $\mathbf{C}_0$ take the isomorphisms in $\mathbf{C}$.

Since we are restricting to a subcollection of objects and a subcollection of morphisms in $\mathbf{C}$, to check that $\mathbf{C}_0$ is a subcategory of $\mathbf{C}$ we need only make sure that $\mathbf{C}_0$ contains the domain and codomain of any morphism in $\mathbf{C}_0$, that the identity morphism of any object in $\mathbf{C}_0$ is a morphism in $\mathbf{C}_0$, and that the composite of any morphisms in $\mathbf{C}_0$ is once again a morphism of $\mathbf{C}_0$.

Clearly the domain and codomain of any morphism in $\mathbf{C}_0$ is an object of $\mathbf{C}_0$. Likewise, since the identity morphisms are isomorphisms, $\mathbf{C}_0$ contains all identity morphisms. Lastly, the composition of isomorphisms is an isomorphism. If $f : x \to y$ and $g : y \to z$ are isomorphisms, then we have inverses f^{-1} and g^{-1}, respectively, and $g \circ f : x \to z$ has inverse $f^{-1} \circ g^{-1} : z \to x$. ■

Exercise 1.1.3. For any category $\mathbf{C}$ and any object $c \in \mathbf{C}$, show that:
(i) There is a category $c/\mathbf{C}$ whose objects are morphisms $f : c \to x$ with domain c and in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes, i.e., so that $g = hf$.
(ii) There is a category $\mathbf{C}/c$ whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the domains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] & & y \\ & c \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, i.e., so that $f = gh$.
The categories $c/\mathbf{C}$ and $\mathbf{C}/c$ are called **slice categories** of $\mathbf{C}$ **under** and **over** c, respectively.

Proof. (i) The objects and morphisms of $c/\mathbf{C}$ have already been specified; each morphism moreover has a specified domain and codomain; an identity morphism for an object $f : c \to x$ in $c/\mathbf{C}$ is

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "f"] & \\ x \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd}$$

simply the identity morphism of x in $\mathbf{C}$, as we have $\mathrm{id}_x f = f$, and for any morphism $h : x \to y$ from $f : c \to x$ to $g : c \to y$ we have that $h\mathrm{id}_x$ and $\mathrm{id}_y h$ equal h, since this holds in $\mathbf{C}$.

Composites in $c/\mathbf{C}$ are inherited from composites in $\mathbf{C}$. That is, if $f : c \to x$, $g : c \to y$, and $h : c \to z$ are objects in $c/\mathbf{C}$ then the diagram

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & & c \arrow[dl, "g"'] \arrow[dr, "h"] & \\ x \arrow[rr, "\psi"] & & y \arrow[rr, "\varphi"] & & z \end{tikzcd}$$

commutes, and so ∎

1.2 Duality

Exercise 1.2.1.

Proof. ∎

Exercise 1.2.2.

Proof. ∎

Exercise 1.2.3.

Proof. ∎

Exercise 1.2.4. What are the monomorphisms in the category of fields?

Proof. Let $f : F \to K$ be a field homomorphism. Since f is, in particular, a homomorphism of unital rings, we know that $\ker f$ is an ideal of F; since F is a field, either $\ker \phi = 0$ or $\ker \phi = F$. If $\ker f = F$ then f is the zero map, which is not a ring homomorphism (nor field homomorphism) since the identity 1 is not preserved by f. Thus the only permissible case is when $\ker f = 0$, i.e., when f

is an injection. In this case, if $\phi, \psi : k \to F$ are two other field homomorphisms such that $f\phi = f\psi$ then we have $f(\phi(x)) = f(\psi(x))$ for all $x \in k$, and since f is injective, this means $\phi(x) = \psi(x)$ for all $x \in k$, whence $\varphi = \psi$. Thus when $\ker f = 0$ we have found that f is a monomorphism; hence any morphism in **Field** is a monomorphism. ∎

Exercise 1.2.5. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category **Ring** of rings. Conclude that a map which is both monic and epic need not be an isomorphism.

Proof. The inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is injective as a map of sets, so if $f, g : R \to \mathbb{Z}$ are two ring homomorphisms such that $if = ig$ then we automatically have $f = g$, since for all $x \in R$ we have $i(f(x)) = i(g(x))$, which means $f(x) = g(x)$ for all $x \in R$. Thus the inclusion i is monic.

On the other hand, suppose $f, g : \mathbb{Q} \to R$ are two ring homomorphisms such that $fi = gi$. Since f and g are ring homomorphisms, $\ker f$ and $\ker g$ are both ideals of $\mathbb{Q}$, hence either zero or all of $\mathbb{Q}$ since $\mathbb{Q}$ is a field. Since the multiplicative identity 1 must be preserved by f and g, we know $\ker f = \ker g = 0$. Thus both f and g are injections; thus if there were to be some element $x \in \mathbb{Q}$ for which $f(x) \neq g(x)$, then $x \neq x$ is implied, which is absurd; hence $f = g$ is required. This, in particular, means that i is epic.

In conclusion, $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is monic and epic in **Ring** but clearly not an isomorphism. Thus, in general categories, morphisms which are monic and epic need not be isomorphisms. ∎

Exercise 1.2.6. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

Proof. Let $f : x \rightarrowtail y$ be a monic split epimorphism; that is, f has a right inverse, call it $g : y \to x$ such that $fg = 1_y$. Note then that $fgf = \mathrm{id}_y f = f = f\mathrm{id}_x$ holds, i.e., we have a commuting diagram

$$\begin{tikzcd} x \arrow[r, shift left, "gf"] \arrow[r, shift right, "\mathrm{id}_x"'] & x \arrow[r, tail, "f"] & y \end{tikzcd}$$

Since f is monic and $f(gf) = f(\mathrm{id}_x)$, we require $gf = \mathrm{id}_x$. Thus g is also a left inverse for f, whence f is an isomorphism. ∎

Exercise 1.2.7. Regarding a poset $(\mathbf{P}, \leq)$ as a category, define the supremum of a subcollection of objects $A \in \mathbf{P}$ in such a way that the dual statement defines the infimum. Prove that the supremum of a subset of objects is unique , whenever it exists, in such a way that the dual proof demonstrates the uniqueness of the infimum.

Proof. ∎

1.3 Functoriality

Exercise 1.3.1. What is a functor between groups, regarded as one object categories?

Proof. Let G and H be groups with one object categories $\mathbf{B}G$ and $\mathbf{B}H$, respectively. Any functor $F : \mathbf{B}G \to \mathbf{B}H$ takes the unique object of $\mathbf{B}G$, call it G, to the unique object of $\mathbf{B}H$, call it H. Moreover, F takes each automorphism $g : G \to G$ to an automorphism $F(g) : H \to H$ in such a way that if $gh : G \to G$ then $F(gh) = F(g)F(h) : H \to H$ by the functor axioms. Likewise, the identity element is preserved by F, again by the functor axioms. Thus, in essence, F is a group homomorphism in the usual sense. ∎

Exercise 1.3.2.

Proof. ∎

Exercise 1.3.3.

Proof. ∎

Exercise 1.3.4.

Proof. ∎

Exercise 1.3.5.

Proof. ∎

Exercise 1.3.6.

Proof. ∎

Exercise 1.3.7. Define functors to construct the slice categories $c/\mathbf{C}$ and $\mathbf{C}/c$ of Exercise 1.1.3 as special cases of comma categories constructed in Exercise 1.3.6. What are the projection functors?

Proof. Let $\mathbf{C}$ be a category with c some $\mathbf{C}$-object. Consider the constant endofunctor $c : \mathbf{C} \to \mathbf{C}$ which takes every $\mathbf{C}$-object to c and every $\mathbf{C}$-morphism to the identity morphism on c. Consider also the identity endofunctor $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$. We claim that the comma category $c \downarrow \mathrm{id}_{\mathbf{C}}$ is equal to the slice category $c/\mathbf{C}$ of Exercise 1.1.3.

Recall that the objects of $c \downarrow \mathrm{id}_{\mathbf{C}}$ are triples (x, y, f) where x and y are $\mathbf{C}$-objects and $f : x \to y$ is a $\mathbf{C}$-morphism. A morphism in the comma category $c \downarrow \mathrm{id}_{\mathbf{C}}$ from (x, y, f) to (x', y', f') is then a pair of $\mathbf{C}$-morphisms $h : x \to x'$ and $k : y \to y'$ such that the square

$$\begin{tikzcd} c(x) \arrow[r, "f"] \arrow[d, "c(h)"'] & \mathrm{id}_{\mathbf{C}}(y) \arrow[d, "\mathrm{id}_{\mathbf{C}}(k)"] \\ c(x') \arrow[r, "f'"] & \mathrm{id}_{\mathbf{C}}(y') \end{tikzcd}$$

commutes, i.e., so that $f'c(h) = \mathrm{id}_{\mathbf{C}} f$. But note that $c(x) = c(x') = c$, and $c(h) = \mathrm{id}_c$ and $\mathrm{id}_{\mathbf{C}}(y) = y$, $\mathrm{id}_{\mathbf{C}}(y') = y'$, and $\mathrm{id}_{\mathbf{C}}(k) = k$. Thus the square above becomes simply

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\mathrm{id}_c"'] & y \arrow[d, "k"] \\ c \arrow[r, "f'"] & y' \end{tikzcd}$$

which we may rewrite as

$$\begin{tikzcd} y \arrow[rr, "k"] & & y' \\ & c \arrow[ul, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

and so the commutativity condition becomes $kf = f'$. Hence the objects of $c \downarrow \mathrm{id}_{\mathbf{C}}$ may be considered as triples (c, x, f) where $f : c \to x$ is a $\mathbf{C}$-morphism, and morphisms in $c \downarrow \mathrm{id}_{\mathbf{C}}$ between (c, x, f) and (c, y, g) are $\mathbf{C}$-morphisms $h : x \to y$ such that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] & & y \\ & c \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, i.e., so that $hf = g$. This data is precisely that of the slice category $c/\mathbf{C}$. The projection functors $\mathrm{dom} : c \downarrow \mathrm{id}_{\mathbf{C}} \to \mathbf{C}$ for this comma category is

$$\mathrm{dom} : c \downarrow \mathrm{id}_{\mathbf{C}} \to \mathbf{C}$$

$$(c, x, f) \mapsto c$$

$$(c, x, f) \to (c, y, g) \mapsto \mathrm{id}_c,$$

which is simply the functor sending all objects of $c \downarrow \mathrm{id}_{\mathbf{C}}$ to c and all morphisms to the identity morphism on c. On the other hand, the projection functor $\mathrm{cod} : c \downarrow \mathrm{id}_{\mathbf{C}} \to \mathbf{C}$ for this comma category is

$$\mathrm{cod} : c \downarrow \mathrm{id}_{\mathbf{C}} \to \mathbf{C}$$

$$(c, x, f) \mapsto x$$

$$(c, x, f) \to (c, y, g) \mapsto h : x \to y$$

where $h : x \to y$ is the $\mathbf{C}$-morphism inherited from the morphism of objects $(c, x, f) \to (c, y, g)$.

To construct the other slice category $\mathbf{C}/c$, simply reverse the order of the identity endofunctor and constant endofunctor on $\mathbf{C}$ above; that is, consider the comma category $\mathrm{id}_{\mathbf{C}} \downarrow c$ and then go through the above process in exactly the same way. ■

Exercise 1.3.8. Lemma 1.3.8 shows that functors preserve isomorphisms. Find an example to demonstrate that functors need not **reflect isomorphisms**: that is, find a functor $F : \mathbf{C} \to \mathbf{D}$ and a morphism f in $\mathbf{C}$ so that Ff is an isomorphism in $\mathbf{D}$ but f is not an isomorphism in $\mathbf{C}$.

Proof. Consider the dihedral group D_6 of order 6 and the cyclic group $\mathbb{Z}/6\mathbb{Z}$ of order 6. Let

$$D_6 = \langle x, y \mid x^3 = y^2 = 1, xy = yx^{-1} \rangle$$

be a group presentation. There is a group homomorphism $\varphi : D_6 \to \mathbb{Z}/6\mathbb{Z}$ which sends $x \mapsto \overline{2}$ and $y \mapsto \overline{3}$; despite this homomorphism being a bijection on the level of sets, it is clearly not an isomorphism, for instance since D_6 is non-abelian while $\mathbb{Z}/6\mathbb{Z}$ is abelian.

Now, if we let $U : \mathbf{Group} \to \mathbf{Set}$ denote the forgetful functor, then U sends the groups D_6 and $\mathbb{Z}/6\mathbb{Z}$ to their underlying sets, and U sends the group homomorphism φ to the set-map $U(\varphi)$. As stated above, φ is a bijection of sets (but not a group isomorphism), and hence the set-map $U(\varphi)$ is an isomorphism of $U(D_6)$ and $U(\mathbb{Z}/6\mathbb{Z})$ in $\mathbf{Set}$, while φ is not an isomorphism of D_6 and $\mathbb{Z}/6\mathbb{Z}$ in $\mathbf{Group}$. ■

Exercise 1.3.9.

Proof. ■

Exercise 1.3.10. Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor Conj : **Group** → **Set**. Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.

Proof. We can construct the functor as follows: given a group G we define an equivalence relation $\sim$ on G by asserting that $g \sim h$ if and only if there exists $k \in G$ such that $kgk^{-1} = h$. It is immediate that $\sim$ is reflexive, symmetric, and transitive; hence an equivalence relation. We denote the set of equivalence classes (which are precisely the set of conjugacy classes of elements of G) by $G/\sim$.

Now, given a group homomorphism $\varphi : G \to H$ we can construct an induced set-map from the set of conjugacy classes of elements of G to the set of conjugacy classes of elements of H. To do this, we define $\widetilde{\varphi} : G/{\sim} \to H/{\sim}$ by $\widetilde{\varphi}([g]_\sim) = [\varphi(g)]_\sim$. This is a well-defined map since if $[g]_\sim = [h]_\sim$ then there exists $k \in G$ for which $kgk^{-1} = h$, and hence $\varphi(h) = \varphi(kgk^{-1}) = \varphi(k)\varphi(g)\varphi(k)^{-1}$ holds since φ is a homomorphism of groups; but this means $\varphi(h) \sim \varphi(g)$, and so $[\varphi(g)]_\sim = [\varphi(h)]_\sim$ holds.

The constructions above allow us to define a functor Conj : **Group** → **Set** by taking a group G to the set $G/\sim$ and by taking a group homomorphism $\varphi : G \to H$ to the induced set-map $\widetilde{\varphi} : G/{\sim} \to H/{\sim}$. To verify that Conj is a functor, we first note that the induced set-map from the identity group homomorphism $\mathrm{id}_G : G \to G$ is precisely the identity set-map on $G/\sim$. What remains then is to check that if $G \xrightarrow{\varphi} H \xrightarrow{\psi} K$ are group homomorphisms then $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$ holds. However this is clear for if $[g]_\sim \in G/\sim$ then

$$(\widetilde{\psi \circ \varphi})([g]_\sim) = [(\psi \circ \varphi)(g)]_\sim = [\psi(\varphi(g))]_\sim = \widetilde{\psi}([\varphi(g)]_\sim) = (\widetilde{\psi} \circ \widetilde{\varphi})([g]_\sim).$$

Thus Conj satisfies the functoriality axioms and hence is a functor from **Group** to **Set**.

Since functors preserve isomorphisms, it thus follows that if two groups are isomorphic in **Group** then their sets of conjugacy classes must be isomorphic in **Set**; that is, there exists a bijection between them, to which they necessarily have the same cardinality. The contrapositive of this statement gives the desired result. ■

1.4 Naturality

1.5 Equivalence of categories

1.6 The art of the diagram chase

Exercise 1.6.1. Show that any map from a terminal object in a category to an initial one is an isomorphism. An object that is both initial and terminal is called a **zero object**.

Proof. Let **C** be a category with initial object 0 and terminal object 1, and suppose $f : 1 \to 0$ is a **C**-morphism. Since 0 is initial in **C** and 1 is, in particular, a **C**-object, there exists a unique morphism $g : 0 \to 1$. Thus we have a diagram

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & 0 \arrow[r, "g"] \arrow[rr, bend left, "fg"] & 1 \arrow[r, "f"'] & 0 \end{tikzcd}$$

Since 1 is terminal, there exists only one **C**-morphism $1 \to 1$, and so this morphism must be the identity morphism id_1 (by the very definition of a category). Likewise, since 0 is initial, there exists only one **C**-morphism $0 \to 0$, and this morphism must be exactly the identity morphism id_0. Now, in view of the above diagram, we have $gf : 1 \to 1$ and $fg : 0 \to 0$ both morphisms in **C** by composition; hence $gf = \mathrm{id}_1$ and $fg = \mathrm{id}_0$ is required, to which f is an isomorphism with inverse morphism g. ■

Exercise 1.6.2. Show that any two terminal objects in a category are connected by a unique isomorphism.

Proof. Let 1 and $1'$ be two terminal objects in some category **C**. We know then that there exists unique **C**-morphisms $f : 1 \to 1'$ and $g : 1' \to 1$. As explained in Exercise 1.6.1 above, the only morphisms $1 \to 1$ and $1' \to 1'$ are id_1 and $\mathrm{id}_{1'}$, respectively. Since $fg : 1' \to 1'$ and $gf : 1 \to 1$ are two such morphisms, we require that $fg = \mathrm{id}_{1'}$ and $gf = \mathrm{id}_1$; hence $1 \cong 1'$. ■

Exercise 1.6.3. Show that any faithful functor reflects monomorphisms . That is, if $F : \mathbf{C} \to \mathbf{D}$ is faithful, prove that if Ff is a monomorphism in **D** then f is a monomorphism in **C**. Argue by duality that faithful functors also reflect epimorphisms. Conclude that in any concrete category, any morphism that defines an injection of underlying sets is a monomorphism and any morphism that defines a surjection of underlying sets is an epimorphism.

Proof. Let $F : \mathbf{C} \to \mathbf{D}$ be a faithful functor, and let $f : x \to y$ be a morphism in **C**. Suppose now that $F(f)$ is a monic.

To prove that f is monic, suppose $g, h : c \to x$ are two morphisms in **C** such that $fg = fh$. Then $F(fg) = F(fh)$ holds in **D**, and by the functoriality axioms we require $F(f)F(g) = F(f)F(h)$. Since $F(f)$ is monic, this means $F(g) = F(h)$.

The fact that F is faithful means we have an injection on the level of hom-sets $\mathrm{Hom}_{\mathbf{C}}(x, y) \hookrightarrow \mathrm{Hom}_{\mathbf{D}}(F(x), F(y))$; equivalently, if $F(f) = F(g)$ in **D**, where $F(f), F(g) : Fx \to Fy$, then we require $f = g$ in **C**.

Thus since $F(g) = F(h)$ holds in **D**, we require that $g = h$ hold in **C**, whence f is monic.

The dual statement to what we have shown above asserts that faithful functors reflect epimorphisms. Therefore, we may conclude that in any concrete category **C**, i.e., a category **C** equipped with a faithful functor $U : \mathbf{C} \to \mathbf{Set}$, if a **C**-morphism $f : x \to y$ defines an injection of sets $F(f) : F(x) \to F(y)$ then $F(f)$ is a monic in **Set**, and since F reflects monomorphisms, f must be a monomorphism as well. Dually, if $f : x \to y$ defines a surjection of sets $F(f) : F(x) \to F(y)$, then $F(f)$ is an epimorphism in **Set**, and since F is faithful it reflects epimorphisms, whence f is epic in **C**. ■

Exercise 1.6.4. Find an example to show that a faithful functor need not preserve epimorphisms. Argue by duality, or by counterexample, that a faithful functor need not preserve monomorphisms.

Proof. Consider the forgetful functor $U : \mathbf{Ring} \to \mathbf{Set}$ which takes a unital ring to its underlying set and which takes a ring homomorphism which preserves the multiplicative identity to its underlying set-map. Now, recall for a moment Exercise 1.2.5, where we proved that the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is both monic and epic in the category **Ring**. We can see that $U(i) : U(\mathbb{Z}) \to U(\mathbb{Q})$ is not epic in **Set** since the set function $U(i)$ is not surjective, and a morphism is epic in **Set** if and only if it is a surjective set-map (Example 1.2.8); thus, faithful functors need not preserve epimorphisms.

Now in the opposite category $\mathbf{Ring}^{\mathrm{op}}$ we have that $i^{\mathrm{op}} : \mathbb{Q} \to \mathbb{Z}$ is both monic and epic. The contravariant forgetful functor $U : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ takes i^{op} to the set-map $U(i^{\mathrm{op}}) : U(\mathbb{Q}) \to U(\mathbb{Z})$.

Since no set-map injects $\mathbb{Q}$ into $\mathbb{Z}$, $U(i^{\mathrm{op}})$ is not injective, hence not a monomorphism in the category **Set**. Thus faithful functors need not preserve monomorphisms. ■

Exercise 1.6.5. More specifically, find a concrete category that contains a monomorphism whose underlying set function is not injective. Find a concrete category that contains an epimorphism whose underlying function is not surjective. Exercise 4.5.5 explains why the latter examples may seem less familiar than the former.

Proof. ■

Exercise 1.6.6. A **coalgebra** for an endofunctor $T : \mathbf{C} \to \mathbf{C}$ is an object $C \in \mathbf{C}$ equipped with a map $\gamma : C \to T(C)$. A morphism $f : (C, \gamma) \to (C', \gamma')$ of coalgebras is a map $f : C \to C'$ so that the square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\ T(C) \arrow[r, "T(f)"'] & T(C') \end{tikzcd}$$

commutes. Prove that if (C, γ) is a **terminal coalgebra**, that is a terminal object in the category of coalgebras, then the map $\gamma : C \to T(C)$ is an isomorphism.

Proof. Let (C, γ) be the terminal coalgebra in the category of coalgebras associated to the endofunctor $T : \mathbf{C} \to \mathbf{C}$. Since we have a map $\gamma : C \to T(C)$, we may apply T to this map and obtain a new map $T(\gamma) : T(C) \to T(T(C))$; hence we have the coalgebra $(T(C), T(\gamma))$. Since (C, γ) is a terminal coalgebra, there exists a unique morphism of coalgebras $f : T(C) \to C$ such that the square

$$\begin{tikzcd} T(C) \arrow[r, "f"] \arrow[d, "T(\gamma)"'] & C \arrow[d, "\gamma"] \\ T(T(C)) \arrow[r, "T(f)"] & T(C) \end{tikzcd}$$

commutes, i.e., such that $\gamma f = T(f)T(\gamma)$. By the functoriality axioms, we know $T(f)T(\gamma) = T(f\gamma)$, and hence we have $\gamma f = T(f\gamma)$.

Now note that $f\gamma : C \to C$ is a map making the outermost rectangle of the following diagram

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[rr, bend left, "f\gamma"] & T(C) \arrow[r, "f"] \arrow[d, "T(\gamma)"'] & C \arrow[d, "\gamma"] \\ T(C) \arrow[r, "T(\gamma)"] & T(T(C)) \arrow[r, "T(f)"] & T(C) \end{tikzcd}$$

commute. Observe that this is true for commutativity asserts an equality $f\gamma\gamma = T(f)T(\gamma)\gamma$, and this is indeed the case for

$$T(f)T(\gamma)\gamma = T(f\gamma)\gamma = (\gamma f)\gamma = \gamma f \gamma$$

since above we found that $\gamma f = T(f\gamma)$ holds. In particular, this means that $f\gamma : C \to C$ is a morphism of coalgebras; but wait, we note that the identity morphism $\mathrm{id}_C : C \to C$ makes the diagram

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, "\gamma"'] & C \arrow[d, "\gamma"] \\ T(C) \arrow[r, "T(\mathrm{id}_C)=T_{T(C)}"] & T(C) \end{tikzcd}$$

commute, and hence is a morphism of coalgebras. But (C, γ) is a terminal coalgebra by assumption, and hence there exists only one morphism of coalgebras $(C, \gamma) \to (C, \gamma)$; thus we require that $f\gamma = \mathrm{id}_C$ hold true.

Moreover, since we have already shown that $\gamma f = T(f\gamma)$, we can use the fact that $f\gamma = \mathrm{id}_C$ to write $\gamma f = T(f\gamma) = T(\mathrm{id}_C) = \mathrm{id}_{T(C)}$ by the functoriality axioms; hence we have proven that $f\gamma = \mathrm{id}_C$ and $\gamma f = \mathrm{id}_{T(C)}$, which proves that γ is an isomorphism with inverse morphism f. ■

1.7 The 2-category of categories

Exercise 1.7.1.

Proof. ■

Exercise 1.7.2. Given a natural transformation $\beta : H \Rightarrow K$ and functors F and L as displayed in

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{E} \arrow[r, "L"] & \mathbf{F} \end{tikzcd}$$

define a natural transformation $L\beta F : LHF \Rightarrow LKF$ by $(L\beta F)_c = L\beta_{F(c)}$. This is the **whiskered composite** of β with L and F. Prove that $L\beta F$ is natural.

Proof. To prove that $L\beta F$ is natural we take an arbitrary morphism $f : x \to y$ in $\mathbf{C}$ and prove that the diagram

$$\begin{tikzcd} LHF(x) \arrow[r, "L\beta F_x = L\beta_{F(x)}"] \arrow[d, "LHF(f)"'] & LKF(x) \arrow[d, "LKF(f)"] \\ LHF(y) \arrow[r, "L\beta F_y = L\beta_{F(y)}"] & LKF(y) \end{tikzcd}$$

commutes, i.e., that $LKF(f)L\beta_{F(x)} = L\beta_{F(y)}LHF(f)$.

Since $f : x \to y$ is a $\mathbf{C}$-morphism, we have $F(f) : F(x) \to F(y)$ a $\mathbf{D}$-morphism; since $\beta : H \Rightarrow K$ is a natural transformation, we thus have a square

$$\begin{tikzcd} HF(x) \arrow[r, "\beta_{F(x)}"] \arrow[d, "HF(f)"'] & KF(x) \arrow[d, "KF(f)"] \\ HF(y) \arrow[r, "\beta_{F(y)}"] & KF(y) \end{tikzcd}$$

which commutes, i.e., such that $KF(f)\beta_{F(x)} = \beta_{F(y)}HF(f)$. Keep in mind that this square is a diagram in the category $\mathbf{E}$, and so we are at liberty to apply the functor L to each object and morphism above. Doing so, we obtain $L(KF(f)\beta_{F(x)}) = L(\beta_{F(y)}HF(f))$. Expanding via the functoriality axioms, we obtain

$$L(KF(f)\beta_{F(x)}) = L(KF(f))L(\beta_{F(x)}) = LKF(f)L\beta_{F(x)}$$

and

$$L(\beta_{F(y)})L(HF(f)) = L\beta_{F(y)}LHF(f)$$

and these values must agree by our application of L above; hence the commutativity of the first square shown indeed holds, and so $L\beta F : LHK \Rightarrow LKF$ is a natural transformation, equivalently $L\beta F$ is natural, as desired. ■

Exercise 1.7.3. Redefine the horizontal composition of natural transformations introduced in Lemma 1.7.4 using vertical composition and whiskering.

Proof. We show that given a pair of natural transformations

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{E} \end{tikzcd}$$

there is a natural transformation $\beta * \alpha : HF \Rightarrow KG$ whose component at $c \in \mathbf{C}$ is given by the common value of $\beta_{G(c)}H\alpha_c = K\alpha_c\beta_{F(c)}$.

We consider the whiskered composite (see Exercise 1.7.2 above)

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] & \mathbf{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{E} \arrow[r, "\mathrm{id}_{\mathbf{E}}"] & \mathbf{E} \end{tikzcd}$$

where $\mathrm{id}_{\mathbf{E}}$ is the identity endofunctor. From this whiskered composite, first dropping the unnecessary identity endofunctor, we obtain a natural transformation $\mathrm{id}_{\mathbf{E}}\,\beta F : HF \Rightarrow KF$. We consider also the whiskered composite

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\mathrm{id}_{\mathbf{C}}"] & \mathbf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{E} \arrow[r, "K"] & \mathbf{E} \end{tikzcd}$$

and obtain a natural transformation (again dropping the identity endofunctors) via $K\alpha\,\mathrm{id}_{\mathbf{C}} : KF \Rightarrow KG$.

With these whiskers firmly in mind, we now can put together a pasting diagram via vertical composition:

$$\begin{tikzcd}[column sep=huge] \mathbf{C} \arrow[r, bend left=70, "HF"{name=U}] \arrow[r, "HG"{description, name=M}] \arrow[r, bend right=70, "KG"'{name=D}] & \mathbf{E} \arrow[from=U, to=M, phantom, "\Downarrow \mathrm{id}_{\mathbf{E}}\beta F"] \arrow[from=M, to=D, phantom, "\Downarrow K\alpha\mathrm{id}_{\mathbf{C}}"] \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=huge] \mathbf{C} \arrow[r, bend left=50, "HF"{name=U}] \arrow[r, bend right=50, "KG"'{name=D}] & \mathbf{E} \arrow[from=U, to=D, phantom, "\Downarrow K\alpha\mathrm{id}_{\mathbf{C}} \cdot \mathrm{id}_{\mathbf{E}}\beta F"] \end{tikzcd}$$

Recall now from Exercise 1.7.2 that for each $\mathbf{C}$-object c we have the components $(\mathrm{id}_{\mathbf{E}}\,\beta F)_c = \mathrm{id}_{\mathbf{E}}\,\beta_{F(c)}$ and $(K\alpha\,\mathrm{id}_{\mathbf{C}})_c = K\alpha_{\mathrm{id}_{\mathbf{C}}}(c)$. Now from the the operation of vertical composition, Lemma 1.7.1, we have

$$(K\alpha\,\mathrm{id}_{\mathbf{C}} \cdot \mathrm{id}_{\mathbf{E}}\,\beta F)_c = (K\alpha\,\mathrm{id}_{\mathbf{C}})_c \cdot (\mathrm{id}_{\mathbf{E}}\,\beta F)_c = K\alpha_c\beta_{F(c)}.$$

This is precisely the common value stated prior, and so indeed we have recovered horizontal composition as defined in the text via whiskered composites and vertical composition. To finish, we may set $\beta * \alpha := \mathrm{id}_{\mathbf{E}}\,\beta F \cdot K\alpha\,\mathrm{id}_{\mathbf{C}}$, so that our notation lines up with that of the text. ■

Exercise 1.7.4.

Proof. ■

Exercise 1.7.5.

Proof. ■

Exercise 1.7.6.

Proof. ■

Exercise 1.7.7.

Proof. ■

2 Universal Properties, Representability, and the Yoneda Lemma

2.1 Representable functors

2.2 The Yoneda Lemma

2.3 Universal properties and universal elements

2.4 The category of elements

3 Limits and Colimits

3.1 Limits and colimits as universal cones

Exercise 3.1.1. For a fixed diagram $F \in \mathbf{C}^{\mathbf{J}}$, describe the actions of the cone functors $\mathrm{Cone}(-, F) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and $\mathrm{Cone}(F, -) : \mathbf{C} \to \mathbf{Set}$ on morphisms in $\mathbf{C}$.

Proof. We know that $\mathrm{Cone}(-, F)$ acts on objects c of $\mathbf{C}$ by taking c to the set of cones over F with summit c. Therefore, given a morphism $f : c \to c'$ in $\mathbf{C}$, $\mathrm{Cone}(-, F)$ must somehow act on f to produce a set-map between the set of cones over F with summit c' and the set of cones over F with summit c (since the functor is contravariant!). There is really only one way to do this, and this is by taking each cone over F with summit c', say $\{\mu_j : c' \to F(j)\}_{j \in J}$, to the cone $\{\lambda_j : c \to F(j)\}_{j \in J}$ where $\lambda_j := \mu_j f$; i.e., we define the legs of the cone over F with summit c to be the legs obtained from the original cone over F with summit c' via pre-composition with f. That is,

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[dr, "\lambda_j := \mu_j f"] & \\ c' \arrow[r, "\mu_j"'] & F(j) \end{tikzcd}$$

It is easy to see that this is indeed a cone over F with summit c; another way to obtain this is by noting that for each $g : i \to j$ in J, we have a commuting square

$$\begin{tikzcd} c' \arrow[r, "\mu_i"] \arrow[d, "\mathrm{id}_{c'}"'] & F(i) \arrow[d, "F(g)"] \\ c' \arrow[r, "\mu_j"'] & F(j) \end{tikzcd}$$

since the cone over F with summit c' is simply a natural transformation $\mu : c' \Rightarrow F$. Now we can take this natural transformation and construct a new one: $\lambda : c \Rightarrow F$ via

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\mathrm{id}_c"'] & c' \arrow[r, "\mu_i"] \arrow[d, "\mathrm{id}_{c'}"'] & F(i) \arrow[d, "F(g)"] \\ c \arrow[r, "f"'] & c' \arrow[r, "\mu_j"'] & F(j) \end{tikzcd} \quad = \quad \begin{tikzcd} c \arrow[r, "\lambda_i"] \arrow[d, "\mathrm{id}_c"'] & F(i) \arrow[d, "F(g)"] \\ c \arrow[r, "\lambda_j"'] & F(j) \end{tikzcd}$$

Thus a morphism $f : c \to c'$ in $\mathbf{C}$ corresponds under the functor $\mathrm{Cone}(-, F)$ to a map from the set of cones over F with summit c' to the set of cones over F with summit c by taking $\{\lambda_j : c' \to F\}_{j \in J}$ to $\{\lambda_j f : c \to F\}_{j \in J}$.

The action of the functor $\mathrm{Cone}(F, -)$ on morphisms $f : c \to c'$ in $\mathbf{C}$ is obtained similarly, however in this case the morphism is taken to the map from the set of cones under F with nadir c to the set of cones under F with nadir c' by taking $\{\lambda_j : F(j) \to c\}_{j \in J}$ to $\{f \lambda_j : F(j) \to c'\}_{j \in J}$. ■

Exercise 3.1.2. For a fixed diagram $F \in \mathbf{C}^{\mathbf{J}}$, show that the cone functor $\mathrm{Cone}(-, F)$ is naturally isomorphic to $\mathrm{Nat}(\Delta(-), F)$, the restriction of the hom functor for the category $\mathbf{C}^{\mathbf{J}}$ along the constant functor embedding defined in 3.1.1.

Proof. ■

Exercise 3.1.3.

Proof. ■

Exercise 3.1.4.

Proof. ■

Exercise 3.1.5.

Proof. ■

Exercise 3.1.6. Prove that if

$$\begin{tikzcd} E \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is an equalizer diagram, then h is a monomorphism.

Proof. Assuming the diagram in question is an equalizer diagram, to prove that h is monic suppose we have two morphisms $k, l : X \to E$ such that $hk = hl$. Note that if $hk = hl$ then we have a commuting diagram

$$\begin{tikzcd} E \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X \arrow[u, shift left, "k"] \arrow[u, shift right, "l"'] \arrow[ur, "hk=hl"'] & & \end{tikzcd}$$

In particular, since by assumption $fh = gh$ (from the equalizer diagram), we know that

$$fhk = (fh)k = (gh)k = ghk$$

In particular, $hk : X \to A$ is a morphism such that $f(hk) = g(hk)$, and so by the universal property of the equalizer, $k : X \to E$ is the unique map making the diagram commute. But completely analogously, since

$$fhl = (fh)l = (gh)l = ghl$$

we have $hl : X \to A$ is a morphism such that $f(hl) = g(hl)$, and so by the universal property of the equalizer, $l : X \to E$ is the unique morphism making the diagram commute. But since $hk = hl$ by assumption, the unique map making the diagram commute must be the same for both, i.e., we require that $k = l$.

Hence we have shown that $hk = hl$ implies $k = l$, and so h is shown to be monic; hence equalizers are monic, as was desired. ■

Exercise 3.1.7. Prove that if

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d, "g"] \\ B \arrow[r, tail, "f"'] & A \end{tikzcd}$$

is a pullback square and f is a monomorphism, then k is a monomorphism.

Proof. To see this, first suppose $l, j : X \rightrightarrows P$ are two morphisms such that $kl = kj$. Note that since the original diagram is a pullback square, we know that $gk = fh$ holds true.

With this equality $gk = fh$ in mind, observe that $hl : X \to B$ and $kl : X \to C$ are two morphisms such that $gkl = fhl$, i.e., we have a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "l"] \arrow[drr, bend left, "kl"] \arrow[ddr, bend right, "hl"'] & & \\ & P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d, "g"] \\ & B \arrow[r, tail, "f"'] & A \end{tikzcd}$$

The universal property of the pullback asserts that there exists a unique morphism $X \to P$ which factors through the maps h and k; but $l : X \to P$ is precisely such a morphism, and hence is unique.

Observe, in a completely analogous manner, that $hj : X \to B$ and $kj : X \to C$ are two morphisms such that $gkj = fhj$, i.e., we have a commuting diagram

$$\begin{tikzcd} X \arrow[dr, "j"] \arrow[drr, bend left, "kj"] \arrow[ddr, bend right, "hj"'] & & \\ & P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d, "g"] \\ & B \arrow[r, tail, "f"'] & A \end{tikzcd}$$

The universal property of the pullback asserts that there exists a unique morphism $X \to P$ which factors through h and k; the morphism $j : X \to P$ is precisely such a morphism and hence is unique.

Now, observe that $hl, hj : X \to B$ are two morphisms such that

$$fhj = gkj = g(kj) = g(kl) = gkl = fhl$$

where the first equality holds from commutativity of the second diagram above, the third equality holds since we initially assumed $kj = kl$, and the final equality holds by commutativity of the first diagram above. In particular, then, we have that $f(hj) = f(hl)$, and since f is by assumption a monomorphism, this implies $hj = hl$.

But now note that by our initial assumption we have $kl = kj$, and we have just shown that $kj = kl$. Now the diagram

$$\begin{tikzcd}
X \arrow[rrd, bend left, "kj=kl"] \arrow[rdd, bend right, "hj=hl"'] \arrow[rd, shift left, "j"] \arrow[rd, shift right, "l"'] & & \\
& P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "g"] \\
& B \arrow[r, tail, "f"'] & A
\end{tikzcd}$$

commutes, but the universal property of the pullback square asserts that there is a unique morphism $x \to P$ making the diagram commute, but indeed both l and j also make the diagram commute; hence we require that $l = j$ hold.

Thus, to summarize, we have shown that if $kl = kj$ then $l = j$; this proves that k is a monomorphism, which was the desired result. ∎

Exercise 3.1.8.

Proof. ∎

Exercise 3.1.9.

Proof. ∎

Exercise 3.1.10.

Proof. ∎

Exercise 3.1.11.

Proof. ∎

Exercise 3.1.12.

Proof. ∎

Exercise 3.1.13. What is the coproduct in the category of commutative rings?

Proof. Let R and S be two objects in the category **CRing**. We claim that the coproduct of R and S is precisely the tensor product of R and S; that is, we claim that $R \coprod S = R \otimes S$. To see this, we note first that $R \otimes S$ satisfies the universal property for coproducts; that is, we have the two inclusion ring homomorphisms $i_R : R \to R \otimes S$ sending $r \mapsto r \otimes 1_S$ and $i_S : S \to R \otimes S$ sending $s \mapsto 1_R \otimes s$. So

$R \otimes S$ satisfies the universal property of coproducts, but the question is, is it the universal commutative ring with respective to this property?

To this end, suppose the commutative ring A with two ring homomorphisms $f_R : R \to A$ and $f_S : S \to A$ is universal with respect to this property. That is, suppose $A = R \coprod S$. Then we have a diagram

$$\begin{tikzcd}
& R \arrow[dr, "f_R"] & \\
R \times S \arrow[ur, "\pi_R"] \arrow[dr, "\pi_S"'] \arrow[rr, dashed, "\exists\, \Phi"] & & A \\
& S \arrow[ur, "f_S"'] &
\end{tikzcd}$$

where Φ is a bilinear (linear with respect to R and with respect to S) ring homomorphism $R \times S \to A$, and where π_R and π_S are the usual projection ring homomorphisms.

Recall now the defining property of the tensor product: to each bilinear ring homomorphism $\Phi : R \times S \to A$, there exists a unique map $\varphi : R \otimes S \to A$ such that $\varphi \circ i = \Phi$, where $i : R \times S \to R \otimes S$ is the canonical ring homomorphism sending $(r, s) \mapsto r \otimes s$. Thus, in our case, we have a commutative diagram

$$\begin{tikzcd}
& R \arrow[dr, "f_R"] & & \\
R \times S \arrow[ur, "\pi_R"] \arrow[dr, "\pi_S"'] \arrow[rr, "\Phi"] \arrow[rrr, bend left=60, "i"] & & A & R \otimes S \arrow[l, "\varphi"] \\
& S \arrow[ur, "f_S"'] & &
\end{tikzcd}$$

Rearranging the diagram just a bit, we have equivalently that

$$\begin{tikzcd}
R \arrow[dr, "i_R"] \arrow[drr, bend left, "f_R"] & & \\
& R \otimes S \arrow[r, "\varphi"] & A \\
S \arrow[ur, "i_S"] \arrow[urr, bend right, "f_S"'] & &
\end{tikzcd}$$

commutes, i.e., that $\varphi i_R = f_R$ and $\varphi i_S = f_S$. But we assumed that A was the coproduct of R and S, and so by the universal property, i.e., since $R \otimes S$ with i_R and i_S also satisfies the universal property for coproducts, there must exist a unique ring homomorphism $\psi : A \to R \otimes S$ such that the diagram

$$\begin{tikzcd}
R \arrow[dr, "f_R"] \arrow[drr, bend left, "i_R"] & & \\
& A \arrow[r, "\psi"] & R \otimes S \\
S \arrow[ur, "f_S"] \arrow[urr, bend right, "i_S"'] & &
\end{tikzcd}$$

commutes, i.e., $\psi f_R = i_R$ and $\psi f_S = i_S$.

Taken together, we have that

$$i_R = \psi f_R = \psi(\varphi i_R) = \psi\varphi i_R,$$

with the same when we replace i_R and f_R by i_S and f_S, as well as

$$f_R = \varphi i_R = \varphi(\psi f_R) = \varphi\psi f_R,$$

with the same when we replace f_R and i_R by f_S and i_S. Now since there exists at least one ring homomorphism between all pairs of distinct objects of **CRing**, we can apply Exercise 3.1.11 to assert that the legs, i.e., the morphisms f_R and f_S are split monomorphisms, hence monomorphisms; thus the fact that $f_R = \varphi\psi f_R$ implies $\mathrm{id}_A = \varphi\psi$. Moreover, since the canonical inclusions $i_R : R \to R \otimes S$ and $i_S : S \to R \otimes S$ are injective ring homomorphisms, they are monomorphisms in **CRing**, and so since $i_R = \psi\varphi i_R$ we have $\mathrm{id}_{R\otimes S} = \psi\varphi$. Thus φ is an isomorphism with inverse ψ, hence $A \cong R \otimes S$ and so up to isomorphism the coproduct of two commutative rings is their tensor product. ■

3.2 Limits in the category of sets

Exercise 3.2.1.

Proof. ■

3.3 Preservation, reflection, and creation of limits and colimits

Exercise 3.3.1.

Proof. ■

Exercise 3.3.2. Prove Lemma 3.3.5, that a full and faithful functor reflects both limits and colimits.

Proof. Let $F : \mathbf{C} \to \mathbf{D}$ be a fully faithful functor, let $K : \mathbf{J} \to \mathbf{C}$ be a diagram in $\mathbf{C}$, and consider the diagram $FK : \mathbf{J} \to \mathbf{D}$ in $\mathbf{D}$. To prove that F reflects limits, we show that if $\lambda : c \Rightarrow F$ is a cone over F in $\mathbf{C}$ such that the image cone $\lambda' : F(c) \Rightarrow FK$ is a limit cone in $\mathbf{D}$, then the original cone λ is a limit cone for F.

So assume $\lambda' : F(c) \Rightarrow FK$ is a limit cone over F. If $\mu : c' \Rightarrow F$ is another cone over F in $\mathbf{C}$ then clearly its image $\mu' : F(c') \Rightarrow FK$ is a cone over FK in $\mathbf{D}$, and so there is a unique morphism of cones $f' : F(c') \to F(c)$ such that for all $\mathbf{J}$-objects j we have

$$\begin{tikzcd} F(c') \arrow[rr, "f'"] \arrow[dr, "\mu'_j"'] & & F(c) \arrow[dl, "\lambda'_j"] \\ & FK(j) & \end{tikzcd}$$

i.e., $\lambda'_j f = \mu'_j$. However, since F is fully faithful, there is a bijection on the level of hom-sets $\mathrm{Hom}_{\mathbf{C}}(c', c) \cong \mathrm{Hom}_{\mathbf{D}}(F(c'), F(c))$, whence we can lift a unique morphism $f : c' \to c$ in $\mathbf{C}$ whose image under F is f', i.e., $F(f) = f'$. Indeed, for each of the components λ'_j and μ'_j we can also lift unique morphisms to get a diagram

$$\begin{tikzcd} c' \arrow[rr, "f"] \arrow[dr, "\mu'_j"'] & & c \arrow[dl, "\lambda_j"] \\ & K(j) & \end{tikzcd}$$

in **C** which commutes. This holds since the the bijection on the level of hom-sets takes the legs of the cone $\lambda : c \Rightarrow F$ to the legs of the cone $\lambda' : F(c) \Rightarrow FK$, and vice-versa. In particular, we recover a morphism of cones $f : c' \to c$, which is unique by the uniqueness of f'. Hence we conclude that $\lambda : c \Rightarrow F$ is a limit cone, whence F reflects limits.

The proof that F reflects colimits is obtained by dualizing the above argument accordingly. ■

Exercise 3.3.3.

Proof. ■

Exercise 3.3.4.

Proof. ■

Exercise 3.3.5.

Proof. ■

Exercise 3.3.6.

Proof. ■

3.4 The representable nature of limits and colimits

Exercise 3.4.1.

Proof. ■

Exercise 3.4.2.

Proof. ■

Exercise 3.4.3.

Proof. ■

Exercise 3.4.4. Prove Lemma 3.4.16.

Proof. We show that the product of two objects c and d in a category $\mathbf{C}$ can be defined using the existence of pullbacks and a terminal object.

So let $\mathbf{C}$ be a category with pullbacks and a terminal object 1, and let c and d be $\mathbf{C}$-objects. Since 1 is terminal, there exist unique morphisms $!_c : c \to 1$ and $!_d : d \to 1$. As such, we can form the pullback square

$$\begin{tikzcd} c \times d \arrow[r, "\pi_c"] \arrow[d, "\pi_d"'] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[d, "!_c"] \\ d \arrow[r, "!_d"'] & 1 \end{tikzcd}$$

where we have suggestively denoted the pullback object $c \times d$ using the usual product notation, and the two pullback morphisms π_c and π_d denoted just like the usual projection maps from the product (note that these are arbitrary labelings for now).

Now, the universal property of the pullback states that for any other $\mathbf{C}$-object e equipped with two morphisms $f_c : e \to c$ and $f_d : e \to d$ such that $!_d f_d = !_c f_c$, there exists a unique morphism $f : e \to c \times d$ such that $\pi_c f = f_c$ and $\pi_d f = f_d$.

Note, however, that the requirement of $!_d f_d = !_c f_c$ holds vacuously for such a $\mathbf{C}$-object e and morphisms f_c, f_d; this is because there exists a unique morphism $!_e : e \to 1$ and since $!_e = !_d f_d$ and $!_e = !_c f_c$, we always have that $!_d f_d = !_c f_c$. Thus the universal property of the pullback (in this instance) reduces to saying that given any $\mathbf{C}$-object e with morphisms $f_c : e \to c$ and $f_d : e \to d$, there exists a unique morphism $f : e \to c \times d$. This is precisely the universal property of the product of the two objects c and d; hence we have recovered the construction of the product from pullback squares and a terminal object. ■

3.5 Complete and cocomplete categories

Exercise 3.5.1.

Proof. ■

3.6 Functoriality of limits and colimits

Exercise 3.6.1.

Proof. ■

3.7 Size matters

Exercise 3.7.1. Complete the proof of Lemma 3.7.1 by showing that an initial object defines a limit of the identity functor $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$.

Proof. Let $\mathbf{C}$ be a category with an initial object 0 and let $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ denote the identity endofunctor. Suppose that $\lambda : x \Rightarrow \mathrm{id}_{\mathbf{C}}$ is a cone over $\mathrm{id}_{\mathbf{C}}$. In particular, then, we know that for any $\mathbf{C}$-object c, we have a unique morphism $!_c : 0 \to c$, and we also have a diagram

$$\begin{tikzcd} x \arrow[r, "\lambda_0"] \arrow[d, "\mathrm{id}_x"'] & 0 \arrow[d, "!_c"] \\ x \arrow[r, "\lambda_c"] & c \end{tikzcd}$$

which commutes, i.e., so that $\lambda_c \, \mathrm{id}_x = !_c \lambda_0$, equivalently $\lambda_c = !_c \lambda_0$. This means , however, that the diagram

$$\begin{tikzcd} x \arrow[rr, "\lambda_0"] \arrow[dr, "\lambda_c"'] & & 0 \arrow[dl, "!_c"] \\ & c & \end{tikzcd}$$

commutes for all objects c, hence that the morphism $\lambda_0 : x \to 0$ defines a morphism of cones over $\mathrm{id}_{\mathbf{C}}$ if we consider $! : 0 \Rightarrow \mathrm{id}_{\mathbf{C}}$ to be the cone over $\mathrm{id}_{\mathbf{C}}$ defined by components $!_c : 0 \to c$ for each $\mathbf{C}$-object c.

Since the cone $\lambda : x \Rightarrow \mathrm{id}_{\mathbf{C}}$ was arbitrary, this shows that $! : 0 \Rightarrow \mathrm{id}_{\mathbf{C}}$ is a limit cone; hence that the initial object 0 is the limit of the identity endofunctor. ■

3.8 Interactions between limits and colimits

Exercise 3.8.1.

Proof. ■

Exercise 3.8.2.

Proof. ■

4 Adjunctions

4.1 Adjoint functors

Exercise 4.1.1. Show that functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and bijections $\mathrm{Hom}_{\mathbf{D}}(F(c), d) \cong \mathrm{Hom}_{\mathbf{C}}(c, G(d))$ for each $c \in \mathbf{C}$ and $d \in \mathbf{D}$ define an adjunction if and only if these bijections induce a bijection between commutative squares (4.1.4). That is, prove Lemma 4.1.3.

Proof. Let F and G be as given. Suppose that for all $c, c' \in \mathbf{C}$, $d, d' \in \mathbf{D}$, and morphisms $h : c \to c'$ and $k : d \to d'$, we have that

$$\begin{tikzcd} F(c) \arrow[r, "f^\#"] \arrow[d, "F(h)"'] & d \arrow[d, "k"] & & c \arrow[r, "f^\dagger"] \arrow[d, "h"'] & G(d) \arrow[d, "G(k)"] \\ F(c') \arrow[r, "g^\#"'] & d' & \leftrightsquigarrow & c' \arrow[r, "g^\dagger"'] & G(d') \end{tikzcd}$$

the left-hand square commutes if and only if the right-hand square commutes. To prove that the pair F, G define an adjunction, we need only show that the isomorphism $\mathrm{Hom}_{\mathbf{D}}(F(c), d) \cong \mathrm{Hom}_{\mathbf{C}}(c, G(d))$ is natural in both c and d.

To show naturality in d, we fix some $c \in \mathbf{C}$ and let $k : d \to d'$ be some morphism in $\mathbf{D}$, and we aim to show that the square

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(F(c), d) \arrow[r, "\cong"] \arrow[d, "k\circ-"'] & \mathrm{Hom}_{\mathbf{C}}(c, G(d)) \arrow[d, "G(k)\circ-"] \\ \mathrm{Hom}_{\mathbf{D}}(F(c), d') \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbf{C}}(c, G(d')) \end{tikzcd}$$

commutes. So given a morphism $f^\# : F(c) \to d$, there is a unique $f^\dagger : c \to G(d)$ under the bijection on the top of the square. Similarly, given $kf^\# : F(c) \to d'$, there is a unique $(kf^\#)^\dagger : c \to G(d')$ under the bottom bijection. Commutativity now amounts to showing that $G(k)f^\dagger = (kf^\#)^\dagger$. But observe that the square

$$\begin{tikzcd} F(c) \arrow[r, "f^\#"] \arrow[d, "\mathrm{id}_{F(c)}"'] & d \arrow[d, "k"] \\ F(c) \arrow[r, "kf^\#"'] & d' \end{tikzcd}$$

obviously commutes, and this occurs (by our assumption above) if and only if the square

$$\begin{tikzcd} c \arrow[r, "f^\dagger"] \arrow[d, "\mathrm{id}_c"'] & G(d) \arrow[d, "G(k)"] \\ c \arrow[r, "(kf^\#)^\dagger"'] & G(d') \end{tikzcd}$$

commutes, i.e., if and only if $G(k)f^\dagger = (kf^\#)^\dagger$. Thus our original square consisting of hom-sets commutes, and so the isomorphism $\mathrm{Hom}_{\mathbf{D}}(F(c), d) \cong \mathrm{Hom}_{\mathbf{C}}(c, G(d))$ is natural in d.

On the other hand, naturality in c amounts to showing that for some morphism $h : c' \to c$, the square

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(F(c), d) \arrow[r, "\cong"] \arrow[d, "- \circ F(h)"'] & \mathrm{Hom}_{\mathbf{C}}(c, G(d)) \arrow[d, "- \circ h"] \\ \mathrm{Hom}_{\mathbf{D}}(F(c'), d) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbf{C}}(c', G(d)) \end{tikzcd}$$

commutes. We can follow an analogous process as to the above; simply take some morphism $f^\# : F(c) \to d$ and note that there is a unique morphism $f^\dagger : c \to G(d)$ undeer the top bijection. Likewise, under the bottom bijection, there is a unique morphism $(f^\# F(h))^\dagger : c' \to G(d)$. Commutativity of this square amounts now to showing that $(f^\# F(h))^\dagger = f^\dagger h$. But the square

$$\begin{tikzcd} F(c') \arrow[r, "f^\# F(h)"] \arrow[d, "F(h)"'] & d \arrow[d, "\mathrm{id}_d"] \\ F(c) \arrow[r, "f^\#"'] & d \end{tikzcd}$$

obviously commutes, and by our assumption this occurs if and only if the square

$$\begin{tikzcd} c' \arrow[r, "(f^\# F(h))^\dagger"] \arrow[d, "h"'] & G(d) \arrow[d, "\mathrm{id}_{G(d)}"] \\ c \arrow[r, "f^\dagger"'] & G(d) \end{tikzcd}$$

commutes, i.e., if and only if $(f^\# F(h))^\dagger = f^\dagger h$, which is precisely what we needed to show to prove commutativity of the hom-square above; hence the isomorphism in question is also natural in c, whence F and G form an adjunction.

For the converse, if we assume F and G form an adjunction, then if the square on the left below commutes

$$\begin{tikzcd} F(c) \arrow[r, "f^\#"] \arrow[d, "F(h)"'] & d \arrow[d, "k"] & & c \arrow[r, "f^\dagger"] \arrow[d, "h"'] & G(d) \arrow[d, "G(k)"] \\ F(c') \arrow[r, "g^\#"'] & d' & \leftrightsquigarrow & c' \arrow[r, "g^\dagger"'] & G(d') \end{tikzcd}$$

i.e., if we have that $kf^\# = g^\# F(h)$, then we can apply the functor G to obtain $G(kf^\#) = G(k)G(f^\#)$ and $G(g^\# F(h)) = G(g^\#)GF(h)$, to which $G(k)G(f^\#) = G(g^\# GF(h)$ holds. Now, via the bijection of hom-sets and naturality in both variables, we require $G(g^\#) = g^\dagger$ and $G(f^\#) = f^\dagger$, and also $GF(h) = h$.

Thus $kf^{\#} = g^{\#}F(h)$ if and only if $G(k)f^{\dagger} = g^{\dagger}h$, which is precisely the statement that the left-hand square above commutes if and only if the right-hand square above commutes. ■

Exercise 4.1.2.

Proof. ■

Exercise 4.1.3. Show that any triple of adjoint functors

$$\begin{tikzcd}[column sep=large] \mathbf{C} \arrow[r, "U" description] & \mathbf{D} \arrow[l, bend right=50, "L"', "\perp" description] \arrow[l, bend left=50, "R", "\perp" description] \end{tikzcd}$$

gives rise to a canonical adjunction $LU \dashv RU$ between the induced endofunctors of $\mathbf{C}$.

Proof. We would like to first show a bijection on the level of hom-sets:

$$\mathrm{Hom}_{\mathbf{C}}(LU(c), d) \cong \mathrm{Hom}_{\mathbf{C}}(c, RU(d)).$$

To do so, we first take a morphism $f^{\#} : LU(c) \to d$ in $\mathbf{C}$ and note, in particular, that under the adjunction $L \dashv U$ we have a bijection $\mathrm{Hom}_{\mathbf{C}}(L(c'), d') \cong \mathrm{Hom}_{\mathbf{D}}(c', U(d'))$ for all c' in $\mathbf{C}$ and d' in $\mathbf{D}$. In our particular case, taking $c' = U(c)$ and $d' = d$, the bijection takes the morphism $f^{\#}$ to a unique morphism $f^{\dagger} : U(c) \to U(d)$ in $\mathbf{D}$. Now, under the adjunction $U \dashv R$, we have a bijection of hom-sets $\mathrm{Hom}_{\mathbf{D}}(U(c'), d') \cong \mathrm{Hom}_{\mathbf{C}}(c', R(d'))$; hence in our case, taking $c' = c$ and $d' = U(d)$, the bijection takes the morphism $f^{\dagger}$ to a unique morphism $f^{\ddagger} : c \to RU(d)$. Tracing back, we have shown that morphisms $f^{\#} : LU(c) \to d$ in $\mathbf{C}$ are in bijective correspondence with morphisms $f^{\ddagger} : c \to RU(d)$; that is, we have exhibited a bijection of sets

$$\mathrm{Hom}_{\mathbf{C}}(LU(c), d) \cong \mathrm{Hom}_{\mathbf{C}}(c, RU(d)).$$

Since $LU : \mathbf{C} \leftrightarrows \mathbf{C} : RU$ are a pair of functors with the isomorphisms above for all $\mathbf{C}$-objects c and d, we can establish naturality of the isomorphism via Lemma 4.1.3 (proved in Exercise 4.1.1 above).

Thus in view of the lemma we need only show, for all morphisms $h : c \to c'$ and $k : d \to d'$ between objects of $\mathbf{C}$, that the square on the left below commutes if and only if the square on the right below commutes.

$$\begin{tikzcd} LU(c) \arrow[r, "f^{\#}"] \arrow[d, "LU(h)"'] & d \arrow[d, "k"] \\ LU(c') \arrow[r, "g^{\#}"'] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f^{\ddagger}"] \arrow[d, "h"'] & RU(d) \arrow[d, "RU(k)"] \\ c' \arrow[r, "g^{\ddagger}"'] & RU(d') \end{tikzcd}$$

Now, since $L \dashv U$, Lemma 4.1.3 asserts that the square on the left below commutes if and only if the square on the right below commutes:

$$\begin{tikzcd}
L(U(c)) \arrow[r, "f^\#"] \arrow[d, "L(U(h))"'] \arrow[dr, phantom, "\circlearrowleft"] & d \arrow[d, "k"] & & U(c) \arrow[r, "f^\dagger"] \arrow[d, "U(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & U(d) \arrow[d, "U(k)"] \\
L(U(c')) \arrow[r, "g^\#"'] & d' & \iff & U(c') \arrow[r, "g^\dagger"'] & U(d')
\end{tikzcd}$$

But similarly, the fact that $U \dashv R$ with the lemma once more implies that the square on the left hand side below commutes if and only if the square on the right commutes:

$$\begin{tikzcd}
U(c) \arrow[r, "f^\dagger"] \arrow[d, "U(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & U(d) \arrow[d, "U(k)"] & & c \arrow[r, "f^\ddagger"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & R(U(d)) \arrow[d, "R(U(k))"] \\
U(c') \arrow[r, "g^\dagger"'] & U(d') & \iff & c' \arrow[r, "g^\ddagger"'] & R(U(d'))
\end{tikzcd}$$

Taken together, these two observations give us the naturality of the original two squares for the composites LU and RU by the lemma; hence $LU \dashv RU$ holds, as desired. ∎

Exercise 4.1.4.

Proof. ∎

Exercise 4.1.5.

Proof. ∎

4.2 The unit and counit as universal arrows

Exercise 4.2.1. Prove that any pair of adjoint functors $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ restrict to define an equivalence between the full subcategories spanned by those objects $c \in \mathbf{C}$ and $d \in \mathbf{D}$ for which the components of the unit η_c and of the counit ϵ_d, respectively, are isomorphisms. XXXX(to do)XXXX

Proof. Let $\mathbf{C}_0$ and $\mathbf{D}_0$ denote the full subcategories of $\mathbf{C}$ and $\mathbf{D}$ for which the components of the unit and counit are isomorphisms, respectively. Let $F_0 := F|_{\mathbf{C}_0}$ and $G_0 := G|_{\mathbf{D}_0}$. It is at once clear that we have a functor $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$, since if c is a $\mathbf{C}_0$-object, then $\eta_c : c \to GF(c)$ is an isomorphism, and hence $F_0(c) = F(c)$ is a $\mathbf{D}_0$-object, since $\epsilon_{F(c)} = \epsilon F(c) : FGF(c) \to c$

To show that the equivalence of categories, we need only show that there exist natural isomorphisms $\alpha : \mathrm{id}_{\mathbf{C}_0} \cong G_0F_0$ and $\beta : F_0G_0 \cong \mathrm{id}_{\mathbf{D}_0}$.

Since the unit of the adjunction is a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow GF$, we have that

$$\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[d, "h"'] & GF(c) \arrow[d, "GF(h)"] \\
c' \arrow[r, "\eta_{c'}"] & GF(c')
\end{tikzcd}$$

commutes for all such morphisms $h : c \to c'$ and objects c and c'; that is, we have $\eta_{c'} h = GF(h)\eta_c$. Note, however, that if we restrict η to the identity endofunctor subcategory $\mathbf{C}_0$, we have a natural transformation $\eta' : \mathrm{id}_{\mathbf{C}_0} \Rightarrow G_0 F_0$. Moreover, since each component η'_c is an isomorphism by assumption, η' is a natural isomorphism $\eta' : \mathrm{id}_{\mathbf{C}_0} \cong G_0 F_0$.

In a completely analogous process, the counit ∎

Exercise 4.2.2.

Proof. ∎

Exercise 4.2.3.

Proof. ∎

Exercise 4.2.4.

Proof. ∎

Exercise 4.2.5.

Proof. ∎

4.3 Contravariant and multivariable adjoint functors

4.4 The calculus of adjunctions

4.5 Adjunctions, limits, and colimits

Exercise 4.5.1.

Proof. ∎

Exercise 4.5.2.

Proof. ∎

Exercise 4.5.3.

Proof. ∎

Exercise 4.5.4.

Proof. ∎

Exercise 4.5.5. Show that a morphism $f : x \to y$ in $\mathbf{C}$ is a monomorphism if and only if the square

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

is a pullback. Conclude that right adjoints preserve monomorphisms, and that left adjoints preserve epimorphisms.

Proof. Given a morphism $f : x \to y$ we can consider the identity morphisms on x to form the square

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

We observe that f is a monomorphism if and only if given two morphisms $g, h : z \rightrightarrows x$ such that $fg = fh$, we know that $g = h$. In other words, given the diagram on the left-hand side below commutes

$$\begin{tikzcd} z \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\ & x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, "f"] \\ & x \arrow[r, "f"'] & y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} z \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] \arrow[dr, "f=g"] & & \\ & x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, "f"] \\ & x \arrow[r, "f"'] & y \end{tikzcd}$$

if and only if the diagram on the right-hand side commutes. This is precisely asserting that x with the identity morphisms satisfies the universal property of the pullback; hence the square in question is a pullback square if and only if f is monic.

Now if we have an adjoint pair F and G, where $F \dashv G$, then since G is a right adjoint we know G preserves limits, and so must preserve pullbacks (since pullbacks are, in particular, limits of the diagram $\bullet \to \bullet \leftarrow \bullet$). Thus, if the pullback of $f : x \to y \leftarrow x : f$ exists, then G preserves it, and by what we have shown above this means G takes monomorphisms to monomorphisms.

Dualizing everything we have done above, we find in particular that a morphism $f : x \to y$ is epic if and only if the square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[d, "\mathrm{id}_y"] \\ y \arrow[r, "\mathrm{id}_y"'] & y \end{tikzcd}$$

is a pushout. Then since F is a left adjoint, it preserves colimits, and pushouts are a special case of colimits (the colimit of the diagram of shape $\bullet \leftarrow \bullet \to \bullet$), so F preserves pushouts, hence preserves epimorphisms. ∎

Exercise 4.5.6. Prove Lemma 4.5.13.

Proof. Suppose we have an adjunction

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

with unit $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow GF$ and counit $\epsilon : FG \Rightarrow \mathrm{id}_{\mathbf{D}}$.

Recall that the component of the counit at the object d is defined to be the transpose of $\mathrm{id}_{G(d)}$, which we denote by $\mathrm{id}^\dagger_{G(d)}$. Note that by naturality of the counit, Lemma 4.1.3 gives us that the diagram on the left-hand side below commutes if and only if the diagram on the right-hand side commutes.

$$FG(d) \xrightarrow{\mathrm{id}^\dagger_{G(d)}} d \underset{g}{\overset{f}{\rightrightarrows}} e \qquad \leftrightsquigarrow \qquad G(d) \xrightarrow{\mathrm{id}_{G(d)}} G(d) \underset{G(g)}{\overset{G(f)}{\rightrightarrows}} G(e)$$

We would like to show that $\mathrm{id}^\dagger_{G(d)}$ is an epimorphism if and only if G is faithful. To do this, suppose G is faithful and assume there exist morphisms $f, g : d \rightrightarrows e$ such that $f\,\mathrm{id}^\dagger_{G(d)} = g\,\mathrm{id}^\dagger_{G(d)}$; this is equivalent to the diagram on the left-hand side above commuting, hence we have that the right-hand side commutes, which means $G(f) = G(g)$. But now, since G is faithful, we have an injection on the level of hom-sets $\mathrm{Hom}_{\mathbf{D}}(d, e) \hookrightarrow \mathrm{Hom}_{\mathbf{C}}(G(d), G(e))$. Thus $G(f) = G(g)$ if and only if $f = g$, i.e., $\epsilon_d = \mathrm{id}^\dagger_{G(d)}$ is an epimorphism.

The converse is easily seen: suppose the components of the counit are epimorphisms. Now if $G(f) = G(g)$ where $G(f), G(g) : G(d) \to G(e)$, then the diagram on the right-hand side above commutes, to which the left one does as well, and since $\mathrm{id}^\dagger_{G(d)}$ is epic, we have $f = g$.

In almost the same manner, we can show that G is full if and only if the components of the counit are split monomorphisms. If we assume that G is full then we have a surjection on the level of hom-sets via $\mathrm{Hom}_{\mathbf{D}}(e, d) \twoheadrightarrow \mathrm{Hom}_{\mathbf{C}}(G(e), G(d))$. If we now assume $f, g : e \rightrightarrows d$ are two morphisms such that $\mathrm{id}^\dagger_{G(d)}\, f = \mathrm{id}^\dagger_{G(d)}\, g$, then

If ϵ_d is a split monomorphism for all $\mathbf{D}$-objects d, then each component ϵ_d has a one-sided inverse ϵ'_d such that $\epsilon'_d \epsilon_d = \mathrm{id}_{FG(d)}$. Now, given some arbitrary morphism $f : G(d) \to G(e)$ in $\mathbf{C}$, XXXXXXXXX ∎

Exercise 4.5.7.

Proof. ∎

Exercise 4.5.8.

Proof. ∎

4.6 Existence of adjoint functors

5 Monads and their Algebras

6 All Categories are Kan Extensions

7 Theorems in Category Theory