

Problems from Atiyah-Macdonald's  
*An Introduction to Commutative Algebra*

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## 1 Rings and Ideals

### 1.1 Exercise 1.1

Let  $x$  be a nilpotent element of a ring  $A$ . Show that  $1 + x$  is a unit of  $A$ . Deduce that the sum of a nilpotent element and a unit is a unit.

*Proof.* Let  $x$  be nilpotent, so that  $x$  is an element of the nilradical of  $A$ . Since the nilradical is the intersection of all prime ideals of  $A$ , we know  $x \in \mathfrak{p}$  for each such prime ideal  $\mathfrak{p}$  of  $A$ . Now assume that  $1 + x$  is not a unit. Then  $1 + x \in \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$  of  $A$ ; since  $\mathfrak{m}$  is, in particular, a prime ideal, we know that  $x \in \mathfrak{m}$ , which implies  $1 = 1 + x - x \in \mathfrak{m}$  by closure, and hence that  $\mathfrak{m} = (1)$ , a contradiction. Thus  $x + 1$  is a unit of  $A$ . The same argument applies if we take the sum of  $x$  and a unit in  $A$ . ■

## 1.2 Exercise 1.4

*In the ring  $A[x]$ , the Jacobson radical is equal to the nilradical.*

*Proof.* In general, we know that the nilradical is contained in the Jacobson radical, the former being the intersection of all prime ideals of the ring and the latter being the intersection of all maximal ideals (each of which is prime). In particular, we immediately know that  $\text{nil}(A[x]) \subseteq \text{Jac}(A[x])$ . So it suffices to prove the reverse inclusion.

To this end, take some  $f(x) \in \text{Jac}(A[x])$ . From Proposition 1.9, this is true if and only if  $1 - f(x)g(x)$  is a unit in  $A[x]$  for all  $g(x) \in A[x]$ . In particular, taking  $g(x) = x$ , we can see that

$$1 - f(x)x = 1 - (a_0 + a_1x + \cdots + a_nx^n)x = 1 - a_0x - a_1x^2 - \cdots - a_nx^{n+1}$$

is a unit in  $A[x]$ . This occurs (by Exercise 1.2(i)) if and only if 1 is a unit and  $a_0, \dots, a_n$  are nilpotent. Then, by part (ii) of the same exercise, this means that

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

is nilpotent (since all of the coefficients are nilpotent in  $A$ ); hence  $f(x) \in \text{nil}(A[x])$  holds true, and since  $f(x)$  was arbitrary in the Jacobson radical, we have the desired result. ■

### 1.3 Exercise 1.6

A ring  $A$  is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element  $e$  such that  $e^2 = e \neq 0$ ). Prove that the nilradical and the Jacobson radical of  $A$  are equal.

*Proof.* Let  $A$  be a ring such as in the problem description. As before, we automatically have that  $\text{nil}(A) \subseteq \text{Jac}(A)$ , so it suffices to prove the reverse inclusion. So take some  $0 \neq a \in \text{Jac}(A)$  and suppose that  $a \notin \text{nil}(A)$ . Since the nilradical of  $A$  is the intersection of all prime ideals of  $A$ , we know that since  $a$  is not an element of the nilradical, there exists some prime ideal, say  $\mathfrak{p}$ , of  $A$ , such that  $a \notin \mathfrak{p}$ .

Before proceeding, we note that since  $a \notin \text{nil}(A)$  we have  $(a) \not\subseteq \text{nil}(A)$ . By assumption then, we know that  $(a)$  contains a non-zero idempotent, say  $e$ . Since  $e \in (a)$  we know  $e = ab$  for some non-zero  $b \in A$ . Since  $e$  is idempotent, we have

$$e^2 = e \iff (ab)^2 = ab \iff a^2b^2 = ab$$

Subtracting  $a^2bb^2$  from both sides of the equation on the right hand side above, we obtain

$$ab - a^2b^2 = 0 \iff (1 - ab)ab = 0$$

Since  $a \in \text{Jac}(A)$  by assumption, and we have Proposition 1.9, which states that  $1 - ax$  is a unit for all  $x \in A$ , we can see that in our particular case we have that  $1 - ab$  is a unit. Since the units of  $A$  form a multiplicative abelian group, we know then that  $ab$  is not a unit, for if it were then the product on the right hand side above (namely, 0) would be a unit, which is clearly absurd.

Note that 0 is contained in all ideals, in particular all prime ideals, of  $A$ . So we have  $0 \in \mathfrak{p}$ , and in particular, we have that

$$(1 - ab)ab = 0 \in \mathfrak{p}$$

which implies that either  $1 - ab$  or  $ab$  lies in  $\mathfrak{p}$ . But note that  $1 - ab$  is a unit in  $A$ , and hence if  $1 - ab$  were contained in  $\mathfrak{p}$  we would have  $\mathfrak{p} = (1)$ , a contradiction; hence we require that  $ab \in \mathfrak{p}$  hold true. But again, this implies that either  $a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$ , and the former case is impossible since we assumed that  $a \notin \mathfrak{p}$ ; thus we require that  $b \in \mathfrak{p}$ . ■

## 1.4 Exercise 1.7

Let  $A$  be a ring in which every element  $x$  satisfies  $x^n = x$  for some  $n > 1$  (depending on  $x$ ). Show that every prime ideal of  $A$  is maximal.

*Proof.* Let  $A$  be such a ring as in the problem description, and let  $\mathfrak{p}$  be a prime ideal of  $A$ . To show that  $\mathfrak{p}$  is maximal, we show that  $A/\mathfrak{p}$  is a field. So take some non-zero  $\bar{x}$  in  $A/\mathfrak{p}$ . We know that the representative  $x \in A$  satisfies  $x^n = x$  for some  $n > 1$ . If  $n = 2$  then  $x^2 = x$  holds and we know that

$$x^2 - x = 0 \iff x(x - 1) = 0$$

and hence descending to the quotient, we have

$$\bar{x} \cdot \overline{(x - 1)} = \bar{0}$$

Now, since  $A/\mathfrak{p}$  is an integral domain (as  $\mathfrak{p}$  is prime), either  $\bar{x} = \bar{0}$  or  $\overline{x - 1} = \bar{0}$  must hold. Since  $\bar{x}$  was assumed non-zero in  $A/\mathfrak{p}$ , we thus have  $\overline{x - 1} = \bar{0}$ , and hence that  $\bar{x} = \bar{1}$ , clearly a unit in  $A/\mathfrak{p}$ .

Now we dispense with the case where  $n > 2$ . So we have the same relations,

$$x^n = x \iff x^n - x = 0 \iff x(x^{n-1} - 1) = 0$$

and again we descend to the quotient to get

$$\bar{x} \cdot \overline{(x^{n-1} - 1)} = \bar{0}$$

and once more since  $A/\mathfrak{p}$  is an integral domain, either  $\bar{x} = \bar{0}$  or  $\overline{x^{n-1} - 1} = \bar{0}$  holds; since  $\bar{x}$  was assumed non-zero in  $A/\mathfrak{p}$ , we require that

$$\overline{x^{n-1} - 1} = \bar{0} \implies \overline{x^{n-1}} = \bar{1}$$

Now we can see that

$$\bar{1} = \overline{x^{n-1}} = \overline{xx^{n-2}} = \bar{x} \cdot \overline{x^{n-2}}$$

where we are allowed to consider  $x^{n-2}$  since  $n > 2$ . In fact, we have found a multiplicative inverse for  $\bar{x}$  in  $A/\mathfrak{p}$ , which is  $\overline{x^{n-2}}$ . Since  $\bar{x}$  was an arbitrary non-zero element of  $A/\mathfrak{p}$ , and we have shown that either  $\bar{x}$  is the identity in  $A/\mathfrak{p}$  or a unit in  $A/\mathfrak{p}$ , we have proved that  $A/\mathfrak{p}$  is a field, hence that  $\mathfrak{p}$  is maximal. ■

### 1.5 Exercise 1.10

Let  $A$  be a ring,  $\text{nil}(A)$  its nilradical. Show that the following are equivalent:

- i)  $A$  has exactly one prime ideal;
- ii) every element of  $A$  is either a unit or nilpotent;
- iii)  $A/\text{nil}(A)$  is a field.

*Proof.* (iii)  $\implies$  (i). If  $A/\text{nil}(A)$  is a field then  $\text{nil}(A)$  is a maximal ideal of  $A$ . Now, if  $\mathfrak{p}$  was any other prime ideal of  $A$ , then  $\text{nil}(A) \subseteq \mathfrak{p}$  since the nilradical is the intersection of all prime ideals of  $A$  by Proposition 1.8; since  $\text{nil}(A)$  is maximal, this means  $\text{nil}(A) = \mathfrak{p}$ , and hence  $\text{nil}(A)$  is the unique prime ideal of  $A$ , proving (i).

(i)  $\implies$  (ii). If  $A$  has exactly one prime ideal, say  $\mathfrak{p}$ , then since the nilradical of  $A$  is the intersection of all prime ideals of  $A$ , it follows that  $\mathfrak{p} = \text{nil}(A)$ , i.e., that the nilradical of  $A$  is a prime ideal of  $A$ .

Let  $x \in A$  be some ring element. Two cases arise: either  $x$  is a unit or  $x$  is not a unit. If  $x$  is not a unit, then by Corollary 1.5 we know that  $x$  is contained in some maximal ideal of  $A$ , which must in fact be  $\text{nil}(A)$  since any maximal ideal is necessarily prime, and  $\text{nil}(A)$  is the unique prime ideal of  $A$ . Thus every ring element is either a unit or a nilpotent element, giving (ii).

(ii)  $\implies$  (iii). We aim to show that  $\text{nil}(A)$  is a maximal ideal of  $A$ , which will give (iii). If  $\text{nil}(A) = (1)$  then  $A/\text{nil}(A) \cong 0$  is a field, so assume  $\text{nil}(A) \neq (1)$ . We have a ring  $A$  and  $\text{nil}(A) \neq (1)$  an ideal of  $A$  such that every  $x \in A \setminus \text{nil}(A)$  is a unit, since by assumption every element of  $A$  is either a unit or nilpotent, so if not nilpotent, then a unit. Proposition 1.6 asserts that  $A$  is a local ring and  $\text{nil}(A)$  its maximal ideal, giving (iii). ■

## 1.6 Exercise 1.11

A ring  $A$  is Boolean if  $x^2 = x$  for all  $x \in A$ . In a Boolean ring  $A$ , show that

- i)  $2x = 0$  for all  $x \in A$ ;
- ii) every prime ideal  $\mathfrak{p}$  is maximal, and  $A/\mathfrak{p}$  is a field with two elements;
- iii) every finitely generated ideal in  $A$  is principal.

*Proof.* (i) For any  $x \in A$  in a Boolean ring  $A$ , we have that  $-x = x$ , which can be seen since

$$-x = (-x)^2 = (-x)(-x) = x^2 = x$$

As such, we have the following result:

$$0 = x - x = x + x = 2x$$

which was the desired statement to be shown.

(ii) Exercise 1.7 with  $n = 2$  gives that every prime ideal of  $A$  is maximal. Let  $\mathfrak{p}$  be such a prime ideal of  $A$ . We want to show that  $A/\mathfrak{p}$ , which we already know is a field since  $\mathfrak{p}$  is maximal, has two elements. To do this, note that any element  $\bar{x} \in A/\mathfrak{p}$  satisfies

$$\bar{0} = \bar{x} - \bar{x} = \bar{x} - \bar{x}^2 = (\bar{1} - \bar{x})\bar{x}$$

and so either  $\bar{x} = \bar{1}$ , and  $x$  does not lie in  $\mathfrak{p}$ , or  $\bar{x} = \bar{0}$ , and hence  $x$  does lie in  $\mathfrak{p}$ . In particular,  $A/\mathfrak{p}$  has two elements.

(iii) We use induction on the number of generators for an ideal of  $A$ . The case for  $n = 1$  is trivial since the ideal is principal; taking this as our base case, suppose every ideal generated by  $n - 1$  elements is principal. Let  $\mathfrak{a} = (a_1, \dots, a_n)$  be a finitely generated ideal of  $A$ . Let  $(b) = (a_1, \dots, a_{n-1})$ . Set

$$c = b + a_n - ba_n$$

Then we have that

$$bc = b^2 + bc - b^2c = b + bc - bc = b$$

$$a_nc = a_nb + a_n^2 - ba_n^2 = a_n + a_nb - a_nb = a_n$$

In particular, both  $b$  and  $a_n$  can be written as multiples of  $c$ , and hence

$$(c) = (b, a_n) = (a_1, \dots, a_n) = \mathfrak{a}$$

is a principal ideal. By induction, the statement is proved. ■

### 1.7 Exercise 1.12

*A local ring contains no idempotent  $\neq 0, 1$ .*

*Proof.* Let  $A$  be a local ring with unique maximal ideal  $\mathfrak{m}$ . Assume  $x \in A$  is an idempotent. Then  $x^2 = x$  holds. As such, we have

$$0 = x - x = x - x^2 = x(x - 1)$$

Two solutions are obviously  $x = 0$  or  $x = 1$ . If  $x \neq 0, 1$ , however, then  $x$  must be a zero divisor, hence not a unit. By Corollary 1.5, we have that  $x$  must be contained in some maximal ideal of  $A$ , of which there is only one,  $\mathfrak{m}$ , since  $A$  is local. Thus  $x \in \mathfrak{m}$  holds.

With the idempotent  $x$  we also have the equation

$$(-x + 1)^2 = (-x)^2 - 2x + 1 = x^2 - 2x + 1 = x - 2x + 1 = -x + 1$$

and hence  $-x + 1$  is also an idempotent in  $A$ . Moreover, we can see that  $-x + 1 \neq 0, 1$ , since we are assuming that  $x \neq 0, 1$ , and hence  $-x + 1$  is a zero divisor, thus not a unit. By the same corollary, we require that  $-x + 1$  be contained in  $\mathfrak{m}$ .

But now we have  $x \in \mathfrak{m}$  and  $-x + 1 \in \mathfrak{m}$ , and so by closure of ideals under addition, we require that

$$x + (-x + 1) = x - x + 1 = 1 \in \mathfrak{m}$$

hold, to which  $(1) = \mathfrak{m}$ , a contradiction. Thus no idempotents exist in  $A$  which are not either 0 or 1. ■

## 1.8 Exercise 1.15

Let  $A$  be a ring and let  $X$  be the set of all prime ideals of  $A$ . For each subset  $E$  of  $A$ , let  $V(E)$  denote the set of all prime ideals of  $A$  which contain  $E$ . Prove that

(i) if  $\mathfrak{a}$  is the ideal generated by  $E$ , then  $V(E) = V(\mathfrak{a}) = V(\text{rad}(\mathfrak{a}))$ .

(ii)  $V(0) = X$ ,  $V(1) = \emptyset$ .

(iii) if  $\{E_i\}_{i \in I}$  is any family of subsets of  $A$ , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i)$$

(iv)  $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$  for any ideals  $\mathfrak{a}, \mathfrak{b}$  of  $A$ . These results show that the sets  $V(E)$  satisfy the axioms for closed sets in a topological space. The resulting topology is called the *\*Zariski topology\**. The topological space  $X$  is called the *prime spectrum* of  $A$ , and is written  $\text{Spec}(A)$ .

*Proof.* (i) We first show that if  $E_1 \subseteq E_2 \subseteq A$  then  $V(E_2) \subseteq V(E_1)$ . This is obvious, for if  $\mathfrak{p} \in V(E_2)$  then  $E_2 \subseteq \mathfrak{p}$  and hence  $E_1 \subseteq E_2 \subseteq \mathfrak{p}$ , to which  $\mathfrak{p} \in V(E_1)$ . Now, if we let  $\mathfrak{a} = \langle E \rangle$ , then clearly  $E \subseteq \mathfrak{a}$ , and hence  $V(\mathfrak{a}) \subseteq V(E)$ . For the reverse containment, if  $\mathfrak{p} \in V(E)$  then  $\sum_i r_i e_i \in \mathfrak{p}$  for all  $r_i \in A$  and  $e_i \in E$  by closure, and this is an arbitrary element of  $\mathfrak{a}$ , and hence  $\mathfrak{a} \subseteq \mathfrak{p}$ , giving  $\mathfrak{p} \in V(\mathfrak{a})$ , and hence  $V(E) = V(\mathfrak{a})$ .

We always have that  $\mathfrak{a} \subseteq \text{rad}(\mathfrak{a})$ , and hence  $V(\text{rad}(\mathfrak{a})) \subseteq V(\mathfrak{a})$ . If  $\mathfrak{p} \in V(\mathfrak{a})$  and  $x \in \text{rad}(\mathfrak{a})$ , then  $x^k \in \mathfrak{a} \subseteq \mathfrak{p}$  for some  $k \in \mathbb{Z}^+$ , and since  $\mathfrak{p}$  is prime, either  $x$  or  $x^{k-1}$  lies in  $\mathfrak{p}$ , and so on until we find that  $x \in \mathfrak{p}$ ; hence  $\text{rad}(\mathfrak{a}) \subseteq \mathfrak{p}$ , proving that  $V(\mathfrak{a}) \subseteq V(\text{rad}(\mathfrak{a}))$ , and giving us set equality.

(ii) Clearly any prime ideal of  $A$  contains 0, and hence  $V(0) = X$  holds. Similarly, no prime ideal of  $A$  contains  $(1) = A$ , since each is proper; hence  $V(1) = \emptyset$ .

(iii) Given a family of subsets  $\{E_i\}_{i \in I}$  of  $A$ , we always have  $E_j \subseteq \bigcup_{i \in I} E_i$  for all  $j \in I$ , and hence  $V(\bigcup_{i \in I} E_i) \subseteq V(E_j)$  for all  $j \in I$ , whence  $V(\bigcup_{i \in I} E_i) \subseteq \bigcap_{i \in I} V(E_i)$ . Now if  $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$  then  $\mathfrak{p} \in V(E_i)$  for all  $i \in I$ , and so  $E_i \subseteq \mathfrak{p}$  for all  $i \in I$ , whence  $\bigcup_{i \in I} E_i \subseteq \mathfrak{p}$ , and so  $\mathfrak{p} \in V(\bigcup_{i \in I} E_i)$ , which gives us the reverse equality.

(iv) For ideals  $\mathfrak{a}, \mathfrak{b}$  of  $A$ , we always have that  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ , hence we require that  $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ . Now, if  $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$  then  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ ; if we assume  $x \in \mathfrak{a} \cap \mathfrak{b}$ , then  $x^2 \in \mathfrak{a}\mathfrak{b}$  and so  $x^2 \in \mathfrak{p}$ , to which  $x \in \mathfrak{p}$  holds, giving  $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$ , and so we require that  $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ , giving  $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ .

Suppose now that  $\mathfrak{p} \in V(\mathfrak{a}) \cap V(\mathfrak{b})$ . Then  $\mathfrak{a} \subseteq \mathfrak{p}$  and  $\mathfrak{b} \subseteq \mathfrak{p}$  and clearly then  $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$ , to which  $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ ; thus  $V(\mathfrak{a}) \cap V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$ . If  $\mathfrak{p} \in V(\mathfrak{ab})$  then  $\mathfrak{ab} \subseteq \mathfrak{p}$ ; now if we assume, for contradiction, that there exists  $a \in \mathfrak{a}$  and  $b \in \mathfrak{b}$  for which  $a, b \notin \mathfrak{p}$ , then  $ab \in \mathfrak{ab} \subseteq \mathfrak{p}$  means  $ab \in \mathfrak{p}$  and hence either  $a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$  since  $\mathfrak{p}$  is a prime ideal, a contradiction. Thus both  $\mathfrak{a}$  and  $\mathfrak{b}$  are contained in  $\mathfrak{p}$ , and hence  $\mathfrak{p} \in V(\mathfrak{a}) \cap V(\mathfrak{b})$ , giving  $V(\mathfrak{ab}) \subseteq V(\mathfrak{a}) \cap V(\mathfrak{b})$ .

In particular, we have shown

$$V(\mathfrak{a}) \cap V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) \subseteq V(\mathfrak{a}) \cap V(\mathfrak{b})$$

whence we have equality throughout, proving the desired statement. ■

## 1.9 Exercise 1.18

For psychological reasons it is sometimes convenient to denote a prime ideal of  $A$  by a letter such as  $x$  or  $y$  when thinking of it as a point of  $X = \text{Spec}(A)$ . When thinking of  $x$  as a prime ideal of  $A$ , we denote it by  $\mathfrak{p}_x$  (logically, of course, it is the same thing). Show that

- i) the set  $\{x\}$  is closed (we say that  $x$  is a "closed point") in  $\text{Spec}(A) \iff \mathfrak{p}_x$  is maximal;
- ii)  $\overline{\{x\}} = V(\mathfrak{p}_x)$ ;
- iii)  $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$ ;
- iv)  $X$  is a  $T_0$ -space (this means that if  $x, y$  are distinct points of  $X$ , then either there is a neighborhood of  $x$  which does not contain  $y$ , or else there is a neighborhood of  $y$  which does not contain  $x$ ).

*Proof.* (i) Suppose  $\mathfrak{p}$  is a maximal ideal; then the only ideals of  $A$  containing  $\mathfrak{p}$  are  $\mathfrak{p}$  and  $A$  itself. In particular, the only prime ideal containing  $\mathfrak{p}$  is  $\mathfrak{p}$ , whence  $V(\mathfrak{p}) = \{\mathfrak{p}\}$ , and so the singleton  $\{\mathfrak{p}\}$  is closed. Conversely, if  $\{\mathfrak{p}\}$  is closed in  $X = \text{Spec}(A)$ , then  $\{\mathfrak{p}\} = V(\mathfrak{a})$  for some ideal  $\mathfrak{a}$  of  $A$ . Now assume  $\mathfrak{b}$  is an ideal containing  $\mathfrak{p}$ . If  $\mathfrak{b} \neq (1) = A$ , then from Corollary 1.4 we know there exists some maximal ideal  $\mathfrak{m}$  containing  $\mathfrak{b}$ ; hence we have

$$\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{b} \subseteq \mathfrak{m}$$

to which (since  $\mathfrak{m}$  is maximal, hence prime) we have  $\mathfrak{m} \in V(\mathfrak{a}) = \{\mathfrak{p}\}$ , which forces  $\mathfrak{m} = \mathfrak{p}$ . Thus  $\mathfrak{p}$  is indeed maximal.

(ii) Clearly  $V(\mathfrak{p})$  is a closed set containing  $\{\mathfrak{p}\}$ , and so we automatically have  $\overline{\{\mathfrak{p}\}} \subseteq V(\mathfrak{p})$  since the closure of a set is the intersection of all closed sets containing that set. For the reverse containment, suppose we have some  $\mathfrak{q} \in V(\mathfrak{p})$ . By definition of the closure of a set, to prove our desired statement it suffices to show that  $\mathfrak{q}$  is contained in all closed sets containing  $\{\mathfrak{p}\}$ , so  $\mathfrak{q} \in V(\mathfrak{a})$  for all  $\mathfrak{a} \subseteq \mathfrak{p}$ . But this is obvious, for if  $\{\mathfrak{p}\} \subseteq V(\mathfrak{a})$  then  $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{q}$  means  $\mathfrak{q} \in V(\mathfrak{a})$  since  $\mathfrak{q}$  is a prime ideal.

(iii) For the forward implication, assume  $\mathfrak{p}_y \in \overline{\{\mathfrak{p}_x\}}$ . Then  $\mathfrak{p}_y$  lies in all closed sets of  $X$  containing the point  $\mathfrak{p}_x$ . In particular, since  $V(\mathfrak{p}_x)$  is a closed set containing  $\mathfrak{p}_x$ , we have that  $\mathfrak{p}_y \in V(\mathfrak{p}_x)$ , hence  $\mathfrak{p}_x \subseteq \mathfrak{p}_y$  by definition. Conversely, if  $\mathfrak{p}_x \subseteq \mathfrak{p}_y$  then for any closed set containing  $\mathfrak{p}_x$ , say  $V(\mathfrak{a})$ , we have that  $\mathfrak{a} \subseteq \mathfrak{p}_x \subseteq \mathfrak{p}_y$ , to which  $\mathfrak{p}_y \in V(\mathfrak{a})$ ; hence  $\mathfrak{p}_y$  lies in all closed sets containing  $\mathfrak{p}_x$ , which means  $\mathfrak{p}_y \in \overline{\{\mathfrak{p}_x\}}$ .

(iv) Let  $\mathfrak{p}_x$  and  $\mathfrak{p}_y$  be two distinct points of  $X$ . Consider the closed set  $V(\mathfrak{p}_x)$ . There are two cases: either  $\mathfrak{p}_y \in V(\mathfrak{p}_x)$  or  $\mathfrak{p}_y \notin V(\mathfrak{p}_x)$ .

In the first case,  $\mathfrak{p}_y \in V(\mathfrak{p}_x)$  implies  $\mathfrak{p}_x \subseteq \mathfrak{p}_y$ , and in fact this is a strict containment since we assumed the points were distinct, so  $\mathfrak{p}_x \subset \mathfrak{p}_y$ . Clearly then we have  $\mathfrak{p}_y \not\subseteq \mathfrak{p}_x$ , and so

$\mathfrak{p}_x \notin V(\mathfrak{p}_y)$ . Thus  $\mathfrak{p}_x \in U_x = X \setminus V(\mathfrak{p}_y)$ , and  $U_x$  is an open set containing  $\mathfrak{p}_x$  but not containing  $\mathfrak{p}_y$  by construction.

In the second case,  $\mathfrak{p}_y \notin V(\mathfrak{p}_x)$  implies that  $\mathfrak{p}_y \in U_y = X \setminus V(\mathfrak{p}_x)$ , and  $U_y$  is an open subset of  $X$  which contains  $\mathfrak{p}_y$  but does not contain  $\mathfrak{p}_x$ .

In any case, we have that  $X$  is a  $T_0$ -space. ■

### 1.10 Exercise 1.19

A topological space  $X$  is said to be irreducible if  $X \neq \emptyset$  and if every pair of non-empty open sets in  $X$  intersect, or equivalently if every non-empty open set is dense in  $X$ . Show that  $\text{Spec}(A)$  is irreducible if and only if the nilradical of  $A$  is a prime ideal.

I use the following exercise from Dummit and Foote in the proof of this exercise:

**Exercise 15.2.2** Let  $I$  and  $J$  be ideals in the ring  $R$ . Prove the following statements:

- (a) If  $I^k \subseteq J$  for some  $k \geq 1$  then  $\text{rad}(I) \subseteq \text{rad}(J)$ .
- (b) If  $I^k \subseteq J \subseteq I$  for some  $k \geq 1$  then  $\text{rad}(I) = \text{rad}(J)$ .
- (c)  $\text{rad}(IJ) = \text{rad}(I \cap J) = \text{rad}(I) \cap \text{rad}(J)$ .
- (d)  $\text{rad}(\text{rad}(I)) = \text{rad}(I)$ .
- (e)  $\text{rad}(I) + \text{rad}(J) \subseteq \text{rad}(I + J)$  and  $\text{rad}(I + J) = \text{rad}(\text{rad}(I) + \text{rad}(J))$ .

*Proof.* (a) If  $I^k \subseteq J$  for some  $k \geq 1$  then if  $x \in \text{rad}(I)$  we have  $x^m \in I$  for some  $m \geq 1$ , and so  $(x^m)^k = x^{mk} \in J$  must hold, to which  $x \in \text{rad}(J)$ ; hence  $\text{rad}(I) \subseteq \text{rad}(J)$ .

(b) If we go further and assume  $I^k \subseteq J \subseteq I$  for some  $k \geq 1$ , then if  $x \in \text{rad}(J)$  we have that  $x^m \in J$  for some  $m \geq 1$ , and hence  $x^m \in I$  since  $J \subseteq I$ ; thus  $\text{rad}(J) \subseteq \text{rad}(I)$ . Combined with part (a) above, this gives us the desired equality.

(c) First, note that if  $x \in \text{rad}(IJ)$  then  $x^k \in IJ$  for some  $k \geq 1$ , and since, in general, we have that  $IJ \subseteq I \cap J$ , this means that  $x^k \in I \cap J$  and hence that  $x \in \text{rad}(I \cap J)$ , giving us that  $\text{rad}(IJ) \subseteq \text{rad}(I \cap J)$ . Secondly, if we assume  $x \in \text{rad}(I \cap J)$  then  $x^k \in I \cap J$  for some  $k \geq 1$ , and in particular  $x^k \in I$  and  $x^k \in J$ , and so  $x \in \text{rad}(I)$  and  $x \in \text{rad}(J)$ , whence  $x \in \text{rad}(I) \cap \text{rad}(J)$ ; hence  $\text{rad}(I \cap J) \subseteq \text{rad}(I) \cap \text{rad}(J)$  holds true. Lastly, if we assume that  $x \in \text{rad}(I) \cap \text{rad}(J)$  then  $x^k \in I$  and  $x^m \in J$  for some  $k, m \geq 1$ . We can easily see that  $x^{k+m} = x^k x^m \in IJ$ , whence  $x \in \text{rad}(IJ)$ , and hence  $\text{rad}(I) \cap \text{rad}(J) \subseteq \text{rad}(IJ)$ . Tracing through the inclusions we have shown proves that the desired equality holds.

(d) If  $x \in \text{rad}(\text{rad}(I))$  then  $x^k \in \text{rad}(I)$  for some  $k \geq 1$ , and hence  $(x^k)^m \in I$  for some  $m \geq 1$  by definition of the radical of an ideal; in particular,  $x^{km} \in I$  and so  $x \in \text{rad}(I)$ ; thus  $\text{rad}(\text{rad}(I)) \subseteq \text{rad}(I)$ . Now, since in general an ideal is contained in its radical, we trivially have that  $\text{rad}(I) \subseteq \text{rad}(\text{rad}(I))$ , whence the equality.

(e) If  $x \in \text{rad}(I) + \text{rad}(J)$  then  $x = y + z$  for some  $y \in \text{rad}(I)$  and  $z \in \text{rad}(J)$ . By definition of the radical, there exists  $k, m \geq 1$  for which  $y^k \in I$  and  $z^m \in J$ . Since  $R$  is a commutative ring

with 1, the binomial theorem holds, see Exercise 7.3.25, and so setting  $n = m + k$  gives:

$$x^n = (y + z)^n = \sum_{i=0}^n \binom{n}{i} y^i z^{n-i} \in I + J$$

Hence  $x \in \text{rad}(I + J)$ , whence  $\text{rad}(I) + \text{rad}(J) \subseteq \text{rad}(I + J)$  holds true.

Now, we claim that  $I + J \subseteq \text{rad}(I) + \text{rad}(J)$ . This can be seen for if  $x \in I + J$  then  $x = y + z$  for  $y \in I$  and  $z \in J$ , and clearly then  $y \in \text{rad}(I)$  and  $z \in \text{rad}(J)$  since in general an ideal is contained in its radical; in particular,  $x \in \text{rad}(I) + \text{rad}(J)$ , proving the claim. Now from part (a) we have that  $\text{rad}(I + J) \subseteq \text{rad}(\text{rad}(I) + \text{rad}(J))$ .

This inclusion, combined with that of the previous paragraph, gives us the desired equality of sets. ■

Now onto the proof of Exercise 1.19:

*Proof.* An equivalent condition for  $X = \text{Spec}(A)$  to be irreducible is for no decomposition of  $X$  into the union of two non-empty proper closed sets of  $X$  to exist. From Exercise 1.17(iv) we know that a proper closed subset  $V(\mathfrak{a})$  satisfies

$$V(\mathfrak{a}) \subset V(0) = X$$

which occurs if and only if we have the following relation amongst the radicals of the ideals:

$$\text{rad}(\mathfrak{a}) \subsetneq \text{rad}(0) = \text{nil}(A)$$

But note that  $\text{rad}(\mathfrak{a}) \subsetneq \text{nil}(A)$  if and only if  $\mathfrak{a} \subsetneq \text{nil}(A)$ , with one direction obvious and the other given by the proof from the exercise in Dummit and Foote above.

Now by Exercise 1.15 we know that  $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{ab})$ , and hence  $X$  is irreducible if and only if for all ideals  $\mathfrak{a}, \mathfrak{b}$  of  $A$  with  $\mathfrak{a}, \mathfrak{b} \not\subseteq \text{nil}(A)$  we have  $\mathfrak{ab} \not\subseteq \text{nil}(A)$ . The contrapositive of this statement shows that  $\mathfrak{ab} \in \text{nil}(A)$  if and only if either  $\mathfrak{a} \subseteq \text{nil}(A)$  or  $\mathfrak{b} \subseteq \text{nil}(A)$ . Considering principal ideals generated by a single element, this is equivalent to  $\text{nil}(A)$  being a prime ideal of  $A$ . ■

### 1.11 Exercise 1.20

Let  $X$  be a topological space.

- i) If  $Y$  is an irreducible (Exercise 19) subspace of  $X$ , then the closure  $\overline{Y}$  of  $Y$  in  $X$  is irreducible.
- ii) Every irreducible subspace of  $X$  is contained in a maximal irreducible subspace.
- iii) The maximal irreducible subspaces of  $X$  are closed and cover  $X$ . They are called the irreducible components of  $X$ . What are the irreducible components in a Hausdorff space?
- iv) If  $A$  is a ring and  $X = \text{Spec}(A)$ , then the irreducible components of  $X$  are the closed sets  $V(\mathfrak{p})$ , where  $\mathfrak{p}$  is a minimal prime ideal of  $A$  (Exercise 8).

*Proof.* (i) If we assume that  $\overline{Y}$  is not irreducible, that is, if  $\overline{Y} = C_1 \cup C_2$  for some non-empty closed sets  $C_1, C_2$  in  $\overline{Y}$ , then since  $Y \subseteq \overline{Y}$  always holds, we have that:

$$Y = Y \cap \overline{Y} = Y \cap (C_1 \cup C_2) = (Y \cap C_1) \cup (Y \cap C_2) = D_1 \cup D_2$$

where  $D_1$  and  $D_2$  are closed sets in the subspace topology on  $Y$  induced from  $\overline{Y}$ ; hence  $Y$  is not irreducible. Taking the contrapositive of this implication yields the desired result.

(ii) Let  $\Sigma$  be the set of all irreducible subspaces of  $X$  ordered by inclusion. For an arbitrary chain of elements in  $\Sigma$ , say the chain  $\{Y_\alpha\}_{\alpha \in \mathcal{I}}$  where  $\mathcal{I}$  is some index set, we know that  $\mathcal{Y} = \bigcup_{\alpha \in \mathcal{I}} Y_\alpha$  is a subspace of  $X$ . Let  $U_1$  and  $U_2$  be any two non-empty open subsets of  $\mathcal{Y}$ . Since  $U_1$  and  $U_2$  are non-empty, it follows that there exists  $\alpha, \beta \in \mathcal{I}$  for which  $Y_\alpha \cap U_1 \neq \emptyset$  and  $Y_\beta \cap U_2 \neq \emptyset$ . Without loss of generality, we may take  $\alpha \leq \beta$  in  $\mathcal{I}$ , and hence  $Y_\alpha \subseteq Y_\beta$  holds, to which  $Y_\beta \cap U_1 \neq \emptyset$ .

Note that now, since  $Y_\beta$  is irreducible, and we have two non-empty open subsets  $Y_\beta \cap U_1$  and  $Y_\beta \cap U_2$  in  $Y_\beta$ , we require that

$$Y_\beta \cap U_1 \cap U_2 = (Y_\beta \cap U_1) \cap (Y_\beta \cap U_2) \neq \emptyset$$

hence  $U_1 \cap U_2 \neq \emptyset$ ; thus we may write that  $\mathcal{Y}$  is an irreducible subspace of  $X$ , one which is clearly an upper bound for the chain  $\{Y_\alpha\}_{\alpha \in \mathcal{I}}$ . By Zorn's lemma, we may write that there exists a maximal element of  $\Sigma$ .

(iii) If  $M$  is a maximal irreducible subspace of  $X$  then since  $M \subseteq \overline{M}$ , and part (i) above tells us that  $\overline{M}$  is also an irreducible subspace of  $X$ , the maximality of  $M$  forces  $M = \overline{M}$ , whence any such maximal irreducible subspace is closed.

For any  $x \in X$ , we have the singleton subset  $\{x\}$  of  $X$ . It is trivial to see that in the subspace topology induced from  $X$ , the space  $\{x\}$  is irreducible. Any two non-empty open subsets of  $\{x\}$  must intersect since they both must contain the point  $x$ . Since  $\{x\}$  is irreducible, there exists a maximal irreducible subspace of  $X$  containing  $\{x\}$ , say  $M_x$ . Clearly then we

have that

$$\bigcup_{x \in X} M_x = X$$

That is, the maximal irreducible subspaces of  $X$  cover  $X$ .

If we now assume that  $X$  is a Hausdorff space, then we claim that the irreducible components of  $X$  are precisely the singleton subsets of  $X$ . Since  $X$  is Hausdorff we know that singleton subsets are closed, and hence that for any  $x \in X$  we have  $\{x\} = \overline{\{x\}}$ . As we saw above, each  $\{x\}$  is irreducible. If we assume  $Y$  is an irreducible component of  $X$ , say containing the point  $x$ , then  $Y$  is closed from above, and so is  $\{x\}$  in  $Y$ , hence  $Y \setminus \{x\}$  is open, and if non-empty, then the irreducibility of  $Y$  implies that  $Y \setminus \{x\}$  is dense in  $Y$ , implying that

$$Y \setminus \{x\} = \overline{Y \setminus \{x\}} = \overline{Y} \setminus \overline{\{x\}} = Y$$

where the second equality holds since  $Y = \overline{Y}$  and  $\{x\} = \overline{\{x\}}$ . Thus  $x \notin Y$  must hold, a contradiction. In particular,  $\{x\}$  is a maximal irreducible subspace of  $X$ , an irreducible component of  $X$ .

(iv) The irreducible components of  $\text{Spec}(A)$  must be of the form  $V(\mathfrak{a})$  for some ideal  $\mathfrak{a}$  of  $A$ , since by part (iii) above they are closed, and all closed subsets of  $\text{Spec}(A)$  have this form. Recall that we have a homeomorphism  $V(\mathfrak{a}) \cong \text{Spec}(A/\mathfrak{a})$  from Exercise 1.21. From Exercise 1.20 and the previous homeomorphism, we know that  $V(\mathfrak{a})$  is irreducible if and only if  $\text{Spec}(A/\mathfrak{a})$  is irreducible if and only if  $\text{nil}(A/\mathfrak{a})$  is a prime ideal of  $A/\mathfrak{a}$ . But we know that  $\text{nil}(A/\mathfrak{a})$  corresponds to  $\text{rad}(\mathfrak{a})$  as ideals, and hence  $\text{rad}(\mathfrak{a})$  is prime, to which  $V(\mathfrak{a}) = V(\mathfrak{p})$  for  $\mathfrak{p}$  a prime ideal of  $A$ . We can see that  $V(\mathfrak{p})$  is a maximal subset if there exists no prime ideals  $\mathfrak{q}$  of  $A$  for which  $V(\mathfrak{p}) \subseteq V(\mathfrak{q})$ , and so no prime ideals  $\mathfrak{q}$  for which  $\mathfrak{q} \subseteq \mathfrak{p}$ . This means that  $\mathfrak{p}$  is a minimal prime ideal of  $A$ ; hence all irreducible components of  $\text{Spec}(A)$  are of this form. ■

## 2 Modules

### 2.1 Exercise 2.5

Let  $A[x]$  be the ring of polynomials in one indeterminate over a ring  $A$ . Prove that  $A[x]$  is a flat  $A$ -algebra.

*Proof.* Let  $A$  be a commutative ring and consider  $A[x]$  as an  $A$ -module. Consider the subset  $\mathcal{A} = \{1, x, x^2, \dots\}$  of  $A[x]$ . An arbitrary element  $p(x) \in A[x]$  looks like

$$p(x) = \sum_{i=0}^n r_i x^i$$

for some  $r_1, \dots, r_n \in A$  and  $n \in \mathbb{Z}^+$ . In particular, we may note that

$$p(x) = r_0 + r_1 x + \dots + r_n x^n$$

is the unique description of  $p(x)$  in terms of elements of the subset  $\mathcal{A}$ ; i.e., we have that  $\mathcal{A}$  is a basis for  $A[x]$ . In particular, it is clear that  $A[x]$  is free on  $\mathcal{A}$ , and so is itself a free  $A$ -module. Since free modules are flat, we have desired statement:  $A[x]$  is a flat  $A$ -module. We have the obvious injective mapping  $A \rightarrow A[x]$  given by  $a \mapsto a$ , and so may also consider  $A[x]$  as a flat  $A$ -algebra. ■

## 2.2 Exercise 2.8

i) If  $M$  and  $N$  are flat  $A$ -modules, then so is  $M \otimes_A N$ .

ii) If  $B$  is a flat  $A$ -algebra and  $N$  is a flat  $B$ -module, then  $N$  is flat as an  $A$ -module.

These exercises are equivalent to Exercises 10.5.21 and 10.5.22 and 10.5.23 in Dummit and Foote; the statements are as follows (and with more generality for the rings).

**Exercise 10.5.21 (DF)** Let  $R$  and  $S$  be rings with 1 and suppose  $M$  is a right  $R$ -module, and  $N$  is an  $(R, S)$ -bimodule. If  $M$  is flat over  $R$  and  $N$  is flat as an  $S$ -module prove that  $M \otimes_R N$  is flat as a right  $S$ -module.

*Proof.* Let  $M$  be a right  $R$ -module and  $N$  an  $(R, S)$ -bimodule. Suppose  $M$  is a flat  $R$ -module and  $N$  is a flat  $S$ -module. To prove the desired statement, that  $M \otimes_R N$  is flat as an  $S$ -module, we use the associativity of the tensor product and the fact that both  $M$  and  $N$  are flat as  $R$ -modules and  $S$ -modules, respectively, to iterate Proposition 40(2).

Let  $L$  and  $L'$  be arbitrary left  $S$ -modules, and suppose that  $\psi : L \rightarrow L'$  is an injective  $S$ -module homomorphism. First, consider the following map of abelian groups:

$$N \otimes_S L \xrightarrow{1 \otimes \psi} N \otimes_S L'$$

Since  $N$  was assumed flat as an  $S$ -module, with  $L$  and  $L'$  left  $S$ -modules, with  $\psi$  an injection of  $S$ -modules, Proposition 40(2) gives us that  $1 \otimes \psi$  is an injection.

Now consider the map of abelian groups given by:

$$M \otimes_R (N \otimes_S L) \xrightarrow{1 \otimes (1 \otimes \psi)} M \otimes_R (N \otimes_S L')$$

Since  $M$  was assumed flat as an  $R$ -module, and both  $N \otimes_S L$  and  $N \otimes_S L'$  are left  $R$ -modules in the natural way (explicitly,  $N$  was assumed an  $(R, S)$ -bimodule, so we may simply define an action of  $R$  on the left by  $r(n_i \otimes l_i) = rn_i \otimes l_i$  for all elements of the group  $N \otimes_S L$  and  $N \otimes_S L'$  both) and we also have that the map  $1 \otimes \psi$  is an injection of  $R$ -modules from above, we can refer to Proposition 40(2) to once more state that  $1 \otimes (1 \otimes \psi)$  is injective.

Now recall that we have an isomorphism of abelian groups

$$(M \otimes_R N) \otimes_S L \cong M \otimes_R (N \otimes_S L)$$

which follows by the associativity of the tensor product, Theorem 14 in Section 10.4. We also have the corresponding isomorphism of abelian groups

$$(M \otimes_R N) \otimes_S L' \cong M \otimes_R (N \otimes_S L')$$

from the same theorem.

In particular, by the above isomorphism of abelian groups, as well as the injection  $1 \otimes (1 \otimes \psi)$ , we have that the mapping

$$(M \otimes_R N) \otimes_S L \xrightarrow{(1 \otimes 1) \otimes \psi = 1 \otimes \psi} (M \otimes_R N) \otimes_S L'$$

is injective. Since the left  $S$ -modules  $L$  and  $L'$  were arbitrary, as well as the injection  $\psi : L \rightarrow L'$ , the above suffices to prove that  $M \otimes_R N$  is a flat  $S$ -module by Proposition 40(2). ■

**Exercise 10.5.22 (DF)** Suppose that  $R$  is a commutative ring and that  $M$  and  $N$  are flat  $R$ -modules. Prove that  $M \otimes_R N$  is a flat  $R$ -module. [Use the previous exercise.]

*Proof.* Let  $R$  be a commutative ring, with  $M$  and  $N$  both  $R$ -modules. Suppose  $M$  and  $N$  are flat  $R$ -modules. In particular,  $M$  and  $N$  are both left and right  $R$ -modules, and in fact both may be considered  $(R, R)$ -bimodules. Since both are flat, we may refer to Exercise 10.5.23 above to write that  $M \otimes_R N$  is flat as a right  $R$ -module, and since  $R$  is commutative, also as a left  $R$ -module. Hence  $M \otimes_R N$  is flat. ■

**Exercise 10.5.23 (DF)** Prove that the (right) module  $M \otimes_R S$  obtained by changing the base from the ring  $R$  to the ring  $S$  (by some homomorphism  $f : R \rightarrow S$  with  $f(1_R) = 1_S$ , cf. Example 6 following Corollary 12 in Section 4) of the flat (right)  $R$ -module  $M$  is a flat  $S$ -module.

*Proof.* Let  $R$  and  $S$  be rings with 1 and let  $f : R \rightarrow S$  be a ring homomorphism with  $f(1_R) = 1_S$ . Then  $S$  becomes an  $R$ -module via base change. Let  $M$  be a right  $R$ -module and suppose that  $M$  is flat as an  $R$ -module.

Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be a short exact sequence of  $S$ -modules. We apply the functor  $(M \otimes_R S) \otimes_S -$  to the sequence to obtain a (not necessarily exact) sequence

$$0 \rightarrow (M \otimes_R S) \otimes_S A \rightarrow (M \otimes_R S) \otimes_S B \rightarrow (M \otimes_R S) \otimes_S C \rightarrow 0$$

which is not exact on the left since the functor is right exact. From Theorem 14 in Section 10.4, the associativity of the tensor product, whose hypothesis is satisfied since  $M$  is a right  $R$ -module and all of  $A$ ,  $B$ , and  $C$  are in particular  $(R, S)$ -bimodules, we have

$$(M \otimes_R S) \otimes_S A \cong M \otimes_R (S \otimes_S A) \cong M \otimes_R A$$

where the last isomorphism follows since extending scalars from  $S$  to  $S$  does not change the  $S$ -module  $A$ , i.e.,  $S \otimes_S A \cong A$ , cf. Example 1 following Corollary 9 in Section 10.4. The above

likewise holds for  $B$  and  $C$ , and so our sequence above becomes

$$0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$$

But, by assumption,  $M$  is a flat  $R$ -module and hence the above sequence is actually exact, and so too must our original sequence be. In particular, since the short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  was arbitrary, this shows that  $M \otimes_R S$  is exact as an  $S$ -module, which was the desired statement. ■

### 2.3 Exercise 2.24

If  $M$  is an  $A$ -module, the following are equivalent:

- (1)  $M$  is flat;
- (2)  $\text{Tor}_n^A(M, N) = 0$  for all  $n > 0$  and all  $A$ -modules  $N$ ;
- (3)  $\text{Tor}_1^A(M, N) = 0$  for all  $A$ -modules  $N$ .

*Proof.* Let  $M$  be an  $A$ -module. First we shall prove (1)  $\implies$  (3). Suppose  $M$  is flat. Then we know that the functor  $M \otimes_A -$  is exact. Let  $N$  be an  $A$ -module and let

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} N \rightarrow 0$$

be a projective resolution of  $N$ . In particular, the above sequence is exact, and so applying the functor  $M \otimes_A -$  gives an exact sequence:

$$\cdots \xrightarrow{1 \otimes d_2} M \otimes_A P_1 \xrightarrow{1 \otimes d_1} M \otimes_A P_0 \xrightarrow{1 \otimes \epsilon} M \otimes_A N \rightarrow 0$$

The exactness of the above sequence implies that  $\text{im}(1 \otimes d_n) = \ker(1 \otimes d_{n-1})$  for all  $n \geq 1$ , and hence that

$$\text{Tor}_n^A(M, N) = \frac{\ker(1 \otimes d_n)}{\text{im}(1 \otimes d_{n+1})} = 0$$

for all  $n \geq 1$ . Since  $N$  was arbitrary, we have proved (1)  $\implies$  (3).

The proof that (3)  $\implies$  (2) is immediate.

Now we prove that (2)  $\implies$  (1) which will finish the proof. Suppose  $\text{Tor}_1^A(M, N) = 0$  for all  $A$ -modules  $N$ . Let  $0 \rightarrow B \rightarrow C \rightarrow D \rightarrow 0$  be an exact sequence of  $A$ -modules. We have a long exact sequence of abelian groups

$$\begin{aligned} \cdots \rightarrow \text{Tor}_2^A(M, D) \xrightarrow{\delta_1} \text{Tor}_1^A(M, B) \rightarrow \text{Tor}_1^A(M, C) \rightarrow \text{Tor}_1^A(M, D) \\ \xrightarrow{\delta_0} M \otimes_A B \rightarrow M \otimes_A C \rightarrow M \otimes_A D \rightarrow 0 \end{aligned}$$

Now by assumption  $\text{Tor}_1^A(M, D) = 0$  since  $N$  is an  $R$ -module, and hence we have an exact sequence of abelian groups

$$0 \xrightarrow{\delta_0} M \otimes_A B \rightarrow M \otimes_A C \rightarrow M \otimes_A D \rightarrow 0$$

Since the exact sequence  $0 \rightarrow B \rightarrow C \rightarrow D \rightarrow 0$  was arbitrary, this suffices to prove that  $M \otimes_A -$  is exact, and hence that  $M$  is flat as an  $A$ -module. ■

## 2.4 Exercise 2.25

Let  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  be an exact sequence, with  $N''$  flat. Then  $N'$  is flat  $\iff N$  is flat.

*Proof.* Let  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  be a short exact sequence of  $A$ -modules and suppose  $N''$  is flat. Then for any  $A$ -module  $D$  there is a long exact sequence in homology given by

$$\begin{aligned} \cdots \rightarrow \operatorname{Tor}_2^A(N'', D) \xrightarrow{\delta_1} \operatorname{Tor}_1^A(N', D) \rightarrow \operatorname{Tor}_1^A(N, D) \rightarrow \operatorname{Tor}_1^A(N'', D) \\ \xrightarrow{\delta_0} A \otimes_A D \rightarrow N \otimes_A D \rightarrow N'' \otimes_A D \rightarrow 0 \end{aligned}$$

which is an exact sequence of abelian groups, the long exact sequence in homology coming from the right exact functor  $- \otimes_A D$ .

Since  $D$  is an  $A$ -module, the flatness of  $N''$  means that  $\operatorname{Tor}_n^A(N'', D) = 0$  for all  $n \geq 1$ . In particular, the long exact sequence above becomes

$$\begin{aligned} \cdots \rightarrow 0 \xrightarrow{\delta_1} \operatorname{Tor}_1^A(N', D) \rightarrow \operatorname{Tor}_1^A(N, D) \rightarrow 0 \\ \xrightarrow{\delta_0} 0 \rightarrow N' \otimes_A D \rightarrow N \otimes_A D \rightarrow N'' \otimes_A D \rightarrow 0 \end{aligned}$$

Now we proceed with the main statement to be proved. Suppose  $N'$  is flat. Then Exercise 3.24 above gives that  $\operatorname{Tor}_n^A(N', D) = 0$  for all  $n \geq 1$ . In particular, the long exact sequence above becomes

$$\cdots \rightarrow 0 \xrightarrow{\delta_1} 0 \rightarrow \operatorname{Tor}_1^A(N, D) \rightarrow 0 \xrightarrow{\delta_0} 0 \rightarrow N' \otimes_A D \rightarrow N \otimes_A D \rightarrow N'' \otimes_A D \rightarrow 0$$

And hence  $\operatorname{Tor}_1^A(N, D) = 0$  since the sequence is exact at  $\operatorname{Tor}_1^A(N, D)$ . Since  $D$  was arbitrary, Proposition 16 once again permits us to write that  $N$  is flat.

A completely symmetric argument to the above works to show that if  $N$  is assumed flat then  $N'$  must also be flat. Simply take the original long exact sequence and then  $\operatorname{Tor}_n^A(N, D) = 0$  for all  $n \geq 1$ , which forces  $\operatorname{Tor}_1^A(N', D) = 0$ . Hence  $N'$  is flat if and only if  $N$  is flat, as desired. ■

### 3 Rings and Modules of Fractions

#### 3.1 Exercise 3.1

Let  $S$  be a multiplicatively closed subset of a ring  $A$ , and let  $M$  be a finitely generated  $A$ -module. Prove that  $S^{-1}M = 0$  if and only if there exists  $s \in S$  such that  $sM = 0$ .

*Proof.* Let  $M = (m_1, \dots, m_k)$  and suppose  $S^{-1}M = 0$ . Then for each  $i \in \{1, \dots, k\}$  we have  $m_i/1 = 0/1$  in  $S^{-1}M$ , whence there exists a corresponding  $s_i \in S$  such that

$$s_i(m_i \cdot 1 - 0 \cdot 1) = s_i m_i = 0$$

Set  $s = s_1 \cdots s_k$  and note that  $s \in S$  since  $S$  is multiplicatively closed and we have each  $s_i \in S$ . Now, for any element  $m = \sum_{i=1}^k a_i m_i$  with  $a_i \in A$ , we can see that

$$sm = s \left( \sum_{i=1}^k a_i m_i \right) = \sum_{i=1}^k a_i (sm_i) = \sum_{i=1}^k a_i 0 = 0$$

In particular, since  $m \in M$  was arbitrary, we have  $sM = 0$ .

Conversely, assume that such an  $s \in S$  exists to make  $sM = 0$ . For an arbitrary  $m/t \in S^{-1}M$ , we have in particular that  $sm = 0$ ; hence  $sm = s(m1 - 0) = s(m1 - 0 \cdot t) = 0$ , which means that  $m/t = 0/1$  in  $S^{-1}M$ , and so  $S^{-1}M = 0$ . ■

### 3.2 Exercise 3.2

Let  $\mathfrak{a}$  be an ideal of a ring  $A$ , and let  $S = 1 + \mathfrak{a}$ . Show that  $S^{-1}\mathfrak{a}$  is contained in the Jacobson radical of  $S^{-1}A$ . Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.

*Proof.* Clearly we have that  $0 \in \mathfrak{a}$ , and hence  $1 = 1 + 0 \in S$ . Moreover, if  $x, y \in \mathfrak{a}$  then

$$(1 + x)(1 + y) = 1 + y + x + xy = 1 + (x + y + xy) \in S$$

since  $x + y + xy \in \mathfrak{a}$  by closure under addition and multiplication. In particular, the above two facts show that  $S$  is a multiplicatively closed subset of  $A$ .

We now show that  $S^{-1}\mathfrak{a} \subseteq \text{Jac}(S^{-1}A)$ . To do so, we recall Proposition 1.9 in the text, where it is shown that for a ring  $R$ , we have  $x \in \text{Jac}(R)$  if and only if  $1 - xy$  is a unit in  $R$  for all  $y \in R$ . Take any element  $x/(1 + y) \in S^{-1}\mathfrak{a}$ , where  $x, y \in \mathfrak{a}$ . Consider the element:

$$z := \frac{1}{1} - \frac{x}{1 + y} = \frac{1 + y - x}{1 + y} = \frac{1 + (y - x)}{1 + y}$$

Since  $y - x \in \mathfrak{a}$  by closure, we have  $1 + (y - x) \in S$ , and clearly  $1 + y \in S$ . In particular, the element  $z$  above is a unit in  $S^{-1}A$  with inverse given by  $(1 + y)/(1 + y - x)$  in  $S^{-1}A$ . Thus, by the proposition,  $S^{-1}\mathfrak{a} \subseteq \text{Jac}(S^{-1}A)$ .

Now we prove Corollary 2.5 in a different way. The statement is as follows: let  $M$  be a finitely generated  $A$ -module and let  $\mathfrak{a}$  be an ideal of  $A$  such that  $\mathfrak{a}M = M$ . Then there exists  $x \equiv 1 \pmod{\mathfrak{a}}$  such that  $xM = 0$ .

For a proof, we consider the localized ring  $S^{-1}A$  and its ideal  $S^{-1}\mathfrak{a}$ , for the ideal  $\mathfrak{a}$  of  $A$ . We use the fact that  $S^{-1}\mathfrak{a} \subseteq \text{Jac}(S^{-1}A)$  which we ascertained above to see, by Nakayama's lemma, that  $(S^{-1}\mathfrak{a})(S^{-1}M) = S^{-1}M$  implies  $S^{-1}M = 0$ . But from Exercise 3.1, this occurs if and only if there exists  $s \in S$  such that  $sM = 0$ , and  $s \in S = 1 + \mathfrak{a}$  has the form  $s = 1 + x$  for some  $x \in \mathfrak{a}$ , hence  $s - 1 = x \in \mathfrak{a}$  and so  $s \equiv 1 \pmod{\mathfrak{a}}$ . ■

### 3.3 Exercise 3.7

A multiplicatively closed subset  $S$  of a ring  $A$  is said to be saturated if

$$xy \in S \iff x \in S \text{ and } y \in S$$

Prove that

- (1)  $S$  is saturated  $\iff A \setminus S$  is a union of prime ideals.
- (2) If  $S$  is any multiplicatively closed subset of  $A$ , there is a unique smallest saturated multiplicatively closed subset  $\overline{S}$  containing  $S$ , and that  $\overline{S}$  is the complement of  $A$  in the union of the prime ideals which do not meet  $S$ . ( $\overline{S}$  is called the saturation of  $S$ .)

If  $S = 1 + \mathfrak{a}$ , where  $\mathfrak{a}$  is an ideal of  $A$ , find  $\overline{S}$ .

*Proof.* We prove (1). Suppose  $S \subseteq A$  is saturated. If  $a \in A \setminus S$ , then  $ab \notin S$  for all  $b \in A$  since  $S$  is saturated (otherwise  $ab \in S$  would imply  $a \in S$ ). Thus we see that  $(a) \subseteq A$  is a principal ideal which doesn't intersect  $S$ ;  $(a) \cap S = \emptyset$ .

Denote by  $\mathcal{A}$  the set of all ideals of  $A$  containing the element  $a$  (and hence the ideal  $(a)$  as well) which are also disjoint from  $S$ . We order  $\mathcal{A}$  by inclusion. Since  $(a)$  itself is such an ideal, we know that  $\mathcal{A} \neq \emptyset$ , and since any chain of elements of  $\mathcal{A}$  has an upper bound (take the union of all such elements in the chain, in other words a union of increasing ideals, which is always an ideal) we may apply Zorn's lemma to assert that  $\mathcal{A}$  has a maximal element under inclusion; call this element  $\mathfrak{P}$ .

We show that  $\mathfrak{P}$  is a prime ideal of  $A$ . To do so we shall use the equivalent definition of a prime ideal: if  $x, y \notin \mathfrak{P}$  then  $xy \notin \mathfrak{P}$ . So suppose  $x, y \notin \mathfrak{P}$ . Then  $(x)$  and  $(y)$  do not lie in  $\mathcal{A}$ , and hence the ideals  $\mathfrak{P} + (x)$  and  $\mathfrak{P} + (y)$  do not lie in  $\mathcal{A}$ . By construction of  $\mathcal{A}$ , this implies that  $\mathfrak{P} + (x)$  and  $\mathfrak{P} + (y)$  intersect  $S$ . That is, there exists  $s, t \in S$  for which  $s \in \mathfrak{P} + (x)$  and  $t \in \mathfrak{P} + (y)$ , which means that

$$st \in (\mathfrak{P} + (x))(\mathfrak{P} + (y)) = \mathfrak{P} + (xy)$$

and since  $st \in S$  since  $S$  is multiplicatively closed, this means that  $\mathfrak{P} + (xy)$  intersects  $S$ , and hence does not lie in  $\mathcal{A}$ , and hence that  $xy \notin \mathfrak{P}$ , which suffices to show that  $\mathfrak{P}$  is a prime ideal. In particular,  $\mathfrak{P}$  is a prime ideal containing  $a$  and disjoint from  $S$ .

Since we can perform an analogous construction for every element of  $A \setminus S$ , we may simply take the union of all such prime ideals corresponding to each element, and rest assured that this union covers all of  $A \setminus S$  and remains disjoint from  $S$ .

We now prove the converse. That is, suppose  $A \setminus S$  is a union of prime ideals. First we

show that  $S$  is a multiplicatively closed subset. We can write  $S \setminus A = \bigcup_{i \in \mathcal{I}} \mathfrak{P}_i$ . Prime ideals do not contain 1, and hence  $1 \in S$  must hold, since  $1 \notin \mathfrak{P}_i$  for all  $i \in \mathcal{I}$ . Moreover, if  $x, y \in S$ , then  $x, y \notin \mathfrak{P}_i$  for all  $i \in \mathcal{I}$ , and hence  $xy \notin \mathfrak{P}_i$  for all  $i$ , to which  $xy \in S$ .

Now, to show that  $S$  is saturated, note that if  $x \notin S$  then there exists some prime ideal  $\mathfrak{P}_i$  in the original decomposition of  $A \setminus S$  for which  $x \in \mathfrak{P}_i$ . Since  $\mathfrak{P}_i$  is an ideal, we have  $xy \in \mathfrak{P}_i$  for any  $y \in A$ , hence  $xy \notin S$  for all  $y \in A$ . A completely symmetric argument shows that the same holds for  $y$ . In particular,  $S$  is saturated, as desired.

Now we move on to (2). Set  $X = \bigcup \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\}$ . From part (1) above we know that the set  $\overline{S} = A \setminus X$  is saturated, as it is comprised of the complement of a union of prime ideals. Moreover, since each of the prime ideals in the union  $X$  does not intersect  $S$ , it follows that  $A \setminus \overline{S} \subseteq A \setminus S$ , and hence  $S \subseteq \overline{S}$ . Now if  $T$  was any other saturated set containing  $S$ , then we may write  $T$  as the complement of a union of prime ideals not intersecting  $S$ , and any such union is contained in the larger union  $X$ , hence  $\overline{S} \subseteq T$ , so that  $\overline{S}$  is the minimal saturated set containing  $S$ .

Finally, set  $S = 1 + \mathfrak{a}$  for an ideal  $\mathfrak{a}$  of  $A$ . Then if  $\mathfrak{p}$  is a prime ideal of  $A$  which intersects  $S$ , we have some  $s \in \mathfrak{p} \cap S$  and hence  $s = 1 + x$  for some  $x \in \mathfrak{a}$ ; hence  $s - x = 1$ , to which  $(1) = (s - x)$ . Clearly then we have  $(1) = \mathfrak{p} + \mathfrak{a}$ , so that  $\mathfrak{p}$  and  $\mathfrak{a}$  are relatively prime. In particular, we have that  $A \setminus \overline{S}$  is the union of prime ideals which are not relatively prime to  $\mathfrak{a}$ . ■

## 4 Primary Decomposition

### 4.1 Exercise 4.2

If  $\mathfrak{a} = r(\mathfrak{a})$ , then  $\mathfrak{a}$  has no embedded prime ideals.

This question appears in a slightly different form in Dummit and Foote as Exercise 15.2.38.

Show that every associated prime ideal for a radical ideal is isolated.

*Proof.* Let  $I$  be a radical ideal of the Noetherian ring  $R$  and suppose  $I = \bigcap_{i=1}^m Q_i$  is a minimal primary decomposition for  $I$  with  $\text{rad}(Q_i) = P_i$  for each  $i$ . Since  $I = \text{rad}(I)$ , we know that

$$I = \text{rad}(I) = \text{rad}\left(\bigcap_{i=1}^m Q_i\right) = \bigcap_{i=1}^m \text{rad}(Q_i) = \bigcap_{i=1}^m P_i$$

from Exercise 15.2.2(c). Assume, for contradiction, that  $I$  has an associated prime which is embedded. That is, without loss of generality, suppose we have  $P_2 = \text{rad}(Q_2) \subseteq \text{rad}(Q_1) = P_1$ . We intend to contradict the minimality of the decomposition. To this end, observe that if  $x \in \bigcap_{j \neq 1}^m Q_j$ , then

$$x \in \bigcap_{j \neq 1}^m Q_j \subseteq \bigcap_{j \neq 1}^m P_j \subseteq P_2 \subseteq P_1$$

Thus  $x \in P_i$  for all  $i \in \{1, \dots, m\}$  and hence  $x \in I$  holds. Thus  $x \in Q_i$  for each  $i \in \{1, \dots, m\}$  and so in particular  $x \in Q_1$ . Note, however, that since  $x$  was arbitrary this implies

$$\bigcap_{j \neq 1}^m Q_j \subseteq Q_1$$

which is a contradiction to the assumption that the primary decomposition for  $I$  was minimal; hence no embedded primes exist for  $I$ ; each associated prime is isolated. ■

## 4.2 Exercise 4.5

In the polynomial ring  $k[x, y, z]$  where  $k$  is a field and  $x, y, z$  are independent indeterminates, let  $\mathfrak{p}_1 = (x, y)$ ,  $\mathfrak{p}_2 = (x, z)$ , and  $\mathfrak{m} = (x, y, z)$ ;  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  are prime, and  $\mathfrak{m}$  is maximal. Let  $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$ . Show that  $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$  is a reduced primary decomposition of  $\mathfrak{a}$ . Which components are isolated and which are embedded?

*Proof.* Let  $\mathfrak{p}_1 = (x, y)$ ,  $\mathfrak{p}_2 = (x, z)$ , and  $\mathfrak{m} = (x, y, z)$ . We note that since  $k[x, y, z]/\mathfrak{p}_1 \cong k[z]$  is an integral domain, we have that  $\mathfrak{p}_1$  is a prime ideal of  $k[x, y, z]$ , hence a  $\mathfrak{p}_1$ -primary ideal. Similarly,  $\mathfrak{p}_2$  is a  $\mathfrak{p}_2$ -primary ideal. Note also that  $\mathfrak{m}$  is a maximal ideal of  $k[x, y, z]$  since  $k[x, y, z]/\mathfrak{m} \cong k$  is a field; from Proposition 4.2 we have that  $\mathfrak{m}^2 = (x, y, z)^2$  is an  $\mathfrak{m}$ -primary ideal.

We see that  $\mathfrak{a} = (x^2, xy, xz, yz) = \mathfrak{p}_1\mathfrak{p}_2$  holds. We would now like to prove that:

$$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$$

is a reduced primary decomposition of  $\mathfrak{a}$ . We can check directly that

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, y) \cap (x, z) = (x, yz)$$

$$\mathfrak{p}_1 \cap \mathfrak{m}^2 = (x, y) \cap (x^2, y^2, z^2, xy, xz, yz) = (x^2, y^2, xy, xz, yz)$$

$$\mathfrak{p}_2 \cap \mathfrak{m}^2 = (x, z) \cap (x^2, y^2, z^2, xy, xz, yz) = (x^2, z^2, xy, xz, yz)$$

Indeed, we see that each of  $\mathfrak{p}_1 \cap \mathfrak{m}^2 \not\subseteq \mathfrak{p}_2$  and  $\mathfrak{p}_2 \cap \mathfrak{m}^2 \not\subseteq \mathfrak{p}_1$  and  $\mathfrak{p}_1 \cap \mathfrak{p}_2 \not\subseteq \mathfrak{m}^2$  holds, so no primary ideal contains the intersection of the remaining primary ideals. The associated primes, moreover, are all distinct, for clearly we have  $\mathfrak{p}_1 \neq \mathfrak{p}_2$  and  $\mathfrak{p}_1 \neq \mathfrak{m}$  and  $\mathfrak{p}_2 \neq \mathfrak{m}$ . We have:

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 = \mathfrak{p}_1 \cap (\mathfrak{p}_2 \cap \mathfrak{m}^2) = (x, y) \cap (x^2, z^2, xy, xz, yz) = (x^2, xy, xz, yz) = \mathfrak{a}$$

Thus the primary decomposition for  $\mathfrak{a}$  above is, in fact, reduced.

We now find  $\text{rad}(\mathfrak{a})$ . We know from properties of radicals that

$$\begin{aligned} \text{rad}(\mathfrak{a}) &= \text{rad}(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2) \\ &= \text{rad}(\mathfrak{p}_1) \cap \text{rad}(\mathfrak{p}_2) \cap \text{rad}(\mathfrak{m}^2) \\ &= \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m} \\ &= \mathfrak{p}_1 \cap \mathfrak{p}_2 \end{aligned}$$

Where the last line follows from the previous since  $\mathfrak{p}_1 \subseteq \mathfrak{m}$  and  $\mathfrak{p}_2 \subseteq \mathfrak{m}$  holds, to which the intersection  $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subseteq \mathfrak{m}$  as well; in addition, we can see that  $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$  since for instance  $y \notin \mathfrak{p}_2$ ,

and that  $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$ , since for instance  $z \notin \mathfrak{p}_1$ . In particular,  $\text{rad}(\mathfrak{a}) = (x, yz)$ . The previous two sentences also proves that  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  are the isolated primes associated to  $\mathfrak{a}$  and that  $\mathfrak{m}$  is an embedded prime associated to  $\mathfrak{a}$ . ■

## 5 Integral Dependence and Valuations

### 5.1 Exercise 5.7

Let  $A$  be a subring of a ring  $B$ , such that the set  $B \setminus A$  is closed under multiplication. Show that  $A$  is integrally closed in  $B$ .

*Proof.* Suppose we have an element  $x \in B \setminus A$  which is integral over  $A$ . Then there exists  $a_i \in A$ ,  $0 \leq i < n$ , such that the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

holds true. Subtracting  $a_0$  from both sides of the above equation, and factoring out  $x$ , yields

$$x(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1) = -a_0$$

Since  $-a_0 \in A$  holds, we know that the product on the left hand side of the above equation resides in  $A$ . Since  $x \notin A$  by assumption, we know that  $x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$  must lie in  $A$ , for otherwise the product of two elements of  $B \setminus A$  would lie once more in  $B \setminus A$  since we assume this set is closed under multiplication. By closure under subtraction, we thus require that

$$(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2x + a_1) - a_1 = x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2x \in A$$

Pulling out a factor of  $x$  once more, we obtain that the product

$$x(x^{n-2} + a_{n-1}x^{n-3} + \cdots + a_1) \in A$$

once more lies in  $A$ . But, again, since  $x \notin A$ , we require that the element on the right above lie in  $A$ . Continuing this process, we eventually arrive at a contradiction, for we will have shown that  $x$  must lie in  $A$ . Therefore no such element  $x \in B \setminus A$  which is integral over  $A$  exists, proving that any integral element must lie in  $A$ ; in other words,  $A$  is integrally closed in  $B$ . ■

## 5.2 Exercise 5.12

Let  $G$  be a finite group of automorphisms of a ring  $A$ , and let  $A^G$  denote the subring of  $G$ -invariants, that is of all  $x \in A$  such that  $\sigma(x) = x$  for all  $\sigma \in G$ . Prove that  $A$  is integral over  $A^G$ .

Let  $S$  be a multiplicatively closed subset of  $A$  such that  $\sigma(S) \subseteq S$  for all  $\sigma \in G$ , and let  $S^G = S \cap A^G$ . Show that the action of  $G$  on  $A$  extends to an action on  $S^{-1}A$ , and that  $(S^G)^{-1}A^G \cong (S^{-1}A)^G$ .

*Proof.* Clearly we have the containment  $A^G \subseteq A$ . We first consider the following polynomial:

$$\phi(x) = \prod_{\sigma \in G} (x - \sigma(x))$$

which lies in  $A[x]$ . Note that each of the coefficients of  $\phi(x)$  when expanded out is an elementary symmetric polynomial in the variable  $\sigma(x)$ ; by the fundamental theorem of symmetric polynomials, the coefficients of each such elementary symmetric polynomial lie in  $A^G$ ; hence  $\phi(x)$  itself lies in  $A^G[x]$ . Now if we suppose that  $a \in A$ , then  $a$  is a root of  $\phi(x)$  since  $G$  is a group and thus has the identity automorphism, which means  $\phi(a) = 0$ . Thus  $a$  is integral over  $A^G$ , to which  $A$  is integral over  $A^G$ .

Now let  $S$  be a multiplicatively closed subset of  $A$  such that  $\sigma(S) \subseteq S$  for all  $\sigma \in G$ . Denote  $S^G$  as in the problem description. We can easily extend the action of  $G$  on  $A$  to one of  $G$  on  $S^{-1}A$  by defining  $\sigma \cdot a/s$  by  $\sigma(a)/\sigma(s)$  for all fractions in  $S^{-1}A$ , where  $\sigma(a), \sigma(s)$  is the usual action of  $G$  on  $A$ .

We must check that this action is well-defined. To this end, suppose that  $a/s = b/t$ , where  $a, b \in A$  and  $s, t \in S$ . Then there exists some  $u \in S$  for which

$$u(at - bs) = 0 \iff \sigma(u)(\sigma(a)\sigma(t) - \sigma(b)\sigma(s)) = \sigma(0) = 0$$

and since  $\sigma(u) \in S$  holds, as does  $\sigma(s), \sigma(t) \in S$  since  $\sigma(S) \subseteq S$  by assumption, we have the equality  $\sigma(a)/\sigma(s) = \sigma(b)/\sigma(t)$  in  $S^{-1}A$ . Thus this extended action is well-defined.

The action of  $G$  on  $S^{-1}A$  is multiplicative, as can easily be seen, for we have

$$\sigma\left(\frac{a}{s} \cdot \frac{b}{t}\right) = \sigma\left(\frac{ab}{st}\right) = \frac{\sigma(ab)}{\sigma(st)} = \frac{\sigma(a)\sigma(b)}{\sigma(s)\sigma(t)} = \frac{\sigma(a)}{\sigma(s)} \cdot \frac{\sigma(b)}{\sigma(t)}$$

and moreover additive, for

$$\sigma\left(\frac{a}{s} + \frac{b}{t}\right) = \sigma\left(\frac{at + bs}{st}\right) = \frac{\sigma(at + bs)}{\sigma(st)} = \frac{\sigma(a)\sigma(t)}{\sigma(s)\sigma(t)} + \frac{\sigma(b)\sigma(t)}{\sigma(s)\sigma(t)} = \frac{\sigma(a)}{\sigma(s)} + \frac{\sigma(b)}{\sigma(t)}$$

Now note that for  $a \in A^G$ , we have  $A^G \subseteq A$ , and hence a sequence of maps

$$A^G \longrightarrow A \longrightarrow S^{-1}A$$

$$a \mapsto a \mapsto a/1$$

Note also that  $\sigma \cdot a/1 = \sigma(a)/\sigma(1) = \sigma(a)/1 = a/1$ , so in particular we have that  $a$  maps into  $(S^{-1}A)^G \subseteq S^{-1}A$ . Thus we have a sequence of maps

$$A^G \longrightarrow (S^{-1}A)^G \longrightarrow S^{-1}A$$

Since we have defined  $S^G = S \cap A^G$ , if we have some  $s \in S^G$  then  $\sigma(s) = s$  since  $s \in A^G$ , and hence  $\sigma \cdot 1/s = \sigma(1)/\sigma(s) = 1/s$  lies in  $(S^{-1}A)^G$ , so

$$S^G \longrightarrow A^G \longrightarrow (S^{-1}A)^G$$

$$s \mapsto s \mapsto s/1$$

Moreover, we have that  $s \in S^G$  becomes a unit in  $(S^{-1}A)^G$  since, in particular,  $s \in S$  and  $(S^{-1}A)^G \subseteq S^{-1}A$ . By the universal property associated to rings of fractions (in particular applied to the multiplicatively closed subset  $S^G$  of  $A^G$ ), we have the existence of a unique homomorphism

$$\Phi : (S^G)^{-1}A^G \longrightarrow (S^{-1}A)^G$$

$$a/s \longmapsto a/s$$

To show the isomorphism we prove that  $\Phi$  is injective and surjective. First, note that if  $a/s = 0$  in  $(S^{-1}A)^G$  then there exists some  $t \in S$  for which  $ta = 0$ . Note now that

$$\left( \prod_{\sigma \in G} \sigma(t) \right) a = \left( \prod_{\sigma \in G \setminus \{\text{id}\}} \sigma(t) \right) ta = 0$$

and since

$$\prod_{\sigma \in G} \sigma(t) \in S^G$$

must hold (clearly invariant under applying any automorphism, just shifts the ordering around), this shows that  $a/s = 0$  in  $(S^G)^{-1}A^G$ ; hence  $\Phi$  is injective.

For surjectivity, take any  $a/s \in (S^{-1}A)^G$ . There are two things we can say: first, that  $a/s$  is fixed by all elements of  $G$ . In addition, if we set

$$t = \prod_{\sigma \in G \setminus \{\text{id}\}} \sigma(s)$$

then  $st/1$  is also fixed by all of  $G$ . In particular, so too must the product

$$\frac{a}{s} \cdot \frac{st}{1} = \frac{at}{1}$$

be fixed by all of  $G$ . In particular, then, we must have that  $\sigma(a)\sigma(t)/1 = a/1t$  in  $(S^{-1}A)^G$ . In other words, for every  $\sigma \in G$ , there exists  $s_\sigma \in S$  such that

$$s_\sigma \sigma(a)\sigma(t) = at$$

We now want to construct an element of  $S^G$  using each  $s_\sigma$  above. Consider the element

$$r = \prod_{\tau \in G} \tau \left( \prod_{\sigma \in G} s_\sigma \right)$$

Clearly  $\sigma(r) = r$  for all  $\sigma \in G$ , since applying any automorphism just shifts the ordering of  $r$  written as a product. Thus  $r \in S^G$  holds. Now we can see that

$$\sigma\left(\frac{rat}{1}\right) = \sigma\left(\frac{r}{1}\right)\sigma\left(\frac{at}{1}\right) = \frac{r}{1} \cdot \frac{at}{1} = \frac{rat}{1}$$

which follows since above we saw that  $at/1$  is fixed under all automorphisms in  $G$ . In particular, the above shows that  $rat \in A^G$ . Now we have

$$\frac{a}{s} = \frac{rt}{rt} \cdot \frac{a}{s} = \frac{rat}{rst}$$

and as we saw before  $rta \in A^G$ , and  $rst \in S^G$ , where the latter holds by construction of  $t$  and  $r$ . Thus  $\Phi(rat/rst) = a/s$  and so  $\Phi$  is surjective. ■

### 5.3 Exercise 5.13

In the situation of Exercise 12, let  $\mathfrak{p}$  be a prime ideal of  $A^G$ , and let  $P$  be the set of prime ideals of  $A$  whose contraction is  $\mathfrak{p}$ . Show that  $G$  acts transitively on  $P$ . In particular,  $P$  is finite.

*Proof.* Let  $\mathfrak{P}_1$  and  $\mathfrak{P}_2$  be two prime ideals of  $A$  lying over the prime ideal  $\mathfrak{p}$  of  $A^G$ . Suppose  $x \in \mathfrak{P}_1$ . Consider the element

$$\prod_{\sigma \in G} \sigma^{-1}(x) \in \mathfrak{P}_1$$

(note that this element lies in  $\mathfrak{P}_1$  since  $\text{id} \in G$  and  $\text{id}(x) = x$ , so ideal properties of  $\mathfrak{P}_1$  guarantee that the product lies in the ideal). We can see that this element is also fixed by all of  $G$ , since applying any automorphism just rearranges the terms in the product as the index runs over all elements of  $G$ , and  $G$  is a group. In particular, we have that

$$\prod_{\sigma \in G} \sigma^{-1}(x) \in \mathfrak{P}_1 \cap A^G = \mathfrak{p} = \mathfrak{P}_2 \cap A^G$$

and since  $\mathfrak{P}_2$  is a prime ideal we know that there is some particular  $\tau \in G$  for which  $\tau^{-1}(x) \in \mathfrak{P}_2$ . Set  $y = \tau^{-1}(x)$ . Then  $\tau(y) = x \in \tau(\mathfrak{P}_2)$ . Since  $x \in \mathfrak{P}_1$  was arbitrary, we know that  $\mathfrak{P}_1$  is contained completely in a collection of  $\sigma(\mathfrak{P}_2)$  for  $\sigma \in G$ . In particular, we have

$$\mathfrak{P}_1 \subseteq \bigcup_{\sigma \in G} \sigma(\mathfrak{P}_2)$$

Since  $\mathfrak{P}_1$  is a prime ideal contained in a union of ideals (each is an ideal since we are working with automorphisms), we know from Proposition 1.11 that there exists some  $\tau \in G$  for which  $\mathfrak{P}_1 \subseteq \tau(\mathfrak{P}_2)$ .

We would like to show the reverse containment; to this end, choosing the specific  $\tau$  making  $\mathfrak{P}_1 \subseteq \tau(\mathfrak{P}_2)$ , we have:

$$\tau(\mathfrak{P}_2) \cap A^G = \tau(\mathfrak{P}_2) \cap \tau(A^G) = \tau(\mathfrak{P}_2 \cap A^G) = \tau(\mathfrak{p}) = \mathfrak{p}$$

where the second equality holds by construction of  $A^G$  and the fourth by the fact that  $\mathfrak{P}_2$  lies above  $\mathfrak{p}$ . The last equality, of course, follows since  $\mathfrak{p}$  is a prime ideal of  $A^G$ , and so each element is fixed by all of  $G$ . In particular, we have shown that  $\tau(\mathfrak{P}_2)$  is a prime ideal of  $A$  lying above  $\mathfrak{p}$ .

Since  $A$  is integral over  $A^G$  by Exercise 12 above, we refer to Corollary 5.9 to write that  $\mathfrak{P}_1 = \tau(\mathfrak{P}_2)$  must hold true. Since  $\mathfrak{P}_1$  and  $\mathfrak{P}_2$  were arbitrary prime ideals of  $A$  lying above  $\mathfrak{p}$ , it follows that  $G$  acts transitively on the set of all such prime ideals. In particular, the orbit

stabilizer theorem asserts that

$$|P| = |\mathrm{Orb}_G(\mathfrak{P})| = \frac{|G|}{|\mathrm{Stab}_G(\mathfrak{P})|}$$

and since  $|G| < \infty$ , so too must  $|P| < \infty$ . In particular,  $P$  is finite. ■

#### 5.4 Exercise 5.14

Let  $A$  be an integrally closed domain,  $K$  its field of fractions and  $L$  a finite normal separable extension of  $K$ . Let  $G$  be the Galois group of  $L$  over  $K$  and let  $B$  be the integral closure of  $A$  in  $L$ . Show that  $\sigma(B) = B$  for all  $\sigma \in G$ , and that  $A = B^G$ .

*Proof.* Let  $A, B, K, L$ , and  $G$  be as in the problem description. If  $x \in B$  then  $x$  is an element of  $L$  which is integral over  $A$ ; hence there exists  $a_i \in A, 0 \leq i < n$ , for which

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

Now, given any automorphism  $\sigma \in G$ , we know that  $\sigma(0) = 0$ , and hence that the element

$$\sigma(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = \sigma(x)^n + a_{n-1}\sigma(x)^{n-1} + \cdots + a_1\sigma(x) + a_0$$

must also equal 0, where above we have used the fact that  $\sigma(a) = a$  for all  $a \in K$  and the multiplicativity of the automorphism  $\sigma$ . In particular, the above expression being equal to 0 implies that  $\sigma(x) \in L$  is integral over  $A$ , and hence that  $\sigma(x) \in B$  holds true, to which  $\sigma(B) \subseteq B$  must follow. Now we are interested in showing that  $B \subseteq \sigma(B)$ . For any  $\sigma \in G$  we have  $\sigma^{-1} \in G$  since  $G$  is a group; hence by the above  $\sigma^{-1}(B) \subseteq B$  holds. Using this fact, we have:

$$\sigma(B) = \sigma(\sigma^{-1}(\sigma(B))) \subseteq \sigma(\sigma^{-1}(B)) = B$$

which is what we wanted; hence  $\sigma(B) = B$  for all  $\sigma \in G$ .

Now we shall show that  $A = B^G$ . Utilizing the fact that  $L^G = K$  since  $G$  is the Galois group of  $L/K$ , and noting that  $A$  is integrally closed in  $K$ , we can see that

$$B^G = B \cap L^G = B \cap K = A$$

giving the desired equality. ■

## 6 Chain Conditions

### 6.1 Exercise 6.1xxx

- i) Let  $M$  be a Noetherian  $A$ -module and  $u : M \rightarrow M$  a module homomorphism. If  $u$  is surjective, then  $u$  is an isomorphism.
- ii) If  $M$  is Artinian and  $u$  is injective, then again  $u$  is an isomorphism.

*Proof.* (i) Suppose  $M$  is a Noetherian  $A$ -module and  $u : M \rightarrow M$  is a surjective homomorphism of  $A$ -modules. Since  $x \in \ker(u)$  means  $u(x) = 0$ , and clearly  $u(0) = 0$ , we have that  $x \in \ker(u^2)$ , and in general that  $\ker(u^n) \subseteq \ker(u^{n+1})$  for all  $n \in \mathbb{Z}^+$ . Now we can see that

$$\ker(u) \subseteq \ker(u^2) \subseteq \cdots \subseteq \ker(u^n) \subseteq \cdots$$

is an ascending chain of submodules of  $M$  which must terminate by the Noetherian condition on  $M$ ; so there is some  $N \in \mathbb{Z}^+$  for which  $\ker(u^n) = \ker(u^N)$  for all  $n \geq N$ . Note that

$$\ker(u^{N+1}) = \ker(u^{N+1}) \cap M = \ker(u^{N+1}) \cap \operatorname{im}(u^N)$$

since  $u$  was assumed surjective. Now if we take some  $y \in \ker(u^{N+1})$  we also have that  $y \in \operatorname{im}(u^N)$ , and hence that there exists  $x \in M$  such that  $u^N(x) = y$ . Note now that  $y \in \ker(u^N)$  since  $\ker(u^N) = \ker(u^{N+1})$ , and hence

■

## 6.2 Exercise 6.2

Let  $M$  be an  $A$ -module. If every non-empty set of finitely generated submodules of  $M$  has a maximal element, then  $M$  is Noetherian.

*Proof.* To prove that such an  $A$ -module  $M$  is Noetherian, we shall show that all submodules of  $M$  are finitely generated. Let  $N$  be an arbitrary (possibly non-finitely generated) submodule of  $M$ . We can decompose  $N$  into a sum of cyclic submodules of  $M$  as follows:

$$N = \sum_{x \in N} xA$$

We note also that each submodule  $xA$  of  $N$  is also trivially a submodule of  $M$ . Clearly each submodule  $xA$  is finitely generated, and hence  $\Sigma = \{xA \mid x \in N\}$  is a non-empty set of finitely generated submodules of  $M$  (non-empty since  $0 \in N$  means  $0 \in \Sigma$ ). Applying the hypothesis, we have a maximal element of  $\Sigma$ , say  $yA$  for some  $y \in N$ . We then have that  $xN \subseteq yA$  for all  $x \in N$ , and hence

$$N = \sum_{x \in N} xA \subseteq yA \subseteq N$$

and hence  $N = yA$  is finitely generated. Since  $N$  was an arbitrary submodule of  $M$ , we refer to Proposition 6.2 to write that  $M$  is Noetherian. ■

### 6.3 Exercise 6.5

A topological space  $X$  is said to be Noetherian if the open subsets of  $X$  satisfy the ascending chain condition (or, equivalently, the maximal condition). Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of  $X$  satisfy the descending chain condition (or, equivalently, the minimal condition). Show that, if  $X$  is Noetherian, then every subspace of  $X$  is Noetherian, and that  $X$  is quasi-compact.

*Proof.* Let  $X$  be a Noetherian topological space. Let  $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$  be an open cover for  $X$ . If there are a finite number of open sets  $U_\alpha$  in the cover  $\mathcal{U}$ , then we're done; so assume there are an infinite number of open sets in  $\mathcal{U}$ . Then, if we fix an ordering on  $\mathcal{I}$ , say index it by  $\mathbb{N}$ , then since the arbitrary union of opens is open in  $X$ , we have:

$$U_1 \subseteq (U_1 \cup U_2) \cup \cdots \cup \left( \bigcup_{i=1}^n U_i \right) \subseteq \cdots$$

an infinite ascending chain of open subsets of  $X$ . By the Noetherian condition on  $X$ , this chain eventually stabilizes, say at index  $N \in \mathbb{N}$ . In other words,

$$\bigcup_{i=1}^N U_i = \bigcup_{i=1}^m U_i$$

for all  $m \geq N$ . Note that this means  $U_m \subseteq \bigcup_{i=1}^N U_i$  for all  $m \geq N$ . In particular, taking more unions,

$$\bigcup_{i=1}^N U_i = \bigcup_{i=1}^{\infty} U_i = X$$

holds, since adding more opens to the union on the left hand side above does not increase the size of the set. Thus we have found a finite subcover  $\{U_1, \dots, U_N\}$  of  $\mathcal{U}$  which covers  $X$ ; hence  $X$  is quasi-compact.

Now take any subspace  $Y$  of  $X$ . We show that  $Y$  is Noetherian. So assume we have some infinite ascending chain of opens in  $Y$ , say given by

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots$$

Since  $Y$  has the subspace topology induced by  $X$ , we know that  $V_i = Y \cap U_i$  for open subsets  $U_i$  of  $X$ . Clearly the ascending chain

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots$$

stabilizes since  $X$  is Noetherian, say at the index  $N$ . Then we have  $U_N = U_m$  for all  $m \geq N$ ,

and hence

$$V_N = Y \cap U_N = Y \cap U_m = V_m$$

holds for all  $m \geq N$ . Thus the original chain of opens in  $Y$  indeed stabilizes as well, meaning  $Y$  is Noetherian. ■

## 6.4 Exercise 6.6

*Prove that the following are equivalent:*

- i)  $X$  is Noetherian;
- ii) Every open subspace of  $X$  is quasi-compact.
- iii) Every subspace of  $X$  is quasi-compact.

*Proof.* (i)  $\implies$  (ii). If  $X$  is Noetherian, then every subspace of  $X$  is Noetherian by Exercise 6.5 above; in the same exercise, we showed that Noetherian topological spaces are quasi-compact. Thus any subspace of  $X$  is also quasi-compact, giving (ii).

(ii)  $\implies$  (iii). Suppose every open subspace of  $X$  is quasi-compact. Let  $Y$  be any subspace of  $X$ . We show that  $Y$  is quasi-compact. Let  $\mathcal{V} = \{V_\alpha\}$  be an open cover for  $Y$ . Then for each  $\alpha$  we have that  $V_\alpha = Y \cap U_\alpha$  for open subsets  $U_\alpha$  of  $X$ .

Taking  $\mathcal{U} = \bigcup_\alpha U_\alpha$ , we know that  $\mathcal{U}$  is an open subset of  $X$  as it is the union of opens in  $X$ ; by assumption  $\mathcal{U}$  is thus quasi-compact, and since the  $U_\alpha$  form an obvious open cover, we know that there exists some finite set  $\{U_1, \dots, U_n\}$  of the collection  $\{U_\alpha\}$  which covers  $\mathcal{U}$ . In sum, we have:

$$Y = \bigcup_\alpha V_\alpha = \bigcup_\alpha (Y \cap U_\alpha) = Y \cap \left( \bigcup_\alpha U_\alpha \right) = Y \cap \mathcal{U} = Y \cap \left( \bigcup_{i=1}^n U_i \right) = \bigcup_{i=1}^n (Y \cap U_i) = \bigcup_{i=1}^n V_i$$

Thus we have proved that there is a finite subcollection of  $\mathcal{V}$ , the original open cover for  $Y$ , which covers all of  $Y$ ; hence  $Y$  is quasi-compact, and hence any subspace of  $X$  is quasi-compact, giving (iii).

(iii)  $\implies$  (i). Assume every subspace of  $X$  is quasi-compact. Take some ascending chain of open subsets of  $X$ , say

$$U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$$

Set  $\mathcal{U}$  equal to the union of all such opens in the above chain. Then  $\mathcal{U}$  is a subspace of  $X$ , and hence is quasi-compact; since the  $U_i$  form an open cover for  $\mathcal{U}$  by construction, this means that there must exist a finite subcover, after reindexing, say given by  $\{U_1, \dots, U_N\}$ . Then we have  $U_i \subseteq U_N$  for all  $1 \leq i < N$ . In particular,  $U_N = \mathcal{U}$  holds. Thus for all  $m \geq N$  we have:

$$U_m \subseteq \mathcal{U} = U_N$$

which implies that the original chain stabilizes; hence that  $X$  is Noetherian, giving (i). ■

## 6.5 Exercise 6.8xx

*If  $A$  is a Noetherian ring then  $\text{Spec}(A)$  is a Noetherian topological space. Is the converse true?*

*Proof.* We prove the first statement of the exercise. Suppose  $A$  is a Noetherian ring. We claim that  $\text{Spec}(A)$  satisfies d.c.c. on closed subsets. So take some decreasing chain of closed subsets, say

$$\cdots \subseteq V(\mathfrak{a}_n) \subseteq \cdots \subseteq V(\mathfrak{a}_2) \subseteq V(\mathfrak{a}_1)$$

where  $\mathfrak{a}_i$  is an ideal of  $A$  for all  $i \geq 1$ .

As for the second statement of the exercise, we claim that the converse is false in general. ■

## 7 Noetherian Rings

### 7.1 Exercise 7.1

Let  $A$  be a non-Noetherian ring and let  $\Sigma$  be the set of ideals in  $A$  which are not finitely generated. Show that  $\Sigma$  has maximal elements and that the maximal elements of  $\Sigma$  are prime ideals.

This exercise is equivalent to the following found in Chapter 15 Section 1 of Dummit and Foote:

**Exercise 15.1.11.** Suppose  $R$  is a commutative ring in which all prime ideals are finitely generated. This exercise proves that  $R$  is Noetherian.

- (a) Prove that if the collection of ideals of  $R$  that are not finitely generated is nonempty, then it contains a maximal element  $I$ , and that  $R/I$  is a Noetherian ring.
- (b) Prove that there are finitely generated ideals  $J_1$  and  $J_2$  containing  $I$  with  $J_1 J_2 \subseteq I$  and that  $J_1 J_2$  is finitely generated. [Observe that  $I$  is not a prime ideal.]
- (c) Prove that  $I/J_1 J_2$  is a finitely generated  $R/I$ -submodule of  $J_1/J_1 J_2$ . [Use Exercise 8.]
- (d) Show that (c) implies the contradiction that  $I$  would be finitely generated over  $R$  and deduce that  $R$  is Noetherian.

*Proof.* (a) Let  $\Sigma$  denote the set of all ideals of  $R$  that are not finitely generated. Order  $\Sigma$  by inclusion. Suppose  $\Sigma \neq \emptyset$ . Let  $\{I_\alpha\}_\alpha$  be a chain of elements of  $\Sigma$ . We know that  $\sum_\alpha I_\alpha$  is an ideal of  $R$  and moreover that  $\sum_\alpha I_\alpha$  is not finitely generated. Since  $\sum_\alpha I_\alpha$  contains all  $I_\alpha$  in the chain and lies in the chain, it is an upper bound, hence by Zorn's lemma the set  $\Sigma$  has a maximal element, call it  $I$ .

We claim that  $R/I$  is a Noetherian ring. Recall that any ideal of the quotient ring  $R/I$  is of the form  $J/I$  for an ideal  $J$  of  $R$  containing  $I$ . To show that  $R/I$  is Noetherian, we show that any ideal is finitely generated. So assume, for contradiction, that we have some non-finitely generated ideal  $J/I$  of  $R/I$ . We know then that  $J$  is a non-finitely generated ideal of  $R$ , for if not then the images of the generators for  $J$  in the quotient  $J/I$  would generate  $J/I$ , a contradiction. But since  $J$  is not finitely generated we have  $J \in \Sigma$ , and since  $I \subseteq J$ , we have a contradiction to the maximality of  $I$  in  $\Sigma$ . Hence it follows that  $J/I$  is finitely generated, hence that  $R/I$  is a Noetherian ring.

(b) In the exercise we are assuming that all prime ideals of  $R$  are finitely generated, and so we know that  $I$  is not a prime ideal of  $R$ . Thus, by definition, there must exist  $x, y \in R$  such that  $xy \in I$  with neither  $x$  nor  $y$  lies in  $I$ . Let  $J_1 = I + (x)$  and  $J_2 = I + (y)$ .

Since  $I \subseteq J_1$  and  $I \subseteq J_2$ , we know that  $J_1$  and  $J_2$  are finitely generated ideals of  $R$ , for if this were not the case then both would belong to  $\Sigma$ , which would contradict the maximality

of  $I$  in  $\Sigma$ .

We also have the following containment:

$$J_1 J_2 = (I + (x))(I + (y)) = I^2 + (y)I + (x)I + (xy) \subseteq I$$

since we are assuming  $xy \in I$  holds.

Lastly, observe also that  $J_1 J_2$  is finitely generated, for we know that both  $J_1$  and  $J_2$  are finitely generated, so taking all possible products of generators for  $J_1$  and  $J_2$  would give a (finite) set of generators for  $J_1 J_2$ .

(c) Since  $J_1 J_2 \subseteq I$  by part (b), we can refer to the third isomorphism theorem for rings to write

$$R/I \cong \frac{R/J_1 J_2}{I/J_1 J_2}$$

Since  $J_1/J_1 J_2$  is an ideal of  $R/J_1 J_2$ , we can consider it is an  $R/J_1 J_2$ -module, and hence also as an  $(R/J_1 J_2)/(I/J_1 J_2)$ -module, hence as an  $R/I$ -module by the isomorphism we found above. Moreover,  $J_1/J_1 J_2$  is finitely generated over  $R/I$  since both  $J_1$  and  $J_1 J_2$  are finitely generated over  $R$ . But  $R/I$  is a Noetherian ring by part (a), and since Exercise 15.1.8 proves that submodules of finitely generated modules over Noetherian rings are finitely generated, we have that  $I/J_1 J_2$  is a finitely generated  $R/I$ -submodule of  $J_1/J_1 J_2$ .

(d) We show how part (c) gives a contradiction to  $I$  being non-finitely generated over  $R$ .

From part (c) we know that  $I/J_1 J_2$  is finitely generated as an  $R/I$ -module, hence also as an  $R$ -module (we have  $\pi : R \rightarrow R/I$  which can be used to extend the action of  $R/I$  on  $I/J_1 J_2$ ). In part (b) we showed that  $J_1 J_2$  is finitely generated as an  $R$ -module, and so by Exercise 10.3.7 we know that  $I$  is finitely generated as an  $R$ -module. But  $I$  is an ideal of  $R$ , and so must too be finitely generated as an ideal of  $R$ , which is a contradiction.

Thus the set  $\Sigma$  from part (a) cannot be non-empty, and since  $\Sigma$  is the set of all non-finitely generated ideals of  $R$ , it follows that all ideals of  $R$  are finitely generated, which is what we aimed to prove. ■

## 7.2 Exercise 7.6

*If a finitely generated ring  $K$  is a field, it is a finite field.*

*Proof.* Let  $K$  be a finitely generated ring, so we have  $K = (a_1, \dots, a_n)$ . If  $K$  is, in addition, a field, then there are two cases: either the characteristic of  $K$  is 0 or is  $p$  for a prime  $p > 0$ . We dispense with the case where  $K$  is a field of characteristic 0.

Recall that the prime subfield of fields of characteristic 0 is  $\mathbb{Q}$ . In particular, we have a sequence of ring homomorphisms:

$$\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow K$$

We know  $K$  is finitely generated over  $\mathbb{Z}$ , for instance by taking  $\mathbb{Z}$ -linear combinations of the generators  $a_1, \dots, a_n$  for  $K$  (the assumption that  $K$  is a finitely generated ring); hence  $K$  is finitely generated over  $\mathbb{Q}$ . Since  $K$  receives an injective ring homomorphism from  $\mathbb{Q}$ , we make  $K$  into a  $\mathbb{Q}$ -algebra, in particular a finitely generated  $\mathbb{Q}$ -module.

Now we have  $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$  all rings with  $\mathbb{Z}$  Noetherian,  $K$  finitely generated as a  $\mathbb{Z}$ -module, and in addition  $K$  finitely generated as a  $\mathbb{Q}$ -module. We apply Proposition 7.8 to conclude that  $\mathbb{Q}$  is finitely generated as a  $\mathbb{Z}$ -algebra, which is clearly absurd, for in particular we would require that  $\mathbb{Q}$  be a finitely generated abelian group.

Thus it cannot be the case that  $K$  has characteristic 0, and hence  $\text{char}(K) = p > 0$  for some prime  $p$ . Thus the prime subfield of  $K$  is  $\mathbb{F}_p$ , hence finitely generated as an  $\mathbb{F}_p$ -algebra, which by Proposition 7.9 would mean that  $K$  is a finite algebraic extension of  $\mathbb{F}_p$ , hence a finite field. ■

### 7.3 Exercise 7.8

*If  $A[x]$  is Noetherian, is  $A$  necessarily Noetherian?*

*Proof.* Suppose  $A[x]$  is Noetherian. Consider the map  $\varphi : A[x] \rightarrow A$  defined by  $\varphi(r) = r$  for all  $r \in A$  and  $\varphi(x) = 0$ . For  $f, g \in A[x]$  we may observe that

$$\varphi(f(x) + g(x)) = \varphi((f + g)(x)) = (f + g)(0) = f(0) + g(0) = \varphi(f(x)) + \varphi(g(x))$$

$$\varphi(f(x)g(x)) = \varphi((fg)(x)) = (fg)(0) = f(0)g(0) = \varphi(f(x))\varphi(g(x))$$

whence  $\varphi$  is ring homomorphism. Note that if  $r \in A$  then  $f(x) = r \in A[x]$  maps to  $r$  trivially under  $\varphi$ , so that  $\varphi$  is surjective, i.e.,  $\varphi(A[x]) = R$ . Since the homomorphic image of a Noetherian ring is Noetherian, we have that  $R$  is Noetherian, as desired. ■

## 7.4 Exercise 7.15

Let  $A$  be a Noetherian local ring,  $\mathfrak{m}$  its maximal ideal and  $k$  its residue field, and let  $M$  be a finitely generated  $A$ -module. Then the following are equivalent:

- (1)  $M$  is free;
- (2)  $M$  is flat;
- (3) the mapping of  $\mathfrak{m} \otimes M$  into  $A \otimes M$  is injective;
- (4)  $\text{Tor}_1^A(k, M) = 0$ .

*Proof.* (1)  $\implies$  (2). Obvious, for if  $M$  is free then  $M$  is a direct sum of copies of  $A$ , and since  $A$  is a flat  $A$ -module trivially, and the direct sum of flat modules is flat, we have that  $M$  is flat.

(2)  $\implies$  (3). If  $M$  is flat, then  $- \otimes M$  is an exact functor, and hence we may tensor the short exact sequence of  $A$ -modules

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$$

with  $M$  on the right to obtain the short exact sequence

$$0 \longrightarrow \mathfrak{m} \otimes M \longrightarrow A \otimes M \longrightarrow k \otimes M \longrightarrow 0$$

whence the mapping of  $\mathfrak{m} \otimes M$  into  $A \otimes M$  is injective (by exactness).

(3)  $\implies$  (4). Suppose the mapping  $\phi : \mathfrak{m} \otimes M \rightarrow A \otimes M$  is injective. Consider once more the short exact sequence of  $A$ -modules  $0 \rightarrow \mathfrak{m} \xrightarrow{\iota} A \rightarrow k \rightarrow 0$ . Taking the Tor long exact sequence, we obtain the exact sequence

$$\cdots \longrightarrow \text{Tor}_1^A(A, M) \longrightarrow \text{Tor}_1^A(k, M) \longrightarrow \mathfrak{m} \otimes M \xrightarrow{\iota \otimes \phi} A \otimes M \longrightarrow k \otimes M \longrightarrow 0$$

But since  $\iota \otimes \phi$  is injective, and the sequence is exact, this forces  $\text{Tor}_1^A(k, M) = 0$ .

(4)  $\implies$  (1). Assume  $\text{Tor}_1^A(k, M) = 0$ . Since  $M/\mathfrak{m}M$  is an  $A$ -module which is annihilated by  $\mathfrak{m}$ , we can consider  $M/\mathfrak{m}M$  as an  $A/\mathfrak{m}$ -module, hence a  $k$ -module, hence a  $k$ -vector space. Moreover, since  $M$  is finitely generated as an  $A$ -module, so too must be  $M/\mathfrak{m}M$  finitely generated as an  $A$ -module, hence finitely generated as an  $k$ -vector space, which means there exists a finite basis for  $M/\mathfrak{m}M$ , say  $\overline{x}_1, \dots, \overline{x}_n$ . Lifting each of these elements to  $M$ , we have that  $x_1, \dots, x_n$  generate  $M$  by Proposition 2.8.

Now let  $F$  be the free  $A$ -module generated by  $e_1, \dots, e_n$ . Define a mapping

$$\phi : F \longrightarrow M$$

$$e_i \longmapsto x_i$$

This is obviously an  $A$ -module homomorphism (mapping generators to generators), hence we have a short exact sequence of  $A$ -modules given via

$$0 \longrightarrow \ker \phi \longrightarrow F \longrightarrow M \longrightarrow 0$$

Taking the Tor long exact sequence with respect to the  $A$ -module  $k$ , we obtain an exact sequence

$$\dots \longrightarrow \operatorname{Tor}_1^A(k, F) \longrightarrow \operatorname{Tor}_1^A(k, M) \longrightarrow k \otimes \ker \phi \longrightarrow k \otimes F \xrightarrow{1 \otimes \phi} k \otimes M \longrightarrow 0$$

and by assumption we have  $\operatorname{Tor}_1^A(k, M) = 0$ , and hence

$$0 \longrightarrow k \otimes \ker \phi \longrightarrow k \otimes F \xrightarrow{1 \otimes \phi} k \otimes M \longrightarrow 0$$

is a short exact sequence of  $A$ -modules. This means, in particular, that  $1 \otimes \phi$  is surjective, and since both  $k \otimes F$  and  $k \otimes M$  are  $k$ -vector spaces of the same dimension, and  $1 \otimes \phi$  is  $k$ -linear, this forces  $1 \otimes \phi$  to be an isomorphism, and hence  $k \otimes \ker \phi = 0$  is required.

We now want to apply Nakayama's lemma. First, note that since  $A$  is Noetherian, and  $F$  is a finitely generated  $A$ -module, we have that  $F$  is Noetherian by Proposition 6.5. Since  $\ker \phi$  is a submodule of  $F$ , Proposition 6.2 asserts that  $\ker \phi$  is finitely generated as an  $A$ -module as well. With this in mind, recall from Exercise 2.2 that we have the following isomorphism of  $A$ -modules:

$$k \otimes_A \ker \phi = A/\mathfrak{m} \otimes_A \ker \phi \cong \ker \phi / \mathfrak{m} \ker \phi$$

and since  $k \otimes \ker \phi = 0$  this means that  $\ker \phi / \mathfrak{m} \ker \phi \cong 0$ , hence  $\ker \phi = \mathfrak{m} \ker \phi$ , and since  $\ker \phi$  is finitely generated as an  $A$ -module, Nakayama's lemma (Proposition 2.6) applies, hence  $\ker \phi = 0$  (note that  $\mathfrak{m}$  is trivially contained in the Jacobson radical of  $A$  since, in particular,  $A$  is a local ring, and the Jacobson radical is thus equal to the unique maximal ideal  $\mathfrak{m}$ ).

Surjectivity of the map  $\phi$  is clear, and so we conclude that  $F \cong M$  as  $A$ -modules, which means that  $M$  is free, giving (1). ■

## 7.5 Exercise 7.16

Let  $A$  be a Noetherian ring,  $M$  a finitely generated  $A$ -module. Then the following are equivalent:

- (1)  $M$  is a flat  $A$ -module;
- (2)  $M_{\mathfrak{p}}$  is a free  $A_{\mathfrak{p}}$ -module, for all prime ideals  $\mathfrak{p}$ ;
- (3)  $M_{\mathfrak{m}}$  is a free  $A_{\mathfrak{m}}$ -module, for all maximal ideals  $\mathfrak{m}$ ;

In other words, flat = locally free.

*Proof.* (1)  $\implies$  (2). From Corollary 7.4, the fact that  $A$  is Noetherian means  $A_{\mathfrak{p}}$  is Noetherian, and moreover, each  $A_{\mathfrak{p}}$  is a Noetherian local ring (with unique maximal ideal  $\mathfrak{p}A_{\mathfrak{p}}$ ).

In addition, we have  $M_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A M$ , extension of scalars, and since  $M$  is finitely generated as an  $A$ -module,  $M_{\mathfrak{p}}$  is finitely generated as an  $A_{\mathfrak{p}}$ -module via  $\{1 \otimes a_i\}_{i=1}^n$ , where  $a_1, \dots, a_n$  generate  $M$  as an  $A$ -module.

If  $M$  is a flat  $A$ -module, then by Proposition 3.10 we know  $M_{\mathfrak{p}}$  is a flat  $A_{\mathfrak{p}}$ -module for all prime ideals  $\mathfrak{p}$  of  $A$ . Above we showed that the hypothesis of Exercise 7.15 above applies, and so  $M_{\mathfrak{p}}$  is flat if and only if  $M_{\mathfrak{p}}$  is free for all primes  $\mathfrak{p}$ .

(2)  $\implies$  (3). Trivial, since maximal ideals are, in particular, prime ideals.

(3)  $\implies$  (1). If  $M_{\mathfrak{m}}$  is a free  $A_{\mathfrak{m}}$ -module for all maximal ideals  $\mathfrak{m}$  of  $A$ , then by Exercise 7.15 we know that each  $M_{\mathfrak{m}}$  is flat as an  $A_{\mathfrak{m}}$ -module, and hence by Proposition 3.10, we have that  $M$  is flat as an  $A$ -module, completing the proof. ■

## 7.6 Exercise 7.26x

Let  $A$  be a Noetherian ring and let  $F(A)$  denote the set of all isomorphism classes of finitely generated  $A$ -modules. Let  $C$  be the free abelian group generated by  $F(A)$ . With each short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of finitely generated  $A$ -modules we associate the element  $(M') - (M) + (M'')$  of  $C$ , where  $(M)$  is the isomorphism class of  $M$ , etc. Let  $D$  be the subgroup of  $C$  generated by these elements, for all short exact sequences. The quotient group  $C/D$  is called the Grothendieck group of  $A$ , and is denoted  $K(A)$ . If  $M$  is a finitely generated  $A$ -module, let  $\gamma(M)$ , or  $\gamma_A(M)$ , denote the image of  $(M)$  in  $K(A)$ .

- (1) Show that  $K(A)$  has the following universal property: for each additive function  $\lambda$  on the class of finitely generated  $A$ -modules, with values in an abelian group  $G$ , there exists a unique homomorphism  $\lambda_0 : K(A) \rightarrow G$  such that  $\lambda(M) = \lambda_0(\gamma(M))$  for all  $M$ .
- (2) Show that  $K(A)$  is generated by the elements  $\gamma(A/\mathfrak{p})$ , where  $\mathfrak{p}$  is a prime ideal of  $A$ .
- (3) If  $A$  is a field, or more generally if  $A$  is a principal ideal domain, then  $K(A) \cong \mathbb{Z}$ .
- (4) Let  $f : A \rightarrow B$  be a finite ring homomorphism. Show that the restriction of scalars gives rise to a homomorphism  $f_! : K(B) \rightarrow K(A)$  such that  $f_!(\gamma_B(N)) = \gamma_A(N)$  for a  $B$ -module  $N$ . If  $g : B \rightarrow C$  is another finite ring homomorphism, show that  $(g \circ f)_! = f_! \circ g_!$ .

*Proof.* (1) Let  $\mathcal{C}$  denote the class of all finitely generated  $A$ -modules. Suppose we have some additive function  $\lambda : \mathcal{C} \rightarrow G$ , where  $G$  is some abelian group.

We construct a map

$$\lambda_0 : K(A) \rightarrow G$$

$$(M) \mapsto \lambda(M)$$

■

## 8 Artin Rings

### 8.1 Exercise 8.2

Let  $A$  be a Noetherian ring. Prove that the following are equivalent:

- i)  $A$  is Artinian;
- ii)  $\text{Spec}(A)$  is discrete and finite;
- iii)  $\text{Spec}(A)$  is discrete.

*Proof.* (i)  $\implies$  (ii). If  $A$  is Artinian, we know that  $A$  has only finitely many prime ideals. To see why, note that every prime ideal in an Artin ring is maximal, and since by Proposition 8.3 there are only finitely many maximal ideals, the claim follows. Thus  $\text{Spec}(A)$  is finite. Moreover, from Exercise 3.11, we know that every prime ideal of  $A$  is maximal if and only if  $\text{Spec}(A)$  is a  $T_1$ -space, i.e., every singleton is closed. But now every subset of  $\text{Spec}(A)$  is a finite union of singleton (closed) sets, and hence  $\text{Spec}(A)$  is discrete, proving (ii).

(iii)  $\implies$  (i). If  $\text{Spec}(A)$  is discrete, we know that every singleton subset is closed; hence by Exercise 1.18(i) we know that each prime ideal of  $A$  is maximal; hence  $\dim(A) = 0$ , and since  $A$  is assumed Noetherian, we refer to Theorem 8.5 to write that  $A$  is Artin, proving (i).

(ii)  $\implies$  (iii). If  $\text{Spec}(A)$  is discrete and finite then obviously (iii) is satisfied. ■

## 9 Discrete Valuation Rings and Dedekind Domains

### 9.1 Exercise 9.5

*Let  $M$  be a finitely-generated module over a Dedekind domain. Prove that  $M$  is flat  $\iff M$  is torsion-free.*

*Proof.* Let  $A$  be a Dedekind domain and  $M$  a finitely-generated  $A$ -module. We know that  $A$  is an integral domain (since Dedekind), and as such, by Exercise 3.13, we know that  $M$  is torsion-free if and only if  $M_{\mathfrak{p}}$  is torsion-free as an  $A_{\mathfrak{p}}$ -module for all prime ideals  $\mathfrak{p}$  of  $A$ .

Since each  $A_{\mathfrak{p}}$  is a discrete valuation ring, we know from Proposition 9.2 that every ideal can be written as a power of an element, and hence  $A_{\mathfrak{p}}$  is a principal ideal domain. From the structure theorem for finitely generated modules over PIDs, we can write each  $M_{\mathfrak{p}}$  as the direct sum of its torsion submodule and its free submodule. In particular, each  $M_{\mathfrak{p}}$  is torsion-free if and only if each  $M_{\mathfrak{p}}$  is free. Now Exercise 7.16 states that, since  $A$  is in particular Noetherian (since Dedekind) and  $M$  is finitely-generated as an  $A$ -module,  $M$  is a flat  $A$ -module if and only if  $M_{\mathfrak{p}}$  is a free  $A_{\mathfrak{p}}$ -module for all primes  $\mathfrak{p}$ .

Tracing back through we find that  $M$  is flat as an  $A$ -module if and only if  $M$  is torsion-free, as desired. ■