Notes on Topology

Kyle Mickelson

Updated February 1st, 2024

Contents

1 Topological Spaces

1.1 Opens Sets and Definition of Topology

Definition (Topological Space)**.** Let X be a set and $\mathscr{T}$ be a collection of subsets of X, whose elements are called open sets. Then $(X, \mathscr{T})$ is called a *topological space* if:

(1) $\emptyset, X$ are both in $\mathscr{T}$ (trivial subsets in topology)

(2) for all $U_1, \ldots, U_n \in \mathscr{T}$, we have $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$ (closed under finite intersection)

(3) for all $\{U_\alpha\}$, we have $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}$ (closed under arbitrary union)

Example 1.1.1 (Trivial Topology)**.** Let X be a set and $\mathscr{T} = \{\varnothing, X\}$. Then $(X, \mathscr{T})$ is a topological space, and $\mathscr{T}$ called the trivial topology on X; smallest topology possible on set X. The open sets of the trivial toplogy are simply $\varnothing$ and X.

Example 1.1.2 (Discrete Topology)**.** X a set and $\mathscr{T} = \mathcal{P}(X)$, the set of all subsets of X. Clearly $\mathscr{T}$ is topology on X, we call it the *discrete topology*; the largest topology possible on X. Every single possible subset of X is an open set in the discrete topology.

Proposition 1.1.1 (Finite Complement Topology)**.** *Take $X = \mathbb{R}$ and let $\mathscr{T}$ be the collection of subsets of X whose complements are finite and including the empty set. The open sets in the *finite complement* or *Zariski* toplogy are precisely those sets whose complements are finite.*

Proof. Clearly $\emptyset, \mathbb{R} \in \mathscr{T}$, for $\mathbb{R}^c = \emptyset$ which is trivially finite. If $U_1, \ldots, U_n$ are elements of $\mathscr{T}$, then if $U_i = \emptyset$ for some $1 \leq i \leq n$, we have $\bigcap_{i=1}^{n} U_i = \emptyset$, which is finite; assume each U_i is non-empty. Then $U_i = \mathbb{R} \setminus F_i$ for each i, where F_i is some finite set. We then have:

$$(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} (\mathbb{R} \setminus F_i) = \mathbb{R} \setminus \bigcup_{i=1}^{n} F_i$$

And clearly the complement of the above set is finite, since $\bigcup_{i=1}^{n} F_i$ is a union of finite sets. So $\mathscr{T}$ is closed under finite intersection of open sets. Now take $\{U_\alpha\}$ a collection of sets of $\mathscr{T}$. Then if each $U_j = \emptyset$, then their union is the empty set and we are done. So assume there is at least one U_j such that $U_j \neq \emptyset$. Then U_j has a finite complement, and so $\bigcup U_i$ has a finite complement. ■

Definition (Fine / Coarse Topologies)**.** If X is a set and $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on X, then if $\mathscr{T}_1 \subseteq \mathscr{T}_2$, we say $\mathscr{T}_1$ is *coarser* than $\mathscr{T}_2$, and $\mathscr{T}_2$ is *finer* than $\mathscr{T}_1$. If proper containment, then we say strictly coarser or strictly finer.

Definition (Neighborhood of Point)**.** Let X be a topological space. If $x \in X$ is contained in some open set $U \subset X$, then U is called a neighborhood of x.

Proposition 1.1.2 (Set is open iff every element has neighborhood in set)**.** *Let X be a topological space and $A \subseteq X$. The subset A is an open set of X if and only if for all $x \in A$ there exists neighborhood U of A such that $x \in U \subset A$.*

Proof. Assume A is open. Then if $x \in A$ then for $A = U$ we have $x \in U \subseteq A$ since A is a neighborhood of x. Conversely, assume for all $x \in A$ we have neighborhood U_x such that $x \in U_x \subset A$. Then since $A = \{x\}_{x \in A} = \bigcup_{x \in A} U_x$, A is clearly the union of open sets of X and so is open. ■

1.2 Basis for Topology

Definition (Basis)**.** A collection of subsets of X, $\mathscr{B}$, is called a *basis* for a topology on X if:

(1) for all $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$.

(2) if $B_1, B_2 \in \mathscr{B}$, and $x \in B_1 \cap B_2$, then $\exists$ $B_3 \in \mathscr{B}$ st. $x \in B_3 \subseteq B_1 \cap B_2$.

-call the elements of $\mathscr{B}$ basis elements. -each point in X has a basis element; every point in intersection of basis elements is in another basis element contained in intersection.

—

Definition (Topology generated by basis)**.** Let $\mathscr{B}$ be a basis for a topology on X. Then the *topology $\mathscr{T}$ generated by $\mathscr{B}$* is obtained by taking the open sets to be equal to be unions of basis elements, along with the empty set.

-the topology generated by a basis has open sets equal to unions of basis elements.

Theorem 1.2.1 (Topology generated by basis is topology)**.** *If $\mathscr{B}$ is a basis for a topology on X, then the topology generated by $\mathscr{B}$ is a topology on X.*

-each basis element itself is also an open set in $\mathscr{T}_{\mathscr{B}}$.

Proof. Let X be a set and $\mathscr{B}$ a basis for a topology on X. Let $\mathscr{T}_{\mathscr{B}}$ be the topology generated by $\mathscr{B}$. By definition, $\emptyset \in \mathscr{T}_{\mathscr{B}}$, and since $X = \{x\}_{x\in X}$, and by definition of a basis, for each $x \in X$ there is $B_x \in \mathscr{B}$ such that $x \subset B_x$, $\{x\}_{x\in X} \subseteq \bigcup_{x\in X} B_x = U \in \mathscr{T}_{\mathscr{B}}$. Thus $X \in \mathscr{T}_{\mathscr{B}}$. Now assume $U_1, \ldots, U_n \in \mathscr{T}_{\mathscr{B}}$. If $\bigcap_{i=1}^n U_i = \emptyset \in \mathscr{T}_{\mathscr{B}}$, so assume $\bigcap_{i=1}^n U_i \neq \emptyset$.

Lemma: if $B_1, \ldots, B_n \in \mathscr{B}$ a basis, and $x \in \bigcap_{i=1}^n B_i$, then there exists $B' \in \mathscr{B}$ such that $x \in B' \subseteq \bigcap_{i=1}^n B_i$.

For $n = 1$ the relation is clear by definition of a basis. Let $n = 2$. Then $B_1 \cap B_2$ has a corresponding B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$ by definition of a basis. Suppose holds for n, take $B_1, \ldots, B_n, B_{n+1}$ and let $x \in \bigcap_{i=1}^{n+1} B_i$. Since

$$\bigcap_{i=1}^{n+1} B_i = \bigcap_{i=1}^{n} B_i \cap B_{n+1}$$

By inductive hypothesis we have B' such that $x \in B' \subseteq \bigcap_{i=1}^n B_i$. But then $x \in B' \cap B_{n+1}$, and by definition of a basis there exists $B'' \in \mathscr{B}$ for which $x \in B'' \subseteq B' \cap B_{n+1}$.

Thus there exists some $x \in \bigcap_{i=1}^n U_i$, and since each U is the union of basis elements, it follows that there exists $x \in B_i \subseteq U_i$ for $1 \leq i \leq n$. But then $x \in \bigcap_{i=1}^n B_i$, so that by Lemma above, there exists B' such that $x \in B' \subseteq \bigcap_{i=1}^n U_i$, and so the finite intersection is in $\mathscr{T}_{\mathscr{B}}$. If $\{U_\alpha\}$ is an arbitrary collection of open sets, then since each U is the union of basis elements, we can clearly see that the union of each U will be in $\mathscr{T}_{\mathscr{T}}$. ■

Example 1.2.1 (Standard topology on $\mathbb{R}$)**.** On $\mathbb{R}$, take $\mathscr{B} = \{(a,b) \subset \mathbb{R} \mid a < b\}$. $\mathscr{B}$ is a basis since $\emptyset, \mathbb{R} \in \mathscr{B}$ and if $x \in \mathbb{R}$, then $x \in (x-1, x+1)$ clearly. The finite intersection of open intervals is open, and the arbitrary union of open intervals is open.

Example 1.2.2 (Basis that generates Discrete Topology)**.** Let X be set and $\mathscr{B} = \{\{x\} \mid x \in X\}$. The topology generated by $\mathscr{B}$ is the discrete topology.

Example 1.2.3 (Lower/Upper Limit Topologies)**.** On $\mathbb{R}$, take $\mathscr{B} = \{(a,b] \mid a < b\}$, the topology generated by $\mathscr{B}$ is the *upper limit topology*. Similarly, $\mathscr{B} = \{[a,b) \mid a < b\}$ generates the *lower limit topology*, denoted $\mathbb{R}_l$.

Example 1.2.4 (Digital Line Topology)**.** On $\mathbb{Z}$, consider $\mathscr{B} = \{B(n) \mid n \in \mathbb{Z}\}$ where $B(n) = \{n\}$ if n is odd and $B(n) = \{n-1, n, n+1\}$ if n is even. The topology generated by this basis is called the *digital line topology* on $\mathbb{Z}$.

—

Open set in topology generated by basis iff contains basis element for each element (1.9) Take X a set and $\mathscr{B}$ a basis. Then U is open in $\mathscr{T}_{\mathscr{B}}$ if and only if for each $x \in U$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset U$.

Proof. Let $(X, \mathscr{T}_{\mathscr{B}})$ be a topological space, and $U \in \mathscr{T}_{\mathscr{B}}$. Assume U is open in this topology. If $U = \emptyset$ then since $\emptyset \in \mathscr{B}$ we have $x \in \emptyset \subseteq \emptyset$ and we are done; so assume $U \neq \emptyset$. Then there is some $x \in U$, and since U is the union of basis elements, it follows that there is some B' in the union for which $x \in B'$, and thus $x \in B' \subseteq U$.

Conversely, suppose for each $x \in U$ there exists $B_x \in \mathscr{B}$ st. $x \in B_x \subseteq U$. Then since $U = \{x\}_{x \in U} = \bigcup_{x \in U} B_x$, it follows that U is the union of basis elements and therefore $U \in \mathscr{T}_{\mathscr{B}}$. ■

—

Definition (Open Balls)**.** Take X a space and d a metric on the space. Then for each $x_0 \in X$ and $\epsilon > 0$:

$$B_{(X,d)}(x_0, \epsilon) = \{x \in X \mid d(x_0, x) < \epsilon\}$$

We call this set the *open ball of radius ϵ centered at x_0.* For the case where we consider $\mathbb{R}$, we have $d(x, y) := |x - y|$, normal distance between two points.

Theorem 1.2.2 (Collection of open balls is basis for a topology in R2)**.** *The collection $\mathcal{B} = \{B(x, \epsilon) \mid x \in \mathbb{R}^2, \epsilon > 0\}$ is a basis for a topology on $\mathbb{R}^2$.*

Proof. **Lemma:** If $y \in \mathbb{R}^2$ and $r > 0$, then for every $x \in B(y, r)$ there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset B(y, r)$.

Proof. Let $x \in B(y, r)$. Then $d(x, y) < r$, and $0 < r - d(x, y)$. Now choose ϵ such that $0 < \epsilon < r - d(x, y)$. Then if $z \in B(x, \epsilon)$, we have $d(x, z) < \epsilon$. But

$$d(y, z) \leq d(y, x) + d(x, z) < d(y, x) + \epsilon < r$$

And thus $z \in B(y, r)$, to which $B(x, \epsilon) \subset B(y, r)$. ■

Let $x \in \mathbb{R}^2$ and $\epsilon > 0$. Then $x \in B(x, \epsilon) \in \mathcal{B}$, so axiom (1) of a basis is satisfied. Now if $x \in B(p, r) \cap B(q, r')$. By our lemma, there exists $\epsilon, \epsilon' > 0$ such that $B(x, \epsilon) \subset B(p, r)$ and $B(x, \epsilon') \subset B(q, r')$. Let $\delta = \min\{\epsilon, \epsilon'\}$. Then:

$$B(x, \delta) \subset B(x, \epsilon) \cap B(x, \epsilon) \subset B(p, r) \cap B(q, r')$$

Thus $B(x, \delta)$ satisfies axiom (2) and therefore $\mathcal{B}$ is basis for a topology on $\mathbb{R}^2$. ■

—

Example 1.2.5 (Standard topology on R2)**.** Take $\mathcal{B} = \{B(x, \epsilon) \mid x \in \mathbb{R}^2, \epsilon > 0\}$. In this case $\mathcal{B}$ is the collection of all open balls with respect to Euclidean metric, d_{l^2}.

Theorem 1.2.3 (Collection of open rectangles form basis for topology on R2)**.** *On plane $\mathbb{R}^2$, let:*

$$\mathcal{D} = \{(a, b) \times (c, d) \subset \mathbb{R}^2 \mid a < b, c < d\}$$

Then $\mathcal{D}$ is the basis for a topology on $\mathbb{R}^2$, and the topology generated by $\mathcal{D}$ is the standard topology on $\mathbb{R}^2$.

Proof. Let $\mathcal{B} = \{(a, b) \times (c, d) \subseteq \mathbb{R}^2 \mid a < b, c < d\}$. We will show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$. Let $(x, y) \in \mathbb{R}^2$ be arbitrary. First, note that we have $(x, y) \in (x-1, y-1)\times(x+1, y+1) \in \mathcal{B}$, which follows since $x \in (x-1, x+1)$ and $y \in (y - 1, y + 1)$. Thus condition (i.) of Definition 1.5 is satisfied.

Now take the two sets $(a, b) \times (c, d), (a', b') \times (c', d') \in \mathcal{B}$. Assume the point (x, y) is contained in their intersection. We may rewrite the intersection of the sets $(a, b) \times (c, d)$ and $(a', b') \times (c', d')$ using properties of sets as follows

$$[(a, b) \times (c, d)] \cap [(a', b') \times (c', d')] = [(a, b) \cap (a', b')] \times [(c, d) \cap (c', d')]$$

Since the intersection of two open intervals of $\mathbb{R}$ is again an open interval of $\mathbb{R}$, we may be assured in writing that there exists $a'', b'', c'', d'' \in \mathbb{R}$ for which $(a'', b'') = (a, b) \cap (a', b')$ and $(c'', d'') = (c, d) \cap (c', d')$. But then we have $(a'', b'') \times (c'', d'') \in \mathcal{B}$. Since this set is equal to the intersection of $(a, b) \times (c, d)$ and $(a', b') \times (c', d')$, and we assumed (x, y) was contained in this set, we have found an element of $\mathcal{B}$ contained in the intersection of $(a, b) \times (c, d)$ and $(a', b')\times(c', d')$ that contains (x, y) as an element. Thus condition (ii.) for a basis is satisfied, and we may write that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Now we prove the topology on $\mathbb{R}^2$ generated by $\mathcal{B}$ is the standard topology on $\mathbb{R}^2$. Let U be an open set in $\mathbb{R}^2$ and $(x, y) \in U$. We may write $U = (a, b)\times(c, d)$. Now take $\epsilon = \min\{x - a, y - b, c - x, d - y\}$. We will use this as the radius of an open ball centered at the point (x, y). Using this ball, we can see that

$$(x, y) \in B((x, y), \epsilon) \subseteq (a, b) \times (c, d)$$

By Theorem 1.13 in the text, we may write that the collection of open balls in $\mathbb{R}^2$ is a basis for the topology on $\mathbb{R}^2$ generated by $\mathcal{B}$. Since the topology generated by the open balls is precisely the standard topology on $\mathbb{R}^2$, this suffices to show that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}^2$. ■

Theorem 1.2.4 (Sub-basis generates same topology)**.** *Let $(X, \mathscr{T})$ be a topological space, and $\mathcal{C}$ be a collection of open sets in X. If, for each open set U in X and for each $x \in U$, there exists an open set V in $\mathcal{C}$ such that $x \in V \subset U$, then $\mathcal{C}$ is a basis that generates the topology $\mathscr{T}$.*

-so essentially taking sub-collections from some topology forms a basis that generates precisely the same topology.

Proof. Take $(X, \mathscr{T})$ and $\mathcal{C}$ as defined. Suppose for each $U \subset X$ and $x \in U$ we have $V \in \mathcal{C}$ for which $x \in V \subset U$. First we show $\mathcal{C}$ is a basis. If $x \in X$ then since X is open by assumption we have $V \in \mathcal{C}$ such that $x \in V \subset X$, satisfying axiom (1). Now assume $x \in V_1 \cap V_2$. But since V_1, V_2 are open sets, their intersection is an open set by definition of a topology, so let $U = V_1 \cap V_2$. But then by assumption there exists $V \in \mathcal{C}$ such that $x \in V \subset U = V_1 \cap V_2$, and so axiom (2) is shown. Therefore $\mathcal{C}$ is a basis for a topology on X.

Now we prove that the topology generated by $\mathcal{C}$ is $\mathscr{T}$. Let $\mathscr{T}_C$ denote the topology on X generated by $\mathcal{C}$. Suppose $U \in \mathscr{T}$. By assumption, for all $x \in U$ there exists $V_x \in \mathcal{C}$ for which $x \in V_x \subset U$. By Theorem (1.9), U is open in $\mathscr{T}_C$, i.e., $\mathscr{T} \subseteq \mathscr{T}_C$. Now suppose $W \in \mathscr{T}_C$. By definition of a topology generated by a basis, W is the union of elements of $\mathcal{C}$, which were assumed to be open sets of X. By definition of a toplogy, the union of open sets of $\mathscr{T}$ is again an element of $\mathscr{T}$, and thus $W \in \mathscr{T}$, so $\mathscr{T}_C \subseteq \mathscr{T}$. Therefore we may conclude $\mathscr{T}_C = \mathscr{T}$. ■

—

1.3 Closed Sets

Definition (Closed Set)**.** A subset A of a topological space X is called a *closed set* if the set $X \setminus A$ is open; i.e., closed sets are those whose complements are open. We can also then see that the complement of an open set is closed.

Definition (Closed Balls Closed Rectangles)**.** For each $x \in \mathbb{R}^2$ and $\epsilon > 0$, we define the *closed ball of radius ϵ centered at x*,

$$\overline{B}(x, \epsilon) = \{y \in \mathbb{R}^2 \mid d(x, y) \leq \epsilon\}$$

Where d is the Euclidean metric. Similarly, we have *closed rectangles*, where if $[a, b]$ and $[c, d]$ are closed intervals in $\mathbb{R}^2$ then $[a, b] \times [c, d] \subset \mathbb{R}^2$.

Theorem 1.3.1 (Closed balls and rectangles are clsoed sets in standard topology on R2)**.** *Closed balls and closed rectangles, as defined above, are closed sets when considered under $(\mathbb{R}^2, \mathscr{T})$, where $\mathscr{T}$ is the standard topology on $\mathbb{R}^2$.*

Proof. TBD ■

Theorem 1.3.2 (Closed sets characterized in topological space)**.** *Let $(X, \mathscr{T})$ be a topological space. If we have a collection of closed sets in X, then: 1. $\emptyset$ and X are closed. 2. the finite union of closed sets is closed. 3. the arbitrary intersection of any collection of closed sets is closed.*

Proof. Note that in $(X, \mathscr{T})$, both $\emptyset$ and X are open. Thus $X \setminus \emptyset = X$ is closed and $X \setminus X = \emptyset$ is closed, proving (1). Take $F_1, \ldots, F_n$ each closed sets. Then $F_1^c, \ldots, F_n^c$ are open by definition, and in a topological space $\bigcap_{i=1}^n F_i^c$ is open also. But $(\bigcap_{i=1}^n F_i^c)^c = \bigcup_{i=1}^n F_i$ is then closed, proving (2). Now take $\{F_\alpha\}$ a collection of closed sets. Then $\{F_\alpha^c\}$ is a collection of open sets, whose arbitrary union is open by definition of X being a topological space. Thus $\bigcap_\alpha F_\alpha$ is closed; (3). ■

—

Definition (Hausdorff)**.** A topological space X is *Haussdorf* if, for every distinct $x, y \in X$, there exists disjoint open sets U and V such that $x \in U$ and $y \in V$; i.e., there exist disjoint neighborhoods of x and y.

Theorem 1.3.3 (Hausdorff spaces have closed singleton sets)**.** *If X is a Haussdorf space, then for any $x \in X$ the set $\{x\}$ is closed in X; i.e., every single-point subset of X is closed.*

Proof. Take X to be Haussdorf. Let $x \in X$ and $y \in X \setminus \{x\}$. There exists disjoint neighborhoods U and V of x and y, respectiviely. In particular, $x \notin V$, and so $V \subset X \setminus \{x\}$. But then we have found $y \in V \subset X \setminus \{x\}$ for arbitrary $y \in X \setminus \{x\}$, to which $X \setminus \{x\}$ is open; this implies $\{x\}$ is closed. ■

2 Interiors, Closures, Boundaries

2.1 Interior and Closure of Set

Definition (Interior / Closure of Set)**.** Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. We call the union of all open sets of X contained in A the *interior* of A, denoted $\text{int}(A)$. We call the intersection of all closed sets of X containing A the *closure* of A, denoted $\overline{A}$.

-from the above it is clear int$(A) \subset A \subset \overline{A}$ holds in general. -also the interior of a subset is the largest open set contained in that subset; since from the definition of topology the union of open sets is open.

Properties of closure and interior in regards to open and closed sets (2.2) Let X be topological space and $A, B \subset X$.

1. if U is open in X and $U \subset A$, then $U \subset$ int(A). 2. if C is closed in X and $A \subset C$, then $\overline{A} \subset C$. 3. if $A \subset B$ then int$(A) \subset$ int(B) 4. if $A \subset B$ then $\overline{A} \subset \overline{B}$. 5. A is open if and only if $A =$ int(A). 6. A is closed if and only if $A = \overline{A}$.

Proof. TBD ■

—

Definition (Dense)**.** Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Then A is called *dense* in X if we have $\overline{A} = X$, i.e., the closure of the subset is X itself.

-easiest example is $\mathbb{Q}$; we have $\overline{\mathbb{Q}} = \mathbb{R}$.

—

Point in interior iff contained in open set in subset Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $y \in X$. Then $y \in$ int(A) if and only if there exists an open set $U \subset A$ such that $y \in U \subset A$.

Proof. Suppose $y \in$ int(A). The interior of a set is open, and so $y \in$ int$(A) \subset$ int(A) holds. Conversely, suppose there exists $U \subset A$ for which $y \in U \subset A$. It follows that U is an open set contained in A, and thus is included in the union of all such open sets, the interior of A. Thus $U \subset$ int(A) to which $y \in$ int(A). ■

Point in closure of subset iff open sets containing point intersect subset (2.5) Take $(X, \mathscr{T})$ a topological space and $A \subset X$, $y \in X$. Then $y \in \overline{A}$ if and only if every open set containing y has a non-empty intersection with A.

Proof. Suppose $y \in \overline{A}$. Assume, for contradiction, there exists an open set U containing y for which $U \cap A = \emptyset$. Then $U^c \cap A = A$, which means $A \subset U^c$. Note U^c is closed. Since U^c is a closed set containing A, it follows that $y \in U^c$ for y is contained in the closure of A, which is the intersection of all closed sets containing A. This contradicts $y \in U$.

Conversely, suppose every open set U containing y has $U \cap A \neq \emptyset$. Assume, by way of contradiction, $y \notin \overline{A}$. In general $A \subset \overline{A}$, and so clearly $\overline{A}^c \cap A = \emptyset$. Recall that the closure of a subset is closed. Then $y \in \overline{A}^c$ is contained in an open set, and so by assumption $\overline{A}^c \cap A \neq \emptyset$. A contradiction. ■

Relationships between interior and closure of sets Let A and B be subsets of a topological space X. Then: 1. $\text{int}(X \setminus A) = X \setminus \overline{A}$ 2. $\overline{X \setminus A} = X \setminus \text{int}(A)$. 3. $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$; equality needn't hold in general. 4. $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$.

-so basically (1) the interior of the complement is the complement of the closure (2) the closure of the complement is the complement of the interior -(3) say that unions of interiors are subsets of interiors of unions. -(4) means intersection of interiors is interior of intersections.

Proof. TBD ■

2.2 Limit Points

Definition (Limit Point)**.** Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. We call a point $x \in X$ a *limit point* of the set A if every neighborhood U of x intersects A at point other than x; i.e., if for all open sets U such that $x \in U$, we have $U \cap A \neq \{x\}$.

-so essentially limit points are those who always have a non-empty intersection with the parent set in question; they always have more points than just x itself. -can always find some point in A other than x in an open set U contains x.

Closure of set equal to set union limit points (2.8) Let $A \subset X$ where X is a topological space. Then $\text{cl}(A) = A \cup A'$, where A' is the set of limit points of A. Equivalently; $\overline{A} = A \cup A'$.

Proof. Let $x \in \overline{A}$. If $x \in A$ then trivially $x \in A \cup A'$, so assume $x \notin A$. Since $x \in \overline{A}$, we have for any $x \in U$ where U is an open set that $U \cap A \neq \varnothing$. Thus there is some point $y \in U \cap A$ for which $y \neq x$. By definition x is a limit point of A, so $x \in A \cup A'$, to which we write $\overline{A} \subseteq A \cup A'$.

Now assume $x \in A \cup A'$. If $x \in A$ then $x \in \overline{A}$ trivially. Take $x \notin A$. Then $x \in A'$ and so for all open sets U containing x, we have $U \cap A$ contains a point other than x. But then $U \cap A \neq \varnothing$ for any U, and so by Theorem 2.5 we have $x \in \overline{A}$. Thus $A \cup A' \subseteq \overline{A}$. These relations imply $\overline{A} = A \cup A'$ as desired. ■

Subset is closed iff contains all limit points Let $A \subset X$ where X is a topological space. Then A is closed if and only if A contains all of its limit points.

Proof. Assume A is closed. Then $A = \overline{A}$ by Theorem 2.2(6). By Theorem 2.8 above, we have $A = A \cup A'$, which implies $A' \subset A$. Thus A contains all of its limit points. Conversely, assume A contains all of its limit points. Then by

Theorem 2.8, we know $\overline{A} = A \cup A' = A$, and so by Theorem 2.2(6) we have A is closed. ■

—

Definition (Convergent Sequence / Limit of Sequence)**.** Let $(x_n)_{n=m}^{\infty}$ be a sequence in a topological space X. We say $(x_n)_{n=m}^{\infty}$ *converges* to $x \in X$ if for every neighborhood U containing x, there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Further, we say x is the *limit* of the sequence $(x_n)_{n=m}^{\infty}$ and write:

$$\lim_{n \to \infty} x_n = x$$

-so given any open set containing limit, we can find a point in the sequence for which the sequence never leaves the open set. -open sets get closer and closer to limit, squish the sequence towards it.

Limit point of subset of Rn is limit of convergent sequences in subset Let $A \subset \mathbb{R}^n$ in the standard topology. If x is a limit point of A, then there exists a sequence in A that converges to x.

Proof. TBD ■

Haussdorf spaces induce uniqueness of limits of sequences If X is Haussdorf space, then every convergent sequence of points in X converges to a unique point in X.

Proof. TBD ■

—

2.3 Boundary of Set

Definition (Boundary of Set)**.** Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Then we call the *boundary* of A, denoted by ∂A, the set $\partial A = \text{cl}(A) \setminus \text{int}(A)$. Basically the closure of A minus the interior of A.

-intuitively this is the set of points on the edge of a structure.

Point in boundary iff every neighborhood of point intersects set and complement of set Let $A \subset X$, where $(X, \mathscr{T})$ is a topological space, and let $x \in A$. Then $x \in \partial A$ if and only if every neighborhood U of x intersects A and $X \setminus A$. Equivalently, if each open set U such that $x \in U$ satisfies $U \cap A \neq \varnothing$ and $U \cap (X \setminus A) \neq \varnothing$.

Proof. Let $A \subset X$. Suppose $x \in \partial A$. Then $x \in \text{cl}(A) \setminus \text{int}(A)$, so in particular $x \in \text{cl}(A)$ and $x \notin \text{int}(A)$. Since $x \in \text{cl}(A)$, Theorem 2.5 states that any open set U containing x satisfies $U \cap A \neq \varnothing$. Similarly, since $x \notin \text{int}(A)$, it follows that there exists no neighborhood U of x such that $x \in U \subset A$. Thus $U \cap (X \setminus A) \neq \varnothing$.

Conversely, assume any open set U for which $x \in U$ satisfies $U \cap A \neq \varnothing$ and $U \cap (X \setminus A) \neq \varnothing$. Since there exists no open set for which $U \subset A$, it follows that $x \notin \text{int}(A)$. But $x \in \text{cl}(A)$ if and only if $U \cap A \neq \varnothing$, by Theorem 2.5, which we indeed have as true. Thus $x \in \text{cl}(A)$ and $x \notin \text{int}(A)$ and so $x \in \text{cl}(A) \setminus \text{int}(A)$, to which $x \in \partial A$. ∎

—

Properties of Boundary of Set

Theorem 2.3.1. *boundary is closed set; ∂A is closed.*

Proof. Take $\partial A = \text{c}(A) \setminus \text{int}(A) = \text{cl}(A) \cap \text{int}(A)^c$. Then the complement of the boundary is the set $(\text{cl}(A) \cap \text{int}(A)^c)^c = \text{cl}(A)^c \cup \text{int}(A)$ by De Morgan's Law. Since the closure of A is closed, its complement is open. Also, the interior is an open set. The union of two open sets is open, and so the complement of the boundary is open. Thus ∂A is closed. ∎

Theorem 2.3.2 (boundary union interior is closure)**.** $\partial A \cup \textit{int}(A) = \textit{cl}(A)$.

Proof. By definition, $\partial A = \text{cl}(A) \setminus \text{int}(A) = \text{cl}(A) \cap \text{int}(A)^c$. Now we take the union of both sides with the interior of A, so we obtain:

$$\partial A \cup \text{int}(A) = (\text{cl}(A) \cap \text{int}(A)^c) \cup \text{int}(A)$$

Followed by the distributive law, we find:

$$= (\text{cl}(A) \cup \text{int}(A)) \cap (\text{int}(A)^c \cup \text{int}(A))$$

$$= \text{cl}(A) \cup \text{int}(A) \cap X$$

$$= \text{cl}(A)$$

Which follows since the interior of a set is contained within the closure, and so their union is simply the closure. Thus we have found the desired relation. ∎

Theorem 2.3.3 (Boundary and interior are disjoint)**.** $\partial A \cap \textit{int}(A) = \varnothing$

Proof. Now take $\partial A = \text{cl}(A) \setminus \text{int}(A) = \text{cl}(A) \cap \text{int}(A)^c$. Now we take the intersection of both sides of the equation, and find:

$$\partial A \cap \text{int}(A) = (\text{cl}(A) \cap \text{int}(A)^c) \cap \text{int}(A)$$

$$= \text{cl}(A) \cap \varnothing = \varnothing$$

And thus we obtained the desired relation, the boundary intersecting with the interior of the set is empty. ■

set contains boundary iff set closed; $\partial A \subseteq A$ if and only if A is closed.

Proof. If $\partial A \subseteq A$ then by the first property, $\partial A \cup \text{int}(A) = \text{cl}(A)$, and the fact that $\text{int}(A) \subset A$ holds in general, we have $\text{cl}(A) \subseteq A$, to which $A = \text{cl}(A)$ clearly follows since in general a set is contained in its closure. But this implies that A is a closed set. Conversely, take A closed. Then $A = \text{cl}(A)$, and so $A = \partial A \cup \text{int}(A)$. Therefore we clearly have $\partial A \subseteq A$. ■

set and boundary disjoint iff open; $A \cap \partial A = \varnothing$ if and only if A is open.

Proof. Take $A \cap \partial A = \varnothing$. This means $A \cap \text{cl}(A) \cap \text{int}(A)^c = \varnothing$. But note that $A \subseteq \text{cl}(A)$ holds in general, and so the above reduces to $A \cap \text{int}(A)^c = \varnothing$, which indeed implies that $A \setminus \text{int}(A) = \varnothing$. Thus $A = \text{int}(A)$, and so A is open, proving the forward direction. For the reverse implication, assume A is open. Then $A = \text{int}(A)$. But we know from property 3 above that the boundary of a set and the interior are disjoint. Thus $A \cap \partial A = \varnothing$. ■

Theorem 2.3.4 (Boundary Empty iff Set Open and Closed)**.** *$\partial A = \varnothing$ if and only if A is both open and closed simultaneously.*

Proof. Take $\partial A = \varnothing$. Then we know by property 2 that $\partial A \cup \text{int}(A) = \text{cl}(A)$, and so we have the previous statement equivalent to $\varnothing \cup \text{int}(A) = \text{int}(A) = \text{cl}(A)$. But we know that in general $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$, and so the previous statement occurs if and only if $A = \text{int}(A)$ and $A = \text{cl}(A)$, which occurs if and only if A is both open and closed. This suffices to show the proof of the proposition in both directions. ■

2.4 Subspace Topology

Definition (Subspace Topology)**.** Let X be a topological space and $Y \subset X$. Define

$$\mathscr{T}_Y = \{U \cap Y \mid U \text{ is open in } X\}$$

We call $\mathscr{T}_Y$ the *subspace topology* on Y. When equipped with this topology, we call Y a *subspace* of X. Further, a set $V \subset Y$ is open in Y if V is an open set in the subspace topology on Y.

-so basically, given an ambient topological space, we can take subsets and restrict open sets to those intersecting the subset to get a kind of sub-topology on the subset.

Theorem 2.4.1 (Subspace topology is a topology)**.** *If X is a topological space and $Y \subset X$, then the set $\mathscr{T}_Y$ defined above is a topology on the set Y.*

Proof. First we have $\varnothing, Y \in \mathscr{T}_Y$ since $\varnothing = \varnothing \cap Y$ and $Y = X \cap Y$. Let $V_1, \ldots, V_n \in \mathscr{T}_Y$. Then $V_j = U_j \cap Y$ for open sets $U_1, \ldots, U_n$ in X by construction. But this means $\bigcap_{j=1}^n V_j = \bigcap_{j=1}^n (U_j \cap Y) = \bigcap_{j=1}^n U_j \cap Y$. Since $\bigcap_{j=1}^n U_j$ is an open set given X, we may write that $\bigcap_{j=1}^n V_j$ is open in $\mathscr{T}_Y$. Now let $\{V_\alpha\}_{\alpha \in I}$ be a collection of open sets in T_Y. Then $V_\alpha = U_\alpha \cap Y$ for each $\alpha \in I$ by construction. This implies that $\bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I}(U_\alpha \cap Y) = \bigcup_{\alpha \in I} U_\alpha \cap Y$. Again, since arbitrary unions of open sets in X are open, we have that $\bigcup_{\alpha \in I} V_\alpha \in \mathscr{T}_Y$. Thus $\mathscr{T}_Y$ is a topology on Y. ∎

—

Definition (Standard topology on subset of $\mathbb{R}^n$)**.** Let $Y \subset \mathbb{R}^n$. The *standard topology* on Y is the topology inherited by Y as a subspace of $\mathbb{R}^n$ equipped with the standard topology.

-this basically means we can take subsets of Euclidean space and induce a topology on them given the standard topology. -also, any familiar shapes or objects; tori, spheres, circles, all have topologies induced by the standard topology.

—

Definition (Closed Sets in Subspace Topology)**.** Let X be a topological space and $Y \subset X$ have the subspace topology. We call a set $C \subset Y$ *closed in* Y if C is closed under the subspace topology in Y.

Closed sets in subspace topology equal intersections of closed sets with the subspace (3.4) Let X be a topological space and $Y \subset X$ have the subspace topology. Then $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X.

Proof. Take $C \subset Y$. Suppose C is closed in Y. Then C^c is an open set in Y. This means that $C^c = A \cap Y$ for some open set A in X. Taking the complement of the previous equation, we find $C = (A \cap Y)^c = A^c \cup Y^c$. But then

$$C \cap Y = (A^c \cup Y^c) \cap Y \iff C = (A^c \cap Y) \cup (Y^c \cap Y) = A^c \cap Y$$

Which follows since we assumed $C \subset Y$. Thus, since A was open in X, A^c is closed in X. This suffices to show that if C is a closed set of Y then $C = D \cap Y$ for some closed set D in X.

Conversely, assume $C = D \cap Y$ for some closed set D in X. Then, taking the complement, we find $C^c = (D \cap Y)^c = D^c \cup Y^c$. Intersecting both sides with Y yields $C^c \cap Y = D^c \cap Y$. But note that D^c is open in X, and so $C^c \cap Y$ is open in the subspace topology by construction. But the complement of $C^c \cap Y$ when considered in Y is simply C. This means C is closed in Y. ■

—

Basis for subspace topology is basis elements intersected with subspace (3.5) Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. If $Y \subset X$, then

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Take X a topological space and $Y \subset X$ with the subspace topology. Assume $\mathcal{B}$ is a basis for the topology on X. Take $y \in Y$. Then, since $y \in Y \subset X$, it follows that there exists some $B \in \mathcal{B}$ for which $y \in B$. But then $y \in B \cap Y \in \mathcal{B}_Y$.

Now assume $B_1, B_2 \in \mathcal{B}_Y$ and that $y \in B_1 \cap B_2$. By construction, $B_1 = B_1' \cap Y$ and $B_2 = B_2' \cap Y$ for some $B_1', B_2' \in \mathcal{B}$. By definition of a basis, there exists $B_3' \in \mathcal{B}$ for which $B_3' \subset B_1' \cap B_2'$. In particular, we have $B_3 = B_3' \cap Y \in \mathcal{B}_Y$ satisfying $y \in B_3 \subset B_1 \cap B_2$. Therefore we may write that $\mathcal{B}_Y$ is a basis for the subspace topology on Y. ■

2.5 Product Topology

Definition (Product Topology)**.** Take X and Y topological spaces and $X \times Y$ their product. We call the topology generated by the following basis the *product topology* on $X \times Y$,

$$\mathcal{B} = \{U \times V \mid U \text{ open in } X, V \text{ open in } Y\}$$

-so basically taking two topological spaces and constructing a basis out of their respective open set products.

Basis of product topology generates product topology The collection $\mathcal{B}$ defined above is a basis for a topology on $X \times Y$.

Proof. TBD ■

Products of two bases is another basis that generates product topology Let X and Y be topological spaces and $\mathcal{C}$ a basis for X and $\mathcal{D}$ a basis for Y. Then

$$\mathcal{E} = \{C \times D \mid C \in \mathcal{C},\ D \in \mathcal{D}\}$$

is a basis for a topology on $X \times Y$; in fact the topology generated by $\mathcal{E}$ is the product topology on $X \times Y$.

Proof. TBD ■

—

Subspace topology and product topology equivalent for subsets of products of topological spaces Let X and Y be topological spaces with $A \subset X$ and $B \subset Y$. Then the subspace topology on $A \times B$ as a subset of $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X and B has the subspace topology inherited from Y.

-in other words, the topological space contained in a product topology can be recovered in two ways; (1) via the subset product as a subspace topology, (2) by the subset product as a product topology where each component inherits subspace topology from their ambient space.

Proof. TBD ■

—

Interior of product is product of interiors; Closure of product is product of closures If A and B are subsets of topological spaces X and Y, respectively, then int$(A \times B) =$ int$(A) \times$ int(B). Similarly, cl$(A \times B) =$ cl$(A) \times$ cl(B).

Proof. TBD ■

—

2.6 Quotient Topology

Definition (Quotient Topology)**.** Let X be a topological space, and A an arbitrary set. Let $p : X \to A$ be a surjective mapping. Define a subset $U \subset A$ to be open in A if its preimage, $p^{-1}(A)$, is open in X.

The resulting collection of open sets is called the *quotient topology induced by p*. The map p is called *quotient map*. The topological space A, equipped with the quotient topology, is called a *quotient space*.

Quotient topology is a topology Let $p : X \to A$ be a quotient map. The quotient topology on A induced by p is a topology on A.

Proof. Take X a topological space and A a set. Let $p : X \to A$ be a surjective map. Note that $p^{-1}(\varnothing) = \varnothing$ is open in X and $p^{-1}(A) = X$ is open in X. This implies both $\varnothing$ and X are in the quotient topology.

Now take $U_1, \ldots, U_n$ in the quotient topology. Since each $p^{-1}(U_j)$ was assumed open in X, it follows that their finite intersection is open in X. Note then that since $\bigcap_{j=1}^{n} p^{-1}(U_j) = p^{-1}(\bigcap_{j=1}^{n} U_j)$, it follows that $\bigcap_{j=1}^{n} U_j$ is open in the quotient topology.

Now take $\{U_\alpha\}_{\alpha \in I}$ a collection in the quotient topology. Then since we have that $\bigcup_{\alpha \in I} p^{-1}(U_\alpha) = p^{-1}(\bigcup_{\alpha \in I} U_\alpha)$, and the union of the preimages is open in X since it is the arbitrary union of open sets, it follows that $\bigcup_{\alpha \in I} U_\alpha$ is open in the quotient topology. ∎

3 Continuity and Homeomorphisms

3.1 Continuity

Definition (Continuous map (Open set def of continuity))**.** Let X and Y be topological spaces. A map $f : X \to Y$ is called *continuous* if $f^{-1}(V)$ is open in X for every open set V in Y.

-this definition is equivalent to the ϵ-δ definition of continuity, however it is stated much more simply here. -the subspace topology is the coarsest topology that makes the inclusion map a continuous map. -the quotient topology is the finest topology that makes the quotient map a continuous map.

Continuous map iff preimage of basis elements open for all basis elements (4.3) Let X and Y be topological spaces and $\mathcal{B}$ a basis for the topology on Y. Then we have $f : X \to Y$ continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

Proof. TBD ∎

Continuous map iff preimage of closed sets are closed Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every closed set $C \subset Y$, we have $f^{-1}(C)$ is closed in X.

Proof. TBD ∎

—

Map continuous iff every open set containing image of element there exists neighborhood around element such that image of neighborhood contained in the open set A map $f : X \to Y$ is continuous in the open set definition of continuity if and only if for every $x \in X$ and open set U of Y containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$.

Proof. TBD ∎

Composition of continuous maps is continuous If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof. TBD ∎

Polynomial functions from R to R are continuous Let $\mathbb{R}$ have the standard topology. Then every polynomial function $p : \mathbb{R} \to \mathbb{R}$ with $p(x) = \sum_{i=0}^{n} a_i x^i$ is continuous.

Proof. TBD ∎

Continuous maps take convergent sequences to convergent sequences Assuming $f : X \to Y$ is continuous, if we have a sequence $(x_n)_{n=0}^{\infty}$ in X converging to x, then $(f(x_n))_{n=0}^{\infty}$ in Y converges to $f(Y)$.

Proof. TBD ∎

—

Continuous maps take points in closure to points in closure of image Let $f : X \to Y$ be continuous, and let $A \subset X$. If $x \in cl(A)$, then $f(x) \in cl(f(A))$.

Proof. TBD ∎

—

Pasting Lemma Let X be a topological space and A, B closed subsets of X such that $A \cup B = X$. Assume we have $f : A \to Y$ and $g : B \to Y$ continuous maps, and furthermore that $f(x) = g(x)$ for all $x \in A \cap B$. Then we have $h : X \to Y$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is a continuous map.

Proof. TBD ■

—-

3.2 Homeomorphisms

Definition (Homeomorphism)**.** Let X and Y be topological spaces, and let $f : X \to Y$ be a bijection with inverse $f^{-1} : Y \to X$. If both of f and f^{-1} are continuous functions, then f is called a *homeomorphism*. If there exists a homeomorphism between X and Y, then X and Y are *homeomorphic* and $X \cong Y$.

-basically if one can find function between spaces whose self and inverse is continuous grants homeomorphicity. -it can be noted that 'is homeomorphic to' is an equivalence relation.

Example 3.2.1 (Plane equivalent to Half Plane equivalent to disk)**.** With the standard topology, the space $\mathbb{R}^2$ is homeomorphic to $H = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ and the open disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.

This can be realized by $f : \mathbb{R}^2 \to H$ by $f(x, y) = (e^x, y)$ for all $(x, y) \in \mathbb{R}^2$. If (x, y) such that $x < 0$, then $0 < e^x < 1$ and y is arbitrary, so the entire left half of the plane gets mapped to the vertical sheet $0 < x < 1$. Note the y-axis becomes the line $x = 1$. The right half of the plane is mapped to the region in H where $x > 1$.

Also, $g : \mathbb{R}^2 \to D$ defined by $g(r, \theta) = (1/(1 + r, \theta)$ is a homeomorphism, contracting the whole plane radially inward to coincide with D.

Example 3.2.2 (Surface of cube homeomorphic to sphere)**.** Let C denote the surface of a cube. Then, if both the cube and S^2 are centered at the origin in $\mathbb{R}^3$, then $f : C \to S^2$ by $f(p) = p/|p|$ is a homeomorphism.

Example 3.2.3 (Punctured sphere homeomorphic to plane (stereographic projections))**.** TBD

Ex - Cylinder quotient space is a sphere

—

Definition (Embedding)**.** A function $f : X \to Y$ that maps X homeomorphically to the subspace $f(X)$ in Y is called an *embedding of X in Y.*

Definition (Arc & Simple closed curves)**.** Let X be a topological space. If $f : [-1, 1] \to X$ is an embedding, then the image of f, $f([-1, 1])$, is called an *arc* in X. If $f : S^1 \to X$ is an embedding, then the image of f, $f(S^1)$, is called a *simple closed curve* in X.

Theorem 3.2.1 (Homeomorphism from Haussdorff space implies Haussdorf)**.** *If $f : X \to Y$ is a homeomorphism, and X is Haussdorff, then Y is Haussdorff.*

-Haussdorff is an example of a topological property; a property about topological spaces that relies on open sets, and is transmitted via homeomorphisms.

Proof. TBD ■

4 Metric Spaces

4.1 Metrics

Definition (Metric)**.** Let X be a set and d a function such that $d : X \times X \to \mathbb{R}$ with the following properties, which we call a *metric*: 1. $d(x, y) \geq 0$ for all $x, y \in X$. (non-negativity) 2. $d(x, y) = d(y, x)$ for all $x, y \in X$. (symmetry) 3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality) We call $d(x, y)$ the *distance between the points x and y*, and the pair (X, d), the set with the metric, a *metric space*.

Example 4.1.1 (Euclidean / Standard Metric)**.** On the space $\mathbb{R}$, we call $d(x, y) := |x - y|$ for all $x, y \in \mathbb{R}$ the *Euclidean* or *standard* metric on $\mathbb{R}$. Similarly for $\mathbb{R}^2$, we have the metric as follows $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for all $x, y \in \mathbb{R}^2$.

-usual distance as the crow flies.

Example 4.1.2 (Taxicab / Manhatten Metric)**.** On the space $\mathbb{R}$, define $d_T(x, y) := |x_1 - x_2| + |y_1 - y_2|$ for all $x, y \in \mathbb{R}^2$.

-like moving along the gridlines of a city, only can move north south east and west.

Example 4.1.3 (Max Metric)**.** On the space $\mathbb{R}^2$, we have $d_M(x, y) := \max\{|x_1 - x_2|, |y_1 - y_2|\}$ for all points $x, y \in \mathbb{R}^2$.

-can think of this like taking the greatest distance between two coordinates of one component.

—

Definition (Open metric ball / Closed metric ball)**.** Let (X, d) be a metric space. For $x \in X$ and $\epsilon > 0$, define the *open ball of radius ϵ centered at x* to be the set

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

And similarly, define the *closed ball of radius ϵ centered at x* to be the set

$$\overline{B}_d(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

Theorem 4.1.1 (Collection of open metric balls is basis for a topology on a metric space)**.** *Let (X, d) be a metric space. The collection of open metric balls $\mathcal{B} = \{B(x, \epsilon) \mid x \in X, \epsilon > 0\}$, is a basis for a topology on X.*

Proof. TBD ■

—

Definition (Metric topology / topology induced by metric)**.** Let (X, d) be a metric space. The topology generated by the basis of open metric balls $\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$, is called *the topology induced by d*, also referred to as a *metric topology*.

-basically, given a metric space, we can form a topology on that space by virtue of the open metric balls which the metric induces on the set X.

Theorem 4.1.2 (Set open in metric topology iff each point in set is contained in some open metric ball)**.** *Let (X, d) be a metric space. A set $U \subset X$ is open in the metric topology if and only if for every $y \in U$ there exists $\delta > 0$ such that $B_d(y, \delta) \subset U$.*

Proof. TBD ■

—

4.2 Properties of Metric Spaces

Theorem 4.2.1 (Every metric space is Haussdorff)**.** *If* (X, d) *is a metric space, then* X *is a Haussdorff space.*

Proof. Let $x, y \in X$ such that $x \neq y$. Let $\epsilon = d(x, y)$. With this in mind, we may construct a metric ball $B_d(x, \epsilon/2)$ such that $y \notin B_d(x, \epsilon/2)$. Similarly, we have $x \notin B_d(y, \epsilon/2)$.

Recall that the metric topology on X is generated by the basis $\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$. Since both $B_d(x, \epsilon/2)$ and $B_d(y, \epsilon/2)$ can easily be seen to be elements of $\mathcal{B}$, and basis elements are open in the toplogy generated by a basis, both $B_d(x, \epsilon/2)$ and $B_d(y, \epsilon/2)$ are open sets in X.

Now we show $B_d(x, \epsilon/2) \cap B_d(y, \epsilon/2) = \varnothing$. Assume this isn't the case, in other words, we have some $z \in B_d(x, \epsilon/2) \cap B_d(y, \epsilon/2)$. Then we would have

$$d(x, y) \leq d(x, z) + d(y, z) < \epsilon/2 + \epsilon/2 = \epsilon$$

Which is a contradiction for we took $\epsilon = d(x, y)$. Thus it must be the case that these metric balls are disjoint. Thus, for any pair of distinct points in X, we have found disjoint open sets containing each element. Therefore, X is a Hausdorff space. ■

—

Theorem 4.2.2 (Continuous function criterion for metric space)**.** *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A function* $f : X \to Y$ *is continuous in the open set definition if and only if for each* $x \in X$ *and* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that if* $x' \in X$ *and* $d_X(x, x') < \delta$*, implies* $d_Y(f(x), f(x')) < \epsilon$*.*

Proof. TBD ■

Definition (Distance between sets)**.** Let (X, d) be a metric space and $A, B \subset X$ define *the distance between the sets A and B* by

$$d(A, B) = \text{glb}\{d(a, b) \mid a \in A, b \in B\}$$

The greatest lower bound between sets is also the inferior of the set.

—

Theorem 4.2.3 (Topology is finer than another iff open metric ball can be placed inside)**.** *Let* d *and* d' *be metrics on a set* X*, and* $\mathcal{T}$ *and* $\mathcal{T}'$ *be the topologies that they induce on* X*. Then* $\mathcal{T}'$ *is finer than* $\mathcal{T}$ *if and only if for each* $x \in X$ *and* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$*.*

-so basically if one can put an open metric ball in another, then the topology induced by that smaller metric is finer than the other.

Proof. TBD ■

Theorem 4.2.4 (Standard, taxicab, and max metric induce same topology on R2)**.** *On $\mathbb{R}^2$, the standard metric and the taxicab metric induce the same topology.*

Proof. TBD ■

—

Definition (Bounded Metrics)**.** Let (X, d) be a metric space, with $A \subset X$. We call A *bounded under d* if there exists $\mu > 0$ such that $d(x, y) \leq \mu$ for all $x, y \in A$.

Further, if X itself is bounder under d, then we call d a *bounded metric*.

-boundedness of a metric does not have an impact on the topology it induces. -if X is bounded under d, then every subset of X is also.

Theorem 4.2.5 (Every metric topology is induced by a bounded metric)**.** *Let (X, d) be a metric space, define $d' : X \times X \to \mathbb{R}$ by $d'(x, y) := \min\{d(x, y), 1\}$. Then d' is a bounded metric that induces the same topology as d.*

Proof. TBD ■

Definition (Isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A bijective function $f : X \to Y$ is called an *isometry* if $d_X(x, x') = d_Y(f(x), f(x'))$ for all $x, x' \in X$. If $f : X \to Y$ is an isometry, then we write that the metric spaces X and Y are *isometric*.

-basically an equivalency between spaces for metric spaces. -refer to homeomorphisms for topological spaces.

—

4.3 Metrizability

Definition (Metrizable)**.** Let X be a topological space. We call X *metrizable* if there exists a metric d on X that induces the topology on X.

-if X is a metric space and Y is a subset of X, then the subspace topology on Y is also metrizable. -on any set X, the discrete topology on X is metrizable. -if X is finite, then every metric on X induces the discrete topology. -the real line $\mathbb{R}$ is metrizable.

Homeomorphism from metrizable topological space implies metrizable If X is a metrizable topological space and $X \cong Y$, then Y is a metrizable topological space.

-basically metrizability is a topological property that is preserved over homeomorphisms betwen topological spaces.

Proof. TBD ■

—

Definition (Regular)**.** Let X be a topological space. We call X *regular* if: 1. one-point sets are closed in X; 2. for every $a \in X$ and closed set $B \subset X$ such that $a \notin B$, there exists disjoint open sets U and V such that $a \in U$ and $B \subset V$.

*-the real line $\mathbb{R}$ with the standard topology is regular. -if a topological space is regular, then it is Haussdorff. -regularity is a strengthening of the Haussdorff property. -one of the *separation axioms*.*

Definition (Normal)**.** Let X be a topological space. We call X *normal* if: 1. one-point sets are closed in X; 2. for every pair of disjoint closed sets A and B in X, there exists disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

-if X is a normal space, then it is regular, and then it is Haussdorff. -any metric space is a normal space.

—

Theorem 4.3.1 (Urysohn Metrization Theorem)**.** *If X is a topological space that is regular and has a countable basis, then X is metrizable.*

—

5 Connectedness

5.1 Connectedness

If X is not connected, then we call X *disconnected*. If X is disconnected and U and V are disjoint open sets whose union is X, then we call U and V a *separation of X*.

-so a set is disconnected if we can split it up into open sets, and connected if we cannot; the open set decomposition of the set, if the set is disconnected, is called a separation of the set.

—

Theorem 5.1.1 (Connected iff no proper non-trivial open and closed sets)**.** *If X is a topological space, then X is connected if and only if there exist no non-trivial, proper subsets of X that are open and closed simultaneously in X.*

Proof. Suppose X is connected. Assume, for contradiction, that U is a subset of X that is open and closed in X. Then U is open and $X \setminus U$ is open, so we have found a separation of X, a contradiction. Conversely, if there exists a subset U of X that is open and closed in X, then U is open and $X \setminus U$ is open, to which we have found a separation of X and thus X is disconnected; proved by contrapositive. ■

-I think i could touch up the above proof to make it more succint, but it captures the truth of the statement. -basically open and closed sets present an easy separation of a set, to which any set with such a subset must be disconnected.

—

Definition (Connected / disconnected Subspaces)**.** Let X be a topological space and $A \subset X$. We write that A is *disconnected in X* if A is disconnected in the subspace topology. If A is not disconnected in X, then we say A is *connected in X*.

-basically how to deal with connectedness when considering subspaces of topological spaces; the natural answer is to consider the subspace topology.

—

Definition (Separation of Subspace in Space)**.** Let $A \subset X$ where X is a topological space. If U and V are open sets in the subspace topology on A such that $A \subset U \cup V$, $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$, and $A \cap U \cap V = \varnothing$, then we call the pair U and V a *separation of A in X*.

-this is just a usual separation when considered for subspaces of topological spaces.

Theorem 5.1.2 (Subspace disconnected iff there exists separation of subspace)**.** *Let X be a topological space and $A \subset X$. Then A is disconnected in X if and only if there exist open sets U and V of X such that $A \subset U \cup V$, $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$, and $A \cap U \cap V = \varnothing$.*

Proof. Suppose A is disconnected in X. Then there exists open sets U and V of A such that $U \cap V = \varnothing$ and $U \cup V = A$. But we know that open sets in the subspace topology on A are of the form $U = A \cap U'$ and $V = A \cap V'$ for open sets U' and V' of X. But then

$$A = U \cup V = (A \cap U') \cup (A \cap V') = A \cap (U' \cup V')$$

To which we have $A \subset U' \cup V'$. Furthermore, $A \cap U' \neq \varnothing$ and $A \cap V' \neq \varnothing$ since by assumption $U, V \neq \varnothing$. Also, since $U \cap V = \varnothing$, we can see that indeed $A \cap U' \cap V' = \varnothing$.

Assume the converse. Then $P = A \cap U$ and $Q = A \cap V$ are both open sets in A with the subspace topology. Also $P \cap Q = \varnothing$ since $A \cap U \cap V = \varnothing$. Since $A \subset U \cup V$, we have $A = P \cup Q$. ■

—

Theorem 5.1.3 (Continuity preserves connectedness)**.** *If X is a connected space and $f : X \to Y$ is a continuous function, then $f(X)$ is connected in Y.*

Proof. Suppose X is connected. Assume, for contradiction, that $f(X)$ is disconnected. Then there exists a separation of $f(X)$ in Y, so U and V disjoint open sets of Y such that $f(X) = U \cup V$. But then since f is continuous, $f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) = X$. Further, if $x \in f^{-1}(U) \cap f^{-1}(V)$, then $f(x) \in U$ and $f(x) \in V$, which is a contradiction since $U \cap V = \varnothing$. Thus $f^{-1}(U) \cap f^{-1}(V) = \varnothing$. Hence w have found a separation of X; a contradiction. ■

—

Theorem 5.1.4 (Connected subsets contained in separation of set)**.** *Let $A, B \subset X$ where X is a topological space. Assmme A is connected and $A \subset B$. If U and V form a separation of B, then $A \subset U$ or $A \subset V$.*

Proof. Take $A \subset B$ with A connected. Assume U and V form a separation of B. Then we know that $U \cup V = B$ and $U \cap B \neq \varnothing$ and $V \cap B = \varnothing$ and $B \cap U \cap V = \varnothing$.

Note that $A \cap U$ and $A \cap V$ are both open sets in the subspace topology on A inherited from B. Since A is connected, and

$$(A \cap U) \cup (A \cap V) = A \cap (U \cup V) = A \cap B = A$$

We know that $(A \cap U) \cap (A \cap V) = \varnothing$. In particular, we have disjoint open sets of A whose union is A. But this is a contradiction to our assumption that A is connected. Thus either $A \cap U$ or $A \cap V$ or both must be empty. Both cannot be empty, for then $A \subset B = U \cup V$ would not hold. Thus either $A \cap U = \varnothing$ or $A \cap V = \varnothing$. Without loss of generality, $A \subset U$ or $A \subset V$ must follow. ■

—

Theorem 5.1.5 (Adding limit points to connected subset maintains connectedness (closure of connected set is connected))**.** *Let C be a connected in a topological space X. Assume $C \subset A \subset \overline{C}$. Then A is connected in X.*

Proof. Assume, for contradiction, that A is disconnected. Then there exists a separation of A given by U and V. By Lemma 6.7, either $C \subset U$ or $C \subset V$. Without loss of generality, take $C \subset U$. Now $C \cap V = \varnothing$. Since $A \cap V \neq \varnothing$, choose $x \in A \cap V$. Then, in particular, $x \in \overline{C}$ and $x \in V$. But then x is contained in an open set V where $C \cap V = \varnothing$, and so x is not a boundary point nor an interior point of C. But then $x \notin \overline{C}$, a contradiction. ■

—

Theorem 5.1.6 (Nonempty intersection of connected sets implies union is connected)**.** *Let X be a topological space and $\{C_\alpha\}_{\alpha \in I}$ a collection of connected subsets of X. If $\bigcap_{\alpha \in I} C_\alpha \neq \varnothing$, then $\bigcup_{\alpha \in I} C_\alpha$ is connected.*

Proof. Assume, for contradiction, that $\bigcup_{\alpha \in I} C_\alpha$ is disconnected. Then there exist open sets U and V of X for which $U \cap V = \varnothing$ and $\bigcup_{\alpha \in I} C_\alpha = U \cup V$. Since $x \in \bigcap_{\alpha \in I} C_\alpha$, it follows that $x \in U \cup V$, to which $x \in U$ or $x \in V$. Without loss of generality, let $x \in U$. Then since $x \in C_\alpha$ for all $\alpha \in I$, we may write that $C_\alpha \subset U$, for if not then $U \cap V \neq \varnothing$. But then $\bigcup_{\alpha \in I} C_\alpha \subset U$, which is a contradiction for we took $V \neq \varnothing$. Thus $\bigcup_{\alpha \in I} C_\alpha$ is connected. ■

—

Theorem 5.1.7 (Product of connected sets is connected)**.** *If $X_1, X_2, \ldots, X_n$ are connected, then $\prod_{i=1}^n X_i$ is connected.*

Proof. Let X and Y be connected. Since $\{x\} \times Y \cong Y$ and $X \times \{y\} \cong X$, we may write that $\{x\} \times Y$ and $X \times \{y\}$ are connected for any $x \in X$ and $y \in Y$. Fix $x_0 \in X$. We know that $(\{x_0\} \times Y) \cup (X \times \{y\})$ is connected since their intersection contains $\{x_0\} \times \{y\}$. In particular, the union of all such sets

$$\bigcup_{y \in Y} (\{x_0\} \times Y) \cup (X \times \{y\}) = X \times Y$$

is connected since the set on the left has a non-empty intersection. Therefore the product $X \times Y$ is connected. ■

—-

Definition (Connected Component of Set)**.** Let X be a topological space. The relation $\sim_C$ defined by $x \sim_C y$ if and only if x and y lie in a connected subset of X is an equivalence relation. The equivalence classes of $X/\sim_C$ are called the *components* of X.

-this is basically partitioning a topological space into connected sets; a kind of decomposition into basic connected building blocks of a space.

Theorem 5.1.8 (Components are connected, closed, and contain any connected subsets of a space)**.** *Let X be a topological space. Then the following statements hold:*

1. Every component of X is connected in X. 2. Every component of X is closed in X. 3. If A is a connected subset of X, then A is contained in a component of X.

Proof. (1) Let C be a component of X, with $x \in C$. Assume for contradiction that C is disconnected. Then there exist U and V open in X such that $U \cup V = C$ and $U \cap V = \varnothing$. By definition of a component, there exists a connected set C_x containing x in C. Since C_x is connected, either $C_x \subset U$ or $C_x \subset V$. Without loss of generality, take $C_x \subset U$. But then if $y \in C$ is another point, we must have $y \in C_x$ since x and y are equivalent under $\sim_C$. Thus $C \subset U$, a contradiction. ■

Homeomorphisms preserve components Let X and Y be topological spaces, with C a component of X. If $f : X \to Y$ is a homeomorphism, then $f(C)$ is a component of Y.

Proof. Let C be a component of X. In particular, we know C is connected in X. Thus $f(C)$ is connected in Y since continuous maps preserve connectedness. Since $f(C)$ is connected in Y, it is contained in some component of Y, call it D, so $f(C) \subset D$. Note that $f^{-1} : Y \to X$ is continuous, and since D is connected in Y we know that $f^{-1}(D)$ is connected in X. In particular, $C \subset f^{-1}(D)$. Since C is a component of X, and $f^{-1}(D)$ is a connected set containing C, we must have that $C = f^{-1}(D)$. Thus $f(C) = D$. ■

—

Definition (Totally disconnected)**.** Let X be a topological space. We call X *totally disconnected* if the connected components of X are singleton subsets of X.

-basically no two points lie in the same connected subset of a space means that the components of that space are singleton.

—

5.2 Distinguishing Topological Spaces via Connectedness

)

Theorem 5.2.1 ($\mathbb{R}$ with standard topology is connected)**.** *The topological space $\mathbb{R}$ equipped with the standard topology is connected.*

Proof. Assume, for contradiction, that $\mathbb{R}$ is disconnected. Thus there exists disjoint open sets U and V of $\mathbb{R}$ that form a separation of $\mathbb{R}$. Take $u \in U$ and $v \in V$. Without loss of generality, take $u < v$. Then define $U' = U \cap [u, v]$ and $V' = V \cap [u, v]$. It follows that $U' \cup V' = [u, v]$.

Since $[u, v]$ is bounded above by v, this interval has a least upper bound, say α. Then $u \leq \alpha \leq v$, to which we can see that $\alpha \in [u, v]$. We derive a contradiction by showing that $\alpha \notin U'$ and $\alpha \notin V'$.

Assume, for contradiction, that $\alpha \in U'$. Since $v \notin U'$, and U' is an open set in $\mathbb{R}$, there exists c for which $[\alpha, c) \subset U'$. But then if $d \in (\alpha, c)$ then $\alpha < d$ and $d \in U'$, a contradiction to α being the least upper bound of U'. Thus $\alpha \notin U'$.

Assume, for contradiction, that $\alpha \in V'$. Since V' is open and $u \notin V'$, we know that there exists c such that $(c, \alpha] \subset V'$. Thus c is an upper bound of U', while $c < \alpha$, a contradiction.

Therefore $\alpha \notin U'$ and $\alpha \notin V'$, to which $\alpha \notin U' \cup V' = [u, v]$. This is a contradiction. Therefore $\mathbb{R}$ is connected. ■

-a consequence of this fact is that (a, b), (a, ∞), $(-\infty, b)$ are all connected spaces; following since $\mathbb{R}$ is homeomorphic to all subsets of this form. -also, intervals like $[a, \infty)$, $(-\infty, b]$, $[a, b)$, and $(a, b]$ are all connected, which follows since adding limit points to a connected set maintains connectedness. -also $[a, b]$ is connected,

—

Theorem 5.2.2 ($\mathbb{R}^n$ with standard topology is connected)**.** *Euclidean n-space, $\mathbb{R}^n$, with the standard topology, is connected.*

Proof. By the fatc that $\mathbb{R}$ is connected, and the product of connected spaces is connected, we may write that $\mathbb{R}^n$ is connected. ■

—

Definition (Cutset / cutpoint)**.** Let X be a connected topological space, with $A \subset X$. We call A a *cutset* of X if we have $X \setminus A$ is disconnected. A *cutpoint* is a point $p \in X$ for which $X \setminus \{p\}$ is disconnected.

When we have a cutpoint or a cutset of X, we say that the cutpoint or cutset *separates* X.

-basically which points and/or subsets of a connected space that, upon removal from the space, make the space disconnected, -think of the punctured plane $\mathbb{R}^2 \setminus \{O\}$, or $S^2 \setminus N$. -for the real line $\mathbb{R}$, every point $p \in \mathbb{R}$ is a cutpoint since $\mathbb{R} \setminus \{p\}$ is disconnected, while $\mathbb{R}$ is connected.

Homeomorphisms preserve cutsets Let X and Y be topological spaces and $f : X \to Y$ a homeomorphism. If S is a cutset of X, then $f(S)$ is a cutset of Y.

Proof. By definition of a cutset, we may write that X is connected. In particular we have $X \cong Y$, so we know Y is connected also. Since S is a cutset of X, it follows that $X \setminus S$ is disconnected.

Let U and V be open sets of X for which $U \cap V = \varnothing$ and $U \cup V = X \setminus S$. But then we have $f(U \cup V) = f(X \setminus S)$, which is equivalent to

$$f(U) \cup f(V) = f(X) \setminus f(S) = Y \setminus f(S)$$

Since the image of an open set is open given the homeomorphism, both $f(U)$ and $f(V)$ are open in Y. Also, $f(U) \cap f(V) = f(U \cap V) = f(\varnothing) = \varnothing$. Therefore we have found a separation of $Y \setminus f(S)$ in Y. In particular, the space $Y \setminus f(S)$ is disconnected; hence $f(S)$ is a cutset of Y. ∎

—

Example 5.2.1. $\mathbb{R}$ not homeomorphic to $\mathbb{R}^2$ We know that every $p \in \mathbb{R}$ is a cutpoint of $\mathbb{R}$. However, no points $q \in \mathbb{R}^2$ are cutpoints of $\mathbb{R}^2$ since the punctured plane is connected, and the punctured plane is homeomorphic to $\mathbb{R}^2 \setminus \{q\}$ for any q. Thus $\mathbb{R} \ncong \mathbb{R}^2$.

Example 5.2.2 (S^1 , S^2 not homeomorphic to $\mathbb{R}$)**.** Since $S^1 \setminus \{q\} \cong \mathbb{R}$ for any $q \in S^1$, and $\mathbb{R}$ is connected, it follows that no point in S^1 is a cutpoint. But every point in $\mathbb{R}$ is a cutpoint, and so $\mathbb{R} \ncong S^1$. Similarly, we know that $S^2 \setminus \{q\} \cong \mathbb{R}^2$ for any $q \in S^2$. It follows that no point in S^2 is a cutpoint. But since all points in $\mathbb{R}$ are cutpoints, we have $\mathbb{R} \ncong S^2$.

—

5.3 The Intermediate Value Theorem

Theorem 5.3.1 (The Generalized Intermediate Value Theorem)**.** *Let X be a connected topological space, with $f : X \to \mathbb{R}$ a continuous function. Suppose $p, q \in f(X)$ and $p \leq r \leq q$. Then $r \in f(X)$.*

Proof. Suppose $f : X \to \mathbb{R}$ is continuous and $p, q \in f(X)$ with $p \leq r \leq q$. If we have $r = p$ or $r = q$, then $r \in f(X)$. So assume $r \neq p$ and $r \neq q$. Then we have $p < r < q$.

Assume, for contradiction, that $r \notin f(X)$. Since $f(X) \subset \mathbb{R}$, we know that $U = (-\infty, r)$ and $V = (r, \infty)$ are disjoint open sets of $\mathbb{R}$ such that $f(X) \subseteq U \cup V$. Since $p < r < q$, we have $p \in U$ and $q \in V$. Thus we have that $f(X) \cap U \neq \varnothing$ and $f(X) \cap V \neq \varnothing$. Since $f(X) \cap U \cap V = \varnothing$, it follows that U and V form a separation of $f(X)$ in $\mathbb{R}$. This is a contradiction, for since X was assumed connected, its image $f(X)$ is connected. Thus $r \in f(X)$. ■

—

Theorem 5.3.2 (One-dimensional Brouwer Fixed Point Theorem)**.** *Let $f : [-1, 1] \to [-1, 1]$ be a continuous function. There exists at least one point $c \in [-1, 1]$ such that $f(c) = c$.*

Proof. Let $f : [-1, 1] \to [-1, 1]$ be a continuous function. Define $g(x) = f(x) - x$ which is a continuous function $g : [-1, 1] \to \mathbb{R}$, following from the fact that f is continuous and x, to which their difference is continuous. In particular, since $[-1, 1]$ is connected, we may invoke the intermediate value theorem to write that there exists some point $c \in [-1, 1]$ for which $g(c) = 0$. But then $g(c) = f(c) - c$, and so $f(c) = c$. ■

—

5.4 Path Connectedness

Definition (Path Connected)**.** A topological space X is *path connected* if for every $x, y \in X$ there exists a *path* in X from x to y. A subset $A \subset X$ is called *path connected in X* if A is path connected in the subspace topology inherited from X.

-recall that a path is a continuous function $[0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Basically can we connected all points in the space via a complete line?

—

Path connected implies connected Let X be a topological space. If X is path connected, then X is connected.

Proof. Assume X is path connected. Take $x, y \in X$ arbitrary, then there exists a path between them by assumption, say $p : [0, 1] \to X$. Since $[0, 1]$ is connected in $\mathbb{R}$ with the standard topology, and p is continuous, the image $p([0, 1])$ is connected in X. In particular, $x, y \in p([0, 1])$ and so x and y lie in the same connected subset of X. Thus there is one connected component of X, and so X is connected. ■

Ex - Topologist's Whirlpool This is the primary counter-example to why path connectedness implies connectedness, but the reverse does not hold.

Ex - Topologist's Sine Curve I do not understand this fucking example. It is horrible and very hard to perform. Fuck.

—

Continuous functions preserve path connectedness Let X and Y be topological spaces, with $f : X \to Y$ a continuous function. If X is path connected, then $f(X)$ is path connected in Y.

Proof. Let $p, q \in f(X)$. Then $f(x) = p$ and $f(y) = q$ for $x, y \in X$. Since X is path connected, there exists a path $p : [0, 1] \to X$ such that $p(0) = x$ and $p(1) = y$. Since the composition of continuous functions is continuous, we may write that $f \circ p : [0, 1] \to Y$ is continuous. In particular, we have that $f \circ p(0) = f(p(0)) = f(x) = p$ and $f(p(1)) = f(y) = q$. Thus we have that $f \circ p$ is a path from p and q. Therefore $f(X)$ is connected in Y. ■

—

Definition (Path Components)**.** Let X be a topological space. We say $x \sim_p y$ if there exists a path in X from x to y. To check this is an equivalence relation, note:

1. $x \sim_p x$ since $p : [0, 1] \to X$ defined by $p(c) = x$ for all $c \in [0, 1]$ is a continuous function. 2. If $x \sim_p y$, say with $p : [0, 1] \to X$, then $q : [0, 1] \to X$ by $q(t) = p(1 - t)$ is a continuous function that is a path from x to y, as we have $q(0) = p(1) = y$ and $q(1) = p(0) = x$. 3. If $x \sim_p y$ and $y \sim_p z$, say f and g are paths between them, respectively. Then $h(t) = f(2t)$ for $0 \leq t \leq 1/2$ and $h(t) = g(2t - 1)$ for $1/2 \leq t \leq 1$ is a path from x to z, by the pasting lemma.

The equivalence classes of X under the equivalence relation $\sim_p$ are called the *path components* of X.

6 Compactness

6.1 Compactness

Definition (Covers / Open covers / Subcovers)**.** Let X be a topological space with $A \subset X$. Let $\mathcal{O}$ be a collection of subsets of X. 1. The collection $\mathcal{O}$ *covers* A or is a *cover for A* if A is contained in the union of sets in $\mathcal{O}$. 2. The collection $\mathcal{O}$ is called an *open cover for A* if each set in $\mathcal{O}$ is open in X. 3. If $\mathcal{O}$ covers A and $\mathcal{O}'$ is a subcollection of $\mathcal{O}$ that also covers A, then we call $\mathcal{O}'$ a *subcover* of A.

Definition (Compact)**.** A topological space X is called *compact* if every open cover for X has a finite subcover.

Ex - $\mathbb{R}$ is not compact The collection of open sets of $\mathbb{R}$ given by $\mathcal{O} = \{(i, i+2) \mid i \in \mathbb{Z}\}$ is an open cover for $\mathbb{R}$. $\mathcal{O}$ has no finite subcover, and so $\mathbb{R}$ is not compact.

Ex - Finite spaces are compact If X is a finite topological space, then $\mathcal{P}(X)$, the power set of X, is finite, and thus any open cover for X is necessarily finite since there are only a finite number of sets, specifically open sets, of X.

—

Definition (Compact subspace)**.** Let X be a topological space and $A \subset X$. Then we say that A is *compact in X* if A is compact in the subspace topology inherited from X.

Compact subspace iff open cover containing subspace has finite subcover Let X be a topological space, with $A \subset X$. Then A is compact in X if and only if every open cover for A of open sets in X has a finite subcover.

Proof. Suppose A is compact in X. Then A is compact in the subspace topology inherited from X. Let $\mathcal{O}$ be an open cover for A consisting of open sets of X, say $\mathcal{O} = \{V_\alpha\}_{\alpha \in I}$. Then $A \cap V_\alpha$ is open in A for all $\alpha \in I$ by construction of the subspace topology. In particular, we know that $\{V_\alpha \cap A\}_{\alpha \in I}$ is an open cover of A consisting of open sets in A, and thus has a finite subcover, say $V_1, \ldots, V_n$. But then $A \subseteq \bigcup_{i=1}^n V_i$. Thus $V_1, \ldots, V_n$ is a finite subcover for A of $\mathcal{O}$ consisting of open sets in X.

Assume the converse. Let $\mathcal{O}$ be an open cover of A consisting of open sets of A, say $\mathcal{O} = \{V_\alpha\}_{\alpha \in I}$. Since open sets in A are of the form $V_\alpha = U_\alpha \cap A$ for open sets U_α of X, we know $\{U_\alpha\}_{\alpha \in I}$ is an open cover of A consisting of open sets of X. By assumption, it has a finite subcover, say $U_1, \ldots, U_n$. But then $A \subseteq \bigcup_{i=1}^n V_i$. Thus $V_1, \ldots, V_n$ is a finite subcover of $\mathcal{O}$. ■

—

Continuous functions preserve compactness Let X be a topological space with $A \subset X$, and $f : X \to Y$ a continuous function. If A is compact in X, then $f(A)$ is compact in Y.

Proof. Assume $\mathcal{O}$ is an open cover for $f(A)$, say $\mathcal{O} = \{V_\alpha\}_{\alpha \in I}$. Then since f is continuous, $\{f^{-1}(V_\alpha)\}_{\alpha \in I}$ is an open cover for A. By assumption, there exists a finite subcover $f^{-1}(V_1), \ldots, f^{-1}(V_n)$. But then the open sets $V_1, \ldots, V_n$ cover $f(A)$, to which we have found a finite subcollection of $\mathcal{O}$ that covers $f(A)$. Thus $f(A)$ is compact in Y. ■

—

Properties of unions and intersection of compact sets Let X be a topological space. 1. If $C_1, \ldots, C_n$ are compact subsets of X, then $\bigcup_{i=1}^n C_i$ is compact in X. 2. If X is Hausdorff, and $\{C_\alpha\}_{\alpha \in I}$ is a collection of compact subsets of X, then $\bigcap_{\alpha \in I} C_\alpha$ is compact in X.

Proof. (1) Let $\mathcal{O}$ be an open cover for $\bigcup_{i=1}^n C_i$. In particular, $\mathcal{O}$ is an open cover for any C_j with $1 \leq j \leq n$. Since each C_j is compact, there is a finite subcover, say $\mathcal{O}_j$, for C_j. Now $\mathcal{O}' = \{V \mid V \in \mathcal{O}_j \text{ for some } 1 \leq j \leq n\}$ can be found to be an open cover for $\bigcup_{i=1}^n C_i$ that is finite since it is the set consisting of a finite number of open sets. Thus $\bigcup_{i=1}^n C_i$ is compact.

(2) Assume X is Hausdorff. Since compact sets in Hausdorff spaces are closed, we may write that C_α is closed in X for each $\alpha \in I$. In particular, $\bigcap_{\alpha \in I} C_\alpha$ is closed since it is the arbitrary intersection of closed sets. But then $\bigcap_{\alpha \in I} C_\alpha \subset C_{\alpha'}$ for any $\alpha' \in I$. Since $C_{\alpha'}$ is compact by assumption, this implies $\bigcap_{\alpha \in I} C_\alpha$ is compact. ■

—

Closed subsets of compact sets are compact Let X be a topological space with A compact in X. If B is a closed set in X such that $B \subset A$, then B is compact in X.

-how do compactness and closedness play together? -turns out they are not one-in-the-same.

Proof. Assume A is compact in X and B is closed in X with $B \subset A$. Let $\mathcal{O}$ be an open cover for B, with $\mathcal{O} = \{V_\alpha \mid \alpha \in I\}$. Since B is closed in X, we know that $A \cap B$ is closed in the subspace topology on A inherited from X. Indeed we have then $A \setminus B$ is open in the subspace topology on A.

Now $\mathcal{O} \cup A \setminus B$ is an open cover for A, for either $a \in B$ and so $a \in V_\alpha$ for some $\alpha \in I$ or $a \in A \setminus B$. By assumption, this open cover has a finite subcover, say $\mathcal{O}' = \{V_1, \dots, V_n\}$. Either $A \setminus B \in \mathcal{O}'$ or $A \setminus B \notin \mathcal{O}'$.

If $A \setminus B \in \mathcal{O}'$ then removing it from $\mathcal{O}'$ has no impact on B, since if $b \in B$ then $b \notin A \setminus B$. Thus $\mathcal{O}' \setminus \{A \setminus B\}$ is a finite subcover of $\mathcal{O}$ covering B. If $A \setminus B \notin \mathcal{O}'$, then we recover the same result. In either case, we have shown the existence of a finite subcover for B of $\mathcal{O}$. Thus B is compact in X. ■

—

Ex - $\mathbb{R}_{fc}$ has compact sets that are not closed Consider $\mathbb{R}_{fc}$ the real line in the finite complement topology. Every subset of $\mathbb{R}_{fc}$ is compact.

Proof. Let A be a subset of $\mathbb{R}_{fc}$ with $\mathcal{O}$ an open cover for A, say $\mathcal{O} = \{V_\alpha\}_{\alpha \in I}$. Since $A \subset \bigcup_{\alpha \in I} V_\alpha$, Choose $V \in \mathcal{O}$. Then we know that $A \setminus V = A \cap V^c$ is finite since V is open in $\mathbb{R}_{fc}$. Take $A \setminus V = \{x_1, \dots, x_n\}$. Since A is covered by $\mathcal{O}$, there exist $V_1, \dots, V_n \in \mathcal{O}$ for which $x_i \in V_i$ for each $1 \leq i \leq n$. Now we have that $\mathcal{O}' = \{V, V_1, \dots, V_n\}$ is a finite subcollection of $\mathcal{O}$ that covers A. In particular, every subset of $\mathbb{R}_{fc}$ is compact, but not every subset is closed. ■

—

Theorem 6.1.1 (Compact sets in Hausdorff space are closed)**.** *Let X be a topological Hausdorff space, and A compact in X. Then A is closed in X.*

-the Hausdorff condition guarantees that compact sets are closed.

Proof. Assume A is compact in X. We show $X \setminus A$ is open. Let $x \in X \setminus A$. For every $y \in A$, there exist disjoint open sets U_y and V_y for which $x \in U_y$ and $y \in V_y$ by the Hausdorff condition. It follows that the collection $\mathcal{O} = \{V_y \mid y \in A\}$ is an open cover for A. By assumption, there exists a finite subcover $\mathcal{O}' = \{V_1, \dots, V_n\}$. Thus $A \subseteq \bigcup_{j=1}^n V_j$. Since $V_y \cap U_y = \varnothing$ by construction, we can see that

$$(\bigcup_{j=1}^n V_j) \cap (\bigcup_{j=1}^n U_j) = \bigcup_{j=1}^n (V_j \cap U_j) = \bigcup_{j=1}^n \varnothing = \varnothing$$

This implies that $(\bigcup_{j=1}^n U_j) \cap A = \varnothing$ also, so $\bigcup_{j=1}^n U_j \subset X \setminus A$. Since each U_j is open in X by assumption, and the finite union of open sets is open, we have $\bigcup_{j=1}^n U_j$ is open in X. Since $x \in \bigcup_{j=1}^n U_j \subset X \setminus A$, we may write that $X \setminus A$ is open, to which A is closed. ■

Theorem 6.1.2 (Tube Lemma)**.** *Let X and Y be topological spaces, and assume Y is compact. If we have $x \in X$, and U an open set in $X \times Y$ containing $\{x\} \times Y$, then there exists a neighborhood W of x in X such that $W \times Y \subset U$.*

Proof. TBD ■

Theorem 6.1.3 (Product of compact spaces is compact)**.** *Let X and Y be compact topological spaces. Then $X \times Y$ is compact.*

Similarly, if $X_1, \ldots, X_n$ are each compact, then $\prod_{i=1}^{n} X_i$ is compact.

Proof. TBD ■

6.2 Compactness in Metric Spaces

How does compactness play a role in the theory of metric spaces? What kinds of intricacies are there in regards to how metric structure of spaces and compactness work in tandem?

Theorem 6.2.1 (Nested Interval Property of $\mathbb{R}$)**.** *Let $\{I_n\}_{n\in\mathbb{N}}$ be a collection of closed and bounded intervals of $\mathbb{R}$, each of the form $I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$. Suppose $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$.*

Proof. We have $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$. Since the set $\{b_n\}_{n\in\mathbb{N}}$ is bounded below by a_1, this set has a greatest lower bound, call it β. Similarly, the set $\{a_n\}_{n\in\mathbb{N}}$ is bounded above by b_1, and so this set has a least upper bound, call it α. Since $\alpha \leq \beta$, which follows for $a_n \leq b_n$ for all $n \in \mathbb{N}$, and $\beta \leq b_n$ and $a_n \leq \alpha$ for all $n \in \mathbb{N}$, we may write that $[\alpha, \beta] \neq \varnothing$. Take $x \in [\alpha, \beta]$. Then $\alpha \leq x \leq \beta$, and so for any $n \in \mathbb{N}$, we have that $x \in I_n$. Thus $x \in \bigcap_{n\in\mathbb{N}} I_n$, proving $[\alpha, \beta] \subseteq \bigcap_{n\in\mathbb{N}} I_n$. Now suppose $x \in \bigcap_{n\in\mathbb{N}} I_n$. Then $a_n \leq x$ and $x \leq b_n$ for all $n \in \mathbb{N}$. Thus x is an upper bound for $\{a_n\}_{n\in\mathbb{N}}$ and a lower bound for $\{b_n\}_{n\in\mathbb{N}}$. Since α is a least upper bound, we know $\alpha \leq x$, and since β is a greatest lower bound, we know $x \leq \beta$. In particular, $x \in [\alpha, \beta]$ and so $\bigcap_{n\in\mathbb{N}} I_n \subseteq [\alpha, \beta]$. Therefore we have $\bigcap_{n\in\mathbb{N}} I_n = [\alpha, \beta] \neq \varnothing$. ■

—

Theorem 6.2.2 (Closed bounded implies compact in $\mathbb{R}$)**.** *Every closed and bounded interval $[a, b]$ is compact in $\mathbb{R}$ with the standard topology.*

Proof. Assume, for contradiction, that $[a, b]$ is not compact. Thus there exists an open cover $\mathcal{O}$ for $[a, b]$ which has no finite subcover. Consider the collection of intervals $[a, (a + b)/2] \cup [(a + b)/2, b]$. Since $\mathcal{O}$ covers $[a, b]$, it must cover this union. For either the first or second interval in the union, there must be an infinite number of open sets of $\mathcal{O}$ that cover it, for if this were not the case then the set $\mathcal{O}$ would be finite.

Without loss of generality, take $[a, (a+b)/2]$ to be the set covered by infinitely many open sets of $\mathcal{O}$. Define $I_n = [a_n, b_n]$ where a_n and b_n are the left and right endpoints, respectively, of the division of the interval as above. By the nested interval property of $\mathbb{R}$, $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$. Thus there is some interval $[x, y]$ in this intersection. ■

—-

Product of closed bounded intervals is compact subset of $\mathbb{R}^n$ *Let* $[a_j, b_j]$ *for* $1 \leq j \leq n$ *be a collection of closed and bounded subsets of* $\mathbb{R}$. *Then* $\prod_{j=1}^{n}[a_j, b_j]$ *is a compact subset of* $\mathbb{R}^n$.

Proof. This follows since each $[a_j, b_j]$ is compact in $\mathbb{R}$ since they are closed and bounded. Since the product of compact sets is compact, it follows that their product is compact in $\mathbb{R}^n$. ■

—

Subset of $\mathbb{R}^n$ *compact iff closed bounded Let* $\mathbb{R}^n$ *have the standard topology and metric* d. *Let* $A \subset \mathbb{R}^n$. *Then* A *is compact in* $\mathbb{R}^n$ *if and only if* A *is closed and bounded.*

Proof. TBD ■

Theorem 6.2.3 (Every sequence in compact subset has convergent subsequence)**.** *Let* (X, d_X) *be a metric space, with* $A \subset X$. *Suppose* A *is compact in* X. *If* (x_n) *is a sequence contained in* A, *then there exists a convergent subsequence* (x_{n_m}) *that converges to a limit in* A.

Proof. TBD ■

—

Definition (Cauchy sequence)**.** Let (X, d_X) be a metric space. A sequence (x_n) in X is called a *Cauchy sequence* if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d_X(x_n, x_m) < \epsilon$.

-pretty standard definition for a type of object in a metric space.

Definition (Complete metric space)**.** Let (X, d_X) be a metric space. We call X a *complete metric space* if every Cauchy sequence in X converges to a limit in X.

Theorem 6.2.4 ($\mathbb{R}^n$ is a complete metric space)**.** *Let (x_n) be a Cauchy sequence in $\mathbb{R}^n$ with the standard metric d. Then (x_n) converges to a limit in $\mathbb{R}^n$.*

Proof. TBD ■

Theorem 6.2.5 (Compact metric spaces are complete)**.** *Let (X, d_X) be a metric space. If X is compact, then X is complete.*

Proof. TBD ■