DOTDiff.m: Diffuse Optical Tomography Based on the Diffusion Model

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Overview. This is a MATLAB code for solving the inverse problem in diffuse optical tomography. Here is the setup for the problem.

Mathematical Model. The mathematical model is the following boundary value problem for the diffusion equation

$$-\nabla \cdot \gamma \nabla u + \sigma u = 0, \quad \text{in} \quad \Omega, \qquad \boldsymbol{\nu} \cdot \gamma \nabla u(\mathbf{x}) + \kappa u = f(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{1}$$

The Data. The data we measure is encoded in the Robin-to-Dirichlet map:

$$\Lambda_\sigma : f(\mathbf{x}) \mapsto g(\mathbf{x}) := u \tag{2}$$

It's clear that this map is equivalent to the Robin-to-Neumann map.

The Domain. The computational domain is $\Omega = (0, 2) \times (0, 2)$.

The Objective. The objective here is to reconstruct the absorption coefficient σ from data encoded in Λ_σ. Note that in our setup of the problem, that is, in the steady state situation, the data available allows us to reconstruct only one of the coefficients (γ, σ). The current code try to reconstruct σ. Time-dependent/frequency-dependent measurement can allow us to reconstruct more coefficients [1, 2]. The code can be easily adapted to do that.

The Algorithm. We assume that we collect data from N_s illuminations $\{f_j\}_{j=1}^{N_s}$. The diffusion equation for source f_j is

$$-\nabla \cdot \gamma \nabla u_j + \sigma u_j = 0, \quad \text{in} \quad \Omega, \qquad \boldsymbol{\nu} \cdot \gamma \nabla u_j(\mathbf{x}) + \kappa u_j = f(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{3}$$

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The corresponding data is $g_j(\mathbf{x}) = \Lambda_\sigma f_j := u_j$. We therefore collected data $\{f_j, g_j(\mathbf{x})\}_{j=1}^{N_s}$.

We solve the inverse problem is in least-square form:

$$\sigma^* = \arg\min_\sigma \Phi(\sigma) := \frac{1}{2}\sum_{j=1}^{N_s}\int_{\partial\Omega}(\Lambda_\sigma f_j - g_j)^2 dS(\mathbf{x}) + \frac{\beta}{2}\int_\Omega |\nabla\sigma|^2 d\mathbf{x} \tag{4}$$

where the parameter β is the strength of the regularization term.

The Gradient Calculation. We use the adjoint state method to calculate the gradient of the objective functional with respect to the absorption coefficient σ. Let w_j $(1 \le j \le N_s)$ be the solution to adjoint equation

$$-\nabla\cdot\gamma\nabla w_j + \sigma w_j = 0, \quad \text{in} \quad \Omega, \qquad \boldsymbol{\nu}\cdot\gamma\nabla\ w_j + \kappa w_j = -(\Lambda_\sigma f_j - g_j), \quad \text{on} \quad \partial\Omega \tag{5}$$

We can then show that the Fréchet derivative of Φ with respect to σ in direction $\delta\sigma$ is given as

$$\Phi'(\sigma)[\delta\sigma] = \sum_{j=1}^{N_s}\int_\Omega u_j w_j \delta\sigma(\mathbf{x})d\mathbf{x} - \beta\Big[\int_\Omega(\Delta\sigma)\delta\sigma(\mathbf{x})d\mathbf{x} - \int_{\partial\Omega}\partial_\nu\sigma\delta\sigma(\mathbf{x})dS(\mathbf{x})\Big]. \tag{6}$$

The Forward/Adjoint Solver. In the minimization process, we solve the forward and adjoint diffusion problems (3) and (5) with a standard P_1 finite element solver of the MATLAB PDE Toolbox.

This code was a simplification of the code we used to generate some of the numerical results in the paper [2]. We removed the frequency-dependence and replaced the optimization algorithm with a MATLAB `fminunc` algorithm.

Remark on the objective function. Due to the fact that optical signals decays fast away from the source location, signals measured on detectors far from the source are very weak. The objective function defined in (4) puts equal weight on mismatch at different detecotr locations. This in practice means that the mismatch information at detectors far away from the source play little role in the functional. In practice, it is often better to normalize the optical signals by rescaling the mismatch with the measured signal as follows:

$$\Phi(\sigma) := \frac{1}{2}\sum_{j=1}^{N_s}\int_{\partial\Omega}\left(\frac{\Lambda_\sigma f_j - g_j}{g_j}\right)^2 dS(\mathbf{x}) + \frac{\beta}{2}\int_\Omega|\nabla\sigma|^2 d\mathbf{x} \tag{7}$$

This objective function often works better when $g_j \ne 0$. When this objective function is used, the adjoint problem becomes

$$-\nabla\cdot\gamma\nabla w_j + \sigma w_j = 0, \quad \text{in} \quad \Omega, \qquad \boldsymbol{\nu}\cdot\gamma\nabla\ w_j + \kappa w_j = -(\Lambda_\sigma f_j - g_j)/g_j^2, \quad \text{on} \quad \partial\Omega \tag{8}$$

Nothing else needs to be changed. The code has both the normalized and unnormalized objective functionals in it.

References

[1] K. Ren, G. Bal, and A. H. Hielscher, *Frequency domain optical tomography based on the equation of radiative transfer*, SIAM J. Sci. Comput., 28 (2006), pp. 1463–1489.

[2] ——, *Transport- and diffusion-based optical tomography in small domains: A comparative study*, Applied Optics, 46 (2007), pp. 6669–6679.