HelmIBVP.m: Inverse boundary value problem for the Helmholtz equation

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Overview. This is a MATLAB code for solving inverse medium problem for the interior Helmholtz problem. Here is the setup for the problem.

Mathematical Model. The mathematical model is the following boundary value problem for the Helmholtz equation

$$\Delta u + k^2(1+n(\mathbf{x}))u = 0, \quad \text{in} \quad \Omega, \qquad u(\mathbf{x}) = f(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{1}$$

The Data. The data we measure is encoded in the Dirichlet-to-Neumann map:

$$\Lambda_n : f(\mathbf{x}) \mapsto g(\mathbf{x}) := \boldsymbol{\nu} \cdot \nabla u \tag{2}$$

The Domain. The computational domain is $\Omega = (0,1) \times (0,1)$.

The Algorithm. We assume that we collect data from N_s illuminations $\{f_j\}_{j=1}^{N_s}$. For source f_j, the Helmholtz equation is

$$\Delta u_j + k^2(1+n(\mathbf{x}))u_j = 0, \quad \text{in} \quad \Omega, \qquad u_j(\mathbf{x}) = f_j(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{3}$$

The corresponding data is $g_j(\mathbf{x}) = \Lambda_n f_j := \boldsymbol{\nu}\cdot\nabla u_j$. We therefore collected data $\{f_j, g_j(\mathbf{x})\}_{j=1}^{N_s}$.

The inverse problem is solved in least-square form:

$$n^* = \arg\min_n \Phi(n) := \frac{1}{2}\sum_{j=1}^{N_s}\int_{\partial\Omega}(\Lambda_n f_j - g_j)^2 dS(\mathbf{x}) + \frac{\beta}{2}\int_\Omega |\nabla n|^2 d\mathbf{x} \tag{4}$$

where the parameter β is the strength of the regularization term.

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We use the adjoint state method to calculate the gradient of the objective functional with respect to the refractive index. Let w_j $(1 \le j \le N_s)$ be the solution to adjoint equation

$$\Delta w_j + k^2(1+n)w_j = 0, \quad \text{in} \quad \Omega, \qquad w_j = -(\Lambda_n f_j - g_j), \quad \text{on} \quad \partial\Omega \tag{5}$$

We can then show that the Fréchet derivative of Φ with respect to n in direction δn is given as

$$\Phi'(n)[\delta n] = k^2 \sum_{j=1}^{N_s} \int_\Omega u_j w_j \delta n(\mathbf{x}) d\mathbf{x} - \beta \Big[\int_\Omega (\Delta n)\delta n(\mathbf{x}) d\mathbf{x} - \int_{\partial\Omega} \partial_\nu n \delta n(\mathbf{x}) dS(\mathbf{x}) \Big]. \tag{6}$$

In the minimization process, we solve the forward and adjoint Helmholtz problems (1) and (5) with a standard P_1 finite element solver.

This code was used to generate some of the numerical tests in the paper [1].

References

[1] K. Ren and Y. Zhong, *Imaging point sources in heterogeneous environments*, Submitted, (2019). arXiv:1901.07189.