HelmIBVP.m: Inverse boundary value problem for the Helmholtz equation

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Overview. This is a MATLAB code for solving inverse medium problem for the interior Helmholtz problem. Here is the setup for the problem.

Mathematical Model. The mathematical model is the following boundary value problem for the Helmholtz equation

$$\Delta u + k^2(1+n(\mathbf{x}))u = 0, \quad \text{in} \quad \Omega, \qquad u(\mathbf{x}) = f(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{1}$$

Measurement. The data we measure is encoded in the Dirichlet-to-Neumann map:

$$\Lambda_n : f(\mathbf{x}) \mapsto g(\mathbf{x}) := \mathbf{n} \cdot \nabla u \tag{2}$$

The code assumes that we collect data from N_s illuminations: $\{f_j, g_j(\mathbf{x})\}_{j=1}^{N_s}$.

The Domain. The computational domain is $\Omega = (0,1) \times (0,1)$.

The Algorithm. The inverse problem is solved in least-square form:

$$n^* = \arg\min_n \Phi(n) := \frac{1}{2}\sum_{j=1}^{N_s}\int_{\partial\Omega}(\Lambda_n f_j - g_j^*)^2 dS(\mathbf{x}) + \frac{\beta}{2}\int_\Omega |\nabla n|^2 d\mathbf{x} \tag{3}$$

where g_j^* is the measured differential data corresponding to the probe source f_j and the parameter β is the strength of the regularization term.

We use the adjoint state method to calculate the gradient of the objective functional with respect to the refractive index. Let w_j $(1 \le j \le N_s)$ be the solution to adjoint equation

$$\Delta w_j + k^2(1+n)w_j = 0, \quad \text{in} \quad \Omega, \qquad w_j = -(\Lambda_n h_j - g_j^*), \quad \text{on} \quad \partial\Omega \tag{4}$$

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We can then show that the Fréchet derivative of Φ with respect to n in direction δn is given as

$$\Phi'(n)[\delta n] = k^2 \sum_{j=1}^{J} \int_\Omega w_j v_j \delta n(\mathbf{x}) d\mathbf{x} - \beta \Big[\int_\Omega (\Delta n) \delta n(\mathbf{x}) d\mathbf{x} - \int_{\partial\Omega} \partial_\nu n \delta n(\mathbf{x}) dS(\mathbf{x}) \Big]. \quad (5)$$

In the minimization process, we solve the forward and adjoint Helmholtz problems (1) and (4) with a standard P_1 finite element solver.

This code was used to generate some of the numerical tests in the paper [1].

References

[1] K. Ren and Y. Zhong, *Imaging point sources in heterogeneous environments*, Submitted, (2019). arXiv:1901.07189.