qTAT.m: Quantitative Thermoacoustic Tomography Based on the Helmholtz Model

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Overview. This is a MATLAB code for solving the inverse problem in quantitative thermoacoustic tomography based on the Helmholtz model for wave propagation.

Mathematical Model. The mathematical model is the following boundary value problem for the diffusion equation

$$\Delta u + k^2(1+n)u + ik\sigma u = 0, \quad \text{in} \quad \Omega, \qquad u = f(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{1}$$

This is one of the models used in [1].

The Data. The data we measure are encoded in the map:

$$\Lambda_{n,\sigma} : f(\mathbf{x}) \mapsto H(\mathbf{x}) := \sigma|u|^2 = \sigma\bar{u}u \tag{2}$$

The Domain. The computational domain is $\Omega = (0,2) \times (0,2)$.

The Objective. The objective here is to reconstruct n or σ or both from the data. The theory developed in [1] indicates that the problem of reconstructing σ is very stable.

The Algorithm. We assume that we collect data from N_s illuminations $\{f_j\}_{j=1}^{N_s}$. The diffusion equation for source f_j is

$$\Delta u_j + k^2(1+n)u_j + ik\sigma u_j = 0, \quad \text{in} \quad \Omega, \qquad u_j = f_j(\mathbf{x}), \quad \text{on} \quad \partial\Omega \tag{3}$$

The corresponding data is $H_j(\mathbf{x}) = \Lambda_{n,\sigma} f_j := \sigma|u_j|^2$. We therefore collected data $\{f_j, H_j(\mathbf{x})\}_{j=1}^{N_s}$.

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We solve the inverse problem is in least-square form. We minimize

$$\Phi(n,\sigma) := \frac{1}{2}\sum_{j=1}^{N_s}\int_\Omega (\sigma|u_j|^2 - H_j)^2 d\mathbf{x} + \frac{\beta}{2}\int_\Omega \left[|\nabla n|^2 + |\nabla\sigma|^2\right] d\mathbf{x} \tag{4}$$

where the parameter β is the strength of the regularization term. Note that, the regularization functional should only contain terms for quantities to be reconstructed. For instance, if we only reconstruct σ, then $|\nabla\sigma|^2$ should be the only term in the regularization functional.

The Gradient Calculation. We use the adjoint state method to calculate the gradient of the objective functional with respect to the absorption coefficient σ. Let w_j $(1 \le j \le N_s)$ be the solution to adjoint equation

$$\Delta w_j + k^2 w_j + ik\sigma w_j = -(\sigma|u_j|^2 - H_j)\sigma\overline{u}, \quad \text{in} \quad \Omega, \qquad w_j = 0, \quad \text{on} \quad \partial\Omega \tag{5}$$

We can then show that the Fréchet derivatives of Φ are given as

$$\Phi'(n,\sigma)[\delta n] = \sum_{j=1}^{N_s}\int_\Omega 2\mathrm{Re}(k^2 u_j w_j)\delta n d\mathbf{x} - \beta\Big[\int_\Omega (\Delta n)\delta n(\mathbf{x})d\mathbf{x} - \int_{\partial\Omega}\partial_\nu n \delta n(\mathbf{x}) dS(\mathbf{x})\Big]. \tag{6}$$

$$\begin{aligned}\Phi'(n,\sigma)[\delta\sigma] = \sum_{j=1}^{N_s}\int_\Omega &\left[(\sigma|u_j|^2 - H_j)|u_j|^2 + 2\mathrm{Re}(iku_j w_j)\right]\delta\sigma d\mathbf{x} \\ &- \beta\Big[\int_\Omega (\Delta\sigma)\delta\sigma(\mathbf{x})d\mathbf{x} - \int_{\partial\Omega}\partial_\nu\sigma\delta\sigma(\mathbf{x})dS(\mathbf{x})\Big].\end{aligned} \tag{7}$$

The boundary integral terms will disappear if we assume that the boundary values of the coefficients are known.

The Forward/Adjoint Solver. In the minimization process, we solve the forward and adjoint diffusion problems (3) and (5) with a standard P_1 finite element solver of the MATLAB PDE Toolbox.

References

[1] G. Bal, K. Ren, G. Uhlmann, and T. Zhou, *Quantitative thermo-acoustics and related problems*, Inverse Problems, 27 (2011). 055007.