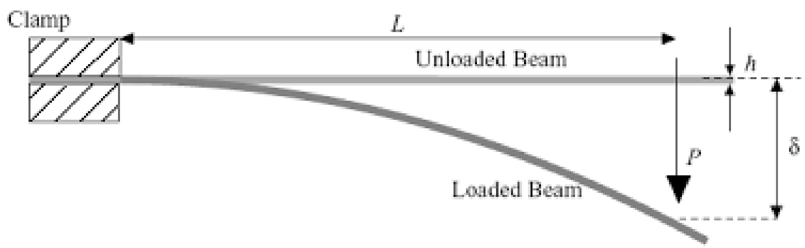
NumPDE course, Skoltech, Term 3, 2015 ¶

Problem Set 1

Modeling a Cantilever Beam

Consider a beam fixed at one end (called a cantilever (http://en.wikipedia.org/wiki/Cantilever) beam) as shown here:



Its deflection from the equilibrium position is described by the deflection $u = u(x)$ which satisfies the boundary-value problem for the Euler-Bernoulli equation (http://en.wikipedia.org/wiki/Euler%E2%80%93Bernoulli_beam_theory)

$$(EIu'')'' = 0$$
$$u(0) = 0 \quad u'(0) = 0$$
$$u''(L) = 0 \quad (EIu'')'(L) = P$$

where P is the force applied to the beam's end, E is the (constant) elastic modulus (a material's property) and I is the second moment of the area of the cross-section. If the cross-section is a rectangle with height $H = H(x)$ and the width is W then $I = I(x) = WH(x)^3/12$ (see more details in Wikipedia under the link above).

Proposition 1.

Assume that $I = I(x)$ is constant.

- **(a)**
$$\frac{u(x-2h) - 4u(x-h) + 6u(x) - 4u(x+h) + u(x+2h)}{h^4}$$
approximates $u''''(x)$ with the second order.
- **(b)** Consider a grid $h = L/N$, $\xi = hi$, $i = 0, \dots, N+2$. The last line of (1) can be approximated by
$$u(L+h) - 2u(L) + u(L-h) = 0$$
$$u(L+2h) - 2u(L+h) + 2u(L-h) - u(L-2h) = 2h^3 P/(EI)$$

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Problem 1 (60pt)

Assume that $I = I(x)$ is constant and the mesh is chosen as in Proposition 1.

- **(10pt)** Prove part (a) of Proposition 1.
- **(10pt)** Formulate the finite difference problem approximating (1) with the second order.
- **(10pt)** Write a code that implements this finite difference problem by constructing an $(N+4) \times (N+4)$ matrix A_h and the right-hand side vector b_h.
- **(10pt)** Given parameters $E = 2.5\,GPa$, $H = 7mm$, $W = 3mm$, $L = 25cm$, $P = 200gm$, $h = L/20$ use your code to compute the solution $u_h = (u_h(-h), \ldots, u_h(L+2h))$. What is the computed deflection of the right end (let us denote it as $u_h(L)$ with a slight abuse of notation)? (Don't know what the abuse of notation (http://en.wikipedia.org/wiki/Abuse_of_notation) is?)
- (Optional.) Upload the values $u(0), u(h), \ldots, u(L)$ to Canvas as a text file named *Your_Name.txt* with numbers separated by a newline. The first one who does it correctly will get a non-material bonus. (The details on uploading to Canvas may be changed later.)
- **(10pt)** Find the solution u in the analytic form. Compare $u(L)$ with $u_h(L)$ for $h = 25, 50, 100$. As you increase h by a factor or 2, by what factor does the difference $|u_h(L) - u(L)|$ decrease?
- **(10pt)** Compute $\lambda_{\min}(A_h)$ and $\lambda_{\max}(A_h)$ for $h = 25, 50, 100$. As you increase h by a factor or 2, by what factor does $cond(A_h)$ increase? Can you find, approximately, for what h your code computes the value $|u_h(L) - u(L)|$ most accurately? Explain why your code's answers are worse if h is less than this value and if h is larger than this value.

Problem 2 (40pt)

Let all parameters be the same as in Problem 1 except for $H(x) = (3 - 2x/L)(2 + \cos(18\pi x/L)) \cdot 6mm$.

- **(10pt)** Formulate the corresponding finite difference problem.
- **(10pt)** Write the corresponding code.
- **(10pt)** Assuming that the beam fractures at a point where the modulus of the quantity $\sigma(x) = u''(x)H(x)$ is largest, find the point where the beam should fracture. Give details on how you compute it (e.g., what value of h you used).

- **(10pt)** Give an example of a smooth function $H = H(x)$, satisfying $3mm \leq H \leq 15mm$ everywhere, such that the beam fractures at $x \approx L/3$.

Problem 3 (bonus problem)

Suppose all parameters, except $H = H(x)$ are the same. You need to find $H(x)$ such that

- The load that can be applied to the beam before it fractures is maximal subject to the constraints below:
- The total material is fixed: $\int_0^L H(x)dx = 2000mm^2$
- $3mm \leq H \leq 15mm$

Take $n = 100$. Upload 101 values $u(0), u(1/n), \ldots, u(1)$ to Stellar as a text file named *PS1P3.txt* with numbers separated by a newline.