Problem Set 4: Burgers' equation

Problem 1 (100pt)

Consider the Burgers' equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ in the computational domain $0 \le x \le 10$, $0 \le t \le 4$.

The initial velocity distribution is $u(x) = \begin{cases} 2, \ x < 1; \\ 2.5 - x/2, \ 1 \le x \le 3; \\ 1, \ x > 3. \end{cases}$

The boundary condition at $x = 0$ is $u(t) = 2$. In the numerical methods, if needed, you can use the condition that at $x = 10$, $u(t) = 1$.

Implement the following three finite-difference methods, the upwind difference (UD), the central difference (CD) and the total variation diminishing (TVD) scheme:

(1)UD: $\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^- u_{i+1}^n - (u_i^n)^- u_i^n}{2\Delta x} + \frac{(u_i^n)^+ u_i^n - (u_{i-1}^n)^+ u_{i-1}^n}{2\Delta x} = 0$, where $(\varphi)^+ = \frac{\varphi + |\varphi|}{2}$, $(\varphi)^- = \frac{\varphi - |\varphi|}{2}$

(2)CD: $\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x} = 0$

(3)TVD (with any limiter function) ψ :

$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x} - \frac{a_{i+1/2}^n (u_{i+1}^n - u_i^n) - a_{i-1/2}^n (u_i^n - u_{i-1}^n)}{2\Delta x} = 0$, where

$a_{i+1/2}^n = |u_{i+1/2}^n| [1 - \psi(1 - |u_{i+1/2}^n| \Delta t / \Delta x)]$;

$r_i = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}$, $u_{i+1/2}^n = \frac{u_i^n + u_{i+1}^n}{2}$.

Questions:

Q1 (20pt). What methods are numerically stable (i.e., give reasonable solutions) and what methods are not? Justify.

Q2 (40pt) At what time and position the shockwave starts to form? Use the definition that there is a shockwave at position $\Delta x(i + 1/2)$ and time $n\Delta t$ if $\frac{|u_{i+1}^n - u_i^n|}{\Delta x} > \Delta x^{-1/2}$. Use different methods and different values for Δx, and Δt, and compare with the exact answer. Comment on the accuracy of different methods.

Q3 (40pt) In what method the shockwave sharpness is better? Use the definition of the shockwave sharpness as $\max_i \frac{|u_{i+1}^n - u_i^n|}{\Delta x}$ at time t=4. The larger is the shockwave sharpness, the better. How does the shockwave sharpness depend on Δx for different schemes?