NumPDE course, Skoltech, Term 3, 2016

Problem Set 2

NOTE: You can use the code that was presented in lectures.

Problem 1 (Image processing) (40pt)

Let us consider a problem of de-noising a color image, by solving the following PDE:

$$\begin{aligned}\frac{\partial}{\partial t} u_i &= \operatorname{div}(\nabla u_i) + \lambda(u^{(0)} - u_i) \\ u_i &= u_i^{(0)} \\ u_i|_{\partial\Omega} &= u_i^{(0)}|_{\partial\Omega}\end{aligned}$$

where u_i, $i = 1, 2, 3$, are the intensity of the red, green, and blue components of the image; $u_i^{(0)}$ represent the original image. The parameter λ should be inversely proportional to the level of the noise, and can be taken, for a start, to be 1/100 or 1/200. The time of integration should be taken so that we can approximate a steady-state solution.

- Take a noisy image from here (https://github.com/oseledets/fastpde/blob/master/fig/color_photo_noisy.png) or generate it yourself. (If you generate yourself then make the noise level to be 64 out of 256.)
- **(10pt)** Write a code that solves the system
- **(10pt)** Set $\lambda = 1/200$ and try getting some results with $\tau = 0.025$, $\tau = 0.25$, and $\tau = 2.5$. Comment on, and explain the results.

Let us consider a more advanced approach:

$$\begin{aligned}\frac{\partial}{\partial t} u_i &= \operatorname{div}(k\nabla u_i) + \lambda(u^{(0)} - u_i) \\ u_i &= u_i^{(0)} \\ u_i|_{\partial\Omega} &= u_i^{(0)}|_{\partial\Omega}\end{aligned}$$

where now $k = \left(1 + \max_i |\nabla u_i|^2\right)^{-1/2}$.

- **(10pt)** Write a code that solves the system
- **(10pt)** Give results for three values of λ, for which the final image is (1) Too noisy, (2) Too blurry, (3) About right

Problem 2 (Elasticity) (60pt)

Consider a domain Ω with the boundary consisting of lines between the following 12 points (given in mm in this very order):

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(250, -2 S - H), (100 - H, -2 S - H), (100 - H, -S), (150 - H, -S), (150 - H, -H), (0, -H), (0, 0), (150, 0), (150, -S - H), (100, -S - H), (100, -2 S), (250, -2 S)
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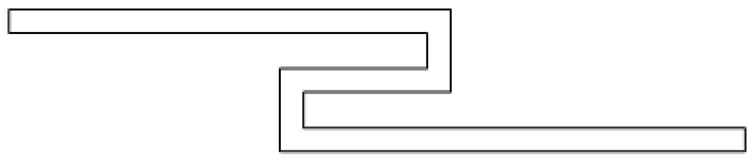
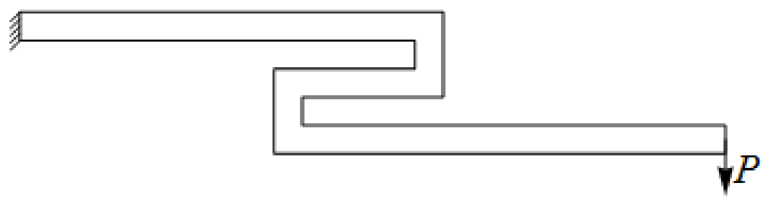
for some values of S and H. The domain is as show here:



Next, we assume a piece of elastic material (similar to a beam) of this shape. It is fixed at the leftmost end and the force P is applied to its rightmost end (the illustration below is for different values of H and S):



We also assume that the material that thickness (width) W in the third dimensions (perpendicular to the plane of the image)

To solve this problem, we consider the linear elasticity equations. The elasticity equations describe the displacement (u, v) of the material in the x and y coordiate, respectively. We split the boudary of Ω into three parts: $\Gamma_1 = \{(x, y) \in \partial\Omega : x = 0\}$ (leftmost end), $\Gamma_2 = \{(x, y) \in \partial\Omega : 0 < x < 250\}$ (middle part), and $\Gamma_3 = \{(x, y) \in \partial\Omega : x = 250\}$ (rightmost part). It is then easy to write the boundary conditions on Γ_1:

$$u = v = 0 \qquad \text{on } \Gamma_1$$

To write down the equations and the rest of the boundary conditions, we need the elasticity tensors which are denoted as

$$C_{11} = \begin{pmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 \\ 0 & \frac{E}{2(1+\nu)} \end{pmatrix}, \qquad C_{12} = \begin{pmatrix} 0 & \frac{E\nu}{(1-2\nu)(1+\nu)} \\ \frac{E}{2(1+\nu)} & 0 \end{pmatrix},$$

$$C_{21} = C_{12}^{\mathrm{T}}, \qquad C_{22} = \begin{pmatrix} \frac{E}{2(1+\nu)} & 0 \\ 0 & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \end{pmatrix}$$

Here E is called the modulus of elasticity and ν is called the Poisson ratio.

The equations are then written as

$$
\begin{aligned}
-\mathrm{div}(C_{11}\nabla u + C_{12}\nabla v) &= 0 && \text{in } \Omega \\
-\mathrm{div}(C_{21}\nabla u + C_{22}\nabla v) &= 0 && \text{in } \Omega \\
u = v &= 0 && \text{on } \Gamma_1 \\
C_{11}u_n + C_{12}v_n = C_{21}u_n + C_{22}v_n &= 0 && \text{on } \Gamma_2 \\
C_{11}u_n + C_{12}v_n &= 0 && \text{on } \Gamma_3 \\
C_{21}u_n + C_{22}v_n &= \frac{P}{HW} && \text{on } \Gamma_3,
\end{aligned}
$$

where u_n and v_n are normal derivatives of the solution. Note that in the last equation $C_{21}\nabla u + C_{22}\nabla v$ is the stress on the boundary and we set it equal to the stress $\frac{P}{HW}$ of the external force P. (The stress has the same dimension as the pressure.)

Let us now convert this to the variational formulation. We denote the test functions by $p = p(x, y)$ and $q = q(x, y)$. We require that the test functions satisfy the Dirichlet boundary conditions (i.e., $p(0, y) = q(0, y) = 0$) and then the weak formulation becomes

$$
\int_\Omega \Big[(C_{11}\nabla u + C_{12}\nabla v)\cdot\nabla p + (C_{21}\nabla u + C_{22}\nabla v)\cdot\nabla q\Big] = \frac{P}{HW}\int_{\Gamma_3} q \qquad \forall p, q
$$

- **(15pt)** Write a code that generates a mesh for this domain (you may, but do not need to, use mesh generation packages)
- **(35pt)** Write a code that solves this problem by the finite element method and outputs the y-component of the deflection of this piece of material, to be precise, $v(250, 2S)$
- **(10pt)** For three sets of parameters, $S = 20$, $W = 3$, $H \in \{7, 8, 9\}$ (all in mm) compute the displacement $v(250, 2S)$, $E = 3.1\cdot 10^9\mathrm{Pa}$, $\nu = 0.35$, $P = m\cdot g$, $m = 200\mathrm{grams}$, $g = -9810\mathrm{mm/s^2}$.
- **(Optional)** Plot the three solutions found. For that you need to change the coordinates of each of the mesh nodes from (x, y) by $(x, y) + (u(x, y), v(x, y))$ and plot the displaced mesh

In [ ]: