

$$\begin{aligned}
& \text{minimize } J(y, u) := \frac{1}{2} \|y - y_\Omega\|^2 + \frac{\lambda}{2} \|u\|^2 \\
& \text{s.t.} \quad -\Delta y = f + u \quad \text{in } \Omega \\
& \quad \quad y = 0 \quad \text{on } \Gamma \\
& \quad \quad -1 \leq u \leq 1
\end{aligned}$$

Harbir's example: $\lambda = 0.001$

$$\bar{p} = \frac{1}{4\pi^2} \sin(2\pi x_1) \sin(2\pi x_2) \quad (\text{Optimal } p)$$

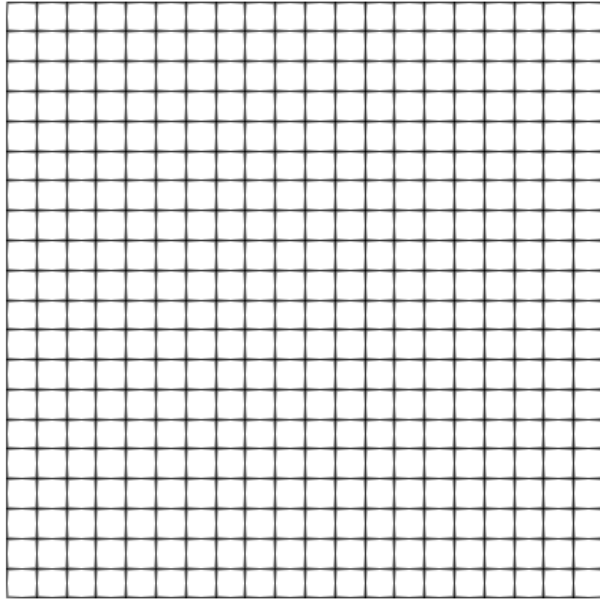
$$\bar{y} = \sin(2\pi x_1) \sin(2\pi x_2) \quad (\text{Optimal } y)$$

$$y_\Omega = -\sin(2\pi x_1) \sin(2\pi x_2)$$

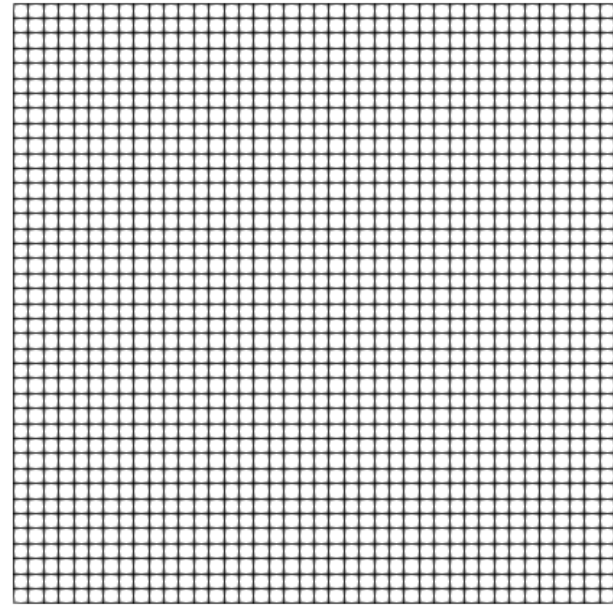
$$\bar{u} = \mathbb{P}_{[-1,1]} \left(-\frac{\bar{p}}{\lambda} \right) \quad (\text{Optimal } u)$$

$$f = 8\pi^2 \sin(2\pi x_1) \sin(2\pi x_2) - \bar{u}$$

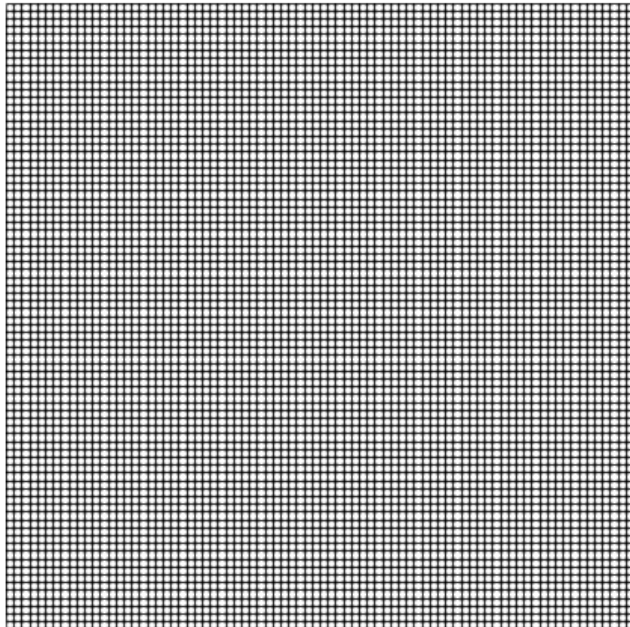
Mesh



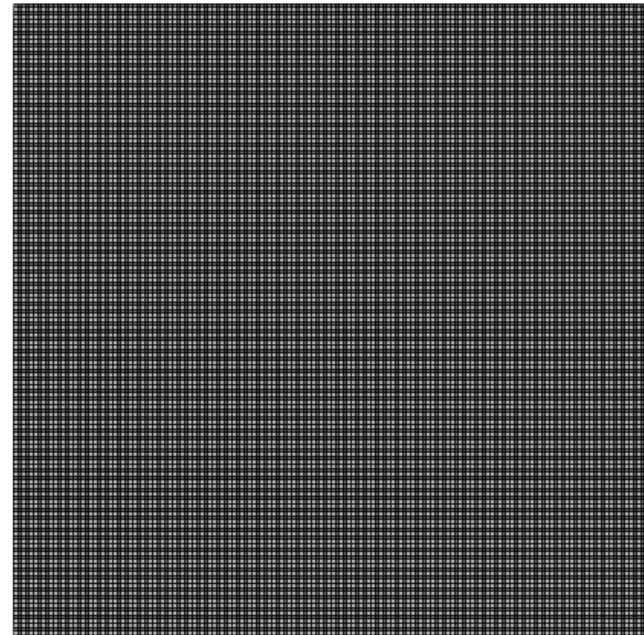
400 cells



1600 cells

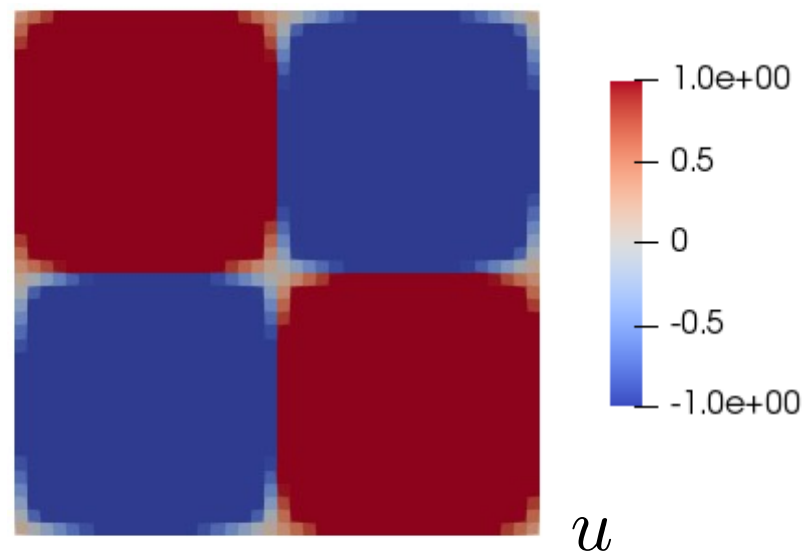
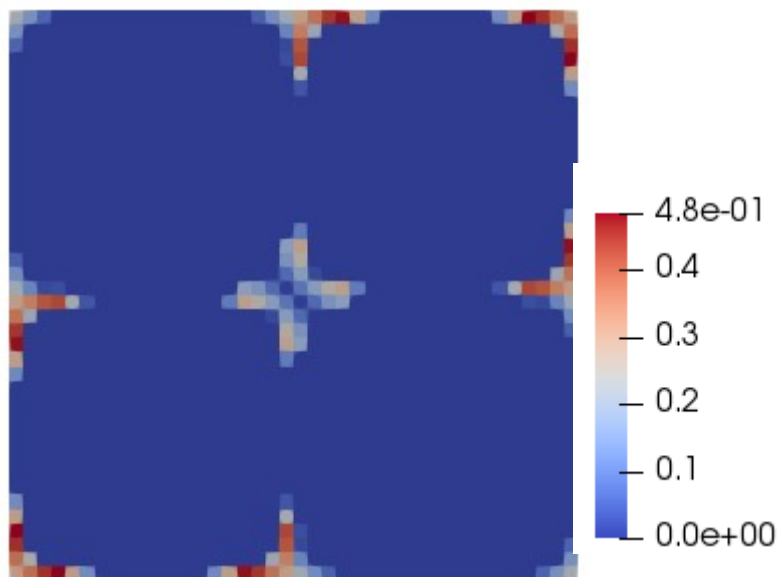


6400 cells

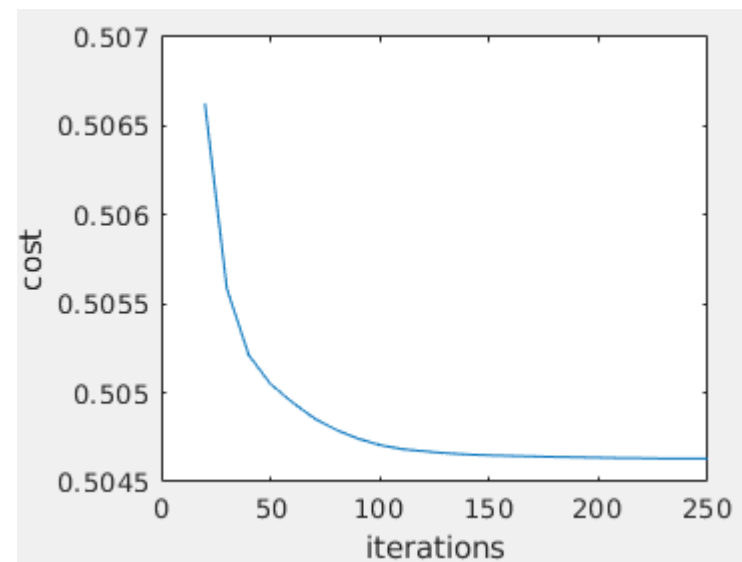
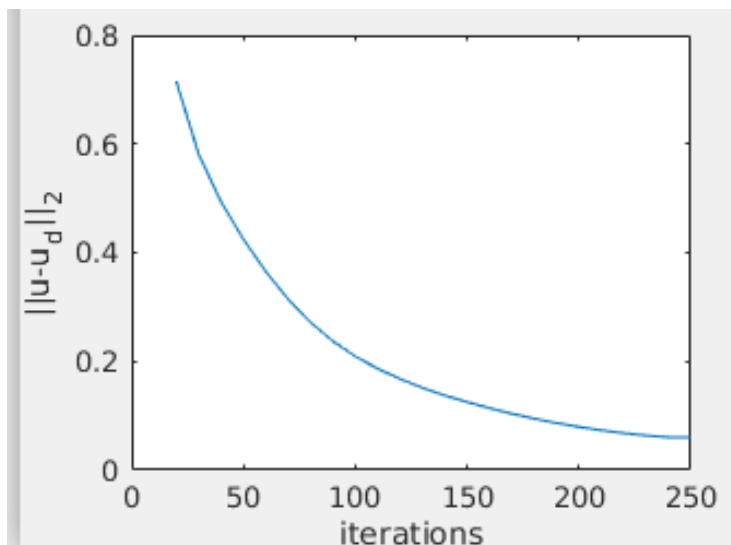


25600 cells

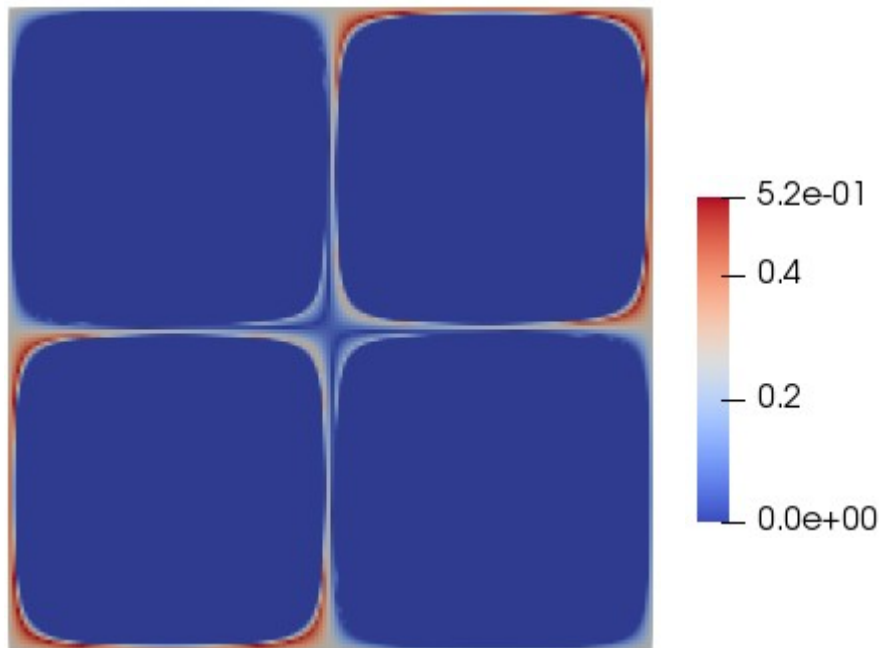
Results plot – mesh – 400 cells



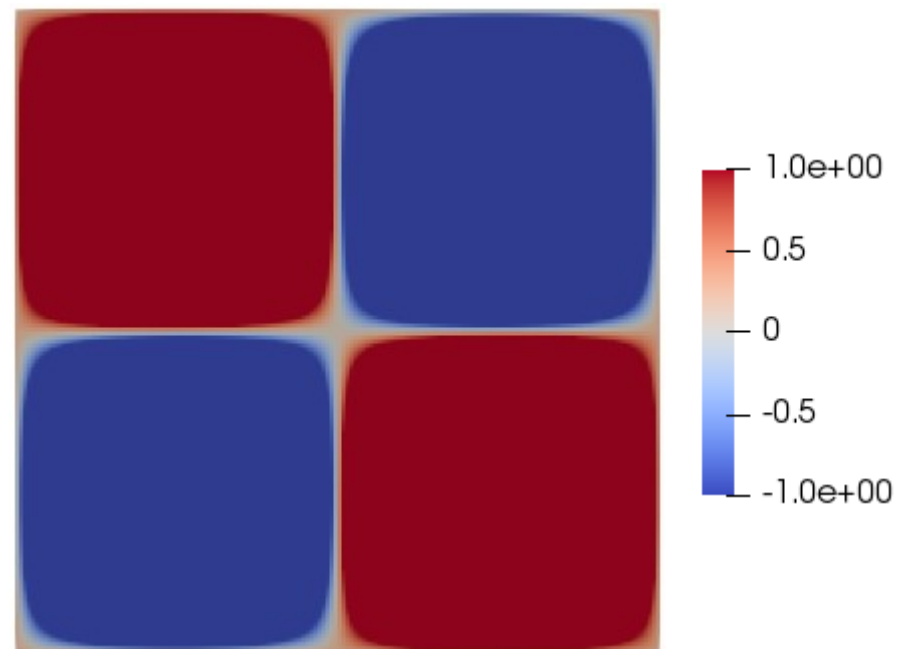
$$|u - \bar{u}|$$



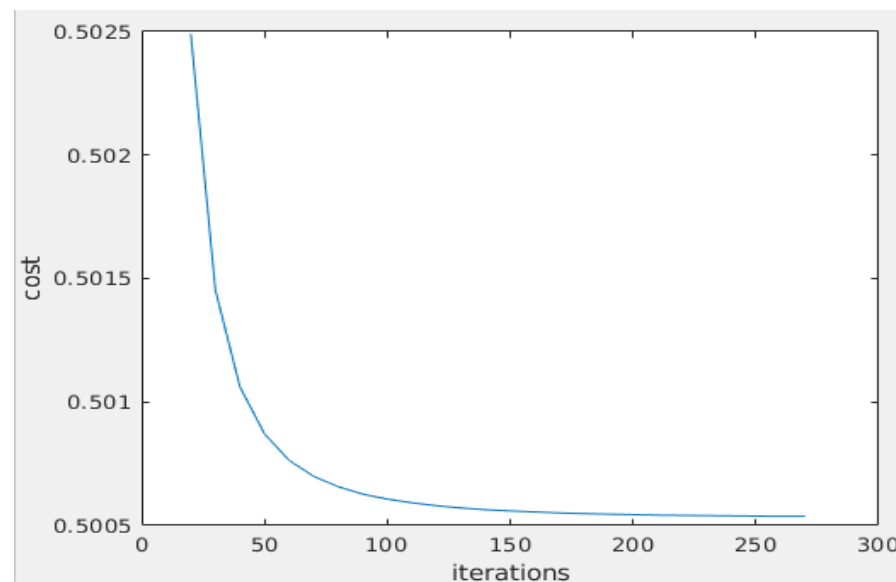
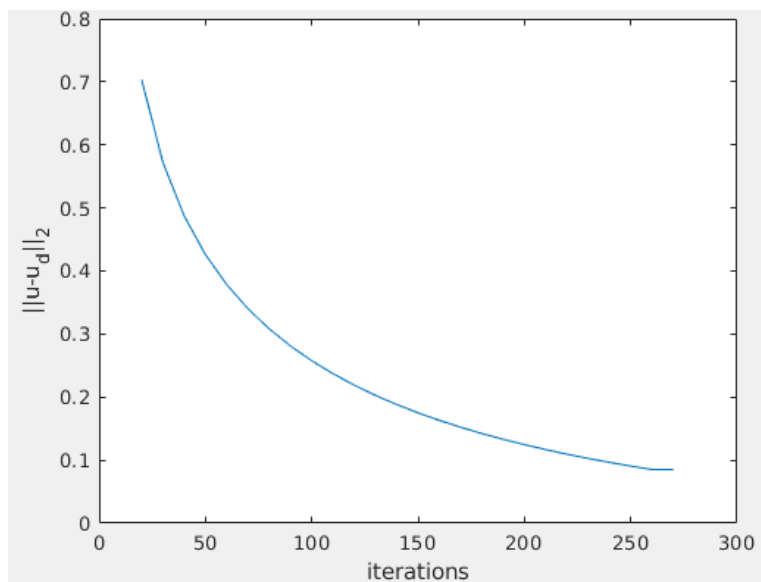
Results plot – mesh – 25600 cells



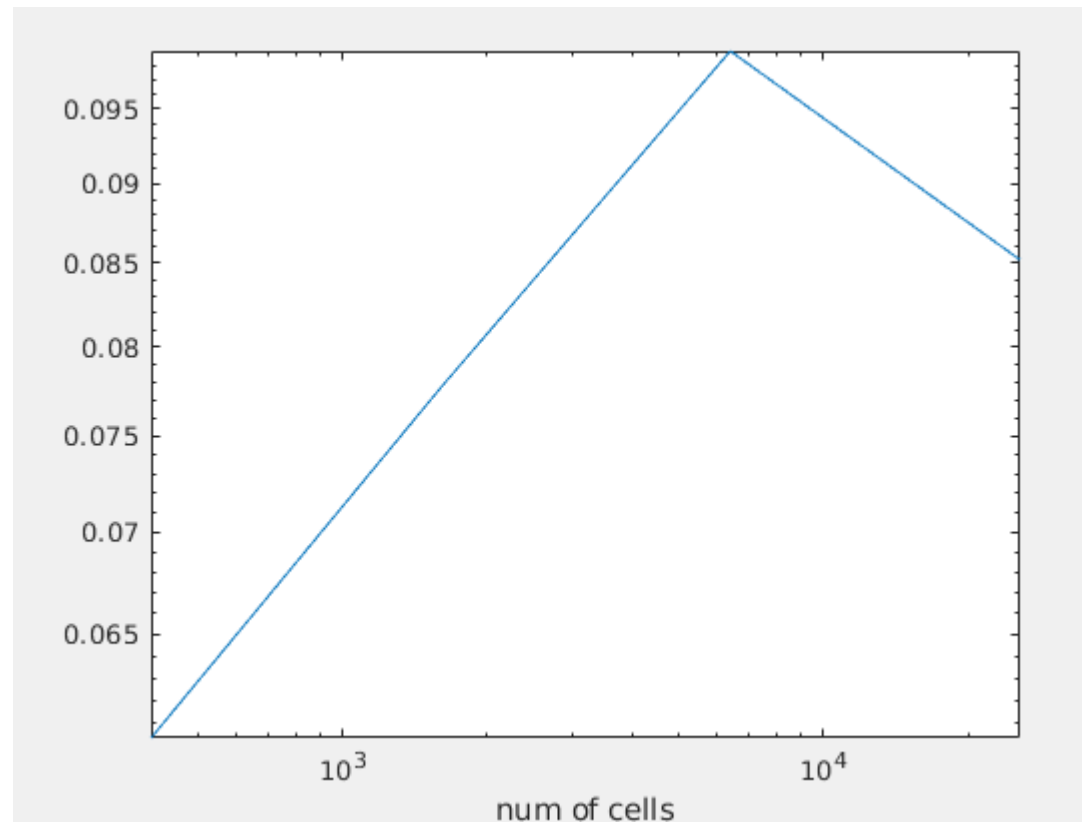
$$|u - \bar{u}|$$



$$u$$



Mesh convergence analysis



X axis: $\log(\text{number of cells})$

y axis: $\log(||u - \bar{u}||_2)$

$$\begin{aligned}
& \text{minimize } J(y, u) := \frac{1}{2} \|y - y_\Omega\|^2 + \frac{\lambda}{2} \|u\|^2 \\
& \text{s.t.} \quad -\Delta y = f + u \quad \text{in } \Omega \\
& \quad \quad y = 0 \quad \text{on } \Gamma \\
& \quad \quad -1 \leq u \leq 1
\end{aligned}$$

Book's example: $\lambda = 0$

$$p(x) = -\frac{1}{128\pi^2} \sin(8\pi x_1) \sin(8\pi x_2) \quad (\text{Optimal } p)$$

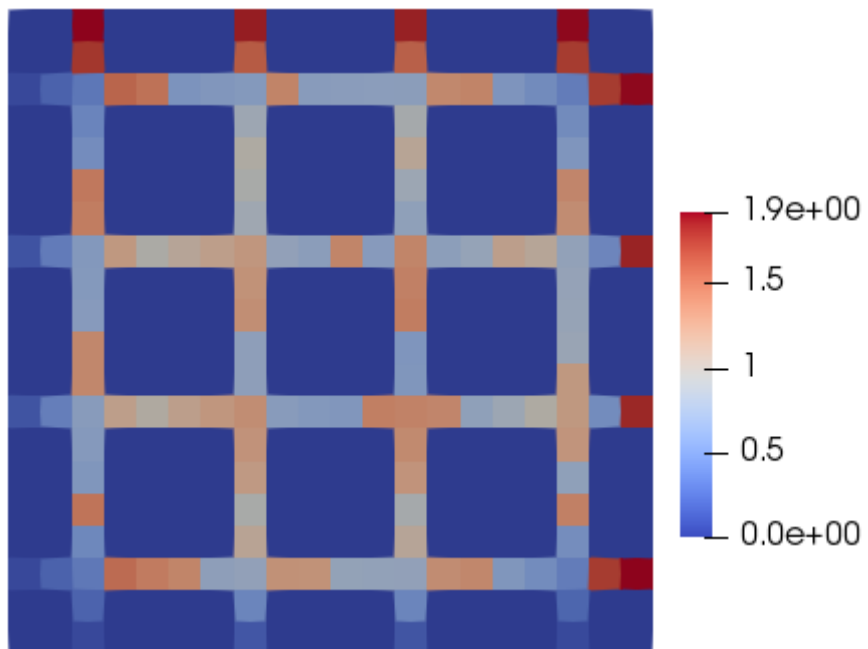
$$\bar{y}(x) = \sin(\pi x_1) \sin(\pi x_2) \quad (\text{Optimal } y)$$

$$y_\Omega(x) = \sin(\pi x_1) \sin(\pi x_2) + \sin(8\pi x_1) \sin(8\pi x_2)$$

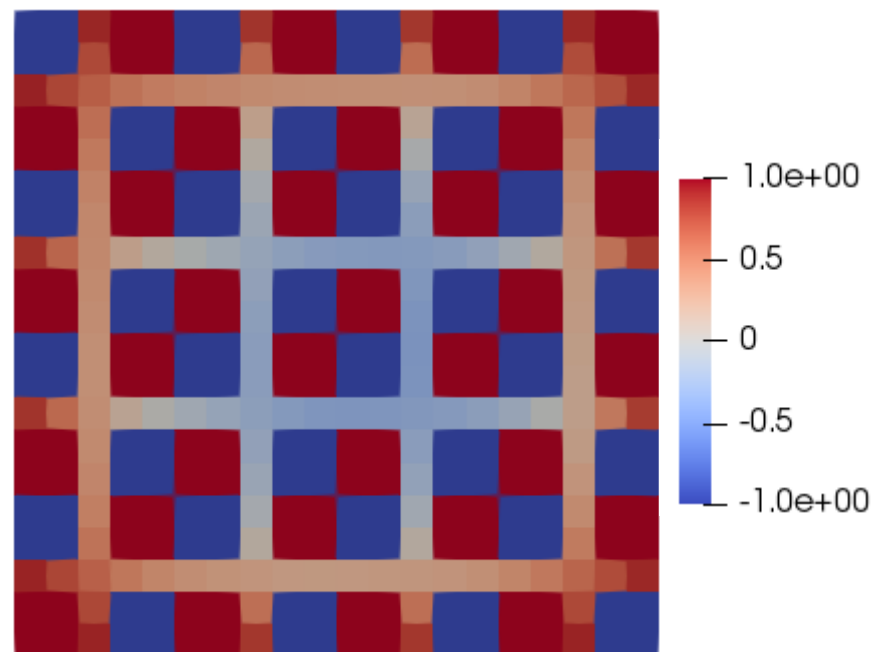
$$\bar{u}(x) = -\text{sign } p(x) \quad (\text{Optimal } u)$$

$$f = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2) + \text{sign}(-\sin(8\pi x_1) \sin(8\pi x_2))$$

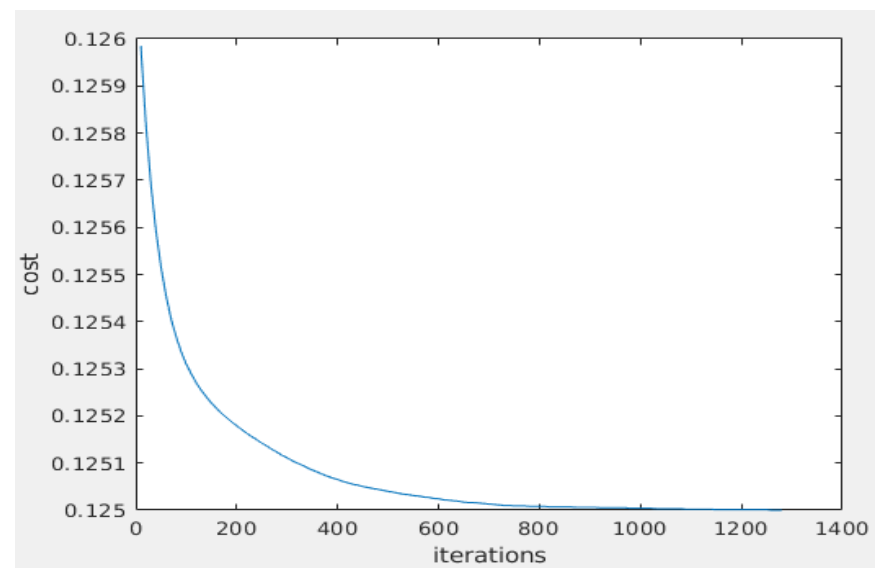
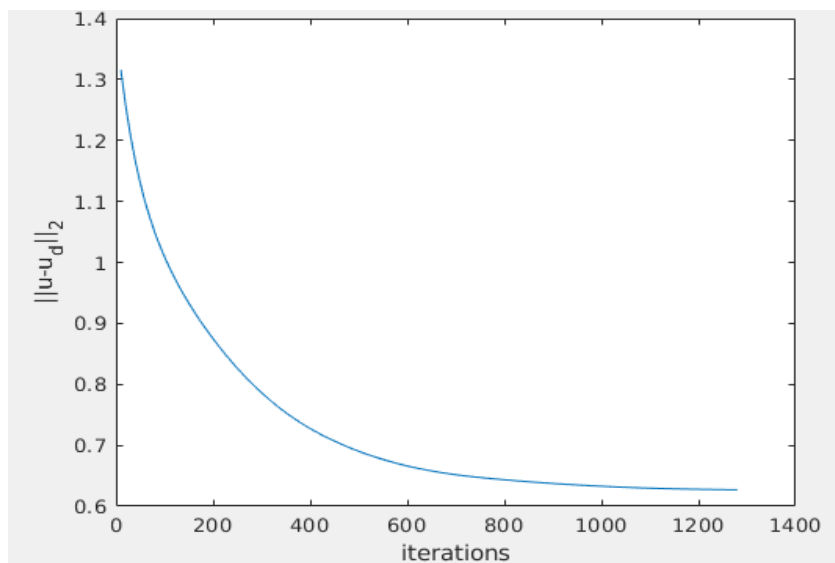
Results plot – mesh – 400 cells



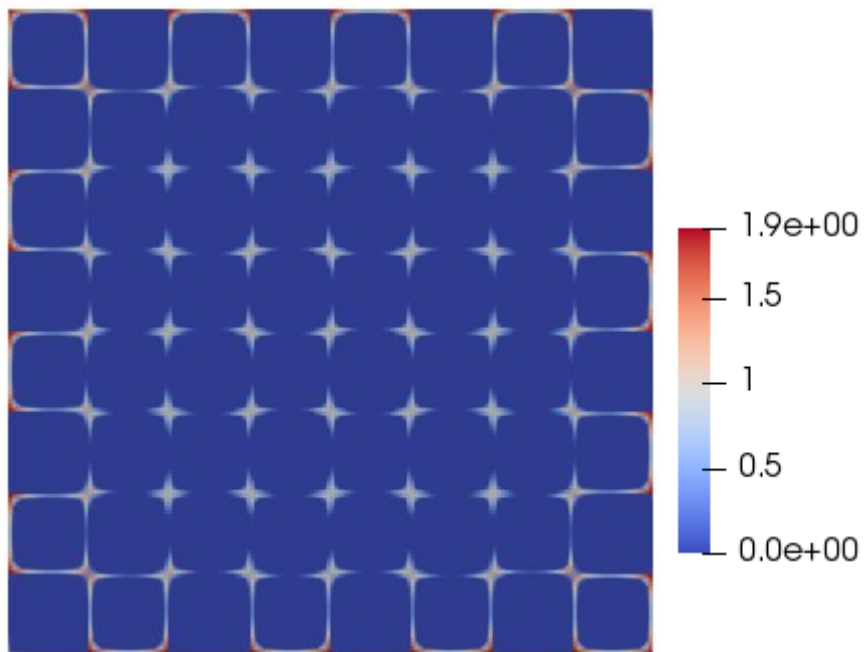
$$|u - \bar{u}|$$



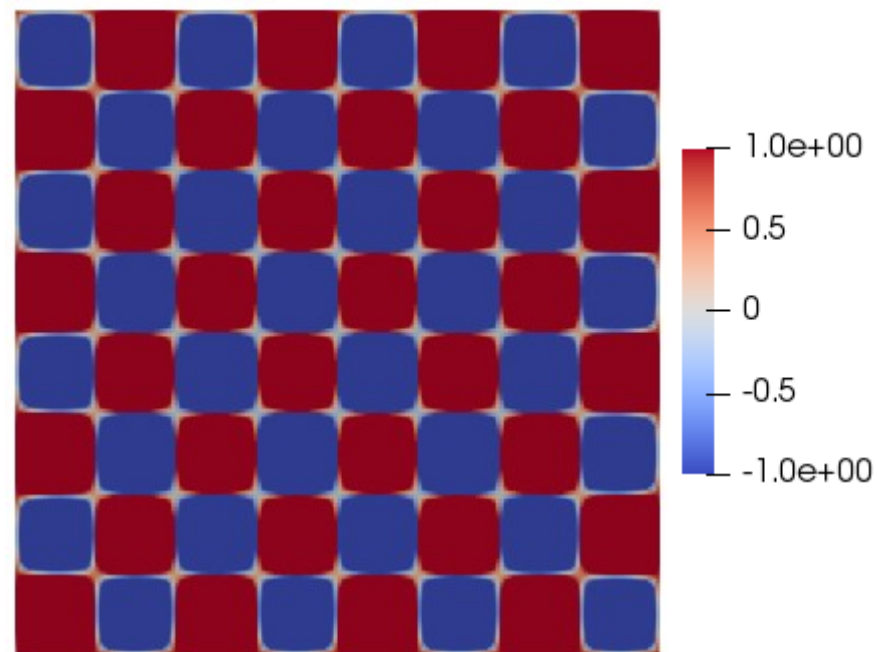
$$u$$



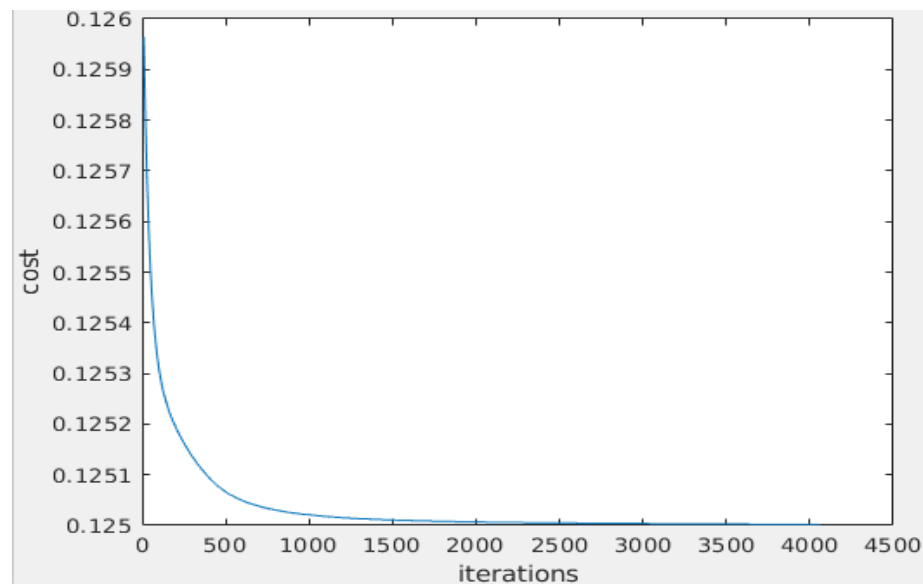
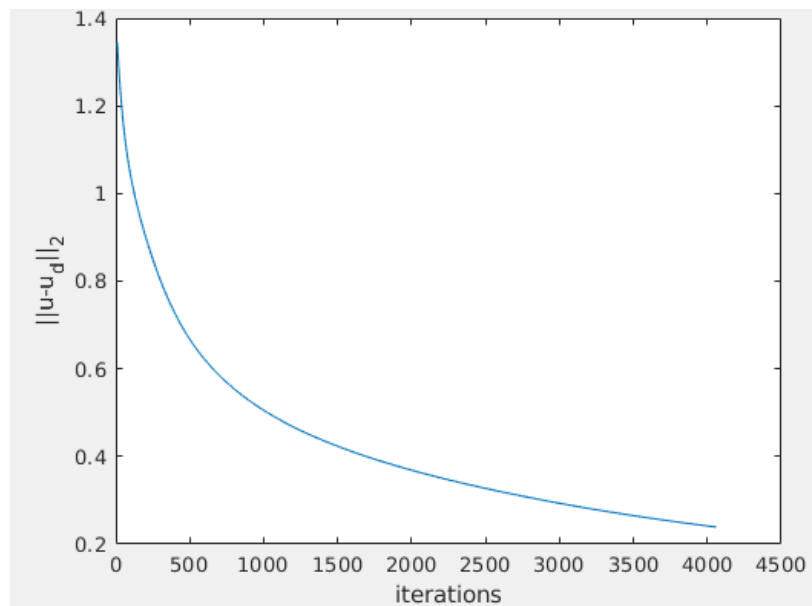
Results plot – mesh – 25600 cells



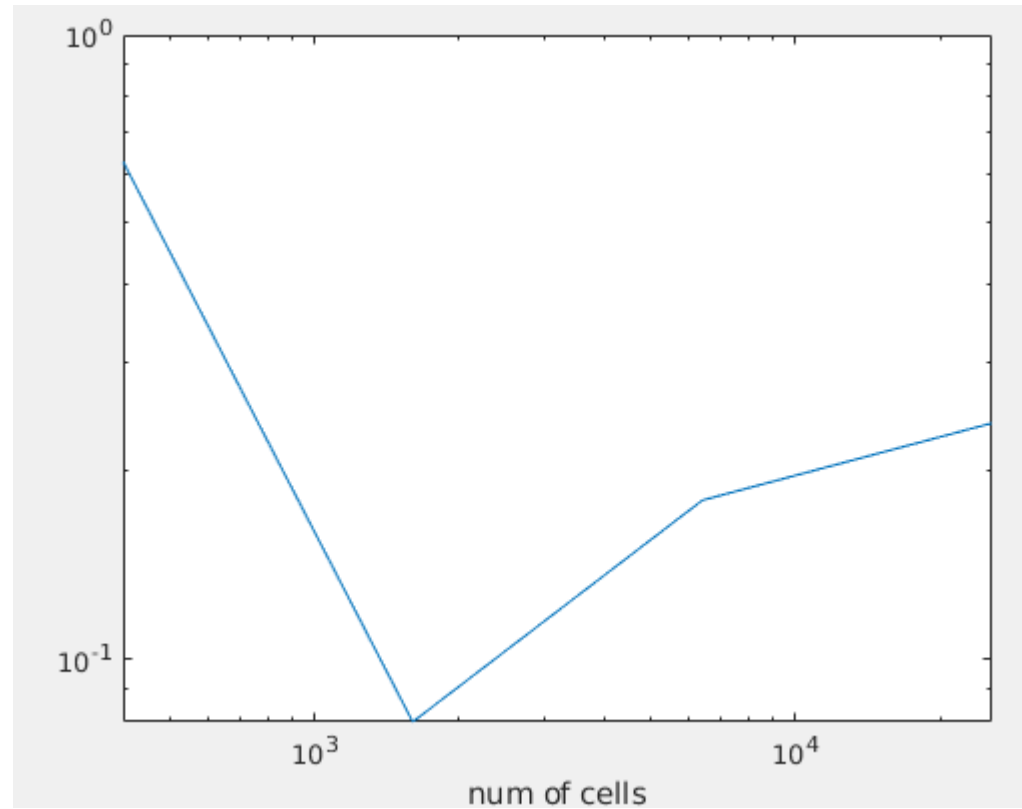
$$|u - \bar{u}|$$



$$u$$



Mesh convergence analysis



X axis: $\log(\text{number of cells})$

y axis: $\log(||u - \bar{u}||_2)$