A non-commuting twist in the partition function

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- Based on the work done with Suresh Govindarajan. "A non-commuting twist in the partition function" arxiv: *yymm.xxxx* (*to appear*)

Outline

CHL models quick review

CHL $\mathbb{Z}_n$ orbifold models[1] with $\mathcal{N} = 4$ supersymmetry in four dimensions.

- These are orbifolds of type II A string theory on $K3 \times T^2$, where the orbifold group G acts as a symplectic automorphism on $K3$ and as shifts on the torus T^2.
- This is dual to the heterotic string theory on T^6 via string-string duality.
- The action of G is determined on $\Gamma_{22,6} \cong \Gamma_{20,4} \oplus \Gamma_{2,2}$ and copied to the Heterotic side by identifying it with the Narain Lattice.
- The result is an asymmetric orbifold of a heterotic string on T^6.

[1] Chaudhuri et.al '95, Aspinwall '95

Motivation I: half-BPS state counting

- The asymptotic degeneracy of electrically charged half-BPS states in the CHL models were computed[2],

$$\tilde{F}(\mu) \simeq \frac{16}{|G| v_{\Lambda_\perp}} e^{4\pi^2/\mu} \left(\frac{\mu}{2\pi}\right)^{12-\frac{k}{2}}$$

- Generating functions for these degeneracies were proposed, [3]

$$g_\rho(\tau) = \prod_{r=1}^{N} \eta(r\tau)^{a_r} = \eta(\tau)^{a_1}\eta(2\tau)^{a_2}\ldots\eta(N\tau)^{a_N}$$

$$\lim_{\mu\to 0} \frac{1}{g_\rho(i\mu/2\pi N)} = \tilde{F}(\mu)$$

- These eta-products where identified with balanced cycle shapes $\rho = 1^{a_1}2^{a_2}\ldots N^{a_N}$,

$$\left(\frac{M}{1}\right)^{a_1}\left(\frac{M}{2}\right)^{a_2}\ldots\left(\frac{M}{N}\right)^{a_N} = \rho \; ; \; M \in \mathbb{Z}$$

- The cycle shapes encode the action of the orbifold and can be used to derive the eta-products.

[2] (Ashoke Sen hep-th/0504005)
[3] (S.Govindarajan, G.Krishna 0907.1410)

Motivation II: Relation to Mathieu representations

- In the type II A on $K3 \times T^2$ the orbifold group acts as a symplectic automorphism on $K3$.
- There is a theorem due to Mukai 88' which relates symplectic automorphisms on $K3$ and Mathieu groups.

Theorem

If there is a finite group G of symplectic automorphisms acting on the $K3$ surface, then

- G is a subgroup of $M_{23} \subset M_{24}$,
- G necessarily has atleast five fixed points, one each from $H^{0,0}(K3), H^{2,0}(K3), H^{1,1}(K3), H^{0,2}(K3), H^{2,2}(K3)$.

- The embedding of M_{23} in M_{24} allows one to use the property that all elements of M_{24} are represented by balanced cycle shapes[4].

[4](Conway and Norton 79')

G	half-BPS Degeneracy	cycle shape	M_{24} conjugacy classes
-	$\eta(\tau)^{24}$	1^{24}	1A
$\mathbb{Z}_2$	$\eta(\tau)^8\eta(2\tau)^8$	$1^8 2^8$	2A
$\mathbb{Z}_3$	$\eta(\tau)^6\eta(3\tau)^6$	$1^6 3^6$	3A
$\mathbb{Z}_4$	$\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4$	$1^4 2^2 4^4$	4B
$\mathbb{Z}_5$	$\eta(\tau)^4\eta(5\tau)^4$	$1^4 5^4$	5A
$\mathbb{Z}_6$	$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$	$1^2 2^2 3^2 6^2$	6A
$\mathbb{Z}_7$	$\eta(\tau)^3\eta(7\tau)^3$	$1^3 7^3$	7A,7B
$\mathbb{Z}_8$	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$	$1^2 2^1 4^1 8^2$	8A

S.Govindarajan G.Krishna arxiv:0907.1410

Motivation III: twisted BPS states

- A twisted index was introduced [5] that counts g twisted BPS states,

$$B_{2n}^{g} = \frac{1}{2n!}\mathrm{Tr}[g(-1)^{2j_3}(2j_3)^{2n}]$$

- It can be computed on moduli subspaces that are compatible with the symmetry group generated by g and by requiring the charges of the theory to be g invariant.
- This twisted index was computed for certain abelian CHL orbifolds[6] and the following results where obtained:

G	H	half-BPS degeneracy	ρ	M_{24} conjugacy classes
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\eta(2\tau)^{12}$	2^{12}	2B
$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\eta(3\tau)^{8}$	3^{8}	3B
$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\eta(4\tau)^{6}$	4^{6}	4C

- Here again, the half-BPS degeneracies are expressible as eta-products that form M_{24} representations.
- What happens when the twist does not commute with orbifold group?

[5] Ashoke Sen 0911.1563

[6] S.Govindarajan 1006.3472

Motivation IV: dihedral groups as symplectic automorphisms

- We will consider $\mathbb{Z}_2$ twists in $\mathbb{Z}_n$ orbifold theories.
- Moduli spaces that admit a dihedral symmetry $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ are compatible with both the twist and orbifold groups.
- If a elliptic $K3$ surface admitted both $\mathbb{Z}_2$ and $\mathbb{Z}_n, 3 \leq n \leq 6$ symmetries as symplectic automorphisms then the dihedral group acts as a symplectic automorphism on $K3$[7].

$$\mathcal{E}_{D_3} : y^2 = x^3 + (a_1\tau + a_0\tau^4 + a_1\tau^7)x + (b_2 + b_1\tau^3 + b_0\tau^6 + b_1\tau^9 + b_2\tau^{12})$$

$$\begin{aligned} \sigma_3 :& (x, y, \tau) \mapsto (\zeta_3^2 x, \zeta_3^3 y, \zeta_3\tau), \\ \varsigma_2 :& (x, y, \tau) \mapsto (\frac{x}{\tau^4}, -\frac{y}{\tau^6}, \frac{1}{\tau}) \end{aligned}$$

- One can choose the charges of the theory Q to take values from the sublattices of $\Gamma_{19,3}$ that are invariant under Dihedral symmetries[8]. This is compatible with both $\mathbb{Z}_2$ twist and $\mathbb{Z}_n$ orbifold projections.

[7] A.Garbagnati 0904.1519
[8] Griess, Lam 0806.2753

Counting degeneracy on non-abelian moduli spaces

- The moduli spaces[9] that admit dihedral symmetries are mapped to the heterotic picture by string-string duality.
- $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ has the commutator subgroup $\mathbb{Z}_n$, which is used to construct the CHL orbifolds.
- After quotienting, there is a residual $\mathbb{Z}_2$ symmetry on the moduli space which is allowed to act as a twist in the partition function of the theory.
- The twist $\mathbb{Z}_2$ does not commute with the orbifold group $\mathbb{Z}_n$, it is a non-commuting twist.
- The degeneracy of $\mathbb{Z}_2$ twisted half-BPS states in $\mathbb{Z}_n$ orbifolds is then computed following Sen's computation for $\mathcal{N} = 4$ CHL models. [10]

[9] A.Garbagnati 0904.1519
[10] Ashoke Sen hep-th/0504005

dihedral groups and twisted partition functions

- The dihedral group of order $2n$ has the following presentation

$$D_n \cong \langle h, g | h^n = e, g^2 = e, ghg = h^{-1} \rangle$$

- The elements of $D_n = \{e, h, \ldots, h^{n-1}, g, gh, \ldots, gh^{n-1}\}$
- The group invariant projector of the $\mathbb{Z}_n$ subgroup has the following property:

$$g.P_{\mathbb{Z}_n} = \frac{1}{n} g \left(\sum_{j=0}^{n-1} h^j \right) = \frac{1}{n} \left(\sum_{j=0}^{n-1} h^j \right) = P_{\mathbb{Z}_n}.g$$

- g commutes with $P_{\mathbb{Z}_n}$ even though it doesn't commute with the individual elements.

Example: Z3

- The $\mathbb{Z}_3$ subgroup of D_3: $\mathbb{Z}_3 = \{e, h, h^2\}$ and $P_{\mathbb{Z}_3} = (e + h + h^2)/3$
- The partition function for $\mathbb{Z}_3$ orbifolds including all twisted sectors

$$Z_{T/\mathbb{Z}_3} = P_{\mathbb{Z}_3} \underset{e}{\boxed{}} + P_{\mathbb{Z}_3} \underset{h}{\boxed{}} + P_{\mathbb{Z}_3} \underset{h^2}{\boxed{}}$$

- Twisting the partition function by $g \in \mathbb{Z}_2$ amounts to insertion of g in the trace,

$$\mathrm{Tr}_{\mathcal{H}_h}(g\ q^H) \equiv {}_g \underset{h}{\boxed{}}$$

- For the g twisted partition function the contribution comes only from the untwisted sector of the orbifold CFT, since the torus boundary conditions are inconsistent when $gh \neq hg$.

$$gX(\tau, \sigma + 2\pi) = ghg^{-1} gX(\tau, \sigma)\ ;\ hX(\tau + 2\pi, \sigma) = hgh^{-1} hX(\tau, \sigma)$$

- Hence, we are left to evaluate

$$Z^{\mathbb{Z}_2}_{T/\mathbb{Z}_3} = \frac{1}{3}\left({}_g \underset{e}{\boxed{}} + {}_{gh} \underset{e}{\boxed{}} + {}_{gh^2} \underset{e}{\boxed{}} \right)$$

Orbifold action: heterotic description

- The action of the orbifold group element $h \in H \equiv \mathbb{Z}_n$

$$P \to R_h P + a_h; \quad P \in \Gamma_{22,6}$$

- $\forall R_h \in R_H$, R_H leaves $22 - k$ of the 22 left moving directions invariant.
- The action of the twist element $g \in G$ on $K3$ leaves 14 of the 22 2-cycles of $K3$ invariant, In the heterotic picture it exchanges the two E_8 components. It is not accompanied by shifts.
- The action of the orbifold and twist leaves the right movers invariant to preserve $\mathcal{N} = 4$ supersymmetry.
- Compatibility with the $\mathbb{Z}_2$ twist, and $\mathbb{Z}_n$ orbifold projection requires the charges Q to take values on a lattice[11] that is invariant under both the symmetries.

[11] Griess, Lam 0806.2753

Orbifold action: on oscillators and lattice

- The complex worldsheet co-ordinates X^j, $j = 1, 2, ..., k/2$ represent the planes of rotation. R_H is characterized by $k/2$ phases $\phi_j(h)$. The effect of the rotation R_H is to multiply the complex oscillators by phases.

$$\alpha^j_{-n} \to e^{2\pi i \phi_j(d)} \alpha^j_{-n} \quad ; \quad \bar{\alpha}^j_{-n} \to e^{-2\pi i \phi_j(d)} \bar{\alpha}^j_{-n}$$

- The Narain Lattice $\Gamma^{(22,6)}$ is embedded in a $22 + 6$ dimensional vector space V.
- The action of the entire group thus separates the vector space V into an invariant subspace $V_\perp$ and its orthogonal complement $V_\|$.
- The invariant sublattice $\Lambda_\perp$ and its orthogonal complement $\Lambda_\|$ are

$$\Lambda_\perp = \Gamma \bigcap V_\perp \quad ; \quad \Lambda_\| = \Gamma \bigcap V_\|.$$

BPS states and level matching

- Momenta in the compact directions take values on the Narain lattice $\Gamma^{(22,6)}$. The (left,right) components of the momentum vector are denoted as $\vec{P} = (\vec{P}_L, \vec{P}_R)$
- $Q = (\vec{Q}_L, \vec{Q}_R)$ to denotes the projection of $\vec{P}$ along $V_\perp$ and $P_\| = (\vec{P}_{\|L}, 0)$ the projection of $\vec{P}$ along $V_\|$.
- The BPS states are picked by keeping the rightmoving oscillators at the lowest eigenvalue allowed by GSO projection, i.e $N_R = 1$.

$$N_L - 1 + \frac{1}{2}\vec{P}_{\|L}^2 = N$$

with $N = \frac{1}{2}(\vec{Q}_R^2 - \vec{Q}_L^2)$ and $\vec{P}_{\|L} = \vec{K}(Q) + \vec{p}$, where $\vec{p} \in \Lambda_\|$ and $\vec{K}(Q) \in V_\|$ is a constant vector that lies in the unit cell of $\Lambda_\|$.

Group invariant projection

- The counting of the number of $\mathbb{Z}_n$ invariant BPS states for a given charge Q is done by implementing the group invariant projection.

$$\frac{1}{n}\sum_{j=o}^{n-1} h^j \underset{e}{\square}$$

- The contribution to the trace with a orbifold group element $h \in \mathbb{Z}_n$ inserted comes only from those $\vec{P}_{\parallel L}$ which are invariant under the action of h, i.e $\vec{P}_{\parallel L} \in V_\perp(h)$.
- When a group element h acts on the vacuum carrying momentum $\vec{P}$ it will produce a phase

$$e^{2\pi i \vec{a}_h . \vec{Q}} e^{-2\pi i \vec{a}_{hL} . (\vec{p} + \vec{K}(Q))}$$

- The twist g does not have shifts, and will not produce these phases.

Degeneracy

- The degeneracy of BPS states in untwisted sector carrying a charge Q is expressed as[12]

$$d(Q) = \frac{16}{|\mathbb{Z}_n|} \sum_{h\in\mathbb{Z}_n} \sum_{N_L=0}^{\infty} d^{osc}(N_L, h) e^{2\pi i \vec{a}_h . \vec{Q}} e^{-2\pi i \vec{a}_{hL} . \vec{K}(Q)}$$
$$\sum_{\substack{\vec{p}\in\Lambda_{||} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL} . \vec{p}} \; \delta_{N_L - 1 + \frac{1}{2}(\vec{p}+\vec{K}(Q))^2, N}$$

  where $d^{osc}(N_L, h)$ is the number of ways one can construct oscillator level N_L from the 24 left-movers weighted by the action of h.

- Treating Q and $\hat{N} \equiv N$ as independent variables, the partition function,

$$\tilde{F}(Q, \mu) = \sum_{\hat{N}} F(Q, \hat{N}) e^{-\mu \hat{N}}$$

[12] (Ashoke Sen hep-th/0504005)

Partition function

- Explicitly, the partition function has the form,

$$\tilde{F}(Q,\mu) = \frac{16}{|\mathbb{Z}_n|} \sum_{h\in\mathbb{Z}_n} e^{2\pi i \vec{a}_h.\vec{Q}} e^{-2\pi i \vec{a}_{hL}.\vec{K}(Q)} \tilde{F}^{osc}(h,\mu) \tilde{F}^{lat}(Q,h,\mu)$$

where, the oscillator and lattice contribution to the partition function are

$$\tilde{F}^{osc}(h,\mu) = \sum_{N_L=0}^{\infty} d^{osc}(N_L,h) e^{-\mu N_L} e^{\mu}$$

$$\tilde{F}^{lat}(Q,h,\mu) = \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL}.\vec{p}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2}$$

- The inverse of the partition function gives the degeneracy

$$F(Q,\tilde{N}) = \frac{1}{2\pi i} \int_{\epsilon-i\pi}^{\epsilon+i\pi} d\mu \; \tilde{F}(Q,\mu) \; e^{\mu\tilde{N}}$$

Oscillator contribution

$$\tilde{F}^{osc}(h,\mu) = q^{-1}\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right)^{24-k_h}\prod_{j=1}^{k_h/2}\left(\prod_{n=1}^{\infty}\frac{1}{1-e^{2\pi i\phi_j(h)}q^n}\frac{1}{1-e^{-2\pi i\phi_j(h)}q^n}\right)$$

- $\phi_j(h)$ and k_h in $\tilde{F}^{osc}(h,\mu)$ depend only on the order of the group element h.
- With a g insertion one evaluates the oscillator contribution for,

$$g\,\underset{e}{\square} + gh\,\underset{e}{\square} + gh^2\,\underset{e}{\square} + \ldots + gh^{n-1}\,\underset{e}{\square}$$

- The elements $g, gh, \ldots, gh^{n-1}$ are each of order 2 and have identical contributions.
- Since g exchanges the E_8 co-ordinates, the number of directions that are rotated $k_g = 8$ and non zero phases $\phi_j(g) = 1/2$

$$\tilde{F}^{osc}(g,\mu) = \frac{1}{\eta(\mu)^8\eta(2\mu)^8}$$

Lattice contribution

- Inclusion of twist: Since the charges are already g invariant, g has no further action on the lattice.
- The lattice contribution from a orbifold group element h is

$$\tilde{F}^{lat}(Q,h,\mu) = \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i a_{\vec{h}L}.\vec{p}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2}.$$

- When h is identity $V_{\perp}(e) = V$. For any other h, we have $dim V_{\perp}(h) < dim(V)$. The dominant contribution would be from

$$\tilde{F}^{lat}(Q,e,\mu) \simeq \sum_{\vec{p}\in\Lambda_{\|}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2} \equiv \Theta^{\|}_{\mathbb{Z}_n}$$

Result

- Combining the oscillator and the lattice contributions, the partition function for g twisted half-BPS states in CHL $\mathbb{Z}_n$ orbifolds is

$$\tilde{F}(Q,\mu) \simeq \frac{16}{|Z_n|} \frac{\Theta^{\parallel}_{\mathbb{Z}_n}}{\eta(\mu)^8 \eta(2\mu)^8}$$

- The resulting modular form has lesser weight than the partition function for the untwisted half-BPS states as can be seen from the asymptotic limit $\mu \to 0$

$$\tilde{F}(\mu) \sim \frac{16}{|Z_n|} \frac{1}{Vol_{\Lambda_{\parallel}}} e^{2\pi^2/\mu} \left(\frac{\mu}{2\pi}\right)^{8-\frac{k_{\mathbb{Z}_n}}{2}}$$

Group	$12 - \frac{k_{\mathbb{Z}_n}}{2}$	$8 - \frac{k_{\mathbb{Z}_n}}{2}$	$k_{\mathbb{Z}_n} = rank(\Lambda_{\parallel})$
$\mathbb{Z}_3$	6	2	12
$\mathbb{Z}_4$	5	1	14
$\mathbb{Z}_5$	4	0	16
$\mathbb{Z}_6$	4	0	16

Discussion: Mathieu representation

- The lattice $\Lambda_{\parallel}$ has euclidean signature, which allows $\Theta^{\parallel}_{\mathbb{Z}_n}$ to be rewritten in terms of Θ functions on the Leech lattice.

$$\frac{1}{g_\rho(\mu)} = \frac{16}{|Z_n|}\frac{\Theta^{\Gamma^{24}}}{\Theta^{\perp L}_{\mathbb{Z}_n}}\frac{1}{\eta(\mu)^8\eta(2\mu)^8}$$

- The Leech lattice is an even unimodular lattice embedded in $\mathbb{R}^{24}$.
- The Mathieu group M_{24} is a subgroup of the automorphism groups of the Leech lattice and it induces a permutation representation.
- The theta functions with momenta sums running over invariant sublattices of the leech lattice are known[13] to form representations of M_{24}.
- Hence, The generating function which counts the degeneracies of $\mathbb{Z}_2$ twisted half-BPS states in $\mathbb{Z}_n$ orbifold theories is a modular form that is a representation of M_{24}.

[13] T.Kondo and T.Tasaka 86'

Summary and future Outlook

- We computed the twisted index for CHL $\mathbb{Z}_n, 3 \leq n \leq 6$ orbifolds when the twist does not commute with the orbifold group.
- This twisted index computes $\mathbb{Z}_2$ twisted 1/2 BPS states in CHL $\mathbb{Z}_n$ orbifolds.
- We derived the generating function that gives the expected asymptotic limit.
- The generating function forms a representation of M_{24}.
- This computation may be extended to 1/4 BPS states.
- It will be useful to consider twists that break supersymmetry, On $K3$ we would have to consider non-symplectic automorphisms.
- It would also be very interesting to see if such a procedure works for any non-abelian group.

Thank You!