Chern-Simons matter theories - 2
"Onshell supersymmetry and the S matrix"

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Plan of the talk

Summary from yesterday

- Summary
- Duality in supersymmetric Chern-Simons theories

Offshell supersymmetry

- $\mathcal{N} = 1$ superspace
- eg:off-shell susy constraints on correlators

Onshell supersymmetry

- Supersymmetric scattering
- S matrix in onshell superspace

Unitarity equations for supersymmetric processes

Summary from yesterday

- In the previous talk, we saw that pure Chern-Simons theories are topological.
- Chern-Simons theories coupled to charged matter exhibit interesting phenomenon.
- Chern-Simons equations of motion ties magnetic flux to charged particles.
- Charge-flux coupling gives rise to physically interesting behavior: anyons and Aharanov-Bohm effect.

Summary from yesterday

- Aharanov-Bohm scattering: S matrix for the non-relativistic scattering of charged particles off a flux tube.

$$h(\theta) = 2\pi(\cos(\pi\nu) - 1)\delta(\theta) + \sin(\pi\nu)\left(Pv\cot\left(\frac{\theta}{2}\right) - i\mathrm{Sgn}(\nu)\right)$$

- The S matrix contains a non-analytic delta function piece modulated by anyonic phases at forward scattering.

- Delta function originally missed by Aharanov-Bohm, observed by Ruijsenaars 20 years later on unitarity grounds.

- We saw that delta function piece is required for unitarity to work.

Summary from yesterday

- We also saw that the pure $U(N)/SU(N)$ Chern-Simons theory has a level-rank transposition duality valid for any N and κ .
- Motivated by AdS/CFT, Vasiliev duality: level-rank duality in CS matter theories.
- 2+1 d bosonization duality

  $U(N)$ CS+fundamental boson at Wilson Fisher limit

  $\Leftarrow$ dual $\Rightarrow$

  $U(N)$ CS+fundamental fermion
- conjectured for all N and κ but tested in planar limit (so far).
- Physical quantities on both sides match under a duality map.

Conjectured Duality for susy matter CS

- Jain, Minwalla, Yokoyama conjectured that $\mathcal{N}=1,2$ supersymmetric matter coupled Chern-Simons theories are self dual

$$Theory(\lambda', w', m') \Longleftrightarrow Theory(\lambda, w, m)$$

- under the map

$$\lambda' = \lambda - \mathrm{Sgn}(\lambda)\ ,\ \ w' = \frac{3-w}{1+w} \quad m_0' = \frac{-2m_0}{1+w}$$
$$N' = |\kappa| - N + 1\ ,\ \ \kappa' = -\kappa$$

- with a pole mass

$$m = \frac{2m_0}{2+(-1+w)\lambda\ \mathrm{Sgn(m)}}$$

- $m' = -m$ under duality and $\lambda m(m_0, w) \geq 0$

Evidence for conjectured duality in susy CS

- S matrices computed in the $\mathcal{N}=2$ Chern-Simons matter theory in the large N limit, to all orders in t'Hooft coupling λ.
- T_B - S matrix for $2 \to 2$ boson scattering, T_F - S matrix for $2 \to 2$ fermion scattering.

$$\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa} ,$$
$$\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa}$$

- Duality easy to see, $\kappa \to -\kappa$ and $m \to -m$. The S matrices map to each other upto an overal unobservable phase. More details on computation in lecture 3.
- Today we will almost derive these S matrices just from supersymmetric ward identities.

Offshell supersymmetry

Summary from yesterday

Summary

Duality in supersymmetric Chern-Simons theories

Offshell supersymmetry

$\mathcal{N} = 1$ superspace

eg:off-shell susy constraints on correlators

Onshell supersymmetry

Supersymmetric scattering

S matrix in onshell superspace

Unitarity equations for supersymmetric processes

$\mathcal{N} = 1$ superspace in 2+1 dimensions

- Superspace : Efficient packaging tool to study supersymmetric theories.

- Manifold: commuting coordinates x^μ, and two component grassmann coordinates θ^α.

$$\int d\theta = 0 \, , \int d\theta\theta = 1 \, , \int d^2\theta\theta^2 = -1$$

- Supersymmetry operator in superspace

$$Q_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\partial_{\alpha\beta}\right)$$

$$\{Q_\alpha, Q_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu \equiv 2i\partial_{\alpha\beta}$$

- Functions in superspace, superfields $\Phi_{\alpha_1,\alpha_2\ldots}$ taylor expanded in a terminating series

$$\Phi_{\alpha_1,\alpha_2\ldots}(x,\theta) = \phi_{\alpha_1\alpha_2\ldots}(x) + \theta^\alpha\psi_{\alpha\alpha_1\alpha_2\ldots}(x) + \theta^2 F_{\alpha_1\alpha_2\ldots}(x)$$

Example: $\mathcal{N}=1$ ungauged theory

- Complex scalar superfield: complex (scalar, fermion and auxilary field)
$$\Phi = \phi + \theta\psi - \theta^2 F$$
- offshell degrees of freedom - bosonic: 2+2, fermionic:4
$$S_E = -\int d^3x d^2\theta \Big(-\frac{1}{2} D^\alpha \bar{\Phi} D_\alpha \Phi + m_0 \bar{\Phi}\Phi + \frac{\eta}{4}(\bar{\Phi}\Phi)^2 \Big)$$
- In components
$$S_E = -\int d^3x \Big(F\bar{F} + \bar{\psi}^\alpha (i\partial_\alpha{}^\beta + m_0 \delta_\alpha{}^\beta)\psi_\beta - \partial\bar{\phi}\partial\phi + m_0(\bar{F}\phi + \bar{\phi}F) + \frac{\eta}{2}\bar{\phi}\phi(\bar{\psi}\psi + \bar{F}\phi + \bar{\phi}F) + \frac{\eta}{4}(\bar{\phi}\psi + \bar{\psi}\phi)^2 \Big)$$

Example: $\mathcal{N} = 1$ ungauged theory

- equations of motion for the auxiliary fields are

$$\bar{F} = -m_0\bar{\phi} - \frac{\eta}{2}(\bar{\phi}\phi)\bar{\phi} \ ,$$
$$F = -m_0\phi - \frac{\eta}{2}(\bar{\phi}\phi)\phi$$

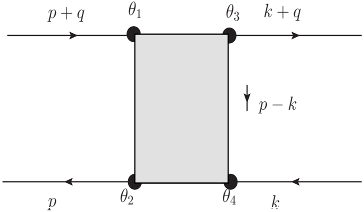
- eliminate auxilary fields

$$S_E = \int d^3x \Big(\partial\bar{\phi}\partial\phi + m_0^2\bar{\phi}\phi - \bar{\psi}^{\alpha}(i\partial_{\alpha}{}^{\beta} + m_0\delta_{\alpha}{}^{\beta})\psi_{\beta} + \eta m_0(\bar{\phi}\phi)^2 + \frac{\eta^2}{4}(\bar{\phi}\phi)^3 - \frac{\eta}{2}(\bar{\phi}\phi)(\bar{\psi}\psi) - \frac{\eta}{4}(\bar{\phi}\psi + \bar{\psi}\phi)^2 \Big)$$

- Onshell degrees of freedom - bosonic: 2 , fermionic 2.

Four point function

- The four point function in superspace



$$V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q) = \langle\bar{\Phi}((p+q),\theta_1)\Phi(-(k+q),\theta_3)\bar{\Phi}(k,\theta_4)\Phi(-p,\theta_2)\rangle$$

- Supersymmetric invariance of the four point function implies

$$(Q_{\theta_1,p+q}+Q_{\theta_2,-p}+Q_{\theta_3,k+q}+Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q)=0$$

$$\sum_{i=1}^{4}\left(\frac{\partial}{\partial\theta_i^\alpha}-p_{\alpha\beta}(\theta_1-\theta_2)^\beta-q_{\alpha\beta}(\theta_1-\theta_3)^\beta-k_{\alpha\beta}(\theta_4-\theta_3)^\beta\right)V(\theta_1,\theta_2,\theta_3,p,q,k)=0$$

Four point function

- Use sum and difference variables

$$
\begin{aligned}
X &= \sum_{i=1}^{4} \theta_i \ , \\
X_{12} &= \theta_1 - \theta_2 \ , \\
X_{13} &= \theta_1 - \theta_3 \ , \\
X_{43} &= \theta_4 - \theta_3 \ .
\end{aligned}
$$

- rewrite derivatives

$$
\begin{aligned}
\frac{\partial}{\partial \theta_1} &= \frac{\partial}{\partial X} + \frac{\partial}{\partial X_{12}} + \frac{\partial}{\partial X_{13}} \ , \\
\frac{\partial}{\partial \theta_2} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{12}}, \\
\frac{\partial}{\partial \theta_3} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{13}} - \frac{\partial}{\partial X_{43}} \ , \\
\sum_{i=1}^{4} \frac{\partial}{\partial \theta_i} &= 4 \frac{\partial}{\partial X}
\end{aligned}
$$

Four point function

- rewrite condition of susy invariance of four point function

$$(4\frac{\partial}{\partial X} - p.X_{12} - q.X_{13} - k.X_{43})V(X, X_{12}, X_{13}, X_{43}, p, q, k) = 0$$

- Supersymmetry determines the θ structure of V upto a shift invariant function.

$$V = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right)F(X_{12}, X_{13}, X_{43}, p, q, k)$$

$$X = \sum_{i=1}^{4}\theta_i \ , \ X_{ij} = \theta_i - \theta_j$$

Scattering a superfield

$$\begin{pmatrix} \Phi(\theta_1, p_1) \\ \bar{\Phi}(\theta_2, p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\Phi}(\theta_3, p_3) \\ \Phi(\theta_4, p_4) \end{pmatrix}$$

$$\mathcal{S}_B : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix} \;,\; \mathcal{S}_F : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix}$$

$$H_1 : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix} \;,\; H_2 : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_3 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \;,\; H_4 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \;,\; H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix}$$

Supersymmetric scattering

- $2 \to 2$ scattering amplitude: transition between free incoming and free outgoing onshell particles.

- Initial and final states of Φ_i are effectively subject to free equations of motion

$$\left(D^2 + m\right)\Phi = 0$$

- Solution

$$\Phi(x,\theta) = \int \frac{d^2p}{\sqrt{2p^0}(2\pi)^2}\Big[\Big(a(\mathbf{p})(1+m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big)e^{ip.x} + \Big(a^{c\dagger}(\mathbf{p})(1+m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})\Big)e^{-ip.x}\Big]$$

- action of off-shell supersymmetry operator on onshell superfields $[Q^{off}_\alpha, \Phi] = Q^{off}_\alpha \Phi = i\left(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\partial_{\beta\alpha}\right)\Phi$

$$-iQ^{on}_\alpha = u_\alpha(\mathbf{p}_i)\left(\alpha\partial_a + \alpha^c\partial_{a^c}\right) + u^*_\alpha(\mathbf{p}_i)\left(a\partial_\alpha + a^c\partial_{\alpha^c}\right) - v^*_\alpha(\mathbf{p}_i)\left(a^\dagger\partial_{\alpha^\dagger} + (a^c)^\dagger\partial_{(\alpha^c)^\dagger}\right) + v_\alpha(\mathbf{p}_i)\left(\alpha^\dagger\partial_{a^\dagger} + (\alpha^c)^\dagger\partial_{(a^c)^\dagger}\right)$$

Onshell superspace

- Introduce creation and annihilation operator superfields

$$A_i(\mathbf{p}) = a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i \ ,$$
$$A_i^\dagger(\mathbf{p}) = a_i^\dagger(\mathbf{p}) + \theta_i\alpha_i^\dagger(\mathbf{p}) \ .$$

- Action of supersymmetry operator

$$[Q_\alpha^{on}, A_i(\mathbf{p}_i,\theta_i)] = Q_\alpha^1 A_i(\mathbf{p}_i,\theta_i)$$
$$[Q_\alpha^{on}, A_i^\dagger(\mathbf{p}_i,\theta_i)] = Q_\alpha^2 A_i^\dagger(\mathbf{p}_i,\theta_i)$$

$$Q_\beta^1 = i\left(-u_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p})\theta\right)$$
$$Q_\beta^2 = i\left(v_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p})\theta\right) \ .$$

Supersymmetry and dual supersymmetry

- Action of bose-fermi duality

$$a^D = \alpha, \quad \alpha^D = a \quad m^D = -m$$

- dual supersymmetry operator has the form

$$(Q^D)^1_\beta = i\left(-u_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p},-m)\theta\right) ,$$

$$(Q^D)^2_\beta = i\left(v_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p},-m)\theta\right)$$

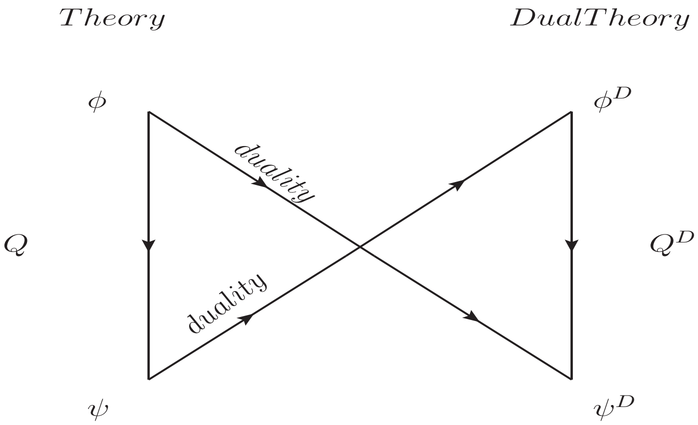
- using $u(m,p) = -v(-m,p), \quad v(m,p) = -u(-m,p)$ and $\theta \leftrightarrow \frac{\partial}{\partial\theta}$

$$(Q^D)^1 \propto Q^1, \quad (Q^D)^2 \propto Q^2$$

- Quantities invariant under usual supersymmetry also invariant under dual supersymmetry.

- Onshell supersymmetry commutes with duality

Supersymmetry and dual supersymmetry



S matrix in onshell superspace

- $2 \to 2$ S matrix: $p_1 + p_2 \to p_3 + p_4$

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\sqrt{(2p_1^0)(2p_2^0)(2p_3^0)(2p_4^0)} = \langle 0|A_4(\mathbf{p}_4, \theta_4)A_3(\mathbf{p}_3, \theta_3)UA_2^\dagger(\mathbf{p}_2, \theta_2)A_1^\dagger(\mathbf{p}_1, \theta_1)|0\rangle$$

- Supersymmetric ward identity for the S matrix

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1, \theta_1) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_2, \theta_2) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_3, \theta_3) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_4, \theta_4)\Big)S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0$$

S matrix in onshell superspace

- S matrix solution (in-states: p_1, p_2 ,out-states p_3, p_4) is determined in terms of two functions $\mathcal{S}_B$ and $\mathcal{S}_F$ of momenta,couplings and mass.

$$
\begin{aligned}
&S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = \mathcal{S}_B + \mathcal{S}_F\ \theta_1\theta_2\theta_3\theta_4 + \\
&\left(\frac{1}{2}C_{12}\mathcal{S}_B - \frac{1}{2}C^*_{34}\mathcal{S}_F\right)\ \theta_1\theta_2 + \left(\frac{1}{2}C_{13}\mathcal{S}_B - \frac{1}{2}C^*_{24}\mathcal{S}_F\right)\ \theta_1\theta_3 \\
&+ \left(\frac{1}{2}C_{14}\mathcal{S}_B + \frac{1}{2}C^*_{23}\mathcal{S}_F\right)\ \theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{S}_B + \frac{1}{2}C^*_{14}\mathcal{S}_F\right)\ \theta_2\theta_3 \\
&+ \left(\frac{1}{2}C_{24}\mathcal{S}_B - \frac{1}{2}C^*_{13}\mathcal{S}_F\right)\ \theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{S}_B - \frac{1}{2}C^*_{12}\mathcal{S}_F\right)\ \theta_3\theta_4
\end{aligned}
$$

- No θ term: four boson scattering, four θ term : four fermion scattering.

- All other processes (two boson to two fermion etc) determined completely in terms of the two independent functions $\mathcal{S}_B$ and $\mathcal{S}_F$.

S matrix in onshell superspace

$$\frac{1}{2}C_{12} = -\frac{1}{4m}v^*(\mathbf{p}_1)v^*(\mathbf{p}_2) \qquad \frac{1}{2}C_{23} = -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_3)$$
$$\frac{1}{2}C_{13} = -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_3) \qquad \frac{1}{2}C_{24} = -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_4)$$
$$\frac{1}{2}C_{14} = -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_4) \qquad \frac{1}{2}C_{34} = -\frac{1}{4m}u^*(\mathbf{p}_3)u^*(\mathbf{p}_4)$$

$$\frac{1}{2}C^*_{12} = \frac{1}{4m}v(\mathbf{p}_1)v(\mathbf{p}_2) \qquad \frac{1}{2}C^*_{23} = \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_3)$$
$$\frac{1}{2}C^*_{13} = \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_3) \qquad \frac{1}{2}C^*_{24} = \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_4)$$
$$\frac{1}{2}C^*_{14} = \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_4) \qquad \frac{1}{2}C^*_{34} = \frac{1}{4m}u(\mathbf{p}_3)u(\mathbf{p}_4)$$

On onshell supersymmetry for $\mathcal{N} = 2$ S matrix

- The $\mathcal{N} = 2$ S matrix is already $\mathcal{N} = 1$ supersymmetric.
- obeys additional constraint from $\mathcal{N} = 2$ supersymmetry.
- Detour to $\mathcal{N} = 2$ superspace,

$$\bar{D}_\alpha \Phi = 0, \qquad D_\alpha \bar{\Phi} = 0.$$

$$\Phi = \phi + \sqrt{2}\theta\psi - \theta^2 F + i\theta\bar{\theta}\partial\phi - i\sqrt{2}\theta^2(\bar{\theta}\partial\!\!\!/\psi) + \theta^2\bar{\theta}^2\partial^2\phi,$$
$$\bar{\Phi} = \bar{\phi} + \sqrt{2}\bar{\theta}\bar{\psi} - \bar{\theta}^2\bar{F} - i\theta\bar{\theta}\partial\bar{\phi} - i\sqrt{2}\bar{\theta}^2(\theta\partial\!\!\!/\bar{\psi}) + \theta^2\bar{\theta}^2\partial^2\bar{\phi}$$

- same degrees of freedom as $\mathcal{N} = 1$ theory.

On onshell supersymmetry for $\mathcal{N} = 2$ S matrix

- Action of off-shell Q_α, $\bar{Q}_\alpha$ on on-shell chiral/anti-chiral superfields determines action of on-shell $\mathcal{N} = 2$ supersymmetry.
- conditions for on-shell $\mathcal{N} = 2$ susy of S matrix, written in $\mathcal{N} = 1$ onshell superspace

$$\left(\sum_{i=1}^{4} Q_\alpha^i(\mathbf{p}_i, \theta_i) + \bar{Q}_\alpha^i(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0$$

$$\left(\sum_{i=1}^{4} Q_\alpha^i(\mathbf{p}_i, \theta_i) - \bar{Q}_\alpha^i(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0$$

On onshell supersymmetry for $\mathcal{N} = 2$ S matrix

- Additional constraint relates the functions $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B \left(C_{13} u_\alpha(\mathbf{p}_3) + C_{14} u_\alpha(\mathbf{p}_4) + C_{12} v_\alpha(\mathbf{p}_2) + v^*_\alpha(\mathbf{p}_1) \right)$$
$$= \mathcal{S}_F (C^*_{24} u_\alpha(\mathbf{p}_3) - C^*_{23} u_\alpha(\mathbf{p}_4) + C^*_{34} v_\alpha(\mathbf{p}_2))$$

- $\mathcal{N} = 2$ S matrix is completely specified by one function.

- eg: $p_1 = p + q, p_2 = -k - q, p_3 = p, p_4 = -k$

$$\mathcal{S}_B = \mathcal{S}_F \frac{-2m(k-p)_- + iq_3(k+p)_-}{2m(k-p)_- + iq_3(k+p)_-} \ .$$

- Already $\mathcal{N} = 2$ supersymmetry is quite constraining.

- Expect all the component S matrices in higher susy cases to be determined by one function.

Unitarity

Unitarity equation

- Define on-shell superfield $S^\dagger$ as

$$S^\dagger(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = S^*(\mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4, \mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2)$$

- Supersymmetric ward identity for $S^\dagger$ implies $S^\dagger$ is supersymmetric if and only if S is supersymmetric.

- The supersymmetric unitarity equation is

$$(S \star S^\dagger - I) = 0$$

- Recall that the superfield expansion for S is completely specified by $\mathcal{S}_B$ and $\mathcal{S}_F$.

- Sufficient to check the LHS for no θ and four θ terms.

- Supersymmetric ward identity guarantees the rest of the terms will obey the unitarity equations.

Product of S matrices

- General multiplication rule for two S matrices

$$S_1 \star S_2 \equiv \int d\Gamma S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2)$$
$$\exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2)$$
$$S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)$$

$$d\Gamma = \frac{d^2 k_3}{2k_3^0(2\pi)^2} \frac{d^2 k_4}{2k_4^0(2\pi)^2} \frac{d^2 k_1}{2k_1^0(2\pi)^2} \frac{d^2 k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4$$

- supersymmetry invariant Identity operator

$$I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \exp(\theta_1\theta_3 + \theta_2\theta_4) I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$$
$$I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)$$

- Multiplication rule with Identity operator

$$S \star I = I \star S = S$$

- More generally product of two supersymmetric S matrices is supersymmetric.

Unitarity equations

- No θ_i term and four θ_i terms

$$
\begin{aligned}
&\int d\tilde{\Gamma}\Big[\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&-Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\
&\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big]=i(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)-\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2))
\end{aligned}
$$

$$
\begin{aligned}
&\int d\tilde{\Gamma}\Big[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\
&\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big]\\
&=4iY(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)-\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\right)
\end{aligned}
$$

$$
d\tilde{\Gamma}=(2\pi)^3\delta^3(p_1+p_2-k_3-k_4)\frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\ .
$$

Unitarity equations

- Unitarity equations made more human readable: go to center of mass frame.

$$p_1 = \left(\sqrt{p^2+m^2}, p, 0\right), \ p_2 = \left(\sqrt{p^2+m^2}, -p, 0\right)$$
$$p_3 = \left(\sqrt{p^2+m^2}, p\cos(\theta), p\sin(\theta)\right), \ p_4 = \left(\sqrt{p^2+m^2}, -p\cos(\theta), -p\sin(\theta)\right)$$

  where θ is the scattering angle between p_1 and p_3.

- Mandelstam variables

$$s = -(p_1+p_2)^2 \ , t = -(p_1-p_3)^2, \ u = (p_1-p_4)^2, \ s+t+u = 4m^2 \ ,$$
$$s = 4(p^2+m^2) \ , t = -2p^2(1-\cos(\theta)) \ , u = -2p^2(1+\cos(\theta))$$

Unitarity equations

- No θ_i term and four θ_i terms

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha))$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_F^S(s,\alpha)-\mathcal{T}_F^{S*}(s,-\alpha))$$

- Under duality $\mathcal{T}_B \to \mathcal{T}_F$ and vice versa; both the equations map to one another.

- Unitarity conditions are compatible with duality.

Unitarity equations

- Consider the general structure ($T(\theta) = i\cot(\theta/2)$.)

$$\mathcal{T}_B = H_B T(\theta) + W_B - iW_2\delta(\theta)\ ,\ \mathcal{T}_F = H_F T(\theta) + W_F - iW_2\delta(\theta)\ ,$$

- first unitarity equation

$$H_B - H_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_B^* - H_B W_2^*)\ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_B H_B^*)\ ,$$

$$W_B - W_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_B^* - W_2^* W_B) - \frac{i}{4\sqrt{s}}(H_B H_B^* - W_B W_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

- Second unitarity equation

$$H_F - H_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_F^* - H_F W_2^*)\ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_F H_F^*)\ ,$$

$$W_F - W_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_F^* - W_2^* W_F) - \frac{i}{4\sqrt{s}}(H_F H_F^* - W_F W_F^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

Thank You!