Attractor mechanism in gauged supergravity

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Sep 2013

Plan of the talk

Background

- Black holes are regions (bounded by an event horizon) in space time where gravity is so strong that even light cannot escape.

- The temperature and entropy of a black hole (in Planck units), Bekenstein-Hawking

$$T_{BH} = \frac{\kappa}{2\pi}\ , \quad S_{BH} = \frac{A}{4}\ .$$

- Black hole is analogous to a thermodynamic system with large microscopic degeneracy.

- Microscopic physics of black holes involve distances of the order of planck scale - expected to be described by a quantum theory of gravity.

Background

- String theory is one of the candidate theories for a quantum description of gravity.
- Succesfully provides microscopic and macroscopic descriptions of supersymmetric black holes.
- Number of BPS states do not change as moduli, coupling constants vary continuously. Witten
- Weak coupling: BPS states in stringy description.
- Logarithm of degeneracy of charged BPS states in string theory agrees with black hole entropy in large charge limit. Strominger-Vafa

Background

- Strong coupling: BPS black hole solutions in supergravity description.
- Attractor mechanism explains origin of black hole entropy in supergravity. Ferrara-Kallosh-Strominger
- Moduli fields in black hole background flow to specific charge dependent values on the horizon.
- Black hole entropy is determined completely in terms of charges and is independent of asymptotic values of moduli - Agrees with microscopic results.
- Attractor mechanism is a consequence of near horizon geometry rather than susy. Ferrara-Gibbons-Kallosh
- Extends to non-susy cases. Goldstein-Iizuka-Jena-Trivedi

Motivation

- Previous studies of attractor mechanism focussed on black holes with flat asymptotics.
- Generalisation to curved spaces, in particular AdS will be valuable.
- In AdS/CFT, black branes are holographic duals to field theories at finite temperature.
- Extremal branes exhibit vanishing entropy density at zero temperature and describe ground states of the field theory.
- Gauged supergravity - Ideal for the study, supports AdS vacuum, describes supergravity regime of AdS/CFT.

Motivation

- Recently, Attractor mechanism generalised for $\mathcal{N} = 2, d = 4$ gauged supergravity.
  Cacciatori-Klemm , Kachru-Kallosh-Shmakova
- Generalised attractors: solutions to equations of motion that reduce to algebraic equations when all the fields, curvature tensors are constants in tangent space.
- Lifshitz, Schrödinger geometries are some examples of generalised attractors.
- Recently, Bianchi attractors: Classification of homogeneous anisotropic extremal black brane horizons in $d = 5$. Iizuka-Kachru-Kundu-Narayan-Sircar-Trivedi
- Embed Bianchi attractors in gauged supergravity, study susy and stability in non susy cases.

Thesis overview

- Exploration of microscopic and macroscopic description of black holes in string theory.
- Microscopic side: Counting of a special class of BPS states in $\mathcal{N} = 4$ supersymmetric string theory.
- Macroscopic side: Generalisation of attractor mechanism to gauged supergravity.

Based on

- Counting BPS states
  - "A non-commuting twist in the partition function", S. Govindarajan , Karthik Inbasekar, arXiv:1201.1628.
- Generalised Attractors
  - "Generalised attractors in five dimensional gauged supergravity", Karthik Inbasekar, P. K.Tripathy , arXiv:1206.3887, JHEP 1209 (2012) 003.
  - "Stability of Bianchi attractors in gauged supergravity", Karthik Inbasekar, P. K.Tripathy , arXiv:1307.1314, to appear in JHEP.

Counting BPS states

Twisted index

- CHL models -theories with $\mathcal{N} = 4$ supersymmetry in four dimensions. Chaudhuri-Howe-Lykken, Aspinwall
- Orbifolds of type II A string theory on $K3 \times T^2$.
- Twisted index counts g twisted BPS states Sen

$$B_{2n}^{g} = \frac{1}{2n!}\mathrm{Tr}[g(-1)^{2j_3}(2j_3)^{2n}]$$

- Counts BPS states that break $4n$, g invariant supersymmetries.
- Computable in special regions of moduli space where g is a symmetry, also require charges to be g invariant.
- Count degeneracy of $1/2$ BPS states in CHL models when twist does not commute with orbifold group.

Non commuting twists

- Consider $\mathbb{Z}_2$ twists in $\mathbb{Z}_n$ orbifold theories.
- Moduli spaces with dihedral symmetry $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$,compatible with both the twist and orbifold groups.

  Garbagnati
- Map the moduli space from $K3 \times T2$ to heterotic picture via string-string duality.
- In $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$, the commutator subgroup $\mathbb{Z}_n$, is used to construct the CHL orbifolds.
- Residual $\mathbb{Z}_2$ symmetry allowed to act as a twist in the partition function.
- compute the twisted partition function. Sen

Results

- Final partition function - product of two terms, oscillator contribution, lattice sum.

$$\tilde{F}(Q,\mu) \simeq \frac{16}{|Z_n|}\frac{\Theta^{\|}_{\mathbb{Z}_n}}{\eta(\mu)^8\eta(2\mu)^8}$$

- Twisted partiton function counts $\mathbb{Z}_2$ invariant BPS states in $\mathbb{Z}_n$ orbifold theory.
- expect the modular form to have lesser weight than untwisted case.
- Check by taking asymptotic limit $\mu \to 0$

$$\tilde{F}(\mu) \sim \frac{16}{|Z_n|}\frac{1}{Vol_{\Lambda_{\|}}}e^{2\pi^2/\mu}\left(\frac{\mu}{2\pi}\right)^{8-\frac{k_{\mathbb{Z}_n}}{2}}$$

- Untwisted $\mathbb{Z}_n$ orbifold partition function has weight $12 - k_{\mathbb{Z}_n}/2$. confirms expectations.

Summary and future outlook

- Computed the twisted index for CHL $\mathbb{Z}_n, 3 \leq n \leq 6$ orbifolds when the twist does not commute with the orbifold group.
- Twisted index computes $\mathbb{Z}_2$ twisted $1/2$ BPS states in CHL $\mathbb{Z}_n$ orbifolds.
- derived the generating function that gives the expected asymptotic limit.
- May be extended to $1/4$ BPS states.
- Useful to consider twists that break supersymmetry, and try to extend the counting problem to non-BPS states - challenging.

Attractor mechanism in gauged supergravity

Gauged Sugra: Symmetries

- The most general $\mathcal{N} = 2, d = 5$ gauged sugra has gravity coupled to vector, tensor and hypermultiplets.

  Ceresole-Dall'Agata

- The scalars in the theory parametrise a manifold that factorises into a direct product of a very special and quaternionic manifold,

$$\mathcal{M}_{scalar} = \mathcal{S}(n_v + n_T) \otimes \mathcal{Q}(n_H).$$

- The R symmetry group is $SU(2)_R$.

Gauged Sugra: Gauging the Symmetries

- Gauging: Suitable subgroup K of the isometry group G of the full scalar manifold $\mathcal{M}_{scalar}$, and the $SU(2)_R$ symmetry group.
- Ordinary derivatives on scalar and fermions are replaced with K-covariant derivatives.

$$\partial_\mu \phi^{\tilde{x}} \to \mathcal{D}_\mu \phi^{\tilde{x}} \equiv \partial_\mu \phi^{\tilde{x}} + g A^I_\mu K^{\tilde{x}}_I(\phi)$$
$$\partial_\mu q^X \to \mathcal{D}_\mu q^X \equiv \partial_\mu q^X + g A^I_\mu K^X_I(q)$$
$$\nabla_\mu B^M_{\nu\rho} \to \mathcal{D}_\mu B^M_{\nu\rho} \equiv \nabla_\mu B^M_{\nu\rho} + g A^I_\mu \Lambda^M_{IN} B^N_{\nu\rho},$$

- Gauging the $SU(2)_R$ Symmetry:

$$\nabla_\mu \psi_{\nu i} \to \nabla_\mu \psi_{\nu i} + g_R A^I_\mu {P_{Ii}}^j(q) \psi_{\nu j}.$$

- Gauging leads to scalar potentials in the theory - possibility of AdS vacuum.

Generalised Attractors: Definition

- In tangent space, all the bosonic fields in the theory take constant values at the attractor points.

$$\phi^{\tilde{z}} = \text{const} \; ; q^{Z} = \text{const} \; ; A^{I}_{a} = \text{const} \; ;$$
$$B^{M}_{ab} = \text{const} \; ; c_{bc}{}^{a} = \text{const}.$$

- The attractor geometries are characterised by constant anholonomy coefficients.

$$[e_a, e_b] = c_{ab}{}^{c} e_c \; ; \quad e_a \equiv e^{\mu}_{a} \partial_{\mu}$$
$$c_{ab}{}^{c} = e^{\mu}_{a} e^{\nu}_{b} (\partial_{\mu} e^{c}_{\nu} - \partial_{\nu} e^{c}_{\mu})$$

Generalised Attractors: Features

- Gauge field, Tensor field and Einstein equations reduce to algebraic equations at the attractor point.
- Scalar field equations reduce to a minimisation condition on an attractor potential.
- The attractor potential is also independently constructed from squares of fermionic shifts.
- Constant anholonomy $\Rightarrow$ regular geometries.

Generalised Attractors: Examples Bianchi types

- Bianchi Attractors: Five dimensional extremal black brane horizons with homogeneous symmetries in spatial directions. Iizuka-Kachru-Kundu-Narayan-Sircar-Trivedi
- Homogeneous symmetries: invariant basis $\tilde{e}_i, i = 1, 2, 3$ that commutes with Killing vectors.

$$[\xi_j, \tilde{e}_i] = 0, \quad [\tilde{e}_i, \tilde{e}_j] = c_{ij}{}^k \tilde{e}_k$$

- Invariant vectors close to form a Lie algebra - isomorphic to Bianchi classification (I-IX) of 3d real Lie algebras Bianchi.
- Metric written in terms of invariant one forms ω^i dual to $\tilde{e}_i$ displays manifest homogeneous symmetries.

$$d\omega^k = \frac{1}{2} c_{ij}{}^k \omega^i \wedge \omega^j$$

Generalised Attractors: Examples Bianchi types

- Additional symmetries: scale invariance, time translation invariance

$$\hat{r} \to \lambda \hat{r} \ , \quad \hat{t} \to \lambda^{-u_0} \hat{t} \ , \quad \omega^i \to \lambda^{-u_i} \omega^i$$

$$ds^2 = L^2 \left[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_i+u_j)} \eta_{ij} \omega^i \otimes \omega^j \right]$$

- Metric has constant anholonomy coefficients.

Generalised Attractors: Example Bianchi II

- One forms, invariant vectors, structure constants,

$$
\begin{aligned}
&c_{23}{}^{1} = 1 = -c_{32}{}^{1}, && && && \\
&\xi_1 = \partial_{\hat{y}}, & \tilde{e}_1 &= \partial_{\hat{y}}, & \omega^1 &= d\hat{y} - \hat{x}d\hat{z}, & d\omega^1 &= \omega^2 \wedge \omega^3, \\
&\xi_2 = \partial_{\hat{z}}, & \tilde{e}_2 &= \hat{x}\partial_{\hat{y}} + \partial_{\hat{z}}, & \omega^2 &= d\hat{z}, & d\omega^2 &= 0, \\
&\xi_3 = \partial_{\hat{x}} + \hat{z}\partial_{\hat{y}}, & \tilde{e}_3 &= \partial_{\hat{x}}, & \omega^3 &= d\hat{x}, & d\omega^3 &= 0
\end{aligned}
$$

- scaling in coordinates,

$$(\hat{x}, \hat{y}, \hat{z}) \to (\lambda^{-u_1}\hat{x}, \lambda^{-(u_1+u_3)}\hat{y}, \lambda^{-u_3}\hat{z})$$

- scaling in one forms,

$$(\omega^1, \omega^2, \omega^3) \to (\lambda^{-(u_1+u_3)}\omega^1, \lambda^{-u_3}\omega^2, \lambda^{-u_1}\omega^3)$$

- metric

$$ds^2 = L^2\left[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_1+u_3)}(\omega^1)^2 + \hat{r}^{2u_3}(\omega^2)^2 + \hat{r}^{2u_1}(\omega^3)^2\right]$$

Bianchi attractors in gauged supergravity

- Choose a model: gauged supergravity model with one vector and two tensor multiplets. Gunaydin-Zagermann

- Moduli space

$$\mathcal{M}_{scalar} = SO(1,1) \times \frac{SO(2,1)}{SO(2)}.$$

- Metric on moduli space $g_{\tilde{x}\tilde{y}}$, $a_{\tilde{I}\tilde{J}}$.

- Gauging: $SO(2)$ subgroup of G using a single vector A^0 (graviphoton).

- R-Symmetry: $A_\mu[U(1)_R] = A^0_\mu V_0 + A^1_\mu V_1$

Bianchi attractors in gauged supergravity

- Potential:

$$\mathcal{V}(\phi) = \frac{g^2}{8}\left[\frac{[(\phi^2)^2 + (\phi^3)^2]}{||\phi||^6}\right] - 2g_R^2\left[2\sqrt{2}\frac{\phi^1}{||\phi||^2}V_0V_1 + ||\phi||^2V_1^2\right].$$

- Conditions for $\mathcal{N} = 2$ supersymmetry and AdS vaccum:

$$\phi^2 = 0, \quad \phi^3 = 0, \quad \phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}}, \quad V_0V_1 > 0, \quad 32\frac{g_R^2}{g^2}V_0^2 \leq 1.$$

- potential evaluated at these values gives the AdS cosmological constant $\mathcal{V}_{AdS} = -6g_R^2(\phi_c^1)^2V_1^2$.

- Bianchi attractors exist for values of scalars for which the theory would also have an *AdS* Vacuum.

Methodology

- Take metric ansatz: Bianchi types,
- gauge field ansatz: time like gauge field

$$A^t = e^t_a A^a = \frac{1}{Lr^u} A^0$$

- Set all tensor fields $B^M_{\mu\nu}$ to zero!
- Use the generalised attractor procedure and solve the algebraic field equations!

Example: Bianchi Type II

Bianchi Type II specified by gauging parameters g, V_0, V_1.

$$ds^2 = L^2\Big[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_1} d\hat{x}^2 + \hat{r}^{2(u_3+u_1)} d\hat{y}^2 - 2\hat{x}\hat{r}^{2(u_3+u_1)} d\hat{y} d\hat{z} + [\hat{r}^{2(u_3+u_1)}\hat{x}^2 + \hat{r}^{2u_3}] d\hat{z}^2 \Big] \, ,$$

$$u_0 = \sqrt{2}, \quad u_3 = u_1 = \frac{1}{2\sqrt{2}}, \quad L = \sqrt{\frac{2}{3}}\frac{(\phi_c^1)^4}{g}, \quad A^0 = \sqrt{\frac{5}{8}}\frac{1}{(\phi_c^1)^2},$$

$$\phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}}, \quad V_0 V_1 > 0, \quad \frac{23}{2(\phi_c^1)^4} \leq 1,$$

Stability analysis

- Generalised attractor analysis does not involve susy, relies on extremisation of an attractor potential.
- Solutions were found at critical points, not at absolute minima of attractor potential.
- Preliminary susy analysis of existing solutions using KSI indicated broken susy.
- Non-susy attractors can be unstable to scalar fluctuations about critical value.
- consider scalar field fluctuations about attractor value,

$$\phi_c + \epsilon \delta\phi(r, t)$$

Stress energy tensor: Backreaction at first order

- For Gauged sugra with generic gauging, trace of Einstein equation,

$$R\frac{(2-d)}{2} = T_{\mu}^{attr\,\mu}|_{\phi_c} + (d-2) g K_{yI}|_{\phi_c} A^{\lambda I}\partial_\lambda(\delta\phi^y) + g^2 \frac{\partial K_{IJ}}{\partial \phi^z}\bigg|_{\phi_c} A^I_\mu A^{J\mu}\delta\phi^z$$

$$T_{\mu}^{\mu attr}|_{\phi_c} = \mathcal{V}_{attr}(\phi_c) D - \left[a_{IJ}|_{\phi_c} F^I_{\mu\nu} F^{\mu\nu J} + g^2 K_{IJ}|_{\phi_c} A^I_\mu A^{\mu J}\right]$$

$$K_{IJ} = g_{xy} K^x_I K^y_J$$

- Scalar fluctuation terms indicate backreaction even at first order perturbation.
- Relevant boundary conditions for scalars should be such that they are well behaved near the horizon.
- For $U(1)_R$ gauging, $g = 0$ and back reaction is absent.

Scalar fluctuation equations

- Scalar fluctuation equation for arbitrary gauged sugra,

$$\nabla_\mu \nabla^\mu \delta\phi^x - g^{zx} \frac{\partial^2 \mathcal{V}_{attr}}{\partial\phi^z \partial\phi^y}\bigg|_{\phi_c} \delta\phi^y + 2g \left(g^{zx} \tilde{\nabla}_y K_{Iz}\right)|_{\phi_c} A^{\mu I} \nabla_\mu \delta\phi^y = 0$$

$\tilde{\nabla}$ - covariant derivative w.r.t g_{xy}.
∇ - covariant derivative w.r.t near horizon metric.

- higher order metric/gauge field fluctuations can be ignored for solving the above equation at lowest order.

- Laplacian for any given 5d Bianchi type metric,

$$\nabla_\mu \nabla^\mu = \frac{1}{L^2}\left[\hat{r}^2 \partial_{\hat{r}}^2 + (m+2)\hat{r}\partial_{\hat{r}} - \frac{1}{\hat{r}^{2u_0}} \partial_{\hat{t}}^2\right]$$

$m = -1 + \sum_I c_I u_I,\ c_I > 0,\ c_0 = 1.$

Scalar fluctuation equations

- For the specific gauged supergravity model fluctuation equation reduce to ,

$$\left[\hat{r}^2\partial_{\hat{r}}^2 + (m+2)\hat{r}\partial_{\hat{r}} - \frac{1}{\hat{r}^{2u_0}}\partial_{\hat{t}}^2 - \lambda\right]\delta\phi^x = 0$$

  λ - Eigenvalue of double derivative of attractor potential.
  Sign of λ - indicates nature of critical point.

- For ansatz $\delta\phi(\hat{r},\hat{t}) = f(\hat{r})e^{ik\hat{t}}$ (with k real), we get Bessel equation

$$\left[\hat{r}^2\partial_{\hat{r}}^2 + (m+2)\hat{r}\partial_{\hat{r}} + (\frac{k^2}{\hat{r}^{2u_0}} - \lambda)\right]f(\hat{r}) = 0$$

Scalar fluctuations

- Scalar fluctuations

$$f(X) = \left(\frac{X}{2}\right)^{\nu_0}\Big[C_1 H^1_{\nu_\lambda}(X)\big[\Gamma(1-\nu_\lambda)e^{i\nu_\lambda\pi} + \Gamma(1+\nu_\lambda)\big] + C_2 H^2_{\nu_\lambda}(X)\big[\Gamma(1-\nu_\lambda)e^{-i\nu_\lambda\pi} + \Gamma(1+\nu_\lambda)\big]\Big]$$

$$X = \frac{k}{u_0 \hat{r}^{u_0}}, \quad \nu_\lambda = \frac{\sqrt{(1+m)^2 + 4\lambda}}{2u_0}, \quad \nu_0 = \frac{(1+m)}{2u_0}$$

- Consistency condition for ν_λ real,

$$\nu_\lambda = \frac{\sqrt{(1+m)^2 + 4\lambda}}{2u_0} = \frac{\sqrt{(\sum_I c_I u_I)^2 + 4\lambda}}{2u_0} \leq 1$$

- implies $\lambda < 0$,

$$-\frac{(\sum_I c_I u_I)^2}{4} \leq \lambda < 0$$

- Scalar fluctuations - well defined for critical points which are maxima of attractor potential.

Conditions for stability

- In our coordinate system horizon is located at $\hat{r} = 0$, $X \simeq 1/\hat{r}$, consider asymptotic expansion of $f(X)$

$$f(X) \sim \left(\frac{X}{2}\right)^{\nu_0 - \frac{1}{2}} \sqrt{\frac{1}{\pi}} \Big[C_1 e^{i(X - \frac{\pi}{2}(\nu_\lambda + \frac{1}{2}))} \left[\Gamma(1-\nu_\lambda) e^{i\nu_\lambda \pi} + \Gamma(1+\nu_\lambda)\right] + C_2 e^{-i(X - \frac{\pi}{2}(\nu_\lambda + \frac{1}{2}))} \left[\Gamma(1-\nu_\lambda) e^{-i\nu_\lambda \pi} + \Gamma(1+\nu_\lambda)\right] \Big]$$

- Leading divergent term is absent only when,

$$\nu_0 = \frac{(1+m)}{2u_0} = \frac{\sum_l c_l u_l}{2u_0} \leq \frac{1}{2}$$

- since $c_0 = 1$,

$$\sum_{l, l \neq 0} c_l u_l \leq 0$$

- But $u_l \geq 0$ for regular horizon, therefore stability conditions are:

$$u_0 \neq 0, \quad u_l = 0 \;\; \forall \; l \neq 0$$

Stable Bianchi metrics

- Bianchi metrics with scale invariance in all directions,

$$ds^2 = L^2 \left[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_i+u_j)} \eta_{ij} \omega^i \otimes \omega^j \right]$$

- Stability condition,

$$u_0 \neq 0, \quad u_I = 0 \ \ \forall \ I \neq 0$$

- Stable Bianchi attractors in gauged supergravity are a subclass with scale invariance only in radial and time directions.

$$ds^2 = L^2 \left(-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + L^2 \left(\eta_{ij} \omega^i \otimes \omega^j \right)$$

- They are of the direct product form $Lif_{u_0}(2) \times M$.

Stability summary

- Unstable generalised attractors

Geometry	λ	u_0	$u_l, l \neq 0$
Lifshitz	-34	3	1
Bianchi II	$-\frac{22}{3}$	$\sqrt{2}$	$u_1 = u_3 = \frac{1}{2\sqrt{2}}$
Bianchi VI $h < 0$	$-1 + \frac{14h}{3} - h^2$	$\frac{1}{\sqrt{2}}(1-h)$	$u_2 = -\frac{1}{\sqrt{2}}h, u_3 = \frac{1}{\sqrt{2}}$

- Stable generalised attractors in direct product form

Geometry	λ	u_0	$u_l, l \neq 0$
$Lif_{u_0}(2) \times M_I$	$-\frac{5u_0^2}{3}$	any $u_0 > 0$	0
$AdS_2 \times M_I$	$-\frac{5}{3}$	1	0
$Lif_{u_0}(2) \times M_{II}$	$-\frac{61}{6}$	$\sqrt{\frac{11}{2}}$	0
$Lif_{u_0}(2) \times M^*$	$\lambda < 0$	any $u_0 > 0$	0

Summary

Results: Generalised attractors: 5d gauged sugra

- Field equations become algebraic at attractor points.
- Scalar moduli - determined as functions of the charges by extremising an attractor potential.
- Attractor potential is constructed independently from fermionic shifts in gauged supergravity.
- Bianchi attractors are examples of generalised attractors in gauged supergravity.
- Explicit examples: $z = 3$ Lifshitz solution, Bianchi II, Bianchi VI solutions in gauged supergravity.
- Explicit examples: $Lif_{u_0}(2) \times M$ for M_I, M_{II} and $AdS_2 \times \mathbb{R}^3$ in $U(1)_R$ gauged sugra.

Results: Stability analysis

- Stress energy tensor in gauged supergravity depends on scalar fluctuations even at first order.
- Instability - Ill behaved fluctuations near the horizon will backreact strongly $\implies$ significant deviation from the attractor geometry.
- Consistency condition on scalar fluctuations: critical point is a maxima of the attractor potential.
- Regularity of the fluctuations near the horizon require the near horizon geometry to factorise as $Lif_{u_0} \times M$,

$$ds^2 = L^2 \left(-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + L^2 \left(\eta_{ij} \omega^i \otimes \omega^j \right)$$

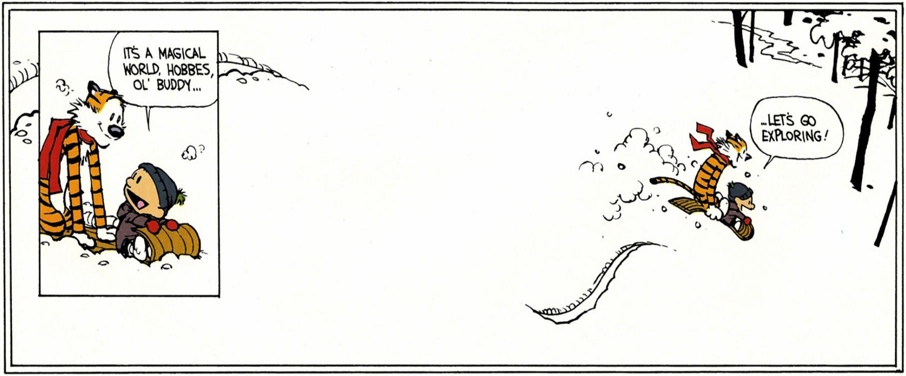
$M = M_I, M_{II} \ldots M_{IX}$ - 3d homogeneous subspaces invariant under the Bianchi type symmetries.

Future outlook

- Completion: search for models to embed the rest of Bianchi attractors.
- SUSY of Bianchi attractors - Try to find susy critical points by solving Killing spinor equations, gaugino, hyperino conditions. Ongoing
- Attractor flow equations: Either analytic/numeric approaches to construct solutions interpolating between the Bianchi types at the IR and AdS_5 in the UV. Will prove attractor mechanism in gauged supergravity.
- String embedding: To understand gauged sugra, generalised attractors from flux compactifications perspective. Particularly gauged sugra with gauging of R symmetries.

Future outlook

- Hyperscale violating Bianchi attractors - Metrics with conformal invariance, different scaling properties Iizuka-Kachru-Kundu-Narayan-Sircar-Trivedi-Wang, Embedding in gauged sugra, susy analysis, string embedding.



Thank You!

Gauged Sugra: Lagrangian

The bosonic part of the five dimensional $\mathcal{N}=2$ gauged supergravity:

$$\begin{aligned}\hat{e}^{-1}Ł^{\mathcal{N}=2}_{Bosonic} =& -\frac{1}{2}R - \frac{1}{4}a_{\tilde{I}\tilde{J}}\mathcal{H}^{\tilde{I}}_{\mu\nu}\mathcal{H}^{\tilde{J}\mu\nu} - \frac{1}{2}g_{XY}\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y \\ &- \frac{1}{2}g_{\tilde{x}\tilde{y}}\mathcal{D}_\mu \phi^{\tilde{x}}\mathcal{D}^\mu\phi^{\tilde{y}} + \frac{\hat{e}^{-1}}{6\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau \\ &+ \frac{\hat{e}^{-1}}{4g}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}B^M_{\mu\nu}\mathcal{D}_\rho B^N_{\sigma\tau} - \mathcal{V}(\phi, q).\end{aligned}$$

$$\begin{aligned}&\mathcal{H}^{\tilde{I}}_{\mu\nu} = (F^I_{\mu\nu}, B^M_{\mu\nu}), && \mu = 0, \dots, 4 \\ &M = 1, \dots, n_T, && I = 0, 1, \dots, n_V \\ &\tilde{x} = 0, 1, \dots n_v + n_T, && X = 1, 2, \dots 4n_H.\end{aligned}$$

Gauged Sugra: Potential and fermionic shifts

$$\mathcal{V}(\phi, q) = 2g^2 W^{\tilde{a}} W^{\tilde{a}} - g_R^2 [2P_{ij}P^{ij} - P_{ij}^{\tilde{a}} P^{\tilde{a}ij}] + 2g^2 \mathcal{N}_{iA}\mathcal{N}^{iA}$$

$$P_{ij} \equiv h^I P_{Iij}, \qquad P_{ij}^{\tilde{a}} \equiv h^{\tilde{a}I} P_{Iij}$$

$$W^{\tilde{a}} \equiv \frac{\sqrt{6}}{4} h^I K_I^{\tilde{x}} f_{\tilde{x}}^{\tilde{a}}, \qquad \mathcal{N}^{iA} \equiv \frac{\sqrt{6}}{4} h^I K_I^X f_X^{Ai}.$$

Bosonic part of supersymmetry transformations:

$$\delta_\epsilon \psi_{\mu i} = \sqrt{6}\nabla_\mu \epsilon_i + \frac{i}{4} h_{\tilde{I}}(\gamma_{\mu\nu\rho}\epsilon_i - 4g_{\mu\nu}\gamma_\rho \epsilon_i)\mathcal{H}^{\nu\rho\tilde{I}} + i g_R P_{ij} \gamma_\mu \epsilon^j$$

$$\delta_\epsilon \lambda_i^{\tilde{a}} = -\frac{i}{2} f_{\tilde{x}}^{\tilde{a}} \gamma^\mu \epsilon_i \mathcal{D}_\mu \phi^{\tilde{x}} + \frac{1}{4} h_{\tilde{I}}^{\tilde{a}} \gamma^{\mu\nu} \epsilon_i \mathcal{H}_{\mu\nu}^{\tilde{I}} + g_R P_{ij}^{\tilde{a}} \epsilon^j + g W^{\tilde{a}} \epsilon_i$$

$$\delta_\epsilon \zeta^A = -\frac{i}{2} f_{iX}^A \gamma^\mu \epsilon^i \mathcal{D}_\mu q^X + g \mathcal{N}_i^A \epsilon^i.$$

The potential can be written as squares of fermionic shifts.

Gauge field equation

- Since ${c_{ab}}^c = const$,

$$F_{ab} = e^\mu_a e^\nu_b (\partial_\mu e^c_\nu - \partial_\nu e^c_\mu) A_c = {c_{ab}}^c A_c$$

- The Gauge field equation of motion,

$$\partial_\mu (\hat{e} a_{I\tilde{J}} \mathcal{H}^{\tilde{J}\mu\nu}) = -\frac{1}{2\sqrt{6}} C_{I\tilde{J}\tilde{K}} \epsilon^{\nu\mu\rho\sigma\tau} \mathcal{H}^{\tilde{J}}_{\mu\rho} \mathcal{H}^{\tilde{K}}_{\sigma\tau} + g\hat{e}\left[g_{XY} K^X_I \mathcal{D}^\nu q^Y + g_{\tilde{x}\tilde{y}} K^{\tilde{x}}_I \mathcal{D}^\nu \phi^{\tilde{y}}\right]$$

in tangent space, is an algebraic equation at the attractor points

$$\hat{e}\ a_{I\tilde{J}} [\omega_{a,}{}^a{}_c \mathcal{H}^{cb\tilde{J}} + \omega_{a,}{}^b{}_c \mathcal{H}^{ac\tilde{J}}] = -\frac{1}{2\sqrt{6}} C_{I\tilde{J}\tilde{K}} \epsilon^{bacde} \mathcal{H}^{\tilde{J}}_{ac} \mathcal{H}^{\tilde{K}}_{de} + g^2 \hat{e}\left[g_{XY} K^X_I K^Y_J + g_{\tilde{x}\tilde{y}} K^{\tilde{x}}_I K^{\tilde{y}}_J\right] A^{Jb}.$$

Tensor field equation

- The tensor field equation is,

$$\frac{1}{g}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MP}\mathcal{D}_\rho B^M_{\mu\nu} + \hat{e}a_{\tilde{I}P}\mathcal{H}^{\tilde{I}\sigma\tau} = 0.$$

- In tangent space,

$$\frac{1}{g}\epsilon^{abcde}\left[c_{ac}{}^f B^M_{fb} + gA^I_c\Lambda^M_{IN}B^N_{ab}\right]\Omega_{MP} + \hat{e}a_{\tilde{I}P}\mathcal{H}^{\tilde{I}de} = 0.$$

is an algebraic equation at the attractor points,

Einstein equation

- The Einstein equation at the attractor point:

$$R_{ab} - \frac{1}{2} R \eta_{ab} = T^{attr}_{ab}$$

- In the absence of torsion, The left handside is algebraic:

$$R_{abc}{}^{d} = \partial_a \omega_{bc}{}^{d} - \partial_b \omega_{ac}{}^{d} - \omega_{ac}{}^{e} \omega_{be}{}^{d} + \omega_{bc}{}^{e} \omega_{ae}{}^{d} - c_{ab}{}^{e} \omega_{ec}{}^{d}$$

$$\omega_{a,bc} = \frac{1}{2}[c_{ab,c} - c_{ac,b} - c_{bc,a}]$$

- The stress energy tensor at the attractor point:

$$T^{attr}_{ab} = \mathcal{V}_{attr}(\phi, q)\eta_{ab} - \Bigg[a_{\tilde{I}\tilde{J}} \mathcal{H}^{\tilde{I}}_{ac} \mathcal{H}_b{}^{c\tilde{J}} + g^2 [g_{XY} K^X_I K^Y_J + g_{\tilde{x}\tilde{y}} K^{\tilde{x}}_I K^{\tilde{y}}_J] A^I_a A^J_b \Bigg].$$

- The Einstein equations are algebraic at the attractor points.

Scalar equation

- The scalar $\phi^{\tilde{x}}$ field equations,

$$\hat{e}^{-1}\partial_\mu\left[\hat{e}\ g_{\tilde{z}\tilde{y}}\mathcal{D}^\mu\phi^{\tilde{y}}\right] - \frac{1}{2}\frac{\partial g_{\tilde{x}\tilde{y}}}{\partial\phi^{\tilde{z}}}\mathcal{D}_\mu\phi^{\tilde{x}}\mathcal{D}^\mu\phi^{\tilde{y}}$$
$$- gA^I_\mu g_{\tilde{x}\tilde{y}}\frac{\partial K^{\tilde{x}}_I}{\partial\phi^{\tilde{z}}}\mathcal{D}^\mu\phi^{\tilde{y}} - \frac{1}{4}\frac{\partial a_{\tilde{I}\tilde{J}}}{\partial\phi^{\tilde{z}}}\mathcal{H}^{\tilde{I}}_{\mu\nu}\mathcal{H}^{\tilde{J}\mu\nu} - \frac{\partial\mathcal{V}(\phi,q)}{\partial\phi^{\tilde{z}}} = 0.$$

- For the quaternion q^Z, the equation of motion is

$$\hat{e}^{-1}\partial_\mu\left[\hat{e}\ g_{ZY}\mathcal{D}^\mu q^Y\right] - \frac{1}{2}\frac{\partial g_{XY}}{\partial q^Z}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y$$
$$- gA^I_\mu g_{XY}\frac{\partial K^X_I}{\partial q^Z}\mathcal{D}^\mu q^Y - \frac{\partial\mathcal{V}(\phi,q)}{\partial q^Z} = 0.$$

Attractor Potential

Using attractor ansatz,

- Equation of motion for $\phi^{\tilde{x}}$ reduces to,

$$\frac{\partial}{\partial\phi^{\tilde{z}}}\left[\mathcal{V}(\phi,q)+\frac{1}{2}g^2\, g_{\tilde{x}\tilde{y}}K_I^{\tilde{x}}K_J^{\tilde{y}}A^{Ia}A_a^J+\frac{1}{4}a_{\tilde{I}\tilde{J}}\mathcal{H}_{ab}^{\tilde{I}}\mathcal{H}^{\tilde{J}ab}\right]=0.$$

- Equation of motion for q^Z reduces to,

$$\frac{\partial}{\partial q^Z}\left[\mathcal{V}(\phi,q)+\frac{1}{2}g^2g_{XY}K_I^XK_J^YA^{aI}A_a^J\right]=0.$$

Attractor Potential

- Scalar field equations reduce to an extremisation condition on an attractor potential.

$$\mathcal{V}_{attr}(\phi, q) = \Big[\mathcal{V}(\phi, q) + \frac{1}{4} a_{\tilde{I}\tilde{J}} \mathcal{H}^{\tilde{I}}_{ab} \mathcal{H}^{\tilde{J}ab} + \frac{1}{2} g^2 [\, g_{\tilde{x}\tilde{y}} K_I^{\tilde{x}} K_J^{\tilde{y}} + g_{XY} K_I^X K_J^Y] A^{Ia} A^J_a \Big]$$

- The attractor potential gives rise to the attractor values of the scalars upon extremisation.

Attractor Potential from fermion shifts

- Susy transformations at attractor points:

$$\delta\psi_{ai} = \sqrt{6}D_a\epsilon_i + (\Sigma_{i|j})^{bc}(\gamma_{abc} - 4\eta_{ab}\gamma_c)\epsilon^j + \gamma_a S_{ij}\epsilon^j$$
$$\delta\lambda_i^{\tilde{a}} = \Sigma^{\tilde{a}}{}_{i|j}\epsilon^j + (\Sigma^{\tilde{a}}{}_{i|j})^a\gamma_a\epsilon^j + (\Sigma^{\tilde{a}}{}_{i|j})^{ab}\gamma_{ab}\epsilon^j$$
$$\delta\zeta^A = (\Sigma^A{}_{|j})\epsilon^j + (\Sigma^A{}_{|j})^a\gamma_a\epsilon^j$$

- Generalised Fermion shifts:

$$\Sigma^{\tilde{a}}{}_{i|j} = g_R P^{\tilde{a}}{}_{ij} - gW^{\tilde{a}}\epsilon_{ij} \quad ; \quad (\Sigma^A{}_{|j}) = g\mathcal{N}_j{}^A$$
$$(\Sigma^{\tilde{a}}{}_{i|j})^a = \frac{i}{2}gf^{\tilde{a}}_{\tilde{x}}K^{\tilde{x}}_I A^{Ia}\epsilon_{ij} \quad ; \quad (\Sigma^A{}_{|j})^a = -\frac{i}{2}gf^A_{jX}K^X_I A^{aI}$$
$$(\Sigma^{\tilde{a}}{}_{i|j})^{ab} = -\frac{1}{4}h^{\tilde{a}}_{\tilde{I}}\mathcal{H}^{\tilde{I}ab}\epsilon_{ij} \quad ; \quad (\Sigma_{i|j})^{bc} = -\frac{i}{4}h_{\tilde{I}}\mathcal{H}^{bc\tilde{I}}\epsilon_{ij}$$
$$S_{ij} = ig_R P_{ij}$$

Attractor Potential from fermion shifts

- The attractor potential can be constructed independently from squares of fermionic shifts

$$
\begin{aligned}
-\mathcal{V}_{attr}\frac{\epsilon^{l}{}_{k}}{4} = \bar{S}^{i}{}_{k} S_{i}{}^{l} - \epsilon^{lj}\Big\{ & \big[(\overline{\Sigma^{A}{}_{|k}})(\Sigma_{A|j}) + \frac{1}{2}(\overline{\Sigma^{\tilde{a}i}{}_{|k}})(\Sigma^{\tilde{a}}{}_{i|j})\big] \\
& + \big[(\overline{\Sigma^{A}{}_{|k}})_{a}(\Sigma_{A|j})^{a} + \frac{1}{2}(\overline{\Sigma^{\tilde{a}i}{}_{|k}})_{a}(\Sigma^{\tilde{a}}{}_{i|j})^{a}\big] \\
& + \big[(\overline{\Sigma^{i}{}_{|k}})_{ab}(\Sigma_{i|j})^{ab} + (\overline{\Sigma^{\tilde{a}i}{}_{|k}})_{ab}(\Sigma^{\tilde{a}}{}_{i|j})^{ab}\big]\Big\},
\end{aligned}
$$

which can be shown to reproduce,

$$
\begin{aligned}
\mathcal{V}_{attr}(\phi, q) = \Big[& \mathcal{V}(\phi, q) + \frac{1}{4} a_{\tilde{I}\tilde{J}} \mathcal{H}^{\tilde{I}}_{ab} \mathcal{H}^{\tilde{J}ab} \\
& + \frac{1}{2} g^{2} [\, g_{\tilde{x}\tilde{y}} K_{I}^{\tilde{x}} K_{J}^{\tilde{y}} + g_{XY} K_{I}^{X} K_{J}^{Y}] A^{Ia} A^{J}_{a} \Big]
\end{aligned}
$$

AdS

$$ds^2 = L^2\left[-\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)\right] ,$$

$$\phi_c^2 = 0 \ , \quad \phi_c^3 = 0 \ , \quad \phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}} , \quad \Lambda = -6 g_R^2 V_1^2 (\phi_c^1)^2 \ ,$$

$$V_0 V_1 > 0 \ , \quad 32\frac{g_R^2}{g^2} V_0^2 \leq 1 \ , \quad L^2 = -\frac{6}{\Lambda} \ ,$$

Lifshitz

$$ds^2 = L^2\left[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)\right] \,,$$

$$u_0 = 3, \quad L = \sqrt{3}\frac{(\phi_c^1)^4}{g} \,, \quad A^{0\bar{0}} = \sqrt{\frac{2}{3}}\frac{1}{(\phi_c^1)^2} \,,$$

$$\phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}} \,, \quad V_0 V_1 > 0 \,, \quad \frac{32}{3(\phi_c^1)^4} \leq 1 \,.$$

Bianchi II

$$
\begin{aligned}
ds^2 = L^2 \Big[& -\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_1} d\hat{x}^2 + \hat{r}^{2(u_3+u_1)} d\hat{y}^2 \\
& - 2\hat{x}\hat{r}^{2(u_3+u_1)} d\hat{y} d\hat{z} + [\hat{r}^{2(u_3+u_1)}\hat{x}^2 + \hat{r}^{2u_3}] d\hat{z}^2 \Big] \; ,
\end{aligned}
$$

$$
u_0 = \sqrt{2} \; , \quad u_3 = u_1 = \frac{1}{2\sqrt{2}} \; , \quad L = \sqrt{\frac{2}{3}} \frac{(\phi_c^1)^4}{g} \; , \quad A^{0\bar{0}} = \sqrt{\frac{5}{8}} \frac{1}{(\phi_c^1)^2} \; ,
$$

$$
\phi_c^1 = \left(\sqrt{2} \frac{V_0}{V_1} \right)^{\frac{1}{3}} \; , \quad V_0 V_1 > 0 \; , \quad \frac{23}{2(\phi_c^1)^4} \leq 1 \; ,
$$

Bianchi VI

$$ds^2 = L^2\left[-\hat{r}^{2u_0}d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + e^{-2\hat{x}}\hat{r}^{2u_2}d\hat{y}^2 + e^{-2h\hat{x}}\hat{r}^{2u_3}d\hat{z}^2\right] ,$$

$$u_0 = \frac{1}{\sqrt{2}}(1-h) , \quad u_2 = -\frac{1}{\sqrt{2}}h , \quad u_3 = \frac{1}{\sqrt{2}} , \quad L = \frac{(\phi_1^c)^4}{\sqrt{6}g}(1-h) ,$$

$$A^{0\bar{0}} = \sqrt{\frac{-2h}{(-1+h)^2}\frac{1}{(\phi_c^1)^2}} , \quad h < 0 , \quad h \neq 0, 1 ,$$

$$\phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1}\right)^{\frac{1}{3}} , \quad V_0 V_1 > 0 , \quad \frac{8(3-h+3h^2)}{(\phi_c^1)^4(-1+h)^2} \leq 1 ,$$

$Lif_{u_0}(2) \times M_I$

$$ds^2 = L^2 \left[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 \right] ,$$

$$A^{0\hat{t}} = \frac{1}{L\hat{r}} A^{0\bar{0}} , \quad A^{1\hat{t}} = \frac{1}{L\hat{r}} A^{1\bar{0}} , \quad \frac{A^{0\bar{0}}}{A^{1\bar{0}}} = \frac{1}{2} \frac{V_1}{V_0} , \quad L^2 = -\frac{u_0^2}{2\Lambda} ,$$

$$\Lambda = -6 g_R^2 V_1^2 (\phi_c^1)^2 , \quad \phi_c^1 = \left(\sqrt{2} \frac{V_0}{V_1} \right)^{\frac{1}{3}} , \quad V_0 V_1 > 0 .$$

$Lif_{u_0} \times M_{II}$

$$ds^2 = L^2\left[-\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{x}^2 + d\hat{y}^2 - 2\hat{x} d\hat{y} d\hat{z} + (\hat{x}^2 + 1) d\hat{z}^2 \right] ,$$

$$A^{0\hat{t}} = \frac{1}{L\hat{r}} A^{0\bar{0}} , \quad A^{1\hat{t}} = \frac{1}{L\hat{r}} A^{1\bar{0}} , \quad \frac{A^{0\bar{0}}}{A^{1\bar{0}}} = \frac{1}{2}\frac{V_1}{V_0} , \quad u_0 = \sqrt{\frac{11}{2}} ,$$

$$L^2 = -\frac{13}{4\Lambda} , \quad \Lambda = -6 g_R^2 V_1^2 (\phi_c^1)^2 , \quad \phi_c^1 = \left(\sqrt{2}\frac{V_0}{V_1} \right)^{\frac{1}{3}} , \quad V_0 V_1 > 0 \ .$$

susy

- The spinors in five dimensions satisfy a symplectic majorana condition:

$$\epsilon^{ij}\bar{\epsilon}_j = (\epsilon^i)^t C$$

- In two component spinor λ notation

$$\epsilon_i = \begin{pmatrix} i\epsilon_{ij}\lambda_j \\ \lambda_i^* \end{pmatrix}$$

  SM spinors have manifest $SU(2)_R$ invariance.

- First one has to check the Killing spinor integrability equation for necessary conditions for supersymmetry.

- Then one has to solve the Killing spinor equations.

SUSY: Killing spinor integrability conditions

- KSI expressible in terms of fermionic shifts. Defining $M_{abc} = \gamma_{abc} - 4\eta_{ab}\gamma_c$,

$$
\begin{aligned}
-\frac{1}{4}R_{ae}{}^{cd}\gamma_{cd}\epsilon_i = &-\frac{1}{\sqrt{6}}(\Sigma_{i|j})^{fc}[\omega_{a,}{}^{b}{}_{f}M_{e[bc]} - \omega_{e,}{}^{b}{}_{f}M_{a[bc]}]\epsilon^j \\
&- \frac{1}{6}\Big\{[(\Sigma_{i|j})^{bc}M_{abc} + \gamma_a S_{ij}][(\Sigma_{k|l})^{gh}M_{egh} + \gamma_e S_{kl}] \\
&- [(\Sigma_{i|j})^{bc}M_{ebc} + \gamma_e S_{ij}][(\Sigma_{k|l})^{gh}M_{agh} + \gamma_a S_{kl}]\Big\}\epsilon^{jk}\epsilon^l
\end{aligned}
$$

- All shifts vanish $\Rightarrow$ Maximal supersymmetry (AdS_5 vacuum, unique).

$$
\frac{1}{4}R_{ae}{}^{cd}\gamma_{cd}\epsilon_i = \frac{1}{6}S_{ij}S_{kl}\gamma_{ae}\epsilon^{jk}\epsilon^l
$$

SUSY: Killing spinor integrability conditions

- Some shifts vanish $\Rightarrow$ partially broken supersymmetry (Lifshitz, Bianchi types)
- cases with only vector multiplets in minimal gauged supergravity : Either 1/2 BPS or 1/4 BPS solutions. [Gauntlett-Gutowski]
- Lifshitz solutions: known to be 1/4 BPS [Cassani-Faedo].
- We expect Bianchi attractors to be 1/4 BPS. (in progress)

Scalar fluctuations time independent

- the scalar fluctuation equation:

$$\left[\hat{r}^2\partial_{\hat{r}}^2 + (m+2)\hat{r}\partial_{\hat{r}} - \frac{1}{\hat{r}^{2u_0}}\partial_{\hat{t}}^2 - \lambda\right]\delta\phi^x = 0$$

  admits a simple solution when the fluctuations $\delta\phi^x$ are time independent.

$$\delta\phi^x = C_1 r^{\left(\sqrt{4\lambda+(1+m)^2}-(1+m)\right)/2} + C_2 r^{\left(-\sqrt{4\lambda+(1+m)^2}-(1+m)\right)/2}$$

- one of the modes vanishes as $r \to 0$ provided λ is positive and it is possible to get stable attractors upon setting $C_2 = 0$.
- none of our examples (possibly model dependent) admit a critical point with $\lambda > 0$. such fluctuations are unstable.

microstate counting details

details microstate counting

CHL $\mathbb{Z}_n$ orbifold models[1] with $\mathcal{N} = 4$ supersymmetry in four dimensions.

- These are orbifolds of type II A string theory on $K3 \times T^2$, where the orbifold group G acts as a symplectic automorphism on $K3$ and as shifts on the torus T^2.
- This is dual to the heterotic string theory on T^6 via string-string duality.
- The action of G is determined on $\Gamma_{22,6} \cong \Gamma_{20,4} \oplus \Gamma_{2,2}$ and copied to the Heterotic side by identifying it with the Narain Lattice.
- The result is an asymmetric orbifold of a heterotic string on T^6.

[1] Chaudhuri et.al '95, Aspinwall '95

dihedral groups as symplectic automorphisms

- Moduli spaces that admit a dihedral symmetry $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ are compatible with both the twist and orbifold groups.
- If a elliptic $K3$ surface admitted both $\mathbb{Z}_2$ and $\mathbb{Z}_n, 3 \leq n \leq 6$ symmetries as symplectic automorphisms then the dihedral group acts as a symplectic automorphism on $K3$[2].

$$\mathcal{E}_{D_3} : y^2 = x^3 + (a_1\tau + a_0\tau^4 + a_1\tau^7)x + (b_2 + b_1\tau^3 + b_0\tau^6 + b_1\tau^9 + b_2\tau^{12})$$

$$\sigma_3 : (x, y, \tau) \mapsto (\zeta_3^2 x, \zeta_3^3 y, \zeta_3 \tau),$$
$$\varsigma_2 : (x, y, \tau) \mapsto (\frac{x}{\tau^4}, -\frac{y}{\tau^6}, \frac{1}{\tau})$$

- One can choose the charges of the theory Q to take values from the sublattices of $\Gamma_{19,3}$ that are invariant under Dihedral symmetries[3]. This is compatible with both $\mathbb{Z}_2$ twist and $\mathbb{Z}_n$ orbifold projections.

[2] A.Garbagnati 0904.1519

[3] Griess, Lam 0806.2753

dihedral groups and twisted partition functions

- The dihedral group of order $2n$ has the following presentation

$$D_n \cong \langle h, g | h^n = e, g^2 = e, ghg = h^{-1} \rangle$$

- The elements of $D_n = \{e, h, \ldots, h^{n-1}, g, gh, \ldots, gh^{n-1}\}$
- The group invariant projector of the $\mathbb{Z}_n$ subgroup has the following property:

$$g.P_{\mathbb{Z}_n} = \frac{1}{n} g \left(\sum_{j=0}^{n-1} h^j \right) = \frac{1}{n} \left(\sum_{j=0}^{n-1} h^j \right) = P_{\mathbb{Z}_n}.g$$

- g commutes with $P_{\mathbb{Z}_n}$ even though it doesn't commute with the individual elements.

Example: Z3

- The $\mathbb{Z}_3$ subgroup of D_3: $\mathbb{Z}_3 = \{e, h, h^2\}$ and $P_{\mathbb{Z}_3} = (e + h + h^2)/3$. The partition function for $\mathbb{Z}_3$ orbifolds including all twisted sectors

$$Z_{T/\mathbb{Z}_3} = P_{\mathbb{Z}_3} \underset{e}{\square} + P_{\mathbb{Z}_3} \underset{h}{\square} + P_{\mathbb{Z}_3} \underset{h^2}{\square}$$

- Twisting the partition function by $g \in \mathbb{Z}_2$ amounts to insertion of g in the trace,

$$\mathrm{Tr}_{\mathcal{H}_h}\big(g\ q^H\big) \equiv g \underset{h}{\square}$$

- For the g twisted partition function-contribution comes only from the untwisted sector of the orbifold CFT, inconsistent bc when $gh \neq hg$.

$$gX(\tau, \sigma+2\pi) = ghg^{-1}gX(\tau, \sigma)\,;\ hX(\tau+2\pi, \sigma) = hgh^{-1}hX(\tau, \sigma)$$

Hence, we are left to evaluate

$$Z^{\mathbb{Z}_2}_{T/\mathbb{Z}_3} = \frac{1}{3}\left(g \underset{e}{\square} + gh \underset{e}{\square} + gh^2 \underset{e}{\square}\right)$$

Orbifold action: heterotic description

- The action of the orbifold group element $h \in H \equiv \mathbb{Z}_n$

$$P \to R_h P + a_h; \quad P \in \Gamma_{22,6}$$

- $\forall R_h \in R_H$, R_H leaves $22 - k$ of the 22 left moving directions invariant.
- The action of the twist element $g \in G$ on $K3$ leaves 14 of the 22 2-cycles of $K3$ invariant, In the heterotic picture it exchanges the two E_8 components. It is not accompanied by shifts.
- The action of the orbifold and twist leaves the right movers invariant to preserve $\mathcal{N} = 4$ supersymmetry.
- Compatibility with the $\mathbb{Z}_2$ twist, and $\mathbb{Z}_n$ orbifold projection requires the charges Q to take values on a lattice[4] that is invariant under both the symmetries.

[4] Griess, Lam 0806.2753

Orbifold action: on oscillators and lattice

- The complex worldsheet co-ordinates X^j, $j = 1, 2, ..., k/2$ represent the planes of rotation. R_H is characterized by $k/2$ phases $\phi_j(h)$. The effect of the rotation R_H is to multiply the complex oscillators by phases.

$$\alpha^j_{-n} \to e^{2\pi i \phi_j(d)} \alpha^j_{-n} \quad ; \quad \bar{\alpha}^j_{-n} \to e^{-2\pi i \phi_j(d)} \bar{\alpha}^j_{-n}$$

- The Narain Lattice $\Gamma^{(22,6)}$ is embedded in a $22 + 6$ dimensional vector space V.
- The action of the entire group thus separates the vector space V into an invariant subspace $V_\perp$ and its orthogonal complement $V_\|$.
- The invariant sublattice $\Lambda_\perp$ and its orthogonal complement $\Lambda_\|$ are

$$\Lambda_\perp = \Gamma \bigcap V_\perp \quad ; \quad \Lambda_\| = \Gamma \bigcap V_\| .$$

BPS states and level matching

- Momenta in the compact directions take values on the Narain lattice $\Gamma^{(22,6)}$. The (left,right) components of the momentum vector are denoted as $\vec{P} = (\vec{P}_L, \vec{P}_R)$
- $Q = (\vec{Q}_L, \vec{Q}_R)$ to denotes the projection of $\vec{P}$ along $V_\perp$ and $P_\parallel = (\vec{P}_{\parallel L}, 0)$ the projection of $\vec{P}$ along $V_\parallel$.
- The BPS states are picked by keeping the rightmoving oscillators at the lowest eigenvalue allowed by GSO projection, i.e $N_R = 1$.

$$N_L - 1 + \frac{1}{2}\vec{P}^2_{\parallel L} = N$$

with $N = \frac{1}{2}(\vec{Q}_R^2 - \vec{Q}_L^2)$ and $\vec{P}_{\parallel L} = \vec{K}(Q) + \vec{p}$, where $\vec{p} \in \Lambda_\parallel$ and $\vec{K}(Q) \in V_\parallel$ is a constant vector that lies in the unit cell of $\Lambda_\parallel$.

Group invariant projection

- The counting of the number of $\mathbb{Z}_n$ invariant BPS states for a given charge Q is done by implementing the group invariant projection.

$$\frac{1}{n}\sum_{j=o}^{n-1} h^j \underset{e}{\Box}$$

- The contribution to the trace with a orbifold group element $h \in \mathbb{Z}_n$ inserted comes only from those $\vec{P}_{\parallel L}$ which are invariant under the action of h, i.e $\vec{P}_{\parallel L} \in V_\perp(h)$.
- When a group element h acts on the vacuum carrying momentum $\vec{P}$ it will produce a phase

$$e^{2\pi i \vec{a}_h.\vec{Q}} e^{-2\pi i \vec{a}_{hL}.(\vec{p}+\vec{K}(Q))}$$

- The twist g does not have shifts, and will not produce these phases.

Degeneracy

- The degeneracy of BPS states in untwisted sector carrying a charge Q is expressed as[5]

$$d(Q) = \frac{16}{|\mathbb{Z}_n|} \sum_{h \in \mathbb{Z}_n} \sum_{N_L=0}^{\infty} d^{osc}(N_L, h) e^{2\pi i \vec{a}_h . \vec{Q}} e^{-2\pi i \vec{a}_{hL} . \vec{K}(Q)} \sum_{\substack{\vec{p} \in \Lambda_{||} \\ \vec{p}+\vec{K}(Q) \in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL} . \vec{p}} \, \delta_{N_L - 1 + \frac{1}{2}(\vec{p}+\vec{K}(Q))^2, N}$$

  where $d^{osc}(N_L, h)$ is the number of ways one can construct oscillator level N_L from the 24 left-movers weighted by the action of h.

- Treating Q and $\hat{N} \equiv N$ as independent variables, the partition function,

$$\tilde{F}(Q, \mu) = \sum_{\hat{N}} F(Q, \hat{N}) e^{-\mu \hat{N}}$$

---

[5] (Ashoke Sen hep-th/0504005)

Partition function

- Explicitly, the partition function has the form,

$$\tilde{F}(Q,\mu) = \frac{16}{|\mathbb{Z}_n|} \sum_{h\in\mathbb{Z}_n} e^{2\pi i \vec{a}_h.\vec{Q}} e^{-2\pi i \vec{a}_{hL}.\vec{K}(Q)} \tilde{F}^{osc}(h,\mu) \tilde{F}^{lat}(Q,h,\mu)$$

  where, the oscillator and lattice contribution to the partition function are

$$\tilde{F}^{osc}(h,\mu) = \sum_{N_L=0}^{\infty} d^{osc}(N_L,h) e^{-\mu N_L} e^{\mu}$$

$$\tilde{F}^{lat}(Q,h,\mu) = \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL}.\vec{p}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2}$$

- The inverse of the partition function gives the degeneracy

$$F(Q,\tilde{N}) = \frac{1}{2\pi i} \int_{\epsilon-i\pi}^{\epsilon+i\pi} d\mu \; \tilde{F}(Q,\mu) \; e^{\mu\tilde{N}}$$

Oscillator contribution

$$\tilde{F}^{osc}(h,\mu) = q^{-1}\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right)^{24-k_h}\prod_{j=1}^{k_h/2}\left(\prod_{n=1}^{\infty}\frac{1}{1-e^{2\pi i\phi_j(h)}q^n}\frac{1}{1-e^{-2\pi i\phi_j(h)}q^n}\right)$$

- $\phi_j(h)$ and k_h in $\tilde{F}^{osc}(h,\mu)$ depend only on the order of the group element h.
- With a g insertion one evaluates the oscillator contribution for,

$$g\,\underset{e}{\square} + gh\,\underset{e}{\square} + gh^2\,\underset{e}{\square} + \ldots + gh^{n-1}\,\underset{e}{\square}$$

- The elements $g, gh, \ldots, gh^{n-1}$ are each of order 2 and have identical contributions.
- Since g exchanges the E_8 co-ordinates, the number of directions that are rotated $k_g = 8$ and non zero phases $\phi_j(g) = 1/2$

$$\tilde{F}^{osc}(g,\mu) = \frac{1}{\eta(\mu)^8\eta(2\mu)^8}$$

Lattice contribution

- Inclusion of twist: Since the charges are already g invariant, g has no further action on the lattice.
- The lattice contribution from a orbifold group element h is

$$\tilde{F}^{lat}(Q,h,\mu) = \sum_{\substack{\vec{p}\in\Lambda_{\|} \\ \vec{p}+\vec{K}(Q)\in V_{\perp}(h)}} e^{-2\pi i a_{\vec{h}L}.\vec{p}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2}.$$

- When h is identity $V_{\perp}(e) = V$. For any other h, we have $dim V_{\perp}(h) < dim(V)$. The dominant contribution would be from

$$\tilde{F}^{lat}(Q,e,\mu) \simeq \sum_{\vec{p}\in\Lambda_{\|}} e^{-\frac{1}{2}\mu(\vec{p}+\vec{K}(Q))^2} \equiv \Theta^{\|}_{\mathbb{Z}_n}$$

Result

- Combining the oscillator and the lattice contributions, the partition function for g twisted half-BPS states in CHL $\mathbb{Z}_n$ orbifolds is

$$\tilde{F}(Q,\mu) \simeq \frac{16}{|Z_n|} \frac{\Theta^{||}_{\mathbb{Z}_n}}{\eta(\mu)^8 \eta(2\mu)^8}$$

- The resulting modular form has lesser weight than the partition function for the untwisted half-BPS states as can be seen from the asymptotic limit $\mu \to 0$

$$\tilde{F}(\mu) \sim \frac{16}{|Z_n|} \frac{1}{Vol_{\Lambda_{||}}} e^{2\pi^2/\mu} \left(\frac{\mu}{2\pi}\right)^{8-\frac{k_{\mathbb{Z}_n}}{2}}$$

Group	$12 - \frac{k_{\mathbb{Z}_n}}{2}$	$8 - \frac{k_{\mathbb{Z}_n}}{2}$	$k_{\mathbb{Z}_n} = rank(\Lambda_{||})$
$\mathbb{Z}_3$	6	2	12
$\mathbb{Z}_4$	5	1	14
$\mathbb{Z}_5$	4	0	16
$\mathbb{Z}_6$	4	0	16