$2 \to 2$ scattering in supersymmetric matter Chern-Simons theories at large N

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Based on

- K.I, S.Jain, S.Mazumdar, S.Minwalla, V.Umesh, S.Yokoyama: Arxiv 1505.06571, *JHEP* **1510** (2015) 176

Related earlier work

- S.Jain, M.Mandlik, S.Minwalla, T.Takimi, S.Wadia, S.Yokoyama: Arxiv 1404.6373, *JHEP* **1504** (2015) 129

Plan of the talk

CS matter theories: Preliminaries

- Non-Abelian $U(N)$ gauge theories in 2+1 dimensions are rich.
- Yang-Mills + Chern-Simons action

$$\frac{i\kappa}{4\pi}\int \mathrm{Tr}\left(AdA+\frac{2}{3}A^3\right)-\frac{1}{4g_{YM}^2}\int d^3x\ \mathrm{Tr}\ F_{\mu\nu}^2$$

- Describes massive gluons with mass $\propto \kappa g_{YM}^2$.
- Low energies : pure Chern-Simons theory, topological.
- Chern-Simons gauge theory coupled to matter gives rise to interesting dynamics.

CS matter theories: Preliminaries

- Equations of motion for abelian theory with scalar matter of unit charge

$$\kappa \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 2\pi J^{\mu}$$

- Chern-Simons interaction ties $\frac{1}{\kappa}$ units of flux to the charged scalar.
- Aharonov-Bohm effect: Exchange of two unit charge particles result in a phase $\frac{\pi}{\kappa}$.
- Chern-Simons gauge field interacting with matter turns them into anyons with anyonic phase $\pi\nu = \frac{\pi}{\kappa}$.
- non-abelian case: for eg exchange of $U(N)$ matter quanta R_1 and R_2 gives a phase operator $(R_1 \times R_2 = \sum_m R_m)$

$$\nu_m = \frac{T_{R_1}.T_{R_2}}{\kappa} = \frac{C_2(R_1) + C_2(R_2) - C_2(R_m)}{2\kappa}$$

CS matter theories: Preliminaries

- Lots of activity in $U(N)$ Chern-Simons theories with fundamental matter.
- Motivations: AdS/CFT, Vasiliev duality, limit of ABJ theory.
- Solvability: The theory is solvable in large N limit.
- 2+1 d bosonization: Bose-fermi duality even in the absence of supersymmetry.

Level rank duality in CS matter theory

- $U(N_B)$ Chern-Simons theory coupled to fundamental boson [Giombi, Minwalla, Prakash, Trivedi, Wadia]

$$S = \int d^3x \Big(i\epsilon^{\mu\nu\rho} \frac{\kappa_B}{4\pi} Tr(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho)$$
$$D_\mu \bar{\phi} D^\mu \phi + \sigma \bar{\phi} \phi + N_B \frac{m_B^2}{b_4} \sigma - N_B \frac{\sigma^2}{2 b_4} \Big)$$

- Wilson-Fisher limit

$$b_4 \to \infty \ , \ m_B \to \infty \ , \ 4\pi \frac{m_B^2}{b_4} = fixed$$

- $U(N_F)$ Chern-Simons theory coupled to fundamental fermion

$$S = \int d^3x \Big(i\epsilon^{\mu\nu\rho} \frac{\kappa_F}{4\pi} Tr(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho)$$
$$+ \bar{\psi} \gamma^\mu D_\mu \psi + m_F \bar{\psi} \psi \Big)$$

Level rank duality in CS matter theory

- Statement of duality

  $U(N_B)$ CS+fundamental boson at Wilson Fisher limit

  $\Leftarrow$ dual $\Rightarrow$

  $U(N_F)$ CS+fundamental fermion

- under the duality map

$$\begin{aligned}\kappa_F =& -\kappa_B\\ N_F =&|\kappa_B| - N_B\\ \lambda_B =&\lambda_F - sgn(\lambda_F)\\ m_F =& -m_B^{cri}\lambda_B\end{aligned}$$

- with condition

$$\lambda_F m_F > 0$$

Evidence for duality

- Spectrum of single trace operators and three point functions on both sides match.
  [Giombi, Minwalla, Prakash, Trivedi, Wadia] ,
  [Aharony, Gur-Ari, Yacoby], [Maldacena, Zhiboedov]

- Thermal partition functions on both sides match.
  [Jain, Trivedi, Wadia, Yokoyama] ,
  [Aharony, Giombi, Gur-Ari, Maldacena,Yacoby], [Takimi]

- Duality follows from a deformation of Giveon-Kutasov duality in supersymmetric theory.
  [Jain,Minwalla,Yokoyama], [Gur-Ari, Yacoby]

- Most recent: $2 \to 2$ S matrices in C.S+bosonic and C.S+fermionic theories map to one another.

Conjectured Duality for susy matter CS

- Jain, Minwalla, Yokoyama conjectured that $\mathcal{N} = 1, 2$ supersymmetric matter coupled Chern-Simons theories are self dual

$$Theory(\lambda', w', m') \Longleftrightarrow Theory(\lambda, w, m)$$

- under the map

$$\lambda' = \lambda - \mathrm{Sgn}(\lambda)\ ,\ \ w' = \frac{3-w}{1+w} \quad m_0' = \frac{-2m_0}{1+w}$$
$$N' = |\kappa| - N + 1\ ,\ \ \kappa' = -\kappa$$

- with a pole mass

$$m = \frac{2m_0}{2 + (-1+w)\lambda\ \mathrm{Sgn(m)}}$$

- $m' = -m$ under duality and $\lambda m(m_0, w) \geq 0$

Scattering in CS matter theories

- The statement of duality is actually a statement of bosonization of fermions.
- Bosonic and fermionic S matrices related by duality is equivalent to a bosonization map.
- Such a mapping is possible in 2+1 dimensions: Dirac equation uniquely determines the polarization spinors as a function of the momentum.
- In large N limit, only planar diagrams contribute. Possible to get exact results as a function of 't Hooft coupling λ.
- It has been shown that the S matrices for $2 \to 2$ processes in the CS+bosonic theory map to the CS+fermionic theory under duality.
  [Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama]

Scattering in CS matter theories: Peculiarities

- Scattering results consistent with duality.
- In singlet channel (particle-Antiparticle) S matrices obtained from naive crossing symmetry rules are inconsistent with unitarity and have incorrect non-relativistic limit.
- Consistency with unitarity requires
  - Delta function term at forward scattering.
  - Modified crossing symmetry rules.
- Conjecture: Singlet channel S matrices have the form

$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\text{naive}}(s,\theta)$$

- $\mathcal{T}^{S;\text{naive}}$ is the matrix obtained from naive analytic continuation of particle-particle scattering.

Nature of the conjecture: Delta function and modified crossing rules

$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\text{naive}}(s,\theta)$$

- The conjectured S matrix has a non-analytic $\delta(\theta)$ piece.
- delta function is already known to be necessary to unitarize non-relativistic Aharanov-Bohm scattering [Ruijsenaars; Bak,Jackiw,Pi].
- $\cos(\pi\lambda)$ is due to the interference of the Aharonov-Bohm phases of the wave packets.
- modified crossing factor $\frac{\sin(\pi\lambda)}{\pi\lambda}$ is the expectation value of circular wilson loop on S^3 in pure Chern-Simons theory [Witten].

Unitarity and anyonic behavior in Singlet Channel

- Unitarity $i(T^\dagger - T) = TT^\dagger$: non-trivial only for singlet channel in the large N limit.

$$T_{Asym} \sim T_{Sym} \sim T_{Adj} \sim O\left(\frac{1}{N}\right) \ , \ T_{sing} \sim O(1)$$

- The anyonic phase operator $\nu_m = \frac{C_2(R_1)+C_2(R_2)-C_2(R_m)}{2\kappa}$

$$\nu_{Asym} \sim \nu_{Sym} \sim \nu_{Adj} \sim O\left(\frac{1}{N}\right) \ , \nu_{Sing} \sim O(\lambda)$$

- symmetric/antisymmetric channels and adjoint channel are effectively non-anyonic in large N.
- Particle-Antiparticle scattering in the singlet channel is effectively anyonic - usual crossing rules fail unitarity.
- Remedy: delta function and modified crossing rules.

Universality and tests

- delta function and modified crossing rules - appear to be universal
- Tests:
  - Unitarity of the S matrix
  - Bose-Fermi duality
  - Non-relativistic limit gives Aharanov-Bohm
- All the tests have been explicitly verified for
  - $U(N)$ Chern-Simons coupled to fundamental bosons
  - $U(N)$ Chern-Simons coupled to fundamental fermions
  - $\mathcal{N} = 1, 2$ Supersymmetric $U(N)$ Chern-Simons matter theories
- Further investigations ongoing for $\mathcal{N} = 3, 4, 5, 6$ susy CS matter theories [K.I, S.Jain, S.Minwalla, S.Yokoyama]

Our work

- Test the conjecture in the most general renormalizable supersymmetric $\mathcal{N} = 1$ Chern-Simons matter theory.
- Superspace - manifest supersymmetry
- Work in large N - only planar diagrams .
- Compute off-shell four point correlator,take on-shell limit and extract the S matrix.
- Provide evidence for duality and subject the conjecture to stringent unitarity test.

Main results

- Results in perfect agreement with duality.
- Unitarity requires the delta function at forward scattering and crossing symmetry rules modified by exactly the same way as conjectured in [Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama]
- Substantial evidence for universality of the conjecture.

Bonus results

- Results of $\mathcal{N} = 2$ theory obtained at special value of quartic scalar coupling.
- Non-renormalization of pole mass and vertex for $\mathcal{N} = 2$ theory - good things happen with more susy .
- $\mathcal{N} = 1$ S matrix has interesting pole structure, with vanishing pole mass on a self-dual codimension one surface in the space of couplings.

Onshell supersymmetry and the S matrix

Supersymmetric scattering

- $2 \to 2$ scattering amplitude: transition between free incoming and free outgoing onshell particles.

- Initial and final states of Φ_i are effectively subject to free equations of motion

$$\left(D^2 + m\right)\Phi = 0$$

- Solution

$$\Phi(x,\theta) = \int \frac{d^2p}{\sqrt{2p^0}(2\pi)^2}\Bigg[\Big(a(\mathbf{p})(1+m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big)e^{ip.x} + \Big(a^{c\dagger}(\mathbf{p})(1+m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})\Big)e^{-ip.x}\Bigg]$$

- action of off-shell supersymmetry operator on onshell superfields $[Q_\alpha^{off}, \Phi] = Q_\alpha^{off}\Phi = i\left(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\partial_{\beta\alpha}\right)\Phi$

$$-iQ_\alpha^{on} = u_\alpha(\mathbf{p}_i)\left(\alpha\partial_a + \alpha^c\partial_{a^c}\right) + u^*_\alpha(\mathbf{p}_i)\left(a\partial_\alpha + a^c\partial_{\alpha^c}\right) - v^*_\alpha(\mathbf{p}_i)\Big(a^\dagger\partial_{\alpha^\dagger} + (a^c)^\dagger\partial_{(\alpha^c)^\dagger}\Big) + v_\alpha(\mathbf{p}_i)\Big(\alpha^\dagger\partial_{a^\dagger} + (\alpha^c)^\dagger\partial_{(a^c)^\dagger}\Big)$$

Onshell superspace

- Introduce creation and annihilation operator superfields

$$A_i(\mathbf{p}) = a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i \ ,$$
$$A_i^\dagger(\mathbf{p}) = a_i^\dagger(\mathbf{p}) + \theta_i\alpha_i^\dagger(\mathbf{p}) \ .$$

- Action of supersymmetry operator

$$[Q_\alpha^{on}, A_i(\mathbf{p}_i, \theta_i)] = Q_\alpha^1 A_i(\mathbf{p}_i, \theta_i)$$
$$[Q_\alpha^{on}, A_i^\dagger(\mathbf{p}_i, \theta_i)] = Q_\alpha^2 A_i^\dagger(\mathbf{p}_i, \theta_i)$$

$$Q_\beta^1 = i\left(-u_\beta(\mathbf{p})\overrightarrow{\frac{\partial}{\partial\theta}} - v_\beta(\mathbf{p})\theta\right)$$
$$Q_\beta^2 = i\left(v_\beta(\mathbf{p})\overrightarrow{\frac{\partial}{\partial\theta}} - u_\beta(\mathbf{p})\theta\right) \ .$$

S matrix in onshell superspace

- $2 \to 2$ S matrix: $p_1 + p_2 \to p_3 + p_4$

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\sqrt{(2p_1^0)(2p_2^0)(2p_3^0)(2p_4^0)} = \langle 0|A_4(\mathbf{p}_4, \theta_4)A_3(\mathbf{p}_3, \theta_3)UA_2^\dagger(\mathbf{p}_2, \theta_2)A_1^\dagger(\mathbf{p}_1, \theta_1)|0\rangle$$

- Supersymmetric ward identity for the S matrix

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1, \theta_1) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_2, \theta_2) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_3, \theta_3) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_4, \theta_4)\Big) S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0$$

S matrix in onshell superspace

- S matrix solution (in-states: p_1, p_2 ,out-states p_3, p_4) is determined in terms of two functions $\mathcal{S}_B$ and $\mathcal{S}_F$ of momenta,couplings and mass.

$$
\begin{aligned}
&S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \mathcal{S}_B + \mathcal{S}_F \ \theta_1\theta_2\theta_3\theta_4 + \\
&\left(\frac{1}{2} C_{12}\mathcal{S}_B - \frac{1}{2} C^*_{34}\mathcal{S}_F\right) \ \theta_1\theta_2 + \left(\frac{1}{2} C_{13}\mathcal{S}_B - \frac{1}{2} C^*_{24}\mathcal{S}_F\right) \ \theta_1\theta_3 \\
&+ \left(\frac{1}{2} C_{14}\mathcal{S}_B + \frac{1}{2} C^*_{23}\mathcal{S}_F\right) \ \theta_1\theta_4 + \left(\frac{1}{2} C_{23}\mathcal{S}_B + \frac{1}{2} C^*_{14}\mathcal{S}_F\right) \ \theta_2\theta_3 \\
&+ \left(\frac{1}{2} C_{24}\mathcal{S}_B - \frac{1}{2} C^*_{13}\mathcal{S}_F\right) \ \theta_2\theta_4 + \left(\frac{1}{2} C_{34}\mathcal{S}_B - \frac{1}{2} C^*_{12}\mathcal{S}_F\right) \ \theta_3\theta_4
\end{aligned}
$$

- No θ term: four boson scattering, four θ term : four fermion scattering.

- All other processes (two boson to two fermion etc) determined completely in terms of the two independent functions $\mathcal{S}_B$ and $\mathcal{S}_F$.

S matrix in onshell superspace

$$
\begin{aligned}
\frac{1}{2}C_{12} &= -\frac{1}{4m}v^*(\mathbf{p}_1)v^*(\mathbf{p}_2) & \frac{1}{2}C_{23} &= -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_3)\\
\frac{1}{2}C_{13} &= -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_3) & \frac{1}{2}C_{24} &= -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_4)\\
\frac{1}{2}C_{14} &= -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_4) & \frac{1}{2}C_{34} &= -\frac{1}{4m}u^*(\mathbf{p}_3)u^*(\mathbf{p}_4)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}C^*_{12} &= \frac{1}{4m}v(\mathbf{p}_1)v(\mathbf{p}_2) & \frac{1}{2}C^*_{23} &= \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_3)\\
\frac{1}{2}C^*_{13} &= \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_3) & \frac{1}{2}C^*_{24} &= \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_4)\\
\frac{1}{2}C^*_{14} &= \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_4) & \frac{1}{2}C^*_{34} &= \frac{1}{4m}u(\mathbf{p}_3)u(\mathbf{p}_4)
\end{aligned}
$$

On onshell supersymmetry for $\mathcal{N} = 2$ S matrix

- The $\mathcal{N} = 2$ S matrix is already $\mathcal{N} = 1$ supersymmetric.

- obeys additional constraint from $\mathcal{N} = 2$ supersymmetry.

- Action of off-shell Q_α, $\bar{Q}_\alpha$ on on-shell chiral/anti-chiral superfields determines action of on-shell $\mathcal{N} = 2$ supersymmetry.

- conditions for on-shell $\mathcal{N} = 2$ susy of S matrix

$$\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i, \theta_i) + \bar{Q}^i_\alpha(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0$$

$$\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i, \theta_i) - \bar{Q}^i_\alpha(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0$$

On onshell supersymmetry for $\mathcal{N} = 2$ S matrix

- Additional constraint relates the functions $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B \left(C_{13} u_\alpha(\mathbf{p}_3) + C_{14} u_\alpha(\mathbf{p}_4) + C_{12} v_\alpha(\mathbf{p}_2) + v^*_\alpha(\mathbf{p}_1) \right)$$
$$= \mathcal{S}_F \left(C^*_{24} u_\alpha(\mathbf{p}_3) - C^*_{23} u_\alpha(\mathbf{p}_4) + C^*_{34} v_\alpha(\mathbf{p}_2) \right)$$

- $\mathcal{N} = 2$ S matrix is completely specified by one function.

- eg: $p_1 = p + q, p_2 = -k - q, p_3 = p, p_4 = -k$

$$\mathcal{S}_B = \mathcal{S}_F \frac{-2m(k-p)_- + iq_3(k+p)_-}{2m(k-p)_- + iq_3(k+p)_-} \ .$$

- Already $\mathcal{N} = 2$ supersymmetry is quite constraining.

- Expect all the component S matrices in higher susy cases to be determined by one function.

Exact computation of all orders S matrix

Theory

- General renormalizable $\mathcal{N}=1$ theory coupled to single fundamental matter multiplet Φ

$$\mathcal{S}_{\mathcal{N}=1} = -\int d^3x d^2\theta \Big[\frac{\kappa}{2\pi} Tr\Big(-\frac{1}{4} D_\alpha \Gamma^\beta D_\beta \Gamma^\alpha \\ -\frac{1}{6} D^\alpha \Gamma^\beta \{\Gamma_\alpha, \Gamma_\beta\} - \frac{1}{24}\{\Gamma^\alpha, \Gamma^\beta\}\{\Gamma_\alpha, \Gamma_\beta\}\Big) \\ -\frac{1}{2}(D^\alpha \bar{\Phi} + i\bar{\Phi}\Gamma^\alpha)(D_\alpha \Phi - i\Gamma_\alpha \Phi) \\ + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \Big]$$

- Φ : complex superfield, Γ_α: real superfield

$$\Phi = \phi + \theta\psi - \theta^2 F \ , \bar{\Phi} = \bar{\phi} + \theta\bar{\psi} - \theta^2 \bar{F} \ ,$$

$$\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta) \ .$$

- Integer parameters N, κ, matter coupling constant w, 't Hooft coupling $\lambda = \frac{N}{\kappa}$.

Supersymmetric light cone gauge

- Supersymmetric generalisation of light cone gauge

$$\Gamma_- = 0 \;\Rightarrow\; A_- = A_1 + iA_2 = 0$$

- Gauge self interactions vanish

$$\begin{aligned} S = -\int d^3x d^2\theta \Big[& -\frac{\kappa}{8\pi} Tr(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2} D^\alpha \bar{\Phi} D_\alpha \Phi \\ & - \frac{i}{2}\Gamma^-(\bar{\Phi} D_- \Phi - D_- \bar{\Phi}\Phi) \\ & + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \Big] \end{aligned}$$

- Susy light cone gauge maintains manifest supersymmetry.

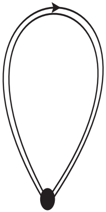
Strategy for computing S matrix

- Use off-shell supersymmetry to constrain the structure of two point and four-point functions in superspace.
- Use these structures to set up a Dyson-Schwinger series for exact propagator and exact off-shell four point function.
- work only with diagrams that contribute to leading order in the large N limit.
- read off S matrices from off-shell four point function by taking on-shell limits.

Exact propagator in large N limit

- Integral equation for self-energy

$\Sigma(p,\theta_1,\theta_2)$ =



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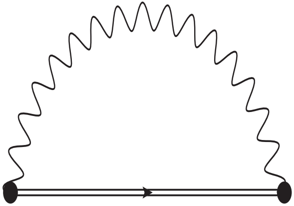


$$
\begin{aligned}
\Sigma(p,\theta_1,\theta_2) = {}& 2\pi\lambda w \int \frac{d^3 r}{(2\pi)^3} \delta^2(\theta_1-\theta_2) P(r,\theta_1,\theta_2) \\
& - 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} D_-^{\theta_2,-p} D_-^{\theta_1,p} \left(\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} P(r,\theta_1,\theta_2) \right) \\
& + 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} \frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} D_-^{\theta_1,r} D_-^{\theta_2,-r} P(r,\theta_1,\theta_2)
\end{aligned}
$$

Exact propagator in large N limit

- Solution to exact propagator is extremely simple

$$P(p, \theta_1, \theta_2) = \frac{D^2 - m}{p^2 + m^2} \delta^2(\theta_1 - \theta_2)$$

- Same structure as the bare propagator with m_0 replaced by m .

- m is the pole mass

$$m = \frac{2m_0}{2 + (-1 + w)\lambda \, \mathrm{Sgn(m)}}$$

is duality invariant, agrees with the pole mass computed by Jain, Minwalla, Yokoyama

- Bonus: In the $\mathcal{N} = 2$ limit ($w = 1$), no mass renormalization for $\mathcal{N} = 2$ theory !

An integral equation for the four point function

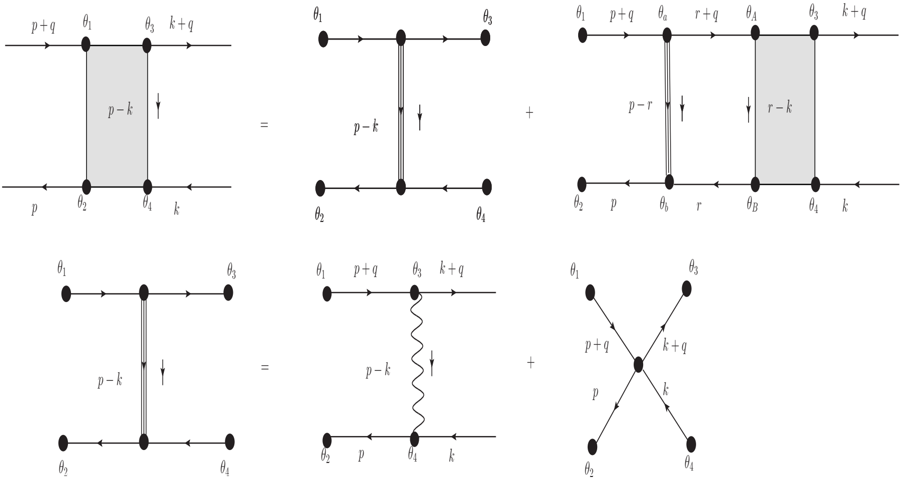


Figure: The diagrams in the first line pictorially represents the Schwinger-Dyson equation for offshell four point function. The second line represents the tree level contributions from the gauge superfield interaction and the quartic interactions.

An integral equation for the four point function

- Schematically the integral equations are of the form

$$V(\theta_i, p_i) = V_0(\theta_i, p_i) + \int \frac{d^3 r}{(2\pi)^3} d^2\theta'_j V_0(\theta_i, \theta'_j, p_i, r) P(\theta'_j, p_i + r) P(\theta'_j, r) V(\theta'_j, \theta_i, p_i)$$

- solved integral equations exactly in large N limit for arbitrary values of the t'Hooft coupling λ and determined the offshell four point function in the kinematic regime $q_\pm = 0$.

- Onshell limit directly gives the S matrix for T (adjoint), U_d (symm) and U_e (Asymm) channels of scattering (q_μ is momentum transfer).

- Impossible to extract S (singlet) channel S matrix since q_μ is center of mass energy (cannot be spacelike).

S matrix in T , U_d, U_e channels of scattering

- *S* matrix: onshell limit of off-shell four point correlator

$$\mathcal{T}_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_B(q,\lambda) \ ,$$
$$\mathcal{T}_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_F(q,\lambda) \ ,$$

$$J_B(q,\lambda) = \frac{4\pi q}{\kappa}\frac{N_1N_2+M_1}{D_1D_2} \ ,$$
$$J_F(q,\lambda) = \frac{4\pi q}{\kappa}\frac{N_1N_2+M_2}{D_1D_2} \ ,$$

S matrix in T , U_d, U_e channels of scattering

$$N_1 = \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (w-1)(2m+iq) + (w-1)(2m-iq) \right) \; ,$$

$$N_2 = \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (q(w+3) + 2im(w-1)) + (q(w+3) - 2im(w-1)) \right) \; ,$$

$$M_1 = -\,8mq((w+3)(w-1) - 4w) \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} \; ,$$

$$M_2 = -\,8mq(1+w)^2 \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} \; ,$$

$$D_1 = \left(i \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (w-1)(2m+iq) - 2im(w-1) + q(w+3) \right) \; ,$$

$$D_2 = \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (-q(w+3) - 2im(w-1)) + (w-1)(q+2im) \right) \; .$$

Bonus: S matrix in T , U_d, U_e channels for $\mathcal{N}=2$ theory

- Remarkable simplification in the $\mathcal{N}=2$ limit (w=1)

$$\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa} ,$$
$$\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa}$$

- All orders S matrix is just tree level - no loop corrections - non renormalization.

Duality invariance of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ S matrices

- Under the duality transformation

$$w' = \frac{3 - w}{w + 1}, \lambda' = \lambda - \mathrm{sgn}(\lambda), m' = -m, \kappa' = -\kappa$$

$$J_B(q, \kappa', \lambda', w', m') = -J_F(q, \kappa, \lambda, w, m) \ ,$$
$$J_F(q, \kappa', \lambda', w', m') = -J_B(q, \kappa, \lambda, w, m) \ .$$

- Duality maps the purely bosonic and purely fermionic S matrices into one another upto overall phase.
- Concrete evidence for duality.
- Supersymmetric ward identity guarantees the duality invariance of all other processes.

S matrix in (singlet) S channel

- We cannot extract the S channel S matrix directly because of kinematic restriction $q_{\pm} = 0$.
- Usual rules of crossing symmetry in quantum field theory predict particle - anti particle scattering from particle particle scattering or vice-versa
- Naive analytic continuation gives a non-unitary S matrix in the S channel as observed in earlier work.
- Any analytic continuation cannot produce the non-analytic delta function piece required for unitarization.
- Remedy: Modify crossing symmetry rules as conjectured in [Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama]

Conjectured S matrix in S channel $\mathcal{N} = 1$ theory

- Conjectured S matrix for the $\mathcal{N} = 1$ theory in center of mass frame

$$\mathcal{S}_B^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right) \ ,$$

$$\mathcal{S}_F^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right) .$$

$$J_B(\sqrt{s},\lambda) = -4\pi i\lambda\sqrt{s}\frac{N_1 N_2 + M_1}{D_1 D_2} \ ,$$

$$J_F(\sqrt{s},\lambda) = -4\pi i\lambda\sqrt{s}\frac{N_1 N_2 + M_2}{D_1 D_2}$$

Conjectured S matrix in S channel $\mathcal{N}=1$ theory

$$
\begin{aligned}
N_1 &= \left((w-1)(2m+\sqrt{s})+(w-1)(2m-\sqrt{s})e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\
N_2 &= \left((-i\sqrt{s}(w+3)+2im(w-1))+(-i\sqrt{s}(w+3)-2im(w-1))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\
M_1 &= 8mi\sqrt{s}((w+3)(w-1)-4w)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \ , \\
M_2 &= 8mi\sqrt{s}(1+w)^2e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \ , \\
D_1 &= \left(i(w-1)(2m+\sqrt{s})-(2im(w-1)+i\sqrt{s}(w+3))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\
D_2 &= \left((\sqrt{s}(w+3)-2im(w-1))+(w-1)(-i\sqrt{s}+2im)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right)
\end{aligned}
$$

Straightforward non-relativistic limit of the $\mathcal{N} = 1$ S matrix

- Non-rel limit: $\sqrt{s} \to 2m$ with all other parameters held fixed.

$$\mathcal{T}_B^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)-1\right) \ ,$$
$$\mathcal{T}_F^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)+1\right) \ .$$

- conjectured S channel S matrix has simple non-relativistic limit leading to known Aharonov-Bohm result.

- Surprisingly this result is also same as the $\mathcal{N} = 2$ S channel S matrix.

- Presumably supersymmetry enhancement in non-relativistic limit.

Unitarity

Unitarity equation

- Define on-shell superfield $S^\dagger$ as

$$S^\dagger(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = S^*(\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4,\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2)$$

- Supersymmetric ward identity for $S^\dagger$ implies $S^\dagger$ is supersymmetric if and only if S is supersymmetric.

- The supersymmetric unitarity equation is

$$(S \star S^\dagger - I) = 0$$

- Recall that the superfield expansion for S is completely specified by $\mathcal{S}_B$ and $\mathcal{S}_F$.

- Sufficient to check the LHS for no θ and four θ terms.

- Supersymmetric ward identity guarantees the rest of the terms will obey the unitarity equations.

Unitarity equations for T, U_d and U_e channels

- Writing $\mathcal{S}_B = I + i\mathcal{T}_B$, $\mathcal{S}_F = I + i\mathcal{T}_f$

- The S matrices in the T, U_d and U_e channels are all $O(1/N)$ - unitarity equation is linear

$$\mathcal{T}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \mathcal{T}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_1, \mathbf{p}_2) \ ,$$
$$\mathcal{T}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \mathcal{T}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_1, \mathbf{p}_2)$$

- Linearity: No branch cuts in the physical domain of scattering in these channels.

- Explicitly checked that unitarity conditions are obeyed using

$$J_B(q,\lambda) = J_B^*(-q,\lambda) \ , \ J_F(q,\lambda) = J_F^*(-q,\lambda)$$

- The S matrix in the S channel is $O(1)$ - the unitarity conditions are non-linear

Product of S matrices

- General multiplication rule for two S matrices

$$S_1 \star S_2 \equiv \int d\Gamma S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2)$$
$$\exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2)$$
$$S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)$$

$$d\Gamma = \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\frac{d^2k_1}{2k_1^0(2\pi)^2}\frac{d^2k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4$$

- supersymmetry invariant Identity operator

$$I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \exp(\theta_1\theta_3 + \theta_2\theta_4) I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$$
$$I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)$$

- Multiplication rule with Identity operator

$$S \star I = I \star S = S$$

- More generally product of two supersymmetric S matrices is supersymmetric.

Unitarity equations in the S channel

- No θ_i term and four θ_i terms

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(- Y(s)(\mathcal{T}_B^S(s,\theta) - \mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta)) - \mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^{S*}(s,-\alpha) - \mathcal{T}_B^S(s,\alpha))$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta) - \mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta)) - \mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_F^S(s,\alpha) - \mathcal{T}_F^{S*}(s,-\alpha))$$

- Under duality $\mathcal{T}_B \to \mathcal{T}_F$ and vice versa; both the equations map to one another.

- Unitarity conditions are compatible with duality.

Unitarity equations in the S channel

- Consider the general structure ($T(\theta) = i\cot(\theta/2)$.)

$$\mathcal{T}_B^S = H_B\,T(\theta) + W_B - iW_2\delta(\theta)\ ,\ \mathcal{T}_F^S = H_F\,T(\theta) + W_F - iW_2\delta(\theta)\ ,$$

- first unitarity equation

$$H_B - H_B^* = \frac{1}{8\pi\sqrt{s}}(W_2H_B^* - H_BW_2^*)\ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2W_2^* + 4\pi^2H_BH_B^*)\ ,$$

$$W_B - W_B^* = \frac{1}{8\pi\sqrt{s}}(W_2W_B^* - W_2^*W_B) - \frac{i}{4\sqrt{s}}(H_BH_B^* - W_BW_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

- Second unitarity equation

$$H_F - H_F^* = \frac{1}{8\pi\sqrt{s}}(W_2H_F^* - H_FW_2^*)\ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2W_2^* + 4\pi^2H_FH_F^*)\ ,$$

$$W_F - W_F^* = \frac{1}{8\pi\sqrt{s}}(W_2W_F^* - W_2^*W_F) - \frac{i}{4\sqrt{s}}(H_FH_F^* - W_FW_F^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

Unitarity equation in the S channel

- Unitarity equations are verified using

$$H_B = H_F = 4\sqrt{s}\sin(\pi\lambda),\ W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda) - 1),\ T(\theta) = i\cot(\theta/2)$$

$$W_B = J_B(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda}\ ,$$
$$W_F = J_F(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda}\ .$$

- Algebraic-miracle: Non-linear unitarity equations obeyed by very complicated functions.

- unitarity is an extremely sensitive test [1] .

- Important to note that the crossing symmetry rules have to be modified exactly as conjectured in [Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama], else the unitarity test fails.

[1] Tagline: one sign is doom

Unitarity in the S channel -$\mathcal{N}=2$ case

- The $\mathcal{N}=2$ T matrix is tree level exact in T, U channels.

- Naive crossing symmetry would imply the same for S channel, unitarity equation $i(T^\dagger - T) = TT^\dagger$ would never be obeyed (LHS would be zero).

- modified crossing rules resolve this puzzle:

$$\mathcal{T}_B^{S;\mathcal{N}=2}(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) - 8m) \ ,$$
$$\mathcal{T}_F^{S;\mathcal{N}=2}(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) + 8m).$$

- Non-analytic piece makes $\mathcal{T}_B, \mathcal{T}_F$ not Hermitian, both LHS and RHS are non-zero and non-linear unitarity equation is obeyed.

Pole structure of the S matrix

Pole structure of the singlet channel S matrix

- Both bosonic and fermionic S matrices have a pole at threshold ($s = 4m^2$) for $w \leq -1$. For $w \leq -1 + \epsilon$ the pole is close to threshold.

- As w is decreased further and as it hits a critical value $w = w_c$ the pole becomes massless!

$$w = w_c(\lambda) = 1 - \frac{2}{|\lambda|}$$

- As w is further decreased and as $w \to -\infty$ the pole approaches threshold once again.

- To summarize, a one parameter tuning of the superpotential interaction parameter w - sufficient to produce massless bound states in a massive theory.

- w can be scaled to w_c - possible decoupled QFT description of light states.

Summary

Summary

- Computations and conjectures for the all orders $2 \to 2$ S matrix in the general renormalizable $\mathcal{N} = 1$ Chern-Simons matter theory with a single fundamental matter multiplet.
- Used supersymmetric ward identity to derive conditions and constraints on off-shell correlators,on-shell S matrices and derive unitarity conditions.
- Computed exact offshell four point correlators in the large N limit in kinematic regime $q_{\pm} = 0$.
- Obtained S matrices by taking onshell limit of offshell four point correlator.
- Conjectured S matrix in the singlet channel.

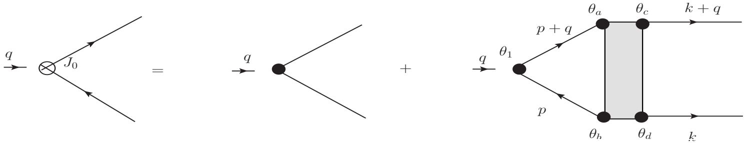
Summary

- Results are consistent with duality.
- Results are consistent with unitarity if and only if we assume that the usual results of crossing symmetry are modified in precisely the manner proposed in [Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama].
- Non-relativistic limit of the S matrix reproduces the known Aharonov-Bohm result.
- The S channel S matrix has an interesting analytic structure. In a certain range of superpotential parameters the S matrix has a bound state pole.
- A one parameter tuning of superpotential parameters can be used to set the pole mass to zero.

Future outlook

- $\mathcal{N} = 2$ S matrices are tree level exact in non-anyonic channels and depend on λ very simple way in the anyonic channel - can it reproduced from general principles and $\mathcal{N} = 2$ supersymmetry?
- Generalisation to higher supersymmetry - mass deformed $\mathcal{N} = 3, 4, 5$, and mass deformed $\mathcal{N} = 6$ ABJ theory - in progress [K.I, S.Jain, S.Minwalla, S. Yokoyama]
- decoupled gapless sector: effective field theory for the massless bound states of the S matrix.
- Four point correlator: useful in computation of 2,3,4 point functions of gauge invariant currents - explicit computation in $\mathcal{N} = 2$ theory?, possible $\mathcal{N} = 2$ generalisation of Maldacena-Zhiboedov solutions - in progress [K.I, S.Jain, P.Nayak]

Current-Current correlators in $\mathcal{N} = 2$ theory



- two point J_0 correlator

$$\langle J_0(\theta_1, q) J_0(\theta_2, -q)\rangle = \frac{N}{8\pi|q|\lambda} \exp(-\theta_1^\alpha \theta_2^\beta q_{\alpha\beta}) \Big(\sin(\pi\lambda) + |q|(1-\cos(\pi\lambda))\delta^2(\theta_1 - \theta_2) \Big)$$

- three point J_0 correlator

$$\begin{aligned}\langle J_0(\theta_1, q) J_0(\theta_1', q') J_0(\theta_1'', -q-q')\rangle = & \left(\frac{N}{72\ q_3 q_3'(q_3+q_3')} \frac{\sin(\pi\lambda)}{\pi\lambda} \right) \Big[-9\cos(\pi\lambda) \\ & + 9i\sin(\pi\lambda)\left(q_3 X^-_{11''} X^+_{11''} + q_3' X^-_{1'1''} X^+_{1'1''}\right) \\ & + 3\cos(\pi\lambda)\,(q_3' - q_3)\left(X^-_{11''} X^+_{1'1''} - X^-_{1'1''} X^+_{11''}\right) \\ & - \cos(\pi\lambda)\,(q_3^2 + 7 q_3 q_3' + q_3'^2)\ X^-_{11''} X^+_{11''} X^-_{1'1''} X^+_{1'1''} \Big] \\ & \times e^{\frac{1}{3} X\cdot(q\cdot X_{11''} + q'\cdot X_{1'1''})}\end{aligned}$$

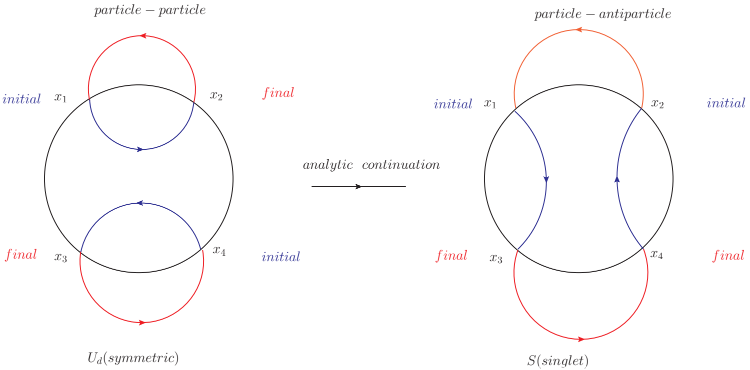
[K.I, S.Jain, P.Nayak]

Open questions -

- Rigorous proof of delta function and modified crossing rules, generalisation to finite N and κ.
- From perturbative pov modified crossing rules could be related to IR divergences.
- IR divergences can be summed up and exponentiated [Grammer,Yennie; Bern,Dickson,Smirnov]
- Modified crossing factor $\frac{\sin(\pi\lambda)}{\pi\lambda}$ is identical to the expectation value of circular wilson loop in pure Chern-Simons theory on S^3.
- To explore: Path integral derivation of Witten's result, crossing and fusion rules in RCFT's.

Modified crossing - A heuristic explanation

- attach Wilson lines to make correlator gauge invariant.



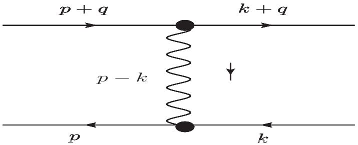
$$T_{U_d} W_{U_d} \to T_S W_S$$

$$\frac{W_{U_d}}{W_S} = \frac{\oint \text{with 2 circular Wilson lines}}{\oint \text{with 1 circular Wilson line}} = \frac{\sin(\pi\lambda)}{\pi\lambda} \quad \text{[Witten]}$$

Thank You!

Need for a conjecture

- $P_i(p_1) + A^j(p_2) \to P_m(p_3) + A^n(p_4)$



- Work in light-cone gauge in the frame $q_\pm = 0$.

- Adjoint channel (from top) - $q_\pm = 0$ is a frame choice, full answer can be obtained by covariantizing.

$$p_1 = p + q \ , \ p_2 = -k - q \ , \ p_3 = p \ , \ p_4 = -k$$

- Singlet channel (from left) , exchange momentum cannot be spacelike!

$$p_1 = p + q \ , \ p_2 = -p \ , \ p_3 = k + q \ , p_4 = -k$$

- Cannot compute singlet channel directly.
- Using usual crossing symmetry gives a non-unitary S matrix for singlet channel.

Supersymmetry and dual supersymmetry

- Action of bose-fermi duality

$$a^D = \alpha, \quad \alpha^D = a \quad m^D = -m$$

- dual supersymmetry operator has the form

$$(Q^D)^1_\beta = i\left(-u_\beta(\mathbf{p}, -m)\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p}, -m)\theta\right) \ ,$$

$$(Q^D)^2_\beta = i\left(v_\beta(\mathbf{p}, -m)\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p}, -m)\theta\right)$$

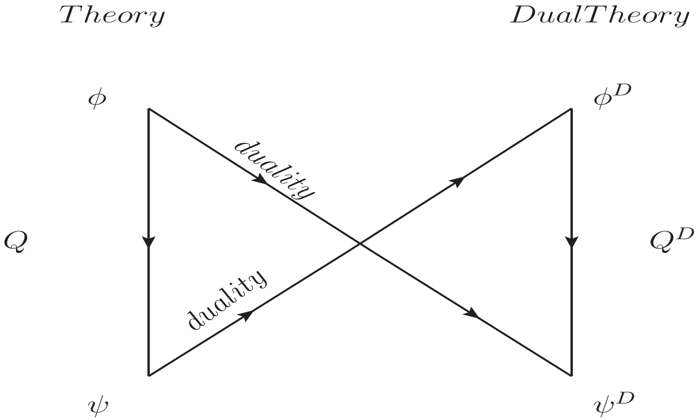
- using $u(m,p) = -v(-m,p), \quad v(m,p) = -u(-m,p)$ and $\theta \leftrightarrow \frac{\partial}{\partial\theta}$

$$(Q^D)^1 \propto Q^1, \quad (Q^D)^2 \propto Q^2$$

- Quantities invariant under usual supersymmetry also invariant under dual supersymmetry.

- Onshell supersymmetry commutes with duality

Supersymmetry and dual supersymmetry



Bare Propagators

- The bare scalar superfield propagator:

$$\langle \bar{\Phi}(\theta_1, p)\Phi(\theta_2, -p')\rangle = \frac{D^2_{\theta_1,p} - m_0}{p^2 + m_0^2}\delta^2(\theta_1-\theta_2)(2\pi)^3\delta^3(p-p')$$

- The gauge superfield propagator:

$$\langle \Gamma^-(\theta_1, p)\Gamma^-(\theta_2, -p')\rangle = -\frac{8\pi}{\kappa}\frac{\delta^2(\theta_1 - \theta_2)}{p_{--}}(2\pi)^3\delta^3(p-p')$$

  where $p_{--} = -(p_1 + ip_2) = -p_-$.

- Gauge field component propagators have same form as non-susy light cone gauge

$$\langle A_+(p)A_3(-p')\rangle = \frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p - p')$$

Susy constraints on two-point correlator

- Supersymmetric ward identity for two point correlator

$$(Q_{\theta_1,p} + Q_{\theta_2,-p})\langle\bar{\Phi}(\theta_1,p)\Phi(\theta_2,-p)\rangle = 0$$

- Exact propagator solves the ward identity

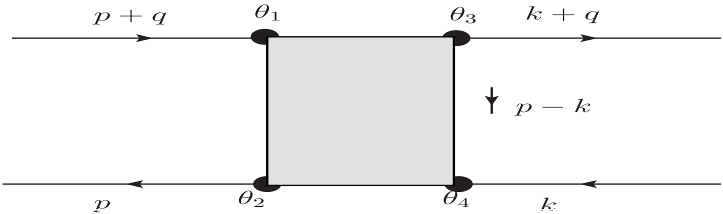
$$\langle\bar{\Phi}(p,\theta_1)\Phi(-p',\theta_2)\rangle = (2\pi)^3\delta^3(p-p')P(\theta_1,\theta_2,p)$$

$$P(\theta_1,\theta_2,p) = \left(C_1(p^\mu)D^2_{\theta_1,p} + C_2(p^\mu)\right)\delta^2(\theta_1-\theta_2)$$

- eg for bare propagator

$$C_1 = \frac{1}{p^2+m_0^2}\ ,\ \ C_2 = \frac{m_0}{p^2+m_0^2}$$

Susy constraints on four-point function



- Supersymmetric ward identity for four point function

$$(Q_{\theta_1,p+q}+Q_{\theta_2,-p}+Q_{\theta_3,-k-q}+Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q)=0$$

$$\langle\bar{\Phi}((p+q+\frac{l}{4}),\theta_1)\Phi(-p+\frac{l}{4},\theta_2)\Phi(-(k+q)+\frac{l}{4},\theta_3)\bar{\Phi}(k+\frac{l}{4},\theta_4)\rangle$$
$$=(2\pi)^3\delta(l)V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q)$$

Susy constraints on four-point function

- Solution of the ward identity

$$V = \exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right)F(X_{12}, X_{13}, X_{43}, p, q, k)$$

$$X = \sum_{i=1}^{4}\theta_i \ , \ X_{ij} = \theta_i - \theta_i \ ,$$

- F is a shift invariant function $\theta_i \to \theta_i + \gamma$.

- V may be taken to be invariant under the Z_2 symmetry

$$p \to k+q, k \to p+q, q \to -q \ ,$$
$$\theta_1 \to \theta_4, \theta_2 \to \theta_3, \theta_3 \to \theta_2, \theta_4 \to \theta_1$$

An integral equation for the four point function

- Most general form of F can be parameterized in terms of 32 bosonic functions of p, k and q.
- leads to 32 coupled integral equations - tedious.
- In the kinematic regime $q_\pm = 0$ the ansatz

$$V = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right)F(X_{12}, X_{13}, X_{43}, p, q, k)$$

$$F = \begin{matrix} X_{12}^+X_{43}^+\Big(A(p,k,q)X_{12}^-X_{43}^-X_{13}^+X_{13}^- + B(p,k,q)X_{12}^-X_{43}^- \\ + C(p,k,q)X_{12}^-X_{13}^+ + D(p,k,q)X_{13}^+X_{43}^-\Big) \end{matrix}$$

is closed under the multiplication rule induced by the RHS of the integral equation.

Product of S matrices

- General multiplication rule for two S matrices

$$S_1 \star S_2 \equiv \int d\Gamma S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2)$$
$$\exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2)$$
$$S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)$$

$$d\Gamma = \frac{d^2k_3}{2k_3^0(2\pi)^2} \frac{d^2k_4}{2k_4^0(2\pi)^2} \frac{d^2k_1}{2k_1^0(2\pi)^2} \frac{d^2k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4$$

- supersymmetry invariant Identity operator

$$I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \exp(\theta_1\theta_3 + \theta_2\theta_4) I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$$
$$I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)$$

- Multiplication rule with Identity operator

$$S \star I = I \star S = S$$

- More generally product of two supersymmetric S matrices is supersymmetric.

Unitarity equation in center of mass frame

- Writing $\mathcal{S}_B = I + i\mathcal{T}_B$, $\mathcal{S}_F = I + i\mathcal{T}_f$
- No theta term:

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B(s,\theta)+4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B(s,\theta)\mathcal{T}_B^*(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^*(s,-\alpha)-\mathcal{T}_B(s,\alpha))$$

- Four theta term:

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B(s,\theta)+4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta)))$$
$$-16Y(s)^2\mathcal{T}_f(s,\theta)\mathcal{T}_f^*(s,-(\alpha-\theta))\Big) = i4Y(s)(-\mathcal{T}_f(s,\alpha)+\mathcal{T}_f^*(s,-\alpha))$$

$$Y(s) = \frac{-s+4m^2}{16m^2}$$

Unitarity equations in the S channel

- The S matrix in the S channel is $O(1)$ - the unitarity conditions are non-linear

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha))$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_F^S(s,\alpha)-\mathcal{T}_F^{S*}(s,-\alpha))$$

- Under duality $\mathcal{T}_B\to\mathcal{T}_F$ and vice versa; both the equations map to one another.

- Unitarity conditions are compatible with duality.

Unitarity equations in the S channel

- In the center of mass frame, the supersymmetric unitarity equations are

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha))$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_F^S(s,\alpha)-\mathcal{T}_F^{S*}(s,-\alpha))$$

- Under duality $\mathcal{T}_B\rightarrow\mathcal{T}_F$ and vice versa; both the equations map to one another.

- Unitarity conditions are compatible with duality.

Poles of the S matrix

- Both bosonic and fermionic S matrix have a pole at threshold for $w \leq -1$. Near $w = -1 - \delta w, y = 1 - \delta y$ the S matrix has the pole structure ($y = \sqrt{s}/2m$)

$$\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}}\ ,\ \mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1+|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}}$$

- As w is decreased further and as it hits a critical value $w = w_c$

$$w = w_c(\lambda) = 1 - \frac{2}{|\lambda|}$$

- the pole becomes massless!. Near $w = w_c - \delta w$ and $y = \delta y$ the poles approach zero mass quadratically

$$\mathcal{T}_B \sim \mathcal{T}_F - \frac{64|m|\sin(\pi\lambda)(-1+|\lambda|)}{|\lambda|\left(\delta w^2\lambda^2 - 4\delta y^2(1-|\lambda|)^2\right)}$$

Poles of the S matrix

- As w is further decreased and as $w \to -\infty$ the pole approaches threshold once again. Near $w = -\frac{1}{\delta w}, y = 1 - \delta y$ the S matrix has the pole structure

$$\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{2-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}\ ,\ \mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}$$

- To summarize, a one parameter tuning of the superpotential interaction parameter w - sufficient to produce massless bound states in our massive theory.

- w can be scaled to w_c - possible decoupled QFT description of light states.

- Is this a $\mathcal{N} = 1$ Wilson-Fischer theory made of single real superfield?

Analytic structure of S channel S matrix

- The S matrix in the singlet channel has an interesting analytic structure.
- As a function of s (at fixed t), there is the expected two particle branch cut starting at $s = 4m^2$.
- For smaller but positive values of s there exist poles in the S matrix for a range of coupling parameters.
- These poles represent bound states that exist at large but finite N.
- At some critical value of the scalar coupling $w = w_c(\lambda)$ the pole becomes massless!
- To summarize, a one parameter tuning of the superpotential interaction parameter w - sufficient to produce massless bound states in a massive theory.