Supersymmetric Bianchi attractors in gauged supergravity

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21 Oct 2016

Based on

- Bidisha Chakrabarty, K.I and Rickmoy Samanta, "On the Supersymmetry of Bianchi attractors in Gauged supergravity", Arxiv 1610.03033.

Related earlier work

- K.I. and Rickmoy Samanta, *JHEP* **1408** (2014) 055.
- K.I. and Prasanta K. Tripathy, *JHEP* **1310** (2013) 163.
- K.I. and Prasanta K. Tripathy, *JHEP* **1209** (2012) 003.

Plan of the talk

Introduction

- Motivation
- Our work

Homogeneous symmetries and Bianchi attractors

Bianchi attractors in 4d gauged supergravity

- $\mathcal{N} = 2, d = 4$ gauged supergravity
- Bianchi I: $AdS_2 \times \mathbb{R}^2$

Bianchi attractors in 5d gauged supergravity

- $\mathcal{N} = 2, d = 5$ gauged supergravity
- Gaugino conditions
- Killing spinor equation
- Bianchi I: AdS_5
- Bianchi I: Anisotropic $AdS_3 \times \mathbb{R}^2$
- Bianchi III and $AdS_3 \times \mathbb{H}^2$

Summary

Motivation: Attractor mechanism

- Black holes in supergravity exhibit a phenomenon known as the attractor mechanism.
- Attractor mechanism plays a crucial role in understanding the origin of black hole entropy in supergravity theories. [Ferrara-Kallosh-Strominger]
- Moduli fields in black hole background flow to fixed point values at the horizon irrespective of their asymptotic values at infinity.
- The fixed point values are determined entirely in terms of black hole charges. BH entropy is determined in terms of the charges.
- Attractor mechanism is a consequence of extremality rather than supersymmetry. [Ferrara-Gibbons-Kallosh] appendix are you happy now
- Extends to non-supersymmetric extremal cases.[Goldstein-Iizuka-Jena-Trivedi]

Motivation: Attractor mechanism

- In recent years enormous effort has gone into generalizing the attractor mechanism to gauged supergravities.
- In gauged supergravity moduli have potential and approach critical points of the potential asymptotically.
- Unless there are flat directions in the potential the asymptotic values of the moduli are fixed in terms of gauge coupling constants.
- The precise statement of the attractor mechanism in gauged supergravity is that "BH entropy is determined entirely by the charges, and is independent of the values of the moduli on the horizon that are not fixed by the charges"
- Significant progress has been made especially for dyonic BPS AdS_4 black holes [Cacciatori-Klemm, Benini-Hristov-Zaffaroni]. Large N index computations in the dual twisted mass deformed ABJM theory "observe" perfect matching of the microstate counting with black hole entropy [Benini-Zaffaroni].

Motivation: From CMT to gravity

- Apart from the attractor mechanism, another reason to study extremal black holes in AdS is due to possible applications for condensed matter physics.
- In gauge-gravity duality, extremal geometries provide the dual gravity description of zero temperature ground states of strongly coupled field theories.
- Many condensed matter systems display novel and diverse phase structures.
- This predicts an equally large number of extremal solutions in the bulk.
- It is an interesting program in (super)gravity to identify and classify such extremal geometries.

Motivation: Near horizon geometries

- Near horizon geometries of extremal branes are particularly interesting.
- It encodes information about low energy dynamics of the dual field theory and is often scale invariant.
- It is often an attractor, with differences from attractor geometry far away dying as one approaches horizon.
- Near horizon geometry is often easier to find analytically and the attractor nature makes it more universal than the full extremal solution.
- An equally interesting program in (super)gravity is to identify and classify near horizon geometries.

Motivation: Bianchi attractors

- Bianchi attractors: Classification of homogeneous anisotropic extremal black brane horizons in $d = 5$. [Iizuka-Kachru-Kundu-Narayan-Sircar-Trivedi]
- The geometries are classified according to the group of homogeneous symmetries along their spatial directions.
- eg: AdS, Lifshitz solutions belong to Bianchi Type I in the classification of geometries.
- Easily constructed in Einstein-Maxwell theories with massive/massless gauge fields and a cosmological constant.
- Non-supersymmetric solutions have also been found in gauged supergravities [K.I-Tripathy, K.I-Samanta]

Motivation: Bianchi attractors

- Non-supersymmetric Bianchi attractors are unstable under linearized fluctuations about their attractor values unless they satisfy the attractor conditions.
- Attractor conditions : There must exist a critical point of the attractor potential and the Hessian of the attractor potential evaluated at the critical points must have positive eigenvalues. [Goldstein-Iizuka-Jena-Trivedi],[K.I-Tripathy]
- The parameters in the non-supersymmetric solutions must be tuned to satisfy the above conditions.
- Supersymmetric solutions always satisfy the attractor conditions.
- BPS near horizon geometries would aid in solving flow equations that determine the full extremal geometry.

Our work: $d = 4, 5$ BPS Bianchi attractors

- We construct supersymmetric Bianchi attractors in $\mathcal{N} = 2, d = 4, 5$ gauged supergravity.
- BPS conditions are obtained when the susy variations of the fermionic fields on the background geometry vanish.
- The Killing spinor equation is obtained by setting the gravitino variation to zero.
- The number of independent components in the constant part of the spinor determines the amount of supersymmetry preserved by the solution.

Our work: $d = 4, 5$ BPS Bianchi attractors

- In $d = 4$ we consider a gauged supergravity coupled to vector and hyper multiplets with a generic gauging of the symmetries of the hyper Kähler manifold.
- In $d = 4$ homogeneous symmetries are along the two spatial directions and the corresponding symmetry groups are of two types namely Bianchi I and Bianchi II.
- In $d = 5$ we consider the $\mathcal{N} = 2$ gauged supergravity coupled to vector and hypermultiplets with a generic gauging of symmetries of the very special manifold and the R symmetry group.
- In $d = 5$ homogeneous symmetries are along the three spatial directions and the corresponding symmetry groups are of nine types namely Bianchi I-IX.

Results: $d = 4$ BPS Bianchi attractors

- In $d = 4$, in the Bianchi I case, we construct a $\frac{1}{4}$ BPS $AdS_2 \times \mathbb{R}^2$ geometry sourced by time like gauge fields.
- The radial spinor preserves 1/4 of the supersymmetry.
- In the Bianchi II case, we construct $AdS_2 \times \mathbb{H}^2$ solution sourced by time like gauge fields.
- In this case the radial spinor breaks all of the supersymmetry.

Results: $d = 5$ BPS Bianchi attractors

- In gauged sugra, fermionic shifts are terms in gaugino and hyperino variation that appear entirely due to the gauging.
- When none of the fermionic shifts vanish there are no susy Bianchi attractor geometries.
- When the central charge satisfies an extremization condition at the attractor point some of the fermionic shifts vanish and BPS solutions are possible.
- The allowed solutions are analogues of Bianchi type solutions in Einstein-Maxwell systems with massless gauge fields and a cosmological constant.

Results: $d = 5$ BPS Bianchi attractors

- In the Bianchi I case, the vacuum AdS_5 geometry is the unique maximally supersymmetric solution.[K.I-Tripathy]
- We also find an anisotropic $AdS_3 \times \mathbb{R}^2$ solution that preserves $1/2$ of the $\mathcal{N} = 2$ supersymmetry.
- In the Bianchi III case, we find a class of $1/2$ BPS solutions labelled by the central charge.
- For a special value of the central charge the Bianchi III solution reduces to the isotropic $AdS_3 \times \mathbb{H}^2$.
- To the best of our knowledge, these are the first examples of anisotropic Bianchi geometries that preserve supersymmetry.

Homogeneous symmetries and Bianchi attractors

Homogeneous Symmetries

- Consider a manifold M endowed with a metric $g_{\mu\nu}$ that is invariant under a given set of isometries.
- The Killing vectors X_i that generate the isometries close to form an algebra

$$[X_i, X_j] = C_{ij}{}^{k} X_k$$

- Homogeneous manifold has identical metric properties at all points in space.
- Any two points on a homogeneous space are connected by a symmetry transformation.
- Symmetry group of a homogeneous space of dimension d is isomorphic to the group corresponding to d dimensional real Lie algebra. [Shepley]
- Real Lie algebras in dim 2: Bianchi I-II ,Real Lie algebras in dim 3: Bianchi I-IX .

Homogeneous Symmetries

- The homogeneous symmetries in the metric can be made manifest by going to an invariant basis of vectors that commute with the Killing vectors

$$[e_i, e_j] = -C_{ij}{}^{k} e_k \ , \ [X_i, e_i] = 0$$

- the metric with homogeneous symmetries can be expressed in terms of one forms ω^i dual to the invariant vectors e_i as

$$ds^2 = g_{ij}\omega^i \otimes \omega^j$$

- The invariant one forms satisfy the relation

$$d\omega^k = \frac{1}{2} C_{ij}{}^{k} \omega^i \wedge \omega^j$$

- eg: Bianchi I:

$$X_i = \partial_{x_i} \equiv e_i \ , \ \omega^i = dx^i \ , \ d\omega^i = 0$$

$$ds^2 = \delta_{ij}\omega^i \otimes \omega^j = dx_1^2 + dx_2^2 + dx_3^2$$

Homogeneous Symmetries

- eg Bianchi III: Killing vectors

$$X_1 = \partial_y \ , \quad X_2 = \partial_z \ , \quad X_3 = \partial_x + y\partial_y \ , \ [X_1, X_3] = X_1$$

- Invariant vectors that commute with the Killing vectors

$$e_1 = e^x\partial_y \ , \quad e_2 = \partial_z \ , \quad e_3 = \partial_x \ , \ [e_1, e_3] = -e_1$$

- The invariant one forms dual to the e_i are

$$\omega^1 = e^{-x}dy \ , \quad \omega^2 = dz \ , \quad \omega^3 = dx \ , \ d\omega^1 = \omega^1 \wedge \omega^3 \ .$$

- The metric with manifest Bianchi III symmetry is

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = e^{-2x}dy^2 + dx^2 + dz^2$$

- in a more familiar form ($x = \ln \rho$) this is $\mathbb{R} \times \mathbb{H}^2$

$$ds^2 = \frac{dy^2 + d\rho^2}{\rho^2} + dz^2$$

- The $\mathbb{H}^2$ part has the symmetries of the Bianchi II group in two dimensions.

Homogeneous Symmetries

- eg Bianchi VII, Killing vectors

$$X_1 = \partial_y \ , \ X_2 = \partial_z \ , \ X_3 = \partial_x + y\partial_z - z\partial_y$$

$$[X_1, X_2] = 0 \ , \ [X_1, X_3] = X_2 \ , \ [X_2, X_3] = -X_1$$

- The invariant vectors that commute with the Killing vectors are

$$e_1 = \cos(x)\partial_y + \sin(x)\partial_z \ , \ e_2 = -\sin(x)\partial_y + \cos(x)\partial_z \ , e_3 = \partial_x$$

- The invariant one forms

$$\omega^1 = \cos(x)dy + \sin(x)dz \ , \ \omega^2 = -\sin(x)dy + \cos(x)dz \ , \omega^3 = dx$$

$$d\omega^1 = -\omega^2 \wedge \omega^3 \ , \ d\omega^2 = \omega^1 \wedge \omega^3 \ , \ d\omega^3 = 0$$

- The metric manifestly invariant under Bianchi VII symmetry can be written as

$$ds^2 = (\omega^1)^2 + \lambda^2(\omega^2)^2 + (\omega^3)^2$$

- at $\lambda = 1$ symmetry is enhanced to Bianchi I.

Bianchi attractors

- Bianchi attractor geometries have the general structure

$$ds^2 = -g_{tt}(r)dt^2 + g_{ij}(r)\omega^i \otimes \omega^j + dr^2$$

- eg: Poincaré AdS in this coordinate system has the form

$$ds^2 = -e^{2r}dt^2 + dr^2 + e^{2r}(dx_i^2)$$

- eg Bianchi III

$$ds^2 = e^{2\beta_t r}dt^2 + dr^2 + e^{2\beta_3 r}(\omega^2)^2 + e^{2\beta_2 r}(\omega^1)^2 + (\omega^3)^2$$

$$\omega^1 = e^{-x^1}dx^2 \ , \omega^2 = dx^3 \ , \omega^3 = dx^1$$

- Exponents are fixed using scaling symmetries

$$ds^2 = r^{2\beta_t}dt^2 + \frac{dr^2}{r^2} + r^{2\beta_3}(\omega^2)^2 + r^{2\beta_2}(\omega^1)^2 + (\omega^3)^2$$

$$(r \to \lambda r \ , t \to \lambda^{-\beta_t} t \ , \ x^3 \to \lambda^{-\beta_3} x^3 \ , \ x^2 \to \lambda^{-\beta_2} x^2$$

- $\beta_t = \beta_3 = 1, \beta_2 = 0$ corresponds to $AdS_3 \times \mathbb{H}^2$.

Bianchi attractors in 4d gauged supergravity

$\mathcal{N} = 2, d = 4$ gauged supergravity

- $\mathcal{N} = 2, d = 4$ gauged supergravity coupled to n_V vector and n_H hyper multiplets.

- Gravity multiplet: $g_{\mu\nu}, A^0_\mu$ and gravitinos $(\psi^A_\mu, \psi_{\mu A})$.

- Vector multiplet: A^Λ_μ, complex scalars z^i and gauginos $(\lambda^{iA}, \lambda^{\bar{i}}_A)$.

- Hyper multiplet: Hyper scalars q^X and hyperinos $(\zeta_\alpha, \zeta^\alpha)$.

- The moduli space of the theory factorizes into a product of a special Kähler manifold and a quaternionic Kähler manifold

$$\mathcal{M} = \mathcal{SK}(n_V) \times \mathcal{Q}(n_H)$$

$\mathcal{N}=2, d=4$ gauged supergravity

- Gauging amounts to replacing ordinary derivatives with gauge covariant derivatives

$$D_\mu z^i = \partial_\mu z^i + g A_\mu^\Lambda K_\Lambda^i(z) \text{ , vector multiplet}$$
$$D_\mu q^X = \partial_\mu q^X + g A_\mu^\Lambda K_\Lambda^X(q) \text{ , hyper multiplet}$$

- $K_\Lambda^i(z)$ and $K_\Lambda^X(q)$ generate isometries on $\mathcal{SK}(n_V)$ and $\mathcal{Q}(n_H)$ respectively.

- Additional terms due to gauging require a potential term for susy completion

$$\mathcal{V}(z,\bar{z},q) = \Big(g^2 (g_{i\bar{j}} K_\Lambda^i K_\Sigma^{\bar{j}} + 4 g_{XY} K_\Lambda^X K_\Sigma^Y) \bar{L}^\Lambda L^\Sigma + (g^{i\bar{j}} f_i^\Lambda f_{\bar{j}}^\Sigma - 3 \bar{L}^\Lambda L^\Sigma) P_\Lambda^x P_\Sigma^x \Big)$$

- The bosonic part of the Lagrangian of the $\mathcal{N}=2$ theory takes the form

$$\mathcal{L} = -\frac{1}{2} R + g_{i\bar{j}} D^\mu z^i D_\mu \bar{z}^{\bar{j}} + g_{XY} D_\mu q^X D^\mu q^Y + \mathcal{V}(z,\bar{z},q)$$
$$+ i (\bar{N}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{-\Lambda} \mathcal{F}^{-\Sigma\mu\nu} - N_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{+\Lambda} \mathcal{F}^{+\Sigma\mu\nu})$$

Bianchi I: $AdS_2 \times \mathbb{R}^2$

- For simplicity, consider gauging of the hypermultiplet manifold only.

- At the attractor point the scalars are constant

$$\partial_\mu z^i = 0\ ,\ \partial_\mu q^X = 0$$

- The effective Lagrangian becomes

$$\mathcal{L}_{eff} = -\frac{1}{2}R + \mathrm{Im}N_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} - \mathcal{V}(z,\bar{z},q) + g^2 g_{XY}K^X_\Lambda K^Y_\Sigma A^\Lambda_\mu A^{\mu\Sigma}$$

- This is similar to an Einstein-Maxwell system with a massive gauge field and a cosmological constant.

- Bianchi I $AdS_2 \times \mathbb{R}^2$ geometry

$$ds^2 = \frac{R_0^2}{\sigma^2}(dt^2 - d\sigma^2) - R_0^2(dy^2 + d\rho^2)$$

$$A^\Lambda = \frac{E^\Lambda}{\sigma}dt$$

Bianchi I: $AdS_2 \times \mathbb{R}^2$

- Gauge field equations of motion

$$g_{XY} K^X_\Lambda K^Y_\Sigma E^\Lambda = 0$$

- Scalar eom reduces to extremization of an effective potential

$$\frac{\partial}{\partial q^X}\mathcal{V}_{eff} = 0 \ , \ \frac{\partial}{\partial z^i}\mathcal{V}_{eff} = 0$$

$$\mathcal{V}_{eff} = \mathcal{V}(z,\bar{z},q) - g_{XY} K^X_\Lambda K^Y_\Sigma \frac{E^\Lambda E^\Sigma}{R_0^2} + \mathrm{Im} N_{\Lambda\Sigma}\frac{E^\Lambda E^\Sigma}{2R_0^4}$$

- Einstein equations

$$0 = R_0^2\mathcal{V}_{eff} + 2 g_{XY} K^X_\Lambda K^Y_\Sigma E^\Lambda E^\Sigma - \mathrm{Im} N_{\Lambda\Sigma}\frac{E^\Lambda E^\Sigma}{R_0^2}$$

$$0 = - R_0^2\mathcal{V}_{eff} + \mathrm{Im} N_{\Lambda\Sigma}\frac{E^\Lambda E^\Sigma}{R_0^2}$$

$$-\frac{1}{R_0^2} = \mathcal{V}_{eff}$$

Bianchi I: $AdS_2 \times \mathbb{R}^2$

- Susy transformations at the attractor point

$$
\begin{aligned}
\delta\psi_{\mu A} =& D_\mu \epsilon_A + i S_{AB}\gamma_\mu \epsilon^B + 2i(\mathrm{Im}N)_{\Lambda\Sigma} L^\Sigma \mathcal{F}^{-\Lambda}_{\mu\nu}\gamma^\nu \epsilon_{AB}\epsilon^B \\
\delta\lambda^{iA} =& - g^{i\bar{j}} \bar{f}^{\Sigma}_{\bar{j}} (\mathrm{Im}N)_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\epsilon_B + W^{iAB}\epsilon_B \\
\delta\zeta_\alpha =& i\mathcal{U}^{B\beta}_X K^X_\Lambda A^\Lambda_\mu \gamma^\mu \epsilon^A \epsilon_{AB}\epsilon_{\alpha\beta} + N^A_\alpha \epsilon_A
\end{aligned}
$$

$$
\begin{aligned}
S_{AB} =& \frac{i}{2}(\sigma^r)_A{}^C \epsilon_{BC} P^r_\Lambda L^\Lambda \\
W^{iAB} =& \epsilon^{AB} k^i_\Lambda \bar{L}^\Lambda + i(\sigma_r)_C{}^B \epsilon^{CA} P^r_\Lambda g^{i\bar{j}} f^\Lambda_{\bar{j}} \\
N^A_\alpha =& 2\mathcal{U}^A_{\alpha X} K^X_\Lambda \bar{L}^\Lambda
\end{aligned}
$$

- Susy invariance requires that the susy variations of the fermions evaluated on the background geometry vanish

$$
\delta\psi_{\mu A} = 0 \ , \ \delta\lambda^{iA} = 0 \ , \ \delta\zeta_\alpha = 0
$$

Bianchi I: $AdS_2 \times \mathbb{R}^2$

- Killing spinor equations on $AdS_2 \times \mathbb{R}^2$

$$
\begin{aligned}
\frac{\gamma^0\sigma}{R_0}\partial_t\epsilon_A - \frac{\gamma^1}{2R_0}\epsilon_A + \frac{iG_A^B\gamma^0}{2R_0}\epsilon_B + iS_{AB}\epsilon^B + \frac{iN}{2R_0^2}\gamma^{01}\epsilon_{AB}\epsilon^B &= 0 \\
\frac{\gamma^1\sigma}{R_0}\partial_\sigma\epsilon_A + iS_{AB}\epsilon^B + \frac{iN}{2R_0^2}\gamma^{01}\epsilon_{AB}\epsilon^B &= 0 \\
\frac{\gamma^2}{R_0}\partial_y\epsilon_A + iS_{AB}\epsilon^B - \frac{N}{2R_0^2}\gamma^{23}\epsilon_{AB}\epsilon^B &= 0 \\
\frac{\gamma^3}{R_0}\partial_\rho\epsilon_A + iS_{AB}\epsilon^B - \frac{N}{2R_0^2}\gamma^{23}\epsilon_{AB}\epsilon^B &= 0
\end{aligned}
$$

$$
N = (\mathrm{Im}N_{\Lambda\Sigma})L^\Sigma E^\Lambda \ , \ \ G_A^B = (\sigma_x)_A{}^B P_\Lambda^x E^\Lambda
$$

- Solved by radial ansatz

$$
\epsilon_A = \frac{1}{\sqrt{\sigma}}\chi_A \ , E^\Lambda P_\Lambda^x = 0
$$

Bianchi I: $AdS_2 \times \mathbb{R}^2$

- χ_A satisfies the 1/4 BPS projection conditions

$$\chi_A = i\epsilon_{AB}\gamma^0\chi^B$$
$$\chi_A = (\sigma_3)_A{}^C\gamma^{10}\chi_C$$

- with the constraints

$$|(\mathrm{Im}N_{\Lambda\Sigma})L^\Sigma E^\Lambda| = \frac{R_0}{2}\ ,\ P^3_\Lambda L^\Lambda = \frac{i}{2R_0}$$

- Gaugino and hyperino conditions

$$K^X_\Lambda\left(\frac{E^\Lambda}{R_0} + 2\bar{L}^\Lambda\right) = 0$$
$$g^{i\bar{j}}\bar{f}^\Sigma_{\bar{j}}\left(-\mathrm{Im}N_{\Lambda\Sigma}\frac{E^\Lambda}{R_0^2} + iP^3_\Sigma\right) = 0$$

- This together with the gauge field equation of motion

$$g_{XY}K^X_\Lambda K^Y_\Sigma E^\Lambda = 0$$

are the necessary conditions for the 1/4 BPS $AdS_2 \times \mathbb{R}^2$ geometry.

$AdS_2 \times \mathbb{H}^2$ geometry

- The $AdS_2 \times \mathbb{H}^2$ metric

$$ds^2 = \frac{R_0^2}{\sigma^2}(dt^2 - d\sigma^2) - \frac{R_0^2}{\rho^2}(dy^2 + d\rho^2)$$

- sourced by time like gauge field

$$A^\Lambda = \frac{E^\Lambda}{\sigma} dt$$

- Scalar eom reduces to extremization of an effective potential

$$\frac{\partial}{\partial q^X}\mathcal{V}_{eff} = 0\ ,\ \ \frac{\partial}{\partial z^i}\mathcal{V}_{eff} = 0$$

$$\mathcal{V}_{eff} = \mathcal{V}(z, \bar{z}, q) - g_{XY} K_\Lambda^X K_\Sigma^Y \frac{E^\Lambda E^\Sigma}{R_0^2} + \mathrm{Im} N_{\Lambda\Sigma} \frac{E^\Lambda E^\Sigma}{2R_0^4}$$

- Gauge field and Einstein equations of motion can be cast in the form

$$\mathcal{V}(z, \bar{z}, q) = -\frac{1}{R_0^2}$$

$$\mathrm{Im} N_{\Lambda\Sigma} E^\Lambda E^\Sigma = 0$$

$$g_{XY} K_\Lambda^X K_\Sigma^Y E^\Lambda E^\Sigma = 0\ .$$

$AdS_2 \times \mathbb{H}^2$ geometry

- The Killing spinor equations on this background

$$\frac{\gamma^0\sigma}{R_0}\partial_t\epsilon_A - \frac{\gamma^1}{2R_0}\epsilon_A + \frac{iG_A^B\gamma^0}{2R_0}\epsilon_B + iS_{AB}\epsilon^B + \frac{iN}{2R_0^2}\gamma^{01}\epsilon_{AB}\epsilon^B = 0$$

$$\frac{\gamma^1\sigma}{R_0}\partial_\sigma\epsilon_A + iS_{AB}\epsilon^B + \frac{iN}{2R_0^2}\gamma^{01}\epsilon_{AB}\epsilon^B = 0$$

$$\frac{\gamma^2\rho}{R_0}\partial_y\epsilon_A - \frac{\gamma^3}{2R_0}\epsilon_A + iS_{AB}\epsilon^B - \frac{N}{2R_0^2}\gamma^{23}\epsilon_{AB}\epsilon^B = 0$$

$$\frac{\gamma^3\rho}{R_0}\partial_\rho\epsilon_A + iS_{AB}\epsilon^B - \frac{N}{2R_0^2}\gamma^{23}\epsilon_{AB}\epsilon^B = 0$$

- Using the radial ansatz $\epsilon_A = \frac{1}{\sqrt{\rho\sigma}}\chi_A$

$$E_\Lambda P_x^\Lambda = 0$$

$$\left(\gamma^1 + \gamma^3\right)\epsilon_A = 4iR_0 S_{AB}\epsilon^B$$

$$\left(\gamma^1 - \gamma^3\right)\epsilon_A = \frac{2iN\gamma^{01}}{R_0}\epsilon_{AB}\epsilon^B.$$

- Simplify the projections using

$$C = \gamma^1\gamma^3\ ,\ \chi^B = -\gamma_0 C(\chi_B)^*$$

$AdS_2 \times \mathbb{H}^2$ geometry

- Simplifying

$$(-1 + C)\,\chi_A = 4iR_0 S_{AB}\gamma^{01} C(\chi_B)^*$$
$$(-C + 1)\,\chi_A = \frac{2iN}{R_0}\epsilon_{AB}(\chi_B)^*$$

- Decomposing in terms of simultaneous eigenstates of $[\gamma_5, C] = 0$

$$\chi_A = \begin{pmatrix} 0 \\ C_A^+|+\rangle \end{pmatrix} + \begin{pmatrix} 0 \\ C_A^-|-\rangle \end{pmatrix}$$

- the projections become

$$(1 - i)C_A^+ = \frac{2iN}{R_0}\epsilon_{AB}(C_B^-)^*$$
$$(1 + i)C_A^- = \frac{2iN}{R_0}\epsilon_{AB}(C_B^+)^*$$

- we see that the projections break supersymmetry

$$C_A^+(1 + \frac{2i|N|^2}{R_0^2}) = 0$$

Bianchi attractors in 5d gauged supergravity

$\mathcal{N} = 2, d = 5$ gauged supergravity

- We consider $\mathcal{N} = 2, d = 5$ gauged supergravity coupled to n_V vector multiplets and n_H hypermultiplets Ceresole-Dall'Agata

- Gravity multiplet contains a graviton, two gravitino (no chirality) and a graviphoton.

- Hyper multiplet same as $d = 4$, vector multiplet scalars ϕ are real.

- The scalars in the theory parametrise a manifold that factorises into a direct product of a very special and quaternionic manifold,

$$\mathcal{M} = \mathcal{S}(n_V) \times \mathcal{Q}(n_H)$$

- The very special manifold is parametrized by $n_V + 1$ functions $h^I(\phi)$ subject to the constraint

$$N \equiv C_{IJK} h^I h^J h^K = 1$$

where C_{IJK} are constant symmetric tensors.

$\mathcal{N} = 2, d = 5$ gauged supergravity

- Gauging the symmetries of the scalar manifold and R symmetry

$$
\begin{aligned}
D_\mu \phi^x =& \partial_\mu \phi^x + g A^I_\mu K^x_I(\phi) \\
D_\mu q^X =& \partial_\mu q^X + g A^I_\mu K^X_I(q) \\
D_\mu \psi_{\nu i} =& \nabla_\mu \psi_{\nu i} + g_R A^I_\mu P_{Ii}{}^j(q) \psi_{\nu j}
\end{aligned}
$$

- Susy closure requires the addition of a potential term

$$\mathcal{V}(\phi, q) = -g_R^2 (2P_{ij}P^{ij} - P^a_{ij}P^{aij}) + 2g^2 N_{iA} N^{iA}$$

$$
\begin{aligned}
& P_{ij} = h^I P_{Iij} \ , \ P^a_{ij} = h^{aI} P_{Iij} \\
& h^{aI} = f^a_x h^{xI} \ , h^I_x = \frac{\partial h^I(\phi)}{\partial \phi^x} \ , \ N^{iA} = \frac{\sqrt{6}}{4} h^I K^X_I f^{Ai}_X
\end{aligned}
$$

- For $SU(2)_R$ gauging, $P_{Iij}(q) = iP^r_I(q)(\sigma^r)_{ij}$.

- For $U(1)_R$ gauging, $n_H = 0$ and $P_{Iij} = -V_I \delta_{ij}$.

$\mathcal{N} = 2, d = 5$ gauged supergravity

- Bosonic part of the Lagrangian

$$\hat{e}^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}a_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\phi)D_\mu\phi^x D^\mu\phi^y + \frac{\hat{e}^{-1}}{6\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau - \mathcal{V}(\phi)$$

- At the attractor point

$$\partial_\mu\phi^x = 0$$

- Susy transformations at the attractor point are

$$\delta_\epsilon\psi_{\mu i} = D_\mu\epsilon_i + \frac{i}{4\sqrt{6}}h_I F^{\nu\rho I}(\gamma_{\mu\nu\rho} - 4g_{\mu\nu}\gamma_\rho)\epsilon_i + \frac{i}{\sqrt{6}}g_R\gamma_\mu\epsilon^j P_{ij}$$

$$\delta_\epsilon\lambda^x_i = -\frac{i}{2}gA^I_\mu K^x_I\gamma^\mu\epsilon_i + \frac{1}{4}h^x_I F^I_{\mu\nu}\gamma^{\mu\nu}\epsilon_i + g_R\epsilon^j P^x_{ij}$$

- The additional terms in the susy transformations due to the gauging are called "fermionic shifts"

Gaugino conditions

- Gaugino conditions when none of the fermionic shifts vanish

$$-\frac{i}{2} g A^I_\mu K^x_I \gamma^\mu \epsilon_i + \frac{1}{4} h^x_I F^I_{\mu\nu} \gamma^{\mu\nu} \epsilon_i + g_R \epsilon^j h^x_I V^I \delta_{ij} = 0$$

- To preserve susy the constant part of the spinor ϵ_i should be a simultaneous eigenspinor of the projection matrices.
- Projection conditions entirely depend on the gauge field configuration.
- eg: time like gauge fields

$$A = A(r)dt$$
$$dA = \partial_r A(r) dr \wedge dt$$

- Projection conditions

$$\gamma_0 \epsilon = \pm i \epsilon$$
$$\gamma_{04} \epsilon = \pm \epsilon$$

- break all susy.

Gaugino conditions

$$-\frac{i}{2} g A^I_\mu K^x_I \gamma^\mu \epsilon_i + \frac{1}{4} h^x_I F^I_{\mu\nu} \gamma^{\mu\nu} \epsilon_i + g_R \epsilon^j h^x_I V^I \delta_{ij} = 0$$

- eg Spacelike gauge fields

$$A = A(x, r)\omega^i$$

$$dA = \partial_r A(x, r) dr \wedge \omega^i + \partial_{x_j} A(x, r) dx^j \wedge \omega^i + \frac{1}{2} A(x, r) C_{jk}{}^i \omega^j \wedge \omega^k$$

- Projection conditions

$$\gamma_i \epsilon = \pm\epsilon$$
$$\gamma_{i4} \epsilon = \pm i\epsilon$$
$$\gamma_{ij} \epsilon = \pm i\epsilon$$

- breaks all susy. There are no supersymmetric Bianchi attractor solutions when the fermionic shifts do not vanish.
- The result is model independent and depends on field configuration that source Bianchi type geometry.

Gaugino conditions: Central charge extremized

$$-\frac{i}{2}gA^I_\mu K^x_I\gamma^\mu\epsilon_i+\frac{1}{4}h^x_I F^I_{\mu\nu}\gamma^{\mu\nu}\epsilon_i+g_R\epsilon^j h^x_I V^I\delta_{ij}=0$$

- central charge is extremized at the attractor point [Larsen, Klemm]

$$\partial_x(Z)=\partial_x(h^I Q_I)=0\ ,\ \ h^I V_I=1$$

- Gaugino conditions become

$$-\frac{i}{2}gA^I_\mu K^x_I\gamma^\mu\epsilon_i=0$$

- Thus susy requires $g=0$ (scalar manifold is ungauged).

- recollect that the effective Lagrangian at the attractor point is

$$\mathcal{L}=-\frac{1}{2}R-\frac{1}{4}a_{IJ}F^I_{\mu\nu}F^{J\mu\nu}-\frac{1}{2}g^2 g_{xy}A^I_\mu A^{\mu J}K^x_I K^y_J$$
$$+\frac{\hat{e}^{-1}}{6\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau-\mathcal{V}(\phi)$$

- possible susy solutions:analogues in Einstein-Maxwell theory sourced by massless gauge fields and a cosmological constant.

Killing spinor equation

- Extremizing the central charge at the attractor point completely solves gaugino condition.

$$\partial_x(Z) = \partial_x(h^I Q_I) = 0 \ , \ h^I V_I = 1 \ , \ g = 0$$

- Amount of susy preserved by all Bianchi attractor solutions of this class is determined entirely by the Killing spinor equation.

$$D_\mu \epsilon_i + \frac{i}{4\sqrt{6}} h_I F^{\nu\rho I} (\gamma_{\mu\nu\rho} - 4 g_{\mu\nu}\gamma_\rho)\epsilon_i + \frac{i}{\sqrt{6}} g_R \gamma_\mu \epsilon_i{}^k \epsilon_k = 0$$

$$D_\mu \epsilon_i \equiv \partial_\mu \epsilon_i + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon_i + g_R A^I_\mu V_I \epsilon_i{}^k \epsilon_k$$

- background geometries are of the form

$$ds^2 = \eta_{ab} e^a e^b = L^2 \left(-e^{2\beta_t r} dt^2 + \eta_{ij}(r)\omega^i \otimes \omega^j + dr^2\right)$$

Bianchi I: AdS_5

- The *AdS* metric

$$ds^2 = L^2\left(-e^{2r}dt^2 + dr^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2\right)$$

- The invariant one forms all commute with one another and satisfy $d\omega^i = 0$ of the Bianchi I algebra.

$$\omega^i = dx^i$$

- The Killing spinor equations in this background

$$e^{-r}\gamma_0\partial_t\epsilon_i - \frac{1}{2}\gamma_4\epsilon_i - \frac{i}{\sqrt{6}}Lg_R{\epsilon_i}^k\epsilon_k = 0$$

$$e^{-r}\gamma_1\partial_{x^1}\epsilon_i + \frac{1}{2}\gamma_4\epsilon_i + \frac{i}{\sqrt{6}}Lg_R{\epsilon_i}^k\epsilon_k = 0$$

$$e^{-r}\gamma_2\partial_{x^2}\epsilon_i + \frac{1}{2}\gamma_4\epsilon_i + \frac{i}{\sqrt{6}}Lg_R{\epsilon_i}^k\epsilon_k = 0$$

$$e^{-r}\gamma_3\partial_{x^3}\epsilon_i + \frac{1}{2}\gamma_4\epsilon_i + \frac{i}{\sqrt{6}}Lg_R{\epsilon_i}^k\epsilon_k = 0$$

$$\gamma_4\partial_r\epsilon_i + \frac{i}{\sqrt{6}}Lg_R{\epsilon_i}^k\epsilon_k = 0$$

Bianchi I: AdS_5

- reduced set of equations

$$\gamma_0 \partial_t \epsilon_i + \gamma_a \partial_{x^a} \epsilon_i = 0$$
$$\gamma_a \partial_{x^a} \epsilon_i - \gamma_b \partial_{x^b} \epsilon_i = 0$$
$$\gamma_4 \partial_r \epsilon_i + e^{-r} \gamma_0 \partial_t \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i = 0$$
$$\gamma_4 \partial_r \epsilon_i - e^{-r} \gamma_a \partial_{x^a} \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i = 0$$

- There are two independent spinor solutions

$$\epsilon_i = e^{\frac{r}{2}} \zeta_i^+ \;,\; \gamma_4 \zeta_i^+ = \zeta_i^+$$
$$\epsilon_i = \left(e^{-\frac{r}{2}} + e^{\frac{r}{2}} (x^m \gamma_m)\right) \zeta_i^- \;,\; \gamma_4 \zeta_i^- = -\zeta_i^-$$

- Each of the spinors $\zeta_i^{\pm}$ preserve 1/2 of the supersymmetry. Together they preserve the full $\mathcal{N} = 2$ susy.

- Substituting back in the KSE we get

$$(1 - \frac{2}{3} L^2 g_R^2) \zeta_i^{\pm} = 0$$

- susy conditions automatically guarantee the eom.

Bianchi I: Anisotropic $AdS_3 \times \mathbb{R}^2$

- The anisotropic $AdS_3 \times \mathbb{R}^2$ metric, $|B| = B^I B_I$.

$$ds^2 = -e^{2r} dt^2 + dr^2 + e^{2r} (\omega^1)^2 + + \frac{|B|^2}{4} \left((\omega^2)^2 + (\omega^3)^2 \right)$$

- Fluxes are turned on in the $\mathbb{R}^2$ direction

$$F^I_{x^2 x^3} = B^I$$

- The Killing spinor equations in the background are of the form

$$\gamma_0 e^{-r} \partial_t \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i - \frac{i}{\sqrt{6}} \left(\frac{Z}{2} \gamma_{23} \epsilon_i + g_R \epsilon_i{}^k \epsilon_k \right) = 0$$

$$\gamma_1 e^{-r} \partial_{x^1} \epsilon_i + \frac{1}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left(\frac{Z}{2} \gamma_{23} \epsilon_i + g_R \epsilon_i{}^k \epsilon_k \right) = 0$$

$$\gamma_2 \partial_{x^2} \epsilon_i + \frac{i}{\sqrt{6}} \frac{|B|}{2} \left(-Z \gamma_{23} \epsilon_i + g_R \epsilon_i{}^k \epsilon_k \right) = 0$$

$$\gamma_3 \partial_{x^3} \epsilon_i + \frac{i}{\sqrt{6}} \frac{|B|}{2} \left(-Z \gamma_{23} \epsilon_i + g_R \epsilon_i{}^k \epsilon_k \right) = 0$$

$$\gamma_4 \partial_r \epsilon_i + \frac{i}{\sqrt{6}} \left(\frac{Z}{2} \gamma_{23} \epsilon_i + g_R \epsilon_i{}^k \epsilon_k \right) = 0$$

where $Z = h_I B^I$ is the central charge.

Bianchi I: Anisotropic $AdS_3 \times \mathbb{R}^2$

- reduced set of equations

$$\gamma_0\partial_t\epsilon_i + \gamma_1\partial_{x^1}\epsilon_i = 0$$
$$\gamma_4\partial_r\epsilon_i + \gamma_0 e^{-r}\partial_t\epsilon_i - \frac{1}{2}\gamma_4\epsilon_i = 0$$
$$\gamma_4\partial_r\epsilon_i - \gamma_1 e^{-r}\partial_{x^1}\epsilon_i - \frac{1}{2}\gamma_4\epsilon_i = 0$$
$$\gamma_2\partial_{x^2}\epsilon_i - \gamma_3\partial_{x^3}\epsilon_i = 0$$

- similar to AdS case, however $\mathbb{R}^2$ directions can scale differently.

- Two independent Killing spinor solutions

$$\epsilon_i = e^{\frac{r}{2}}\zeta_i^+ \ , \ \gamma_4\zeta_i^+ = \zeta_i^+$$
$$\epsilon_i = \left(e^{-\frac{r}{2}} + e^{\frac{r}{2}}(t\gamma_0 + x^1\gamma_1 + \alpha(x^2\gamma_2 + x^3\gamma_3))\right)\zeta_i^- \ , \ \gamma_4\zeta_i^- = -\zeta_i^-$$

Bianchi I: Anisotropic $AdS_3 \times \mathbb{R}^2$

- substituting back in the Killing spinor equations we get $\alpha = 0$, the Killing spinor does not depend on the $\mathbb{R}^2$ direction.

$$\epsilon_i = e^{\frac{r}{2}} \zeta_i^+ \ , \ \gamma_4 \zeta_i^+ = \zeta_i^+$$
$$\epsilon_i = \left(e^{-\frac{r}{2}} + e^{\frac{r}{2}} (t\gamma_0 + x^1 \gamma_1)\right) \zeta_i^- \ , \ \gamma_4 \zeta_i^- = -\zeta_i^-$$

- Above projection breaks 1/2 of the supersymmetry in each of $\zeta^\pm$.
- also get additional projection conditions from KSE

$$\gamma_{23} \zeta_i^\pm = \epsilon_i{}^k \zeta_k^\pm$$
$$|Z| = |g_R| = \frac{\sqrt{6}}{3}$$

- The projection above breaks half of the remaining supersymmetries in each of $\zeta_\pm$.
- Thus each of $\zeta_\pm$ generate $\frac{1}{4}$ of the supersymmetry. Thus the anisotropic $AdS_3 \times \mathbb{R}^2$ is a $\frac{1}{2}$ BPS solution.

Bianchi III

- Bianchi III metric

$$ds^2 = L^2\left(e^{2\beta r}dt^2 + dr^2 + e^{2\beta r}(\omega^2)^2 + (\omega^1)^2 + (\omega^3)^2\right)$$

- the invariant one forms are

$$\omega^1 = e^{-x^1}dx^2 \;,\omega^2 = dx^3 \;,\omega^3 = dx^1$$

- Gauge field $A^I = B^I L\omega^1$

- Killing spinor equations in the above background are

$$e^{-\beta r}\gamma_0\partial_t\epsilon_i - \frac{\beta}{2}\gamma_4\epsilon_i - \frac{i}{\sqrt{6}}\left(\frac{1}{2}Z\gamma_{13}\epsilon_i + Lg_R{\epsilon_i}^k\epsilon_k\right) = 0$$

$$e^{x^1}\gamma_1\partial_{x^2}\epsilon_i - \frac{\gamma_3}{2}\epsilon_i + Lg_R B^I V_I\gamma_1{\epsilon_i}^k\epsilon_k + \frac{i}{\sqrt{6}}\left(-Z\gamma_{13}\epsilon_i + Lg_R{\epsilon_i}^k\epsilon_k\right) = 0$$

$$e^{-\beta r}\gamma_2\partial_{x^3}\epsilon_i + \frac{\beta}{2}\gamma_4\epsilon_i + \frac{i}{\sqrt{6}}\left(\frac{1}{2}Z\gamma_{13}\epsilon_i + Lg_R{\epsilon_i}^k\epsilon_k\right) = 0$$

$$\gamma_3\partial_{x^1}\epsilon_i + \frac{i}{\sqrt{6}}\left(-Z\gamma_{13}\epsilon_i + Lg_R{\epsilon_i}^k\epsilon_k\right) = 0$$

$$\gamma_4\partial_r\epsilon_i + \frac{i}{\sqrt{6}}\left(\frac{1}{2}Z\gamma_{13}\epsilon_i + Lg_R{\epsilon_i}^k\epsilon_k\right) = 0$$

Bianchi III

- The reduced equations are

$$\gamma_0\partial_t\epsilon_i + \gamma_2\partial_{x^3}\epsilon_i = 0$$
$$e^{-\beta r}\gamma_0\partial_t\epsilon_i - \frac{\beta\gamma_4}{2}\epsilon_i + \gamma_4\partial_r\epsilon_i = 0$$
$$e^{-\beta r}\gamma_2\partial_{x^3}\epsilon_i + \frac{\beta\gamma_4}{2}\epsilon_i - \gamma_4\partial_r\epsilon_i = 0$$
$$e^{x^1}\gamma_{13}\partial_{x^2}\epsilon_i + \partial_{x^1}\epsilon_i + \frac{\epsilon_i}{2} + Lg_R B^I V_I\gamma_{13}\epsilon_i{}^k\epsilon_k = 0$$

- The AdS_3 part of the Killing spinor will preserve some supersymmetry provided we assume that the Killing spinor does not depend on the x^2, x^3 coordinates.

- The independent Killing spinor solutions are

$$\epsilon_i = e^{\frac{\beta r}{2}}\zeta_i^+ \ , \ \gamma_4\zeta_i^+ = \zeta_i^+$$
$$\epsilon_i = \left(e^{-\frac{\beta r}{2}} + e^{\frac{\beta r}{2}}(t\gamma_0 + x^3\gamma_2)\right)\zeta_i^- \ , \ \gamma_4\zeta_i^- = -\zeta_i^-$$
$$\gamma_{13}\zeta_i^\pm = \epsilon_i{}^k\zeta_k^\pm \ , \ 4L^2 g_R^2 (B_I V^I)^2 = 1$$

Bianchi III

- Substituting back in the KSE, we get additional constraints on the parameters and no new projections

$$Lg_R = Z \ , \ \beta = \sqrt{\frac{3}{2}} Z$$

- As in the $AdS_3 \times \mathbb{R}^2$ case, the Bianchi III solution is 1/2 BPS.

- These are one parameter family of solutions labelled by the central charge Z.

- When Z takes the special value $Z = g_R = \sqrt{\frac{2}{3}}$ ($AdS_3 \times \mathbb{R}^2$ value) we get

$$L = 1 \ , \beta = 1$$

- The metric becomes the isotropic $AdS_3 \times \mathbb{H}^2$

$$ds^2 = \left(e^{2r} dt^2 + dr^2 + e^{2r} (dx^3)^2 + e^{-x^1} (dx^2)^2 + (dx^1)^2 \right)$$

Summary

Summary

- We analyzed the supersymmetry of Bianchi attractors in $\mathcal{N} = 2$ $d = 4, 5$ gauged supergravity.
- In $d = 4$ we considered a gauged supergravity coupled to vector and hyper multiplets with a generic gauging of the symmetries of the hyper Kähler manifold.
- In $d = 4$ the homogeneous symmetries are along the two spatial directions and the corresponding symmetry groups are of two types namely Bianchi I and Bianchi II.
- In the Bianchi I case, we constructed a BPS $AdS_2 \times \mathbb{R}^2$ geometry sourced by time like gauge fields.
- The radial spinor preserves $1/4$ of the supersymmetry.
- In the Bianchi II case, we constructed $AdS_2 \times \mathbb{H}^2$ solution sourced by time like gauge fields and find that the radial spinor breaks all of the supersymmetry.

Summary

- In $d = 5$ we considered $\mathcal{N} = 2, d = 5$ gauged supergravity coupled to n_V vector multiplets and n_H hypermultiplets and a generic gauging of scalar manifold and R symmetry.
- From gaugino condition: when none of the fermionic shifts vanish there are no susy Bianchi attractor geometries.
- When the central charge satisfies an extremization condition at the attractor point some of the fermionic shifts vanish and BPS solutions are possible.
- The allowed solutions are analogues of Bianchi type solutions in Einstein-Maxwell systems with massless gauge fields and a cosmological constant.

Summary

- In the Bianchi I case, we studied the maximally supersymmetric $\mathcal{N} = 2$ AdS_5 solution.
- We also find a new anisotropic $\frac{1}{2}$ BPS $AdS_3 \times \mathbb{R}^2$ solution.
- In the Bianchi III case, we find a new class of 1/2 BPS solutions labelled by the central charge.
- For a special value of the central charge the Bianchi III solution reduces to $AdS_3 \times \mathbb{H}^2$.
- To the best of our knowledge, these are the first examples of anisotropic Bianchi geometries that preserve supersymmetry.

Future work

- The gaugino conditions can be analyzed with a blackening factor for the near horizon geometries and using the extremization principle for the central charge one can contruct analytic interpolating solutions.
- These would represent new extremal black hole geometries in gauged supergravity.
- The central charge Z determined from the supergravity side can be calculated in the CFT side and matched.
- It will be interesting to find Bianchi solutions in theories with more supersymmetry.
- It may be possible to uplift these solutions to type IIB supergravity.

Thank You!

appendix

Life is not complicated lets not complicate it. go back