Scattering amplitudes in Chern-Simons matter theories

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Based on

References:

K.I, S.Jain, S.Mazumdar, S.Minwalla, V.Umesh, S.Yokoyama, **arXiv: 1505.06571**, JHEP 1510 (2015) 176.

S.Jain, M.Mandlik, S.Minwalla, T.Takimi, S.Wadia, S.Yokoyama, **arXiv:1404.6373,** JHEP 1504 (2015) 129.

K.I, S.Jain, P.Nayak, V.Umesh, **arXiv :1710.04227**

K.I, S.Jain, S.Majumdar, P.Nayak, T.Neogi, T.Sharma, R.Sinha, V.Umesh, **arXiv : 1711.02672**

Introduction

Crossing symmetry

Chern-Simons matter theories

Aharanov-Bohm scattering and unitarity

Bosonization duality

Scattering in Large N Chern-Simons gauge theories

Aharanov-Bohm phases and anyonic channels

Modified crossing rules

Dyson-Schwinger Series and Off-shell Correlators

Large N counting

Correlators exact to all loops

S matrices in Supersymmetric theories

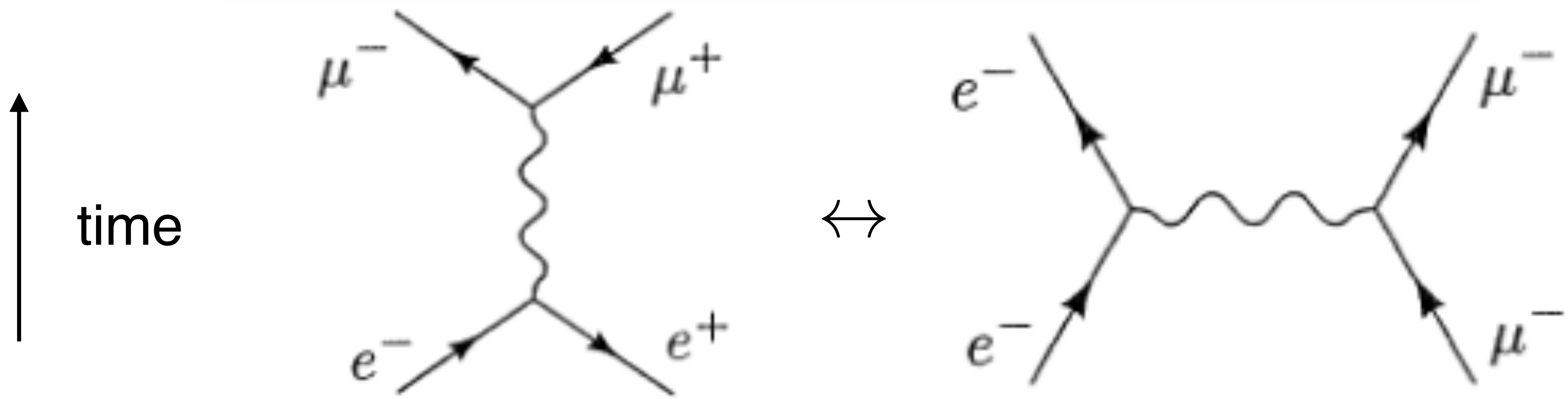
Summary

Crossing symmetry

Crossing Symmetry

- According to traditional wisdom in a relativistic QFT the **scattering amplitude has crossing symmetry**.
- Crossing symmetry is a **statement of analytic continuation of amplitudes**. Eg:

$$S(e^+e^- \to \mu^+\mu^-) \leftrightarrow S(e^-\mu^- \to e^-\mu^-)$$



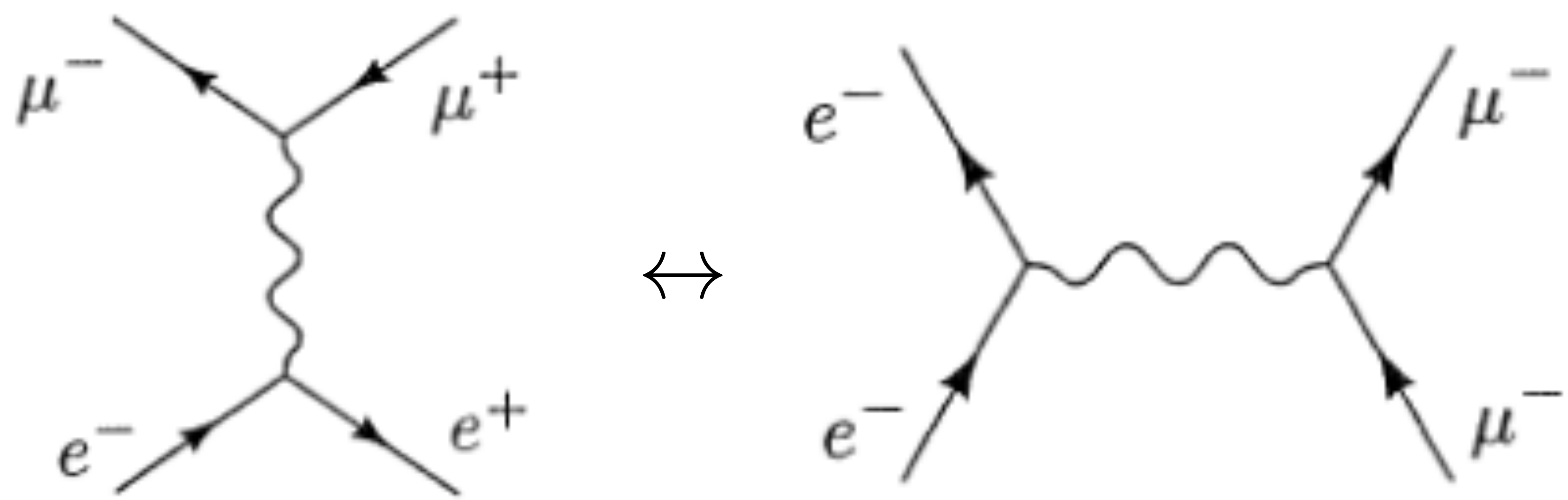
- The S matrix for any process involving a **particle of momentum p**, **in the initial state** is equal to the S matrix for a process with **an anti-particle of momentum k=-p in the final state**.

Crossing Symmetry

- For 2 to 2 scattering it is convenient to express the amplitude in terms of **Mandelstam variables**

$$s = (p_1 + p_2)^2 \ , \ t = (p_1 - p_3)^2 \ , \ u = (p_1 - p_4)^2$$
$$s + t + u = 4m^2$$

Peskin & Schroeder



$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{8e^2}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right] \qquad s \leftrightarrow t \qquad \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{8e^2}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]$$

- Under $s \leftrightarrow t$ the two amplitudes **analytically continue** into one another.

Crossing Symmetry

- In a general QFT Crossing symmetry can be shown from the following general principles **Gribov**.
  - S matrix is a **function of kinematical invariants**.

$$S(s, t, u, g_i)$$

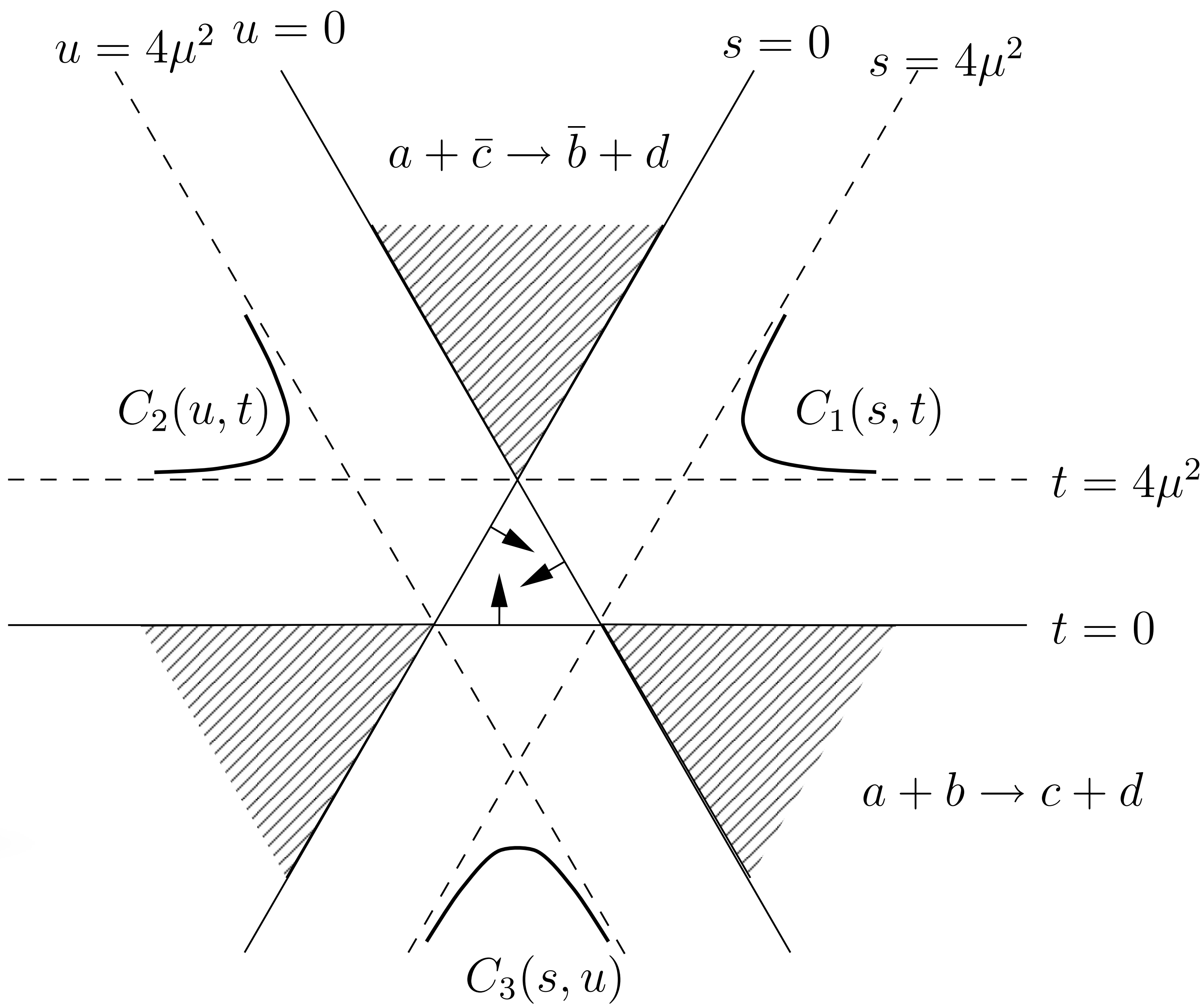
  - It is an **analytic function of its arguments**. Terms like $\theta(p_{i0})$ do not appear.
  - All **singularities** are determined by **physical masses of intermediate state particles**
- A related principle is unitarity, i.e the total probability of all possible processes at any given energy is unity.

$$SS^{\dagger} = 1$$

Crossing in the Mandelstam plane

$$S \propto \frac{1}{t - m_i}$$

t channel



$$S \propto \frac{1}{u - m_i}$$

u channel

$$S \propto \frac{1}{s - m_i}$$

s channel

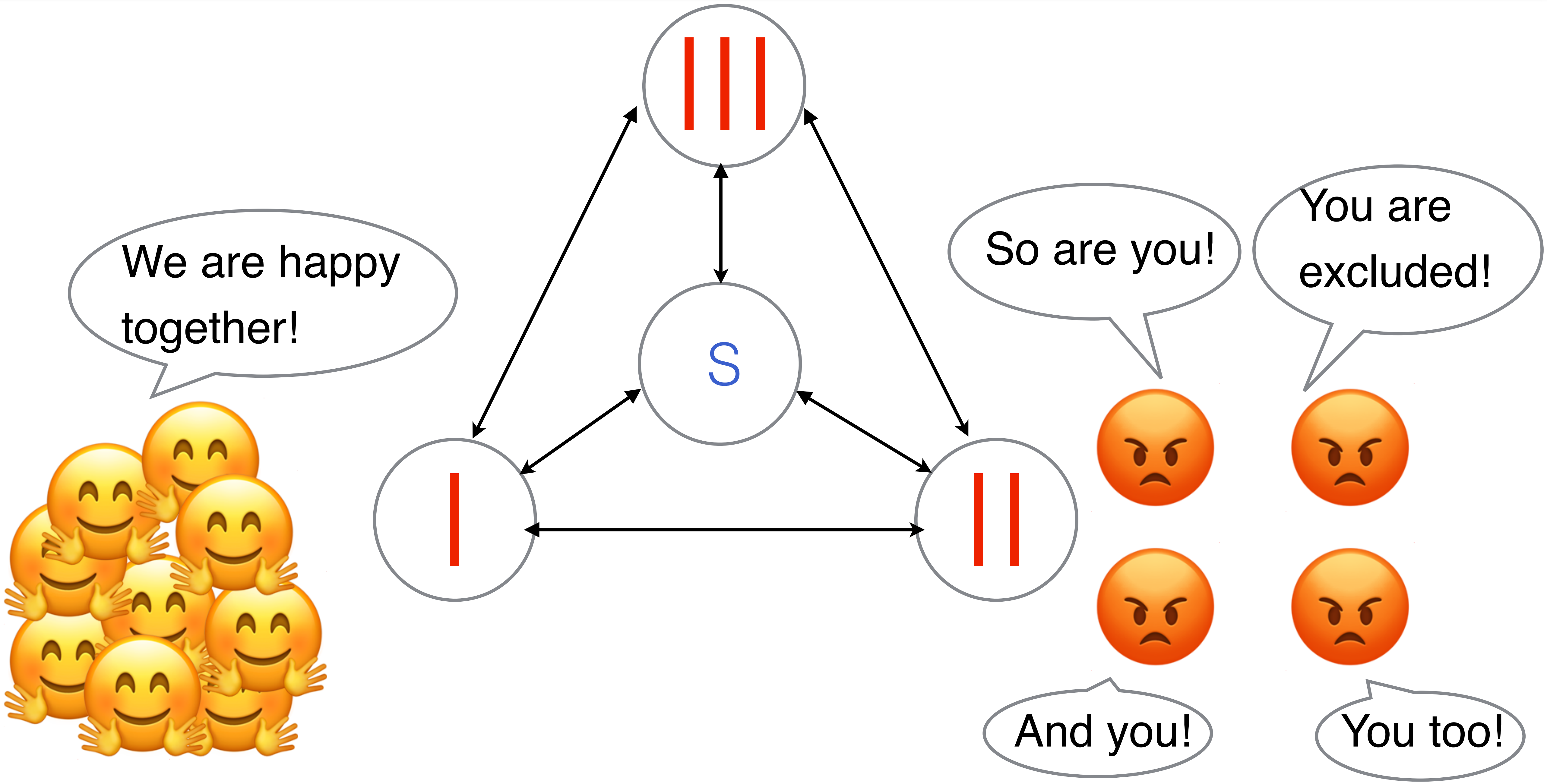
$$s + t + u = 4m^2$$

Fig. 1.2. Crossing reactions on the Mandelstam plane

Gribov

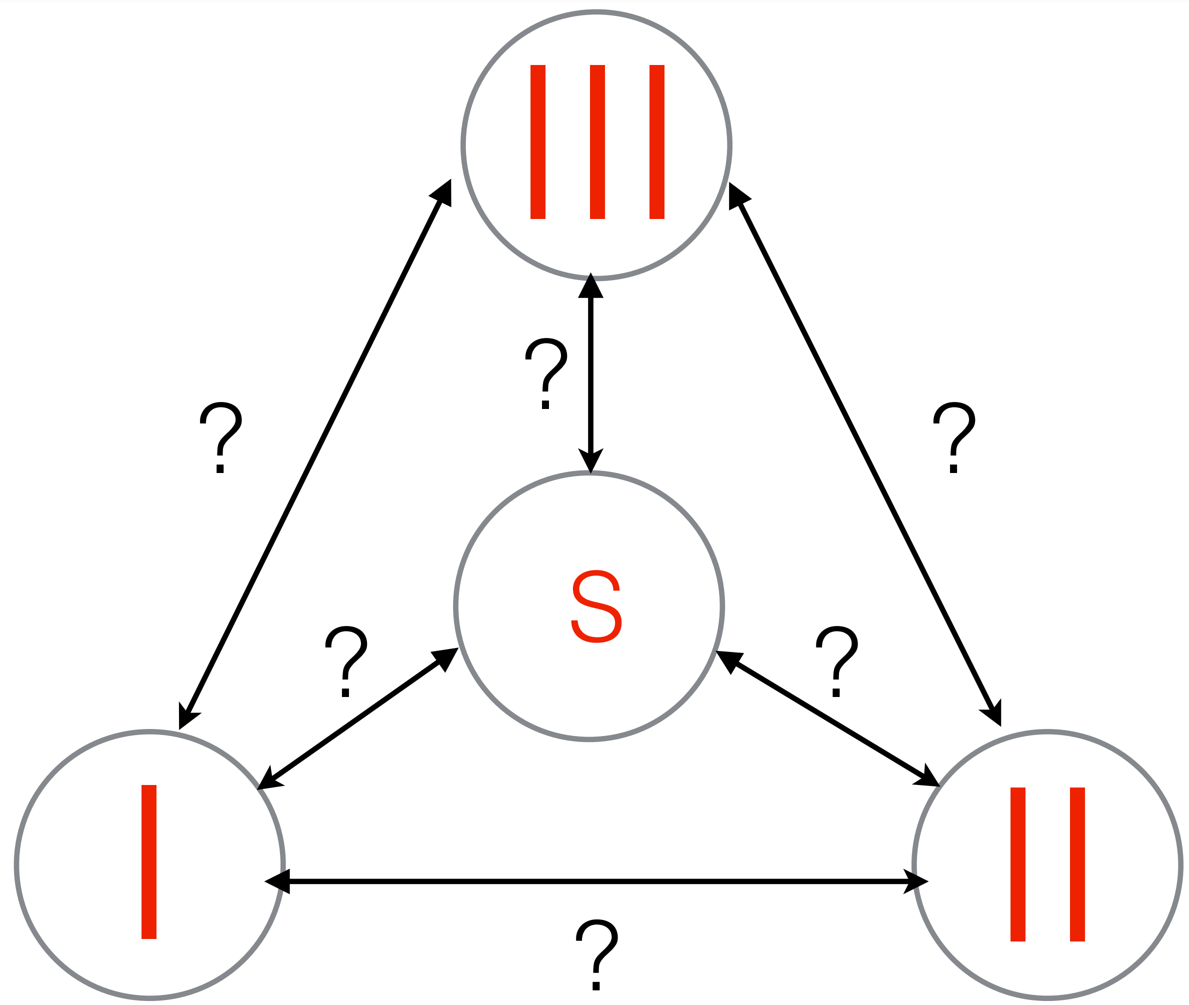
There exists **a single function that can be analytically continued to different channels of scattering**.

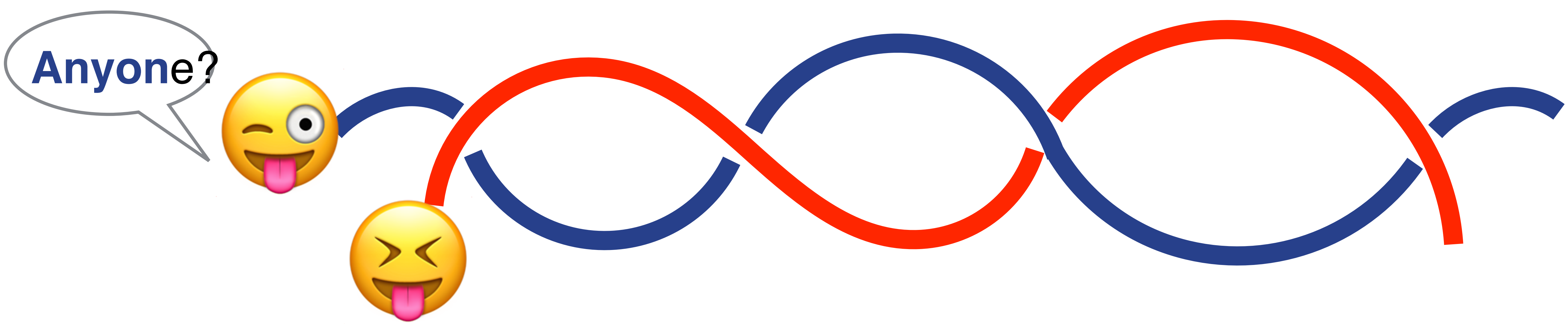
Crossing Symmetry in a usual QFT



Depending on whether the scattering quanta is bosonic or fermionic the **amplitude analytically continues up to an overall phase**.

Crossing Symmetry in the presence of anyons





Does crossing symmetry hold when the scattering particles have anyonic statistics?

Modified Crossing rules

- The anyonic channel S matrix in **ANY** Chern-Simons matter theory takes the following form

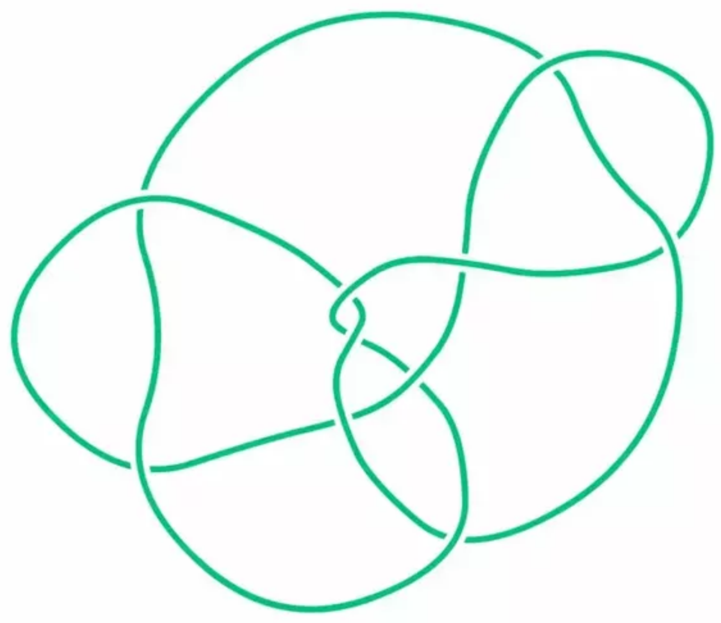
$$\mathcal{S}(s,\theta) = 8\pi\sqrt{s}\cos(\pi\nu)\delta(\theta) + i\frac{\sin(\pi\nu)}{\pi\nu}\mathcal{T}^{S;naive}(s,\theta)$$

- $\mathcal{T}^{S;\text{naive}}$ is the matrix obtained from **naive analytic continuation** from any of the non-anyonic channels.

S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama

How do we answer this question?

Chern-Simons matter theories



Chern-Simons theories

- **Physics in 2+1 dimensions** has interesting features and intriguing surprises.
- There exists a new type of gauge theory completely different from the usual Maxwell theory called **Chern-Simons theory**.
- The Pontryagin density in 3+1 dimensions can be written as a **total derivative**

$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) = 4\partial_\sigma(\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho)$$

- The boundary term has the same form as the Chern-Simons Lagrangian.
- **Pure Chern-Simons theory** is **topological**. The source free classical equations of motion

$$L = \frac{k}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

$$\frac{k}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0$$

- Solutions in the free theory are pure gauge.

Non-abelian Chern-Simons theories

- Non-abelian Chern-Simons action

$$S = \int d^3x \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho)$$

- under a general **gauge transformation**

$$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

- the action changes by a **boundary term**

$$S \to S - 8\pi^2 \kappa w(g) \ , \ w(g) \in \mathcal{Z}$$

- Thus gauge invariance of the quantum amplitude e^{iS} requires

$$\boxed{\kappa = \frac{\mathrm{Integer}}{2\pi}}$$

- The **Chern-Simons level is quantized**, hence the theory has no running parameters.

Chern-Simons matter theories: Aharanov-Bohm phase

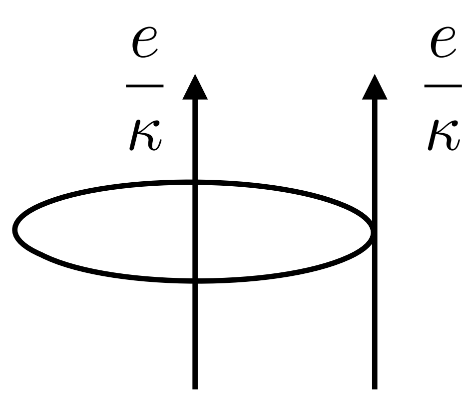
- When coupled to charged matter

$$\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu_{matter}$$

- the **CS gauge field attaches magnetic fluxes to the particles**

$$\rho = \kappa B \ , \ \ J^i = \kappa \epsilon^{ij} E_j$$

- **Adiabatic excursion** of such particles leads to the **Aharanov-Bohm effect**



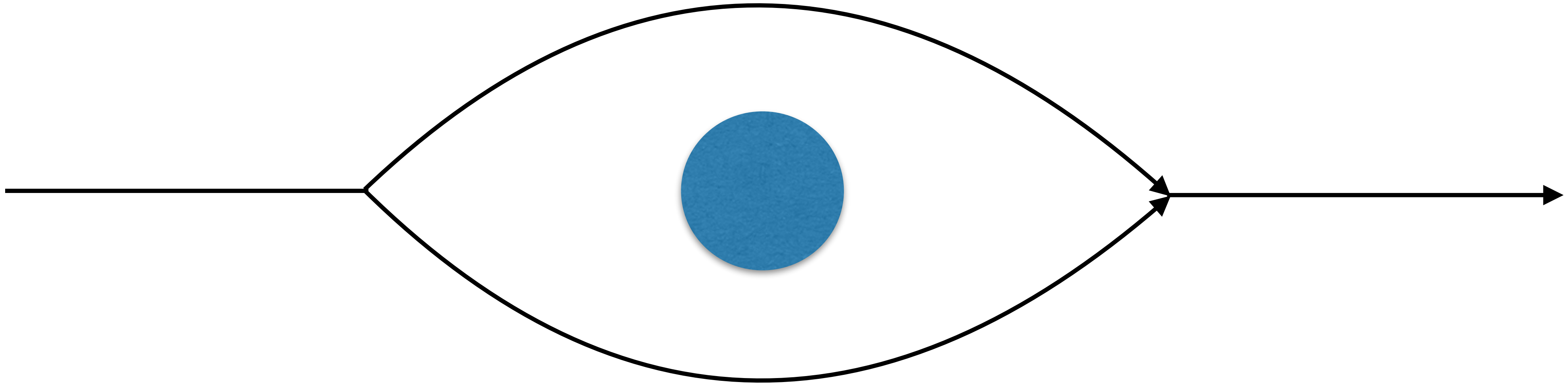
$$\text{AB phase} = e^{ie\int A.dx} = e^{i\frac{e^2}{\kappa}}$$

- **Adiabatic excursion** of such particles leads to the **Aharanov-Bohm effect**

- This phase is interpreted as the **point particle explanation of anyonic statistics**.

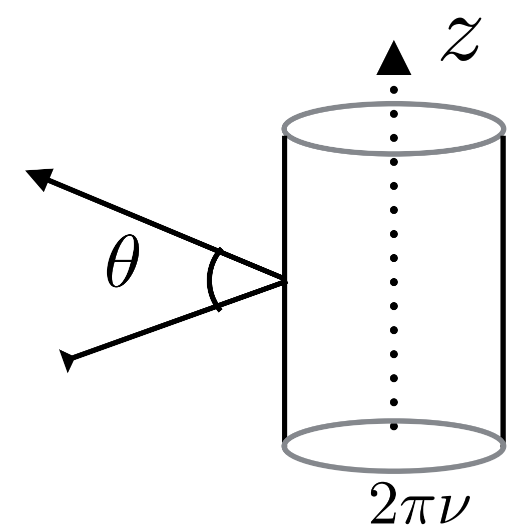
The role of Aharanov-Bohm phase in scattering

Aharanov-Bohm scattering



Aharanov, Bohm; Ruijsennars; Jackiw, Pi

Non-Relativistic Aharanov- Bohm scattering



- Scattering of a unit charged particle of a flux tube

- Schrödinger problem

$$\left(-\frac{1}{2m}(\nabla + 2\pi i G\nu)^2 - \frac{\kappa^2}{2m}\right)\psi = 0$$

$$G_{ij} = \frac{\epsilon_{ij}}{2\pi}\partial_j \ln r$$

- Boundary conditions for the wavefunction:
  - **regularity at the origin**
  - **at large r, reduces to the incoming wave**

- The scattering amplitude is read off by writing the wave function as a **plane wave and a scattered part**.

$$\psi(r,\theta) = e^{-ikx} + \frac{h(\theta) e^{-\frac{i\pi}{4}} e^{ikr}}{\sqrt{2\pi k r}}$$

Aharanov-Bohm scattering and unitarity

$$h(\theta) = 2\pi(\cos\pi\nu - 1)\delta(\theta) + \sin\pi\nu(\mathrm{Pv}\cot\frac{\theta}{2} - i\mathrm{Sgn}(\nu))$$

- Notice the **peculiar delta function piece modulated by the anyonic phase**.
- The delta function piece was **originally missed** by **Aharanov-Bohm!!**
- **Without the delta function** term, the **scattering amplitude fails unitarity!!** **Ruijsennars; Bak, Jackiw, Pi**
- Unitarity equation for a general S matrix: $S(s,\theta) = I + iT(s,\theta)$

$$-i(T(s,\theta) - T^*(s,-\theta)) = \frac{1}{8\pi\sqrt{s}}\int d\alpha T(s,\alpha)T^*(s,-(\alpha-\theta))$$

- Let us decompose the T matrix as follows

$$T(\sqrt{s},\theta) = H(\sqrt{s})\ x(\theta) + W_1(\sqrt{s}) - iW_2(\sqrt{s})\delta(\theta)$$

Aharanov-Bohm scattering and unitarity

- The **unitarity conditions** take the form

$$H - H^* = \frac{1}{8\pi\sqrt{s}}(W_2 H^* - H W_2^*)$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H H^*)$$

$$W_1 - W_1^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_1^* - W_2^* W_1) - \frac{i}{4\sqrt{s}}(H H^* - W_1 W_1^*)$$

- For Aharanov-Bohm scattering

$$H = 4\sqrt{s}\sin(\pi\nu)\ ,\ W_1 = -4\sqrt{s}\sin(\pi\nu)\mathrm{Sgn}(\nu)\ ,\ W_2 = 8\pi\sqrt{s}(\cos(\pi\nu) - 1)$$

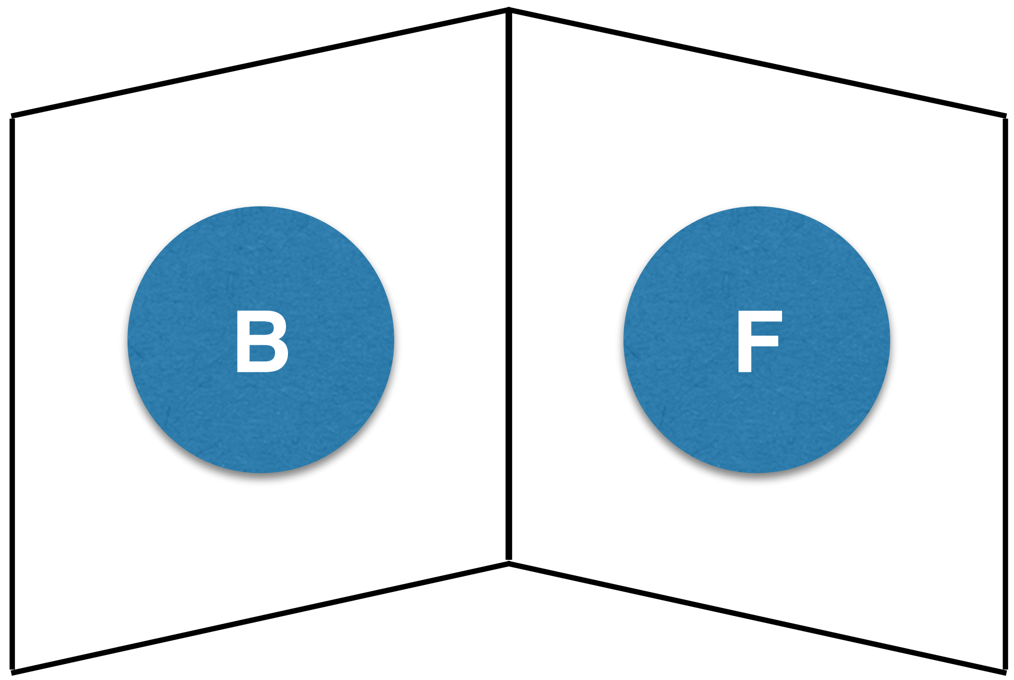
- The first and third equations are **trivially obeyed**.

- The second equation is obeyed due to the trignometric identity.

$$2(1 - \cos(\pi\nu)) = (1 - \cos(\pi\nu))^2 + \sin^2(\pi\nu)$$

- **without the $\cos\pi\nu$ term unitarity condition is not satisfied**.

Large N gauge theories in three dimensions



Bosonization duality

Aharony,Bardeen,Benini,Chang,Frishman,Giombi,Giveon,Gur-Ari,Gurucharan,K.I, Karch,Kutasov,Jain,Maldacena,Mandlik,Minwalla,Moshe,Sharma,Prakash, Takimi,Trivedi,Seiberg,Sonnenschein,Tong,Yacoby,Yin,Yokoyama,Wadia,Witten, Zhiboedov

Chern-Simons coupled to fermions

- Start with an example of a gauge theory: $U(N_F)$ **Chern-Simons coupled to fundamental fermions (regular fermion)**

$$S = \int d^3x \left(i\epsilon^{\mu\nu\rho} \frac{\kappa_F}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \bar{\psi}^i \gamma^\mu D_\mu \psi_i + m_f \bar{\psi}^i \psi_i \right)$$

Complex fermion ψ_i $\longrightarrow$ Transforms as a vector under $U(N_F)$

(Fundamental representation)

$\bar{\psi}^i$ $\longrightarrow$ **(Anti-Fundamental representation)**

Gauge field $A^a (T^a)_i^{\ j}$ $\longrightarrow$ **(Adjoint representation)**

Chern-Simons coupled to bosons

- $U(N_B)$ **Chern-Simons coupled to fundamental bosons**

$$S = \int d^3x \Big(i\epsilon^{\mu\nu\rho}\frac{\kappa_B}{4\pi}\mathrm{Tr}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) + D_\mu\bar{\phi}D^\mu\phi$$
$$+\sigma\bar{\phi}\phi + N_B\frac{m_B^2}{b_4}\sigma - N_B\frac{\sigma^2}{2b_4}\Big)$$

Complex scalar $\quad \phi_i \longrightarrow$ Transforms as a vector under $U(N_B)$

(Fundamental representation)

$\bar{\phi}^i \longrightarrow$ (Anti-Fundamental representation)

Gauge field $\quad A^a(T^a)_i^{\ j} \longrightarrow$ (Adjoint representation)

$\sigma \longrightarrow$ **(Auxiliary field)**

- **Wilson-Fisher limit (critical boson)**

$$b_4 \to \infty \ , \ m_B \to \infty \ , \ 4\pi\frac{m_B^2}{b_4} = \text{fixed}$$

Large N Bosonization duality

t' Hooft large N limit

$$\lambda_B = \lim_{\substack{N_B \to \infty \\ \kappa_B \to \infty}} \frac{N_B}{\kappa_B} \qquad \lambda_F = \lim_{\substack{N_F \to \infty \\ \kappa_F \to \infty}} \frac{N_F}{\kappa_F}$$

- $U(N_B)$ Chern-Simons coupled to fundamental bosons at Wilson-Fisher limit

dual

- $U(N_F)$ Chern-Simons coupled to fundamental Fermions

duality map

$$\begin{cases} \kappa_F = -\kappa_B \\ N_F = |\kappa_B| - N_B \\ \lambda_B = \lambda_F - \mathrm{Sign}(\lambda_F) \\ m_F = -m_B^{Crit} \lambda_B \end{cases}$$

- **Physical observables computed on one side , match with observables on the other side under the duality map.**

- Note the **strong-weak** nature of the duality!

Large N bosonization duality: supersymmetric version

- $d = 3, \mathcal{N} = 1$ supersymmetric Chern-Simons matter theory

$$S_{\mathcal{N}=1} = \int d^3x \Big[-\frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) - D^\mu\bar{\phi}D_\mu\phi - m_0^2\bar{\phi}\phi + i\bar{\psi}(\not{D} + m_0)\psi$$
$$-\frac{4\pi^2 w^2}{\kappa^2}(\bar{\phi}\phi)^3 - \frac{4\pi w m_0}{\kappa}(\bar{\phi}\phi)^2 + \frac{2\pi}{\kappa}(1+w)(\bar{\phi}\phi)(\bar{\psi}\psi)$$
$$-\frac{\pi}{\kappa}(1-w)\left((\bar{\phi}\psi)(\bar{\phi}\psi) + (\bar{\psi}\phi)(\bar{\psi}\phi)\right)\Big]$$

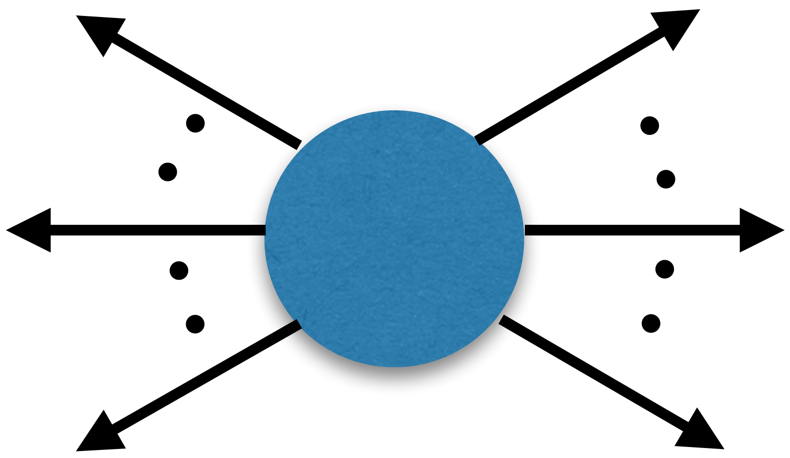
- The theory exhibits a **strong-weak self duality** under the duality map

$$\kappa' = -\kappa\ , \ \ N' = |\kappa| - N + 1\ , \ \ \lambda' = \lambda - \mathrm{Sgn}(\lambda)$$
$$m' = -m \qquad w' = \frac{3-w}{1+w}$$

$$\lambda = \frac{N}{\kappa}\ , N \to \infty, \kappa \to \infty$$

- **at w=1, the theory exhibits an enhanced $\mathcal{N} = 2$ supersymmetry.**

Scattering in Large N Chern-Simons gauge theories



Scattering in Large N Chern-Simons gauge theories

- Consider $2 \to 2$ scattering of quanta in representations R_1 and R_2 in U(N).
- The tensor product of two vector representations can be decomposed as

$$R_1 \times R_2 = \sum_m R_m$$

- Let us choose the convention that particles transform in the fundamental representation of U(N), it follows that anti-particles transform in the anti-fundamental representation.
- Easy to see from the mode expansion

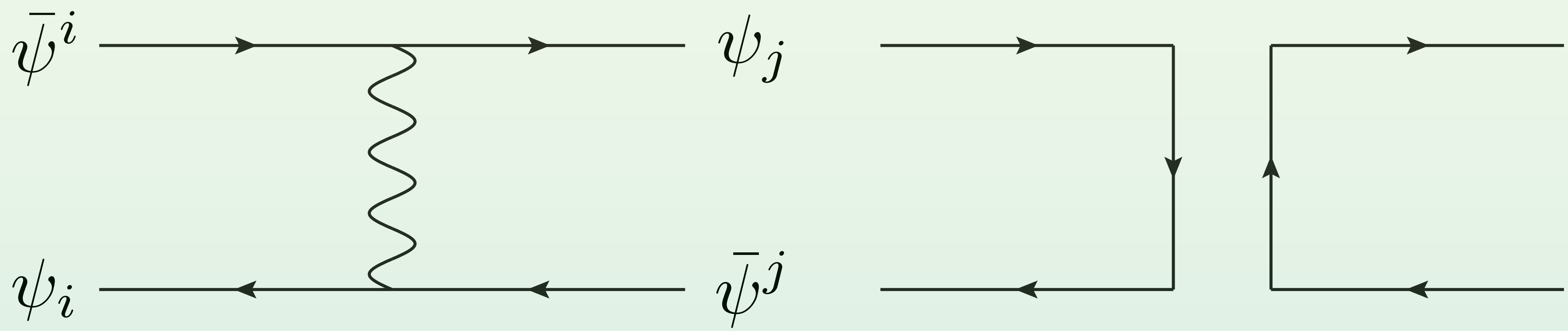
$$\phi_i(x) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left(a_i e^{ip.x} + (a_i)^\dagger_c e^{-ip.x}\right)$$

$$\bar{\phi}^i(x) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left((a^i)^\dagger e^{-ip.x} + (a^i)_c e^{ip.x}\right)$$

a_i particles $\qquad$ $(a^i)_c$ anti-particles

Double line notation

$$S = \int d^3x \left(i\epsilon^{\mu\nu\rho} \frac{\kappa_F}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \bar{\psi}^i \gamma^\mu D_\mu \psi_i + m_f \bar{\psi}^i \psi_i \right)$$



Tree level amplitude for four fermion scattering

- Note that this amplitude is of the order of

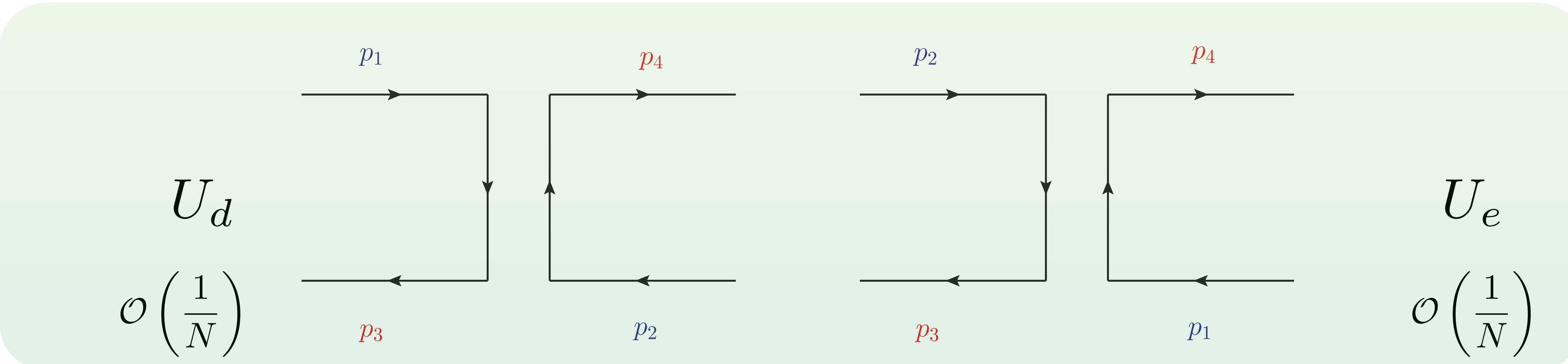
$$\frac{1}{\kappa_F} = \frac{\lambda_F}{N_F} \qquad \mathcal{O}\left(\frac{1}{N}\right)$$

- General feature of all planar 4 point amplitudes in Chern-Simons theories.

Channels of scattering

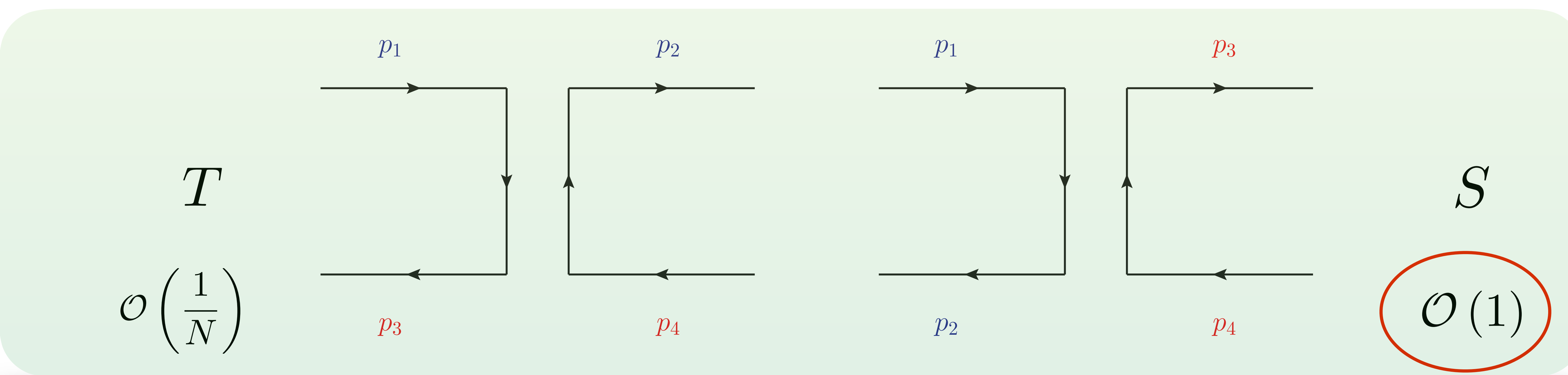
- Particle - Particle scattering = scattering of two fundamentals

$$F_i \otimes F_j = (\text{Symm})_{ij} \oplus (\text{ASymm})_{ij}$$



- Particle - Anti-Particle scattering = scattering of a fundamental and A.F

$$F_i \otimes AF^j = (\text{Adjoint})_i{}^j \oplus (\text{Singlet})\delta_i{}^j$$

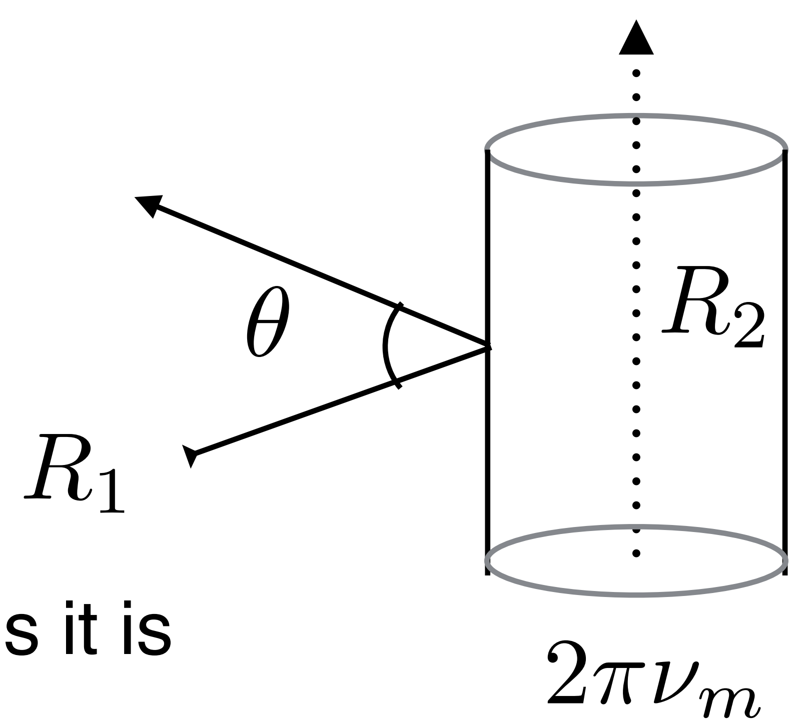


Aharanov-Bohm phase in each of the channels

- The Aharanov-Bohm phase of the quanta R_1 as it circles around quanta R_2 is $2\pi\nu_m$

$$\nu_m = \frac{4\pi}{\kappa} T_1^a T_2^a = \frac{2\pi}{\kappa}\left(C_2(R_1) + C_2(R_2) - C_2(R_m)\right)$$

- **Scattering amplitude in the m'th exchange channel: Aharanov-Bohm scattering of a unit charge particle off a flux tube of flux $2\pi\nu_m$**



If the phase vanishes it is like regular scattering.

If the phase does not vanish channel, it is like scattering of anyons.

Aharanov-Bohm phases in all the channels

- For U(N) the quadratic Casimirs are

$$C_2(F) = C_2(A) = \frac{N^2-1}{2N} \ , \ C_2(Sym) = \frac{N^2+N-2}{N}$$
$$C_2(ASym) = \frac{N^2-N-2}{N} \ , \ C_2(Adj) = N \ , \ C_2(Sing) = 0$$

- The anyonic phases are

$$\nu_{Sym} = \frac{1}{\kappa} - \frac{1}{N\kappa} \ , \ \nu_{Asym} = -\frac{1}{\kappa} - \frac{1}{N\kappa}$$
$$\nu_{Adj} = \frac{1}{N\kappa} \ , \ \nu_{Sin} = -\frac{N}{\kappa} + \frac{1}{N\kappa}$$

- At finite N all channels are anyonic, we do not know (yet..) how to compute such amplitudes in QFT!

Aharanov-Bohm phases in the large N limit

- The large N Aharanov-Bohm phases are

$$\nu_{Asym} \sim \nu_{Sym} \sim \nu_{Adj} \sim O\left(\frac{1}{N}\right), \nu_{Sing} \sim O(\lambda)$$

- For planar diagrams, the symmetric, anti-symmetric and adjoint channels are like regular scattering. (non-anyonic channels)

- Whereas the singlet channel is effectively anyonic!

- The T matrices themselves have the large N behavior

$$T_{Asym} \sim T_{Sym} \sim T_{Adj} \sim O\left(\frac{1}{N}\right) \quad , \quad T_{sing} \sim O(1)$$

- the unitarity equation $i(T^\dagger - T) = TT^\dagger$ in large N limit

$$i(T^\dagger_{non-anyonic} - T_{non-anyonic}) \sim \mathcal{O}\left(\frac{1}{N^2}\right)$$ **Linear**

$$i(T^\dagger_{anyonic} - T_{anyonic}) = T_{anyonic}T^\dagger_{anyonic}$$ **Non-linear**

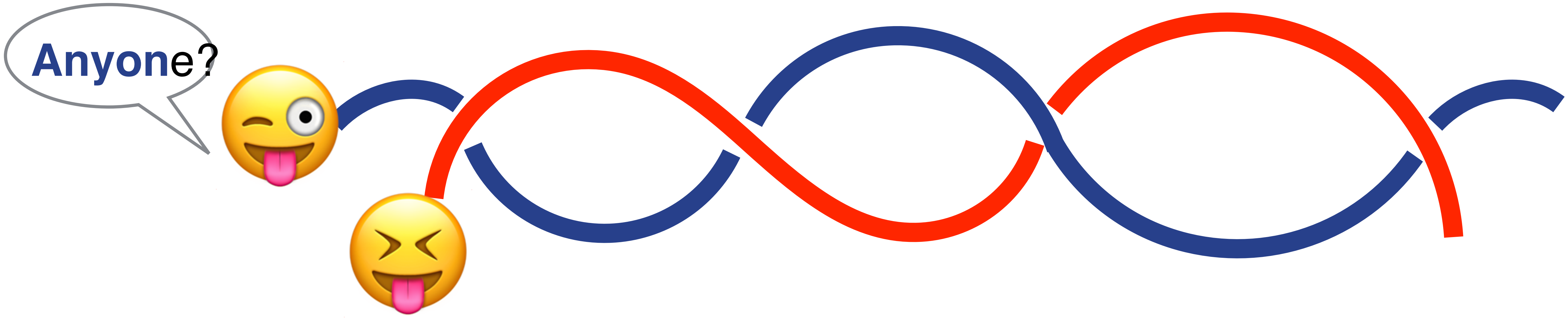
What does this teach us?

- In any U(N) Chern-Simons matter theory, **the symmetric, anti-symmetric and adjoint channels are non-anyonic in the large N limit**.
- The **singlet channel is effectively anyonic in the large N limit**.
- We **cannot compute the anyonic channel directly**!
- QFT says that the amplitude is an analytic function and **all channels of scattering are related by analytic continuation**.
- If so, what happens when we apply naive crossing rules to cross the S matrix from the non-anyonic channel to the anyonic channel?
- Observation: **Naive crossing symmetry rules** from **any of the non-anyonic** channels to the **anyonic channel** leads to a **non unitary S matrix**.

S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama

Modified Crossing rules

A conjecture



Modified Crossing rules

- The singlet (anyonic) channel S matrix in **ANY** Chern-Simons matter theory takes the following form

$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\text{naive}}(s,\theta)$$

- $\mathcal{T}^{S;\text{naive}}$ is the matrix obtained from **naive analytic continuation** from any of the non-anyonic channels.

S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama

Modified Crossing rules

- ***Tests of Universality***
  - **Unitarity of the S matrix**
  - **3d Bosonization duality**
  - **Consistency of non-relativistic limit with Aharanov-Bohm result.**

- All the tests have been explicitly verified for
  - U(N) Chern-Simons coupled to fundamental bosons.
  - U(N) Chern-Simons coupled to fundamental fermions.

    S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama
  - $\mathcal{N}=1,2$ supersymmetric Chern-Simons matter theories.

    K.I, S.Jain, S.Mazumdar, S.Minwalla, V. Umesh, S.Yokoyama
- Preliminary computations in $\mathcal{N}=3$ theories already show consistency with unitarity and 3d bosonization duality. **K.I, S.Jain, L.Janagal, S.Minwalla, A.Shukla**

- Further checks for $\mathcal{N}=3$ and computations for $\mathcal{N}=4,5,6$ are in progress.

Physical explanation of the conjecture

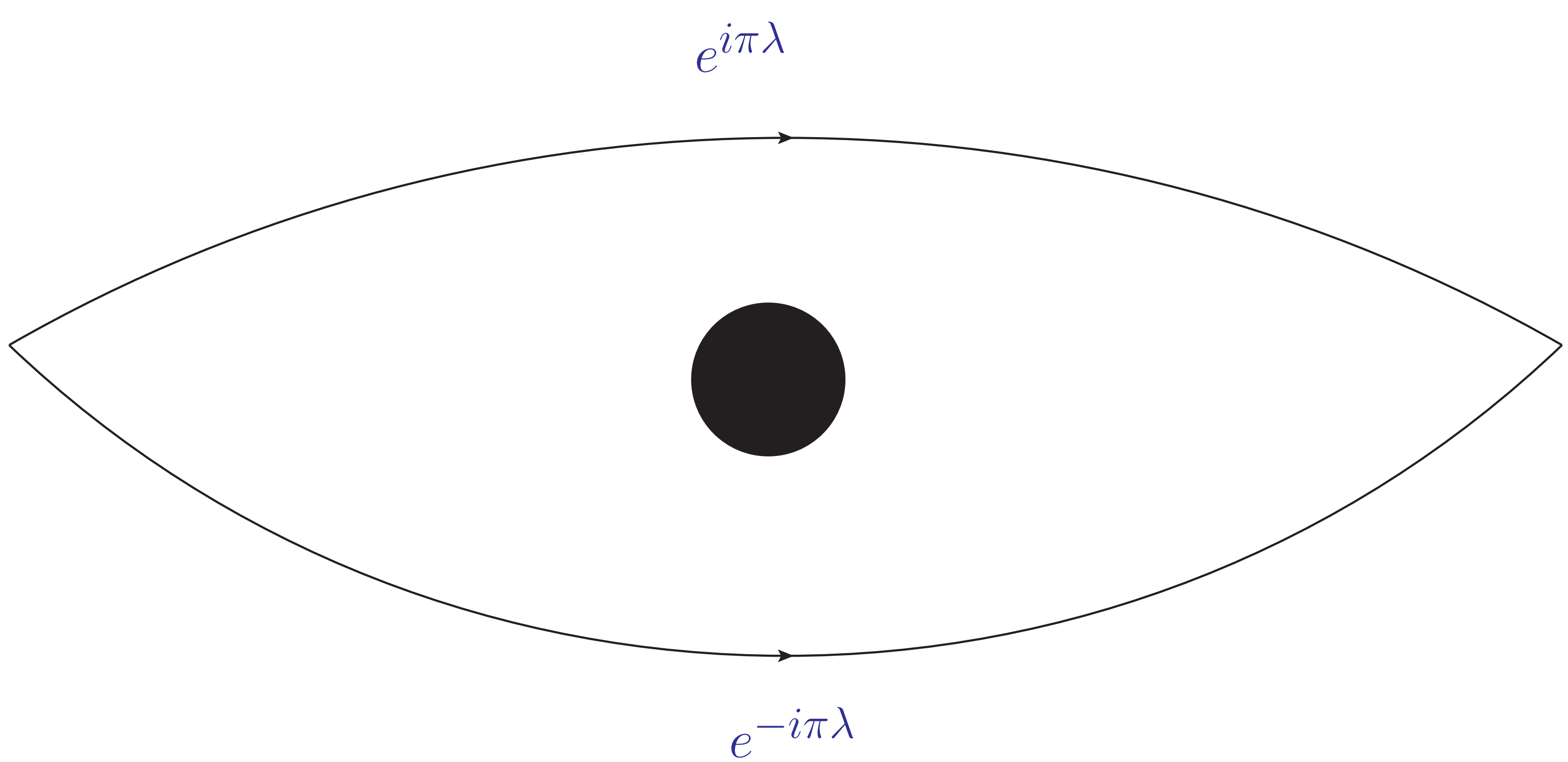
- Aharanov-Bohm result

$$h(\theta) = 2\pi(\cos\pi\nu - 1)\delta(\theta) + \sin\pi\nu(\mathrm{Pv}\cot\frac{\theta}{2} - i\mathrm{Sgn}(\nu))$$

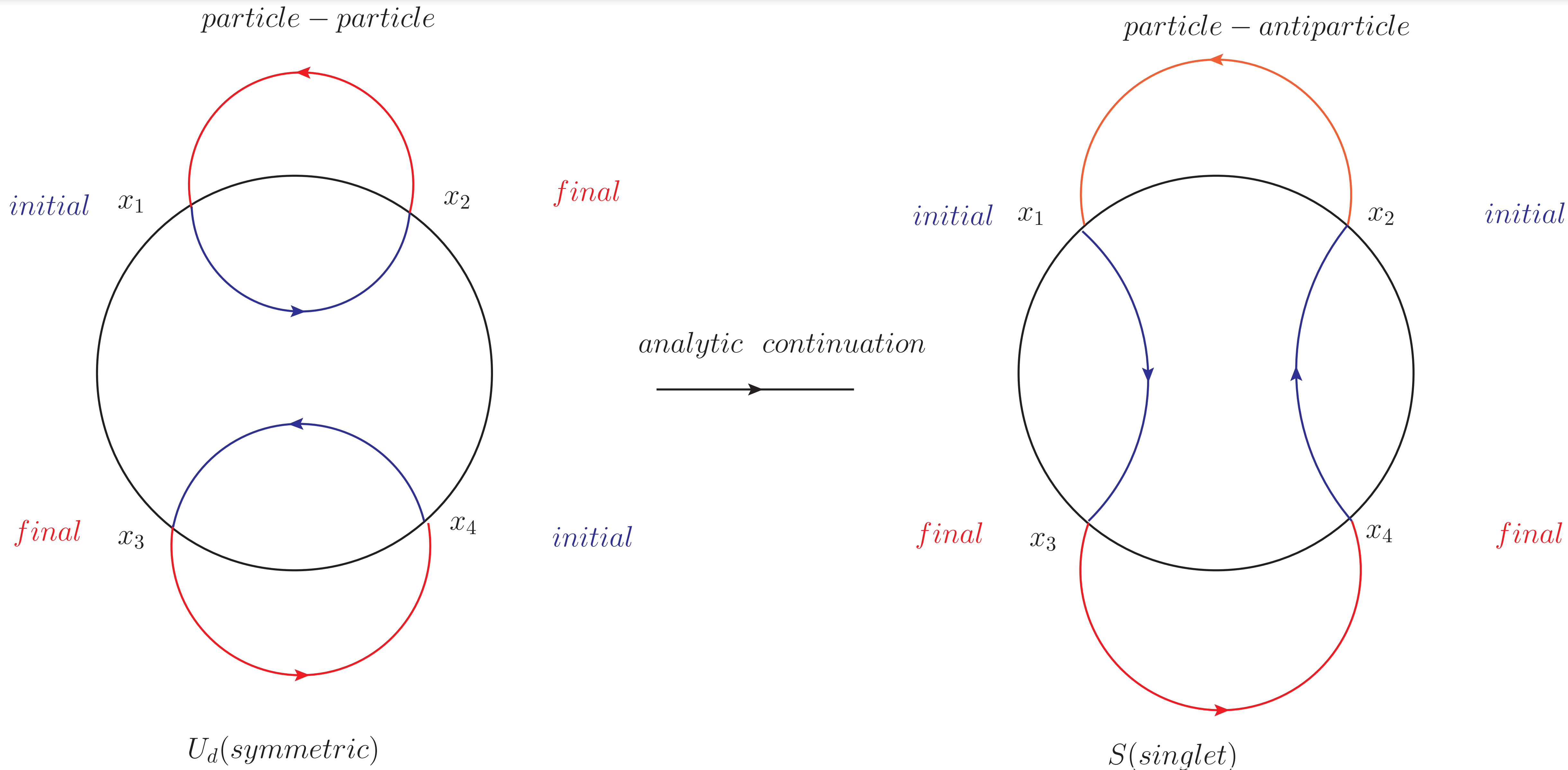
- Conjectured S matrix for anyonic channel

$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\mathrm{naive}}(s,\theta)$$

- $\cos(\pi\lambda)$ in the identity term is due to the **interference of the Aharanov-Bohm phases** of the incoming wave packets.



Physical explanation of the conjecture



- Attach Wilson lines to make correlators gauge invariant

$$T_{U_d} W_{U_d} \to T_S W_S$$

$$\frac{W_{U_d}}{W_S} = \frac{\oint \text{with 2 circular Wilson lines}}{\oint \text{with 1 circular Wilson line}} = \frac{\sin(\pi\lambda)}{\pi\lambda}$$ Witten

Dyson-Schwinger Series, Off-shell Correlators and exact all loop S matrices in large N

Aharony, Gur-Ari, Yacoby

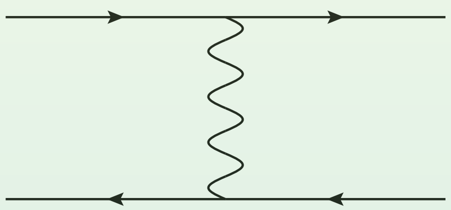
S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama

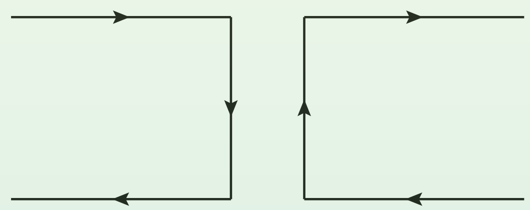
K.I, S.Jain, S.Mazumdar, S.Minwalla, V. Umesh, S.Yokoyama

K.I, S.Jain, L.Janagal, S.Minwalla, A.Shukla

Large N counting and planar diagrams

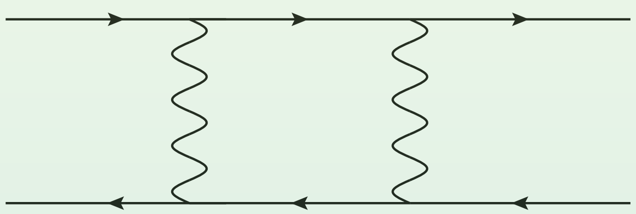
Tree

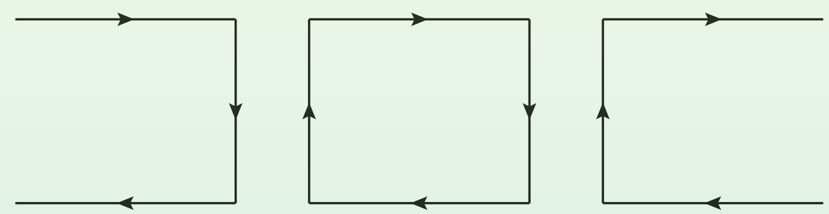




$$\frac{1}{\kappa} = \frac{\lambda}{N} \qquad \mathcal{O}\left(\frac{1}{N}\right)$$

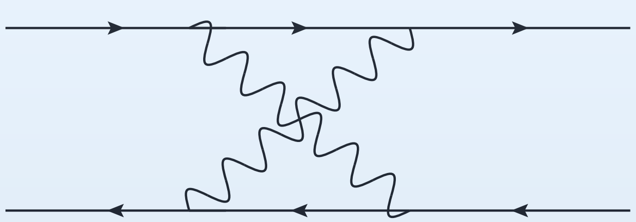
one loop (planar)

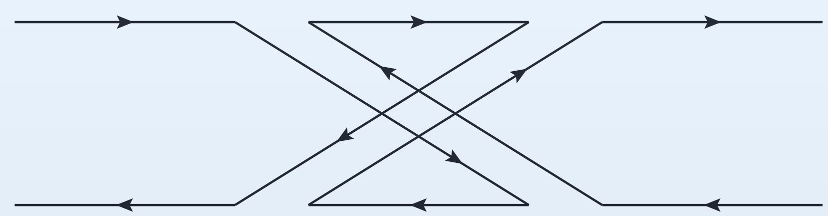




$$\frac{N}{\kappa^2} = \frac{\lambda^2}{N} \qquad \mathcal{O}\left(\frac{1}{N}\right)$$

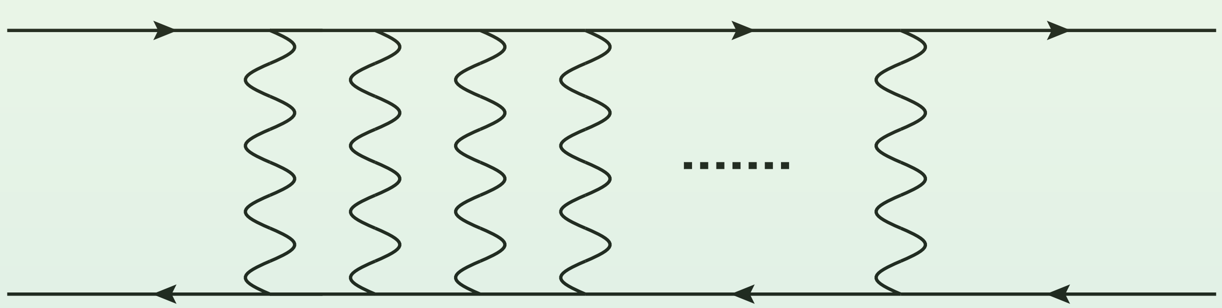
one loop (non planar)





$$\frac{1}{\kappa^2} = \frac{\lambda^2}{N^2} \qquad \mathcal{O}\left(\frac{1}{N^2}\right)$$

All loops (planar)



$$\frac{N^l}{k^{l+1}} = \frac{\lambda^{l+1}}{N} \qquad \mathcal{O}\left(\frac{1}{N}\right)$$

Summing up all loops

- Although, planar diagrams look simple, summing up an infinite set of Feynman diagrams is still a formidable task.

- the S matrix can be defined in terms of an onshell limit of off-shell correlators.

$$\langle \bar{\phi}(p_1)\phi(p_2)\bar{\phi}(p_3)\phi(p_4)\rangle|_{offshell} \xrightarrow{p_i^2+m^2=0} \langle \bar{\phi}(p_1)\phi(p_2)\bar{\phi}(p_3)\phi(p_4)\rangle|_{onshell}$$

- The off-shell correlator can be computed to all loops using the Dyson-Schwinger series in a suitable gauge.

- Then taking the onshell limit, gives the S matrix exact to all loops.

- We will see an explicit example in supersymmetric theories, the general idea is applicable to other theories as well.

Supersymmetric Chern-Simons matter theory

- General renormalizable $\mathcal{N}=1,2$ theory with one fundamental multiplet

$$\mathcal{S}_{\mathcal{N}=1} = -\int d^3x d^2\theta \Big[\frac{\kappa}{2\pi} Tr \Big(-\frac{1}{4} D_\alpha \Gamma^\beta D_\beta \Gamma^\alpha - \frac{1}{6} D^\alpha \Gamma^\beta \{\Gamma_\alpha, \Gamma_\beta\} - \frac{1}{24} \{\Gamma^\alpha, \Gamma^\beta\}\{\Gamma_\alpha, \Gamma_\beta\} \Big) - \frac{1}{2} (D^\alpha \bar{\Phi} + i \bar{\Phi} \Gamma^\alpha)(D_\alpha \Phi - i \Gamma_\alpha \Phi) + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa} (\bar{\Phi}\Phi)^2 \Big]$$

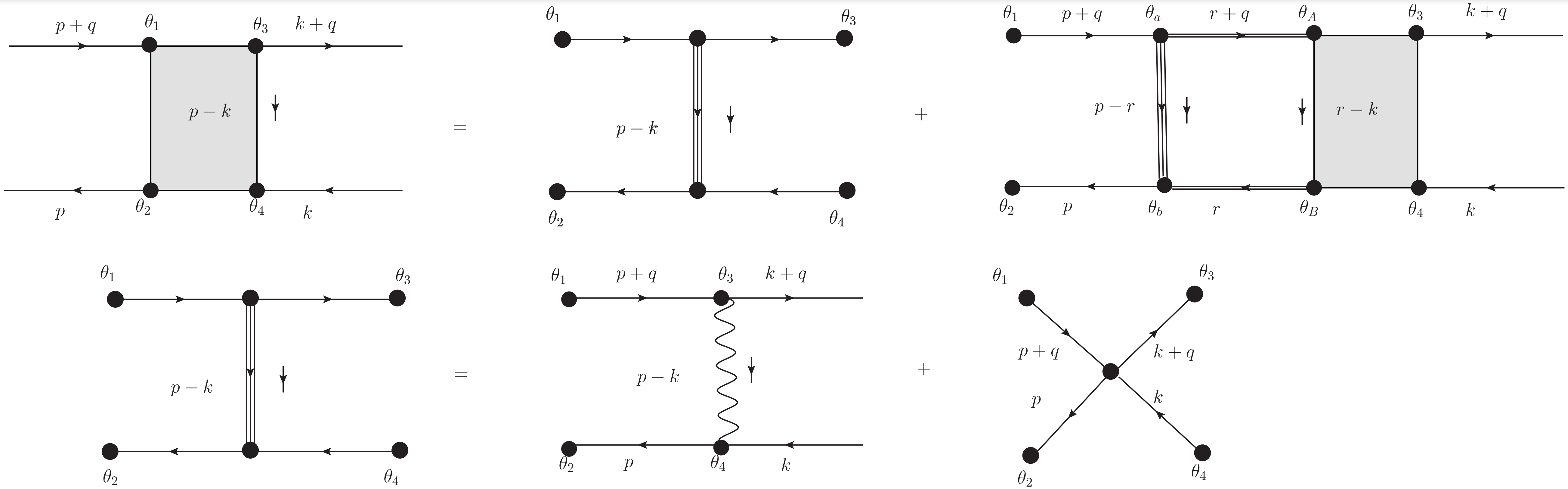
- Φ is a complex scalar superfield, Γ^α is a real superfield. $\alpha = +, -$

$$\Phi = \phi + \theta\psi - \theta^2 F \ , \bar{\Phi} = \bar{\phi} + \theta\bar{\psi} - \theta^2 \bar{F} \ ,$$
$$\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2 (2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta) \ .$$

- Integer parameters N, κ, matter self coupling w, bare mass m_0. At $w=1$ the supersymmetry is enhanced to $\mathcal{N}=2$

$$\mathcal{S}^L_{\mathcal{N}=2} = \int d^3x \Big[-\frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr} \Big(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \Big) + \bar{\psi} i \not{\mathcal{D}} \psi - \mathcal{D}^\mu \bar{\phi} \mathcal{D}_\mu \phi + \frac{4\pi^2}{\kappa^2} (\bar{\phi}\phi)^3 + \frac{4\pi}{\kappa} (\bar{\phi}\phi)(\bar{\psi}\psi) + \frac{2\pi}{\kappa} (\bar{\psi}\phi)(\bar{\phi}\psi) \Big]$$

Exact four point correlator to all loops



- The **integral equation** can be schematically written as

$$V(\theta_i, p_i) = V_0(\theta_i, p_i) + \int \frac{d^3 r}{(2\pi)^3} d^2\theta'_j V_0(\theta_i, \theta'_j, p_i, r)$$
$$P(\theta'_j, p_i + r) P(\theta'_j, r) V(\theta'_j, \theta_i, p_i)$$

- **Solved the integral equations exactly in the large N limit, for arbitrary values of the 't Hooft coupling λ** and determined the off-shell four point function.

Scattering a superfield

$$\begin{pmatrix} \Phi(\theta_1, p_1) \\ \bar{\Phi}(\theta_2, p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\Phi}(\theta_3, p_3) \\ \Phi(\theta_4, p_4) \end{pmatrix}$$

$$\mathcal{S}_B : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix} \, , \, \mathcal{S}_F : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix}$$

$$H_1 : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix} \, , \, H_2 : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_3 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \, , \, H_4 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \, , \, H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix}$$

- Supersymmetric ward identities relate the some of the processes in terms of others. **Not all processes are independent**. For N=1, there are 2 independent processes, for N=2 there is only one independent process!

Let us see some S matrices

S matrices in the non-anyonic channels for $\mathcal{N}=1$

- The **onshell limit directly gives the S matrix** for the T (adjoint), U_d symmetric and U_e (anti-symmetric) channels of scattering.

- The final answer can be covariantized and is gauge invariant.

- $$\mathcal{T}_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu (p-k)^\nu (p+k)^\rho}{(p-k)^2} + J_B(q,\lambda) \ ,$$
$$\mathcal{T}_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu (p-k)^\nu (p+k)^\rho}{(p-k)^2} + J_F(q,\lambda) \ ,$$

$$J_B(q,\lambda) = \frac{4\pi q}{\kappa}\frac{N_1 N_2 + M_1}{D_1 D_2} \ ,$$
$$J_F(q,\lambda) = \frac{4\pi q}{\kappa}\frac{N_1 N_2 + M_2}{D_1 D_2} \ ,$$

S matrices in the non-anyonic channels for $\mathcal{N}=1$

$$N_1 = \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (w-1)(2m+iq) + (w-1)(2m-iq) \right) \ ,$$

$$N_2 = \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (q(w+3) + 2im(w-1)) + (q(w+3) - 2im(w-1)) \right) \ ,$$

$$M_1 = -8mq((w+3)(w-1) - 4w) \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} \ ,$$

$$M_2 = -8mq(1+w)^2 \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} \ ,$$

$$D_1 = \left(i \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (w-1)(2m+iq) - 2im(w-1) + q(w+3) \right) \ ,$$

$$D_2 = \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (-q(w+3) - 2im(w-1)) + (w-1)(q+2im) \right) \ .$$

S matrices in the anyonic channels for $\mathcal{N}=1$

- **Naive crossing symmetry rules give rise to a non-unitary S matrix.**

- Applying the conjectured crossing rules

$$\mathcal{S}_B^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right) ,$$
$$\mathcal{S}_F^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right) .$$

$$J_B(\sqrt{s},\lambda) = -4\pi i\lambda\sqrt{s}\frac{N_1N_2+M_1}{D_1D_2} ,$$
$$J_F(\sqrt{s},\lambda) = -4\pi i\lambda\sqrt{s}\frac{N_1N_2+M_2}{D_1D_2}$$

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A non-trivial check - unitarity $\quad \mathcal{N}=1$

- the unitarity equations can be written using $T(\theta) = i\cot(\theta/2)$.

$$\mathcal{T}_B^S = H_B T(\theta) + W_B - iW_2\delta(\theta)\ ,\ \mathcal{T}_F^S = H_F T(\theta) + W_F - iW_2\delta(\theta)\ ,$$

- bosonic unitarity equation

$$H_B - H_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_B^* - H_B W_2^*)\ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_B H_B^*)\ ,$$

$$W_B - W_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_B^* - W_2^* W_B) - \frac{i}{4\sqrt{s}}(H_B H_B^* - W_B W_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

- Fermionic unitarity equation

$$H_F - H_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_F^* - H_F W_2^*)\ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_F H_F^*)\ ,$$

$$W_F - W_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_F^* - W_2^* W_F) - \frac{i}{4\sqrt{s}}(H_F H_F^* - W_F W_F^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

A non-trivial check - unitarity $\mathcal{N}=1$

- Unitarity equations are verified using

$$H_B = H_F = 4\sqrt{s}\sin(\pi\lambda),\ W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda)-1),\ T(\theta) = i\cot(\theta/2)$$

$$\begin{aligned} W_B &= J_B(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda}\ , \\ W_F &= J_F(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda}\ . \end{aligned}$$

- Important fact is that the S matrix in the singlet channel must have the **delta function modulated by the anyonic phase** and **modified crossing rules** **exactly** as conjectured. **Else unitarity fails**.

Consistency check - non relativistic limit $\mathcal{N}=1$

- **Non relativistic limit** of the singlet channel S matrix $\sqrt{s} \to 2m$ while keeping other parameters fixed.

$$\mathcal{T}_B^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)-1\right) \ ,$$
$$\mathcal{T}_F^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)+1\right) \ .$$

- This has **the same structure as the Aharanov-Bohm result**.

$$h(\theta) = 2\pi(\cos\pi\nu - 1)\delta(\theta) + \sin\pi\nu(\mathrm{Pv}\cot\frac{\theta}{2} - i\mathrm{Sgn}(\nu))$$

Duality invariance $\mathcal{N}=1$

- Under the duality transformation

$$w' = \frac{3-w}{w+1}, \lambda' = \lambda - \mathrm{sgn}(\lambda), m' = -m, \kappa' = -\kappa$$

$$J_B(q,\kappa',\lambda',w',m') = -J_F(q,\kappa,\lambda,w,m)\ ,$$
$$J_F(q,\kappa',\lambda',w',m') = -J_B(q,\kappa,\lambda,w,m)\ .$$

- **Duality maps the bosonic and fermionic S matrices into one another upto an overall phase**.

- **Concrete evidence for duality**.

- Supersymmetric ward identity **guarantees duality invariance** of all other processes in the theory.

S matrices in the non-anyonic channels for $\mathcal{N}=2$

- Remarkable simplification in the $\mathcal{N}=2$ theory (w=1)

$$\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa} \ ,$$
$$\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa}$$

- **The S matrix is tree level exact to all orders in λ.**

K.I, S.Jain, S.Mazumdar, S.Minwalla, V. Umesh, S.Yokoyama

- No loop corrections in the planar limit: **non-renormalization**.

- Recall that $T_{non-anyonic} \sim \mathcal{O}\left(\frac{1}{N}\right)$ and hence unitarity is guaranteed by hermiticity!

$$i(T - T^\dagger) \sim 0 + \mathcal{O}\left(\frac{1}{N^2}\right)$$

- **Duality maps the bosonic and fermionic S matrices into one another upto an overall phase.** $\kappa' = -\kappa \ , \ m' = -m$

S matrices in the anyonic channels for $\mathcal{N}=2$

- Recall that $T_{anyonic} \sim \mathcal{O}(1)$ and hence unitarity is non-linear.

$$i(T - T^\dagger) = TT^\dagger$$

- So **if the anyonic channels did not get renormalized in this theory then unitarity fails**!

- **Modified crossing rules resolve this puzzle!**

$$\mathcal{T}_B^{S;\mathcal{N}=2}(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) - 8m)\ ,$$
$$\mathcal{T}_F^{S;\mathcal{N}=2}(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) + 8m).$$

- The S matrix **continues to be simple** but is **not tree level exact in the singlet channel**.

- Thus unitarity equation is satisfied in the singlet channel for the N=2 theory as well.

S matrices in the anyonic channels for $\mathcal{N}=2$

- It is also very easy to demonstrate in an explicit calculation

$$H_B - H_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_B^* - H_B W_2^*) \ ,$$

$$W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_B H_B^*) \ ,$$

$$W_B - W_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_B^* - W_2^* W_B) - \frac{i}{4\sqrt{s}}(H_B H_B^* - W_B W_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)$$

- For the N=2 theory

$$H_B = H_F = 4\sqrt{s}\sin(\pi\lambda) \ , \ W_B = -8m\sin(\pi\lambda) \ ,$$

$$W_F = 8m\sin(\pi\lambda) \ , \ W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda) - 1) \ , \ Y(s) = \frac{-s + 4m^2}{16m^2}$$

- The **first equation is trivially satisfied**, the **third one is satisfied**

$$(H_B H_B^* - W_B W_B^*) = -16\sin^2(\pi\lambda)(-s + 4m^2)$$

$$Y(W_B - W_F)(W_B^* - W_F^*) = 16\sin^2(\pi\lambda)(-s + 4m^2)$$

- The **second equation is satisfied due to the same trignometric identity as in the Aharanov-Bohm case**.

Summary

Summary

- In QFT **crossing symmetry is the statement that there exists a single function from which all the channels of scattering can be obtained by analytic continuation**.

- **Naive crossing symmetry** works when the scattering particles are of **bosonic or fermionic statistics**.

- When the scattering quanta are **anyonic, the usual crossing rules lead to non-unitary S matrices**.

- General structure of S matrices in the anyonic channels is of the form

$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\text{naive}}(s,\theta)$$

  - S matrix has an **identity term modulated by the anyonic phase**.
  - The usual **crossing rules are modified**.
  - modifications appear to be **universal** for all CS matter theories.

Summary

- We presented **evidence for universality** by studying $2 \to 2$ scattering in supersymmetric Chern-Simons theories at large N.
- The **symmetric, antisymmetric and adjoint channels are non-anyonic in the large N and the usual crossing rules hold**.
- The **singlet channel is anyonic in the large N limit** and the **usual crossing rules lead to non-unitary S matrices**.
- The **conjectured modifications lead to S matrices that are unitary**, **invariant under bosonization duality and have the correct non-relativistic limit**.
- We presented our tests in $\mathcal{N} = 1, 2, 3$ supersymmetric Chern-Simons matter theories, and find **substantial evidence for the conjecture** of

  S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama

Summary

- For the $\mathcal{N}=2,3$ theories, the **S matrices in the non-anyonic channel are not renormalized to all orders in 't Hooft coupling λ.**

- In the **anyonic channel the S matrix is not tree level exact**, but continues to be simple. We saw how the **modifications suggested resolve the issue of unitarity** in this connection.

- It would be very interesting and tempting to **conjecture** that:
  - **In CS matter theories with $\mathcal{N}\geq 2$ susy, the S matrices in the non-anyonic channels are not renormalized to all orders in λ.**

  - **In the anyonic channel, the identity term gets modulated by anyonic phase and modified crossing rules apply as expected.**

  - **However for $\mathcal{N}\geq 4$ the theories have bi-fundamental matter, and care has to be taken while extrapolating any statement.**

Summary

- It would be worthwhile to **rigorously prove the delta function term and modified crossing rules**.

- **The modified crossing factor** $\frac{\sin \pi\lambda}{\pi\lambda}$ **appears all over the place in CS matter theories**. It is in fact the **expectation value of a circular Wilson loop** on S^3 . Understanding how this factor appears in the S matrix through an honest computation in the singlet channel may be key.

- It is important to observe that **at finite N and κ all the scattering channels are anyonic**!

$$\nu_{Sym} = \frac{1}{\kappa} - \frac{1}{N\kappa} \ , \ \nu_{Asym} = -\frac{1}{\kappa} - \frac{1}{N\kappa}$$
$$\nu_{Adj} = \frac{1}{N\kappa} \ , \ \nu_{Sin} = -\frac{N}{\kappa} + \frac{1}{N\kappa}$$

- **Understanding or even conjecturing the crossing rules in this case, may have practical applications**!

Some other cool stuff happening..

$$\mathcal{N} = 2$$

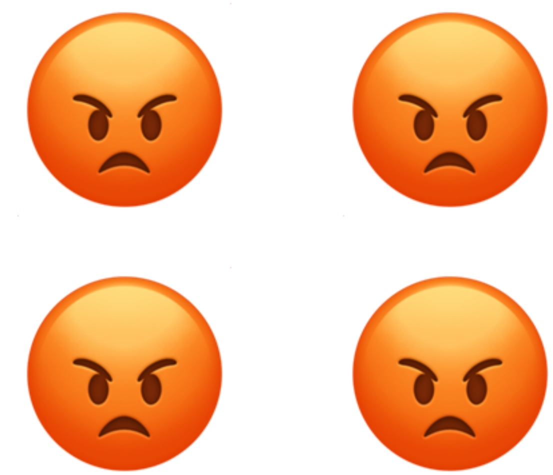
- Using BCFW recursion relations we have computed arbitrary n particle tree level amplitudes!

  K.I, S.Jain, P.Nayak, V.Umesh arXiv :1710.04227

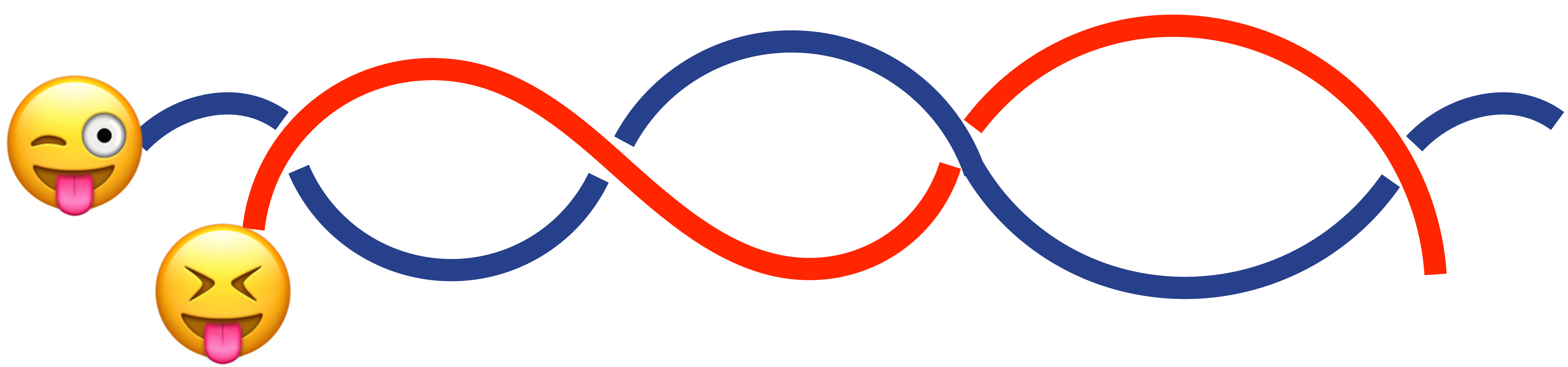
- The non-renormalization in $\mathcal{N} = 2$ theory can be explained using a hidden symmetry in the amplitude: Dual superconformal symmetry!

  K.I, S.Jain, S.Majumdar, P.Nayak, T.Neogi, T.Sharma, R.Sinha, V.Umesh, arXiv : 1711.02672

- The superconformal and dual superconformal symmetries combine to form an infinite dimensional Yangian symmetry that suggests that this theory may be **integrable**!



תודה רבה!



Computing exact propagator

Susy light cone gauge

- There exists a **supersymmetric light cone gauge**

$$\Gamma_- = 0 \;\Rightarrow\; A_- = A_1 + iA_2 = 0$$

- All **gauge self interactions (in superspace) vanish**

$$S = -\int d^3x d^2\theta \Big[-\frac{\kappa}{8\pi} Tr(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2} D^\alpha \bar{\Phi} D_\alpha \Phi \\ - \frac{i}{2}\Gamma^-(\bar{\Phi} D_- \Phi - D_-\bar{\Phi}\Phi) + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \Big]$$

- **Susy light cone gauge maintains manifest susy**.

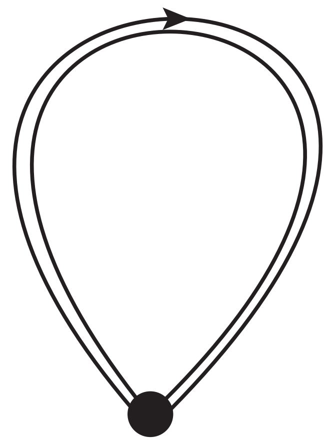
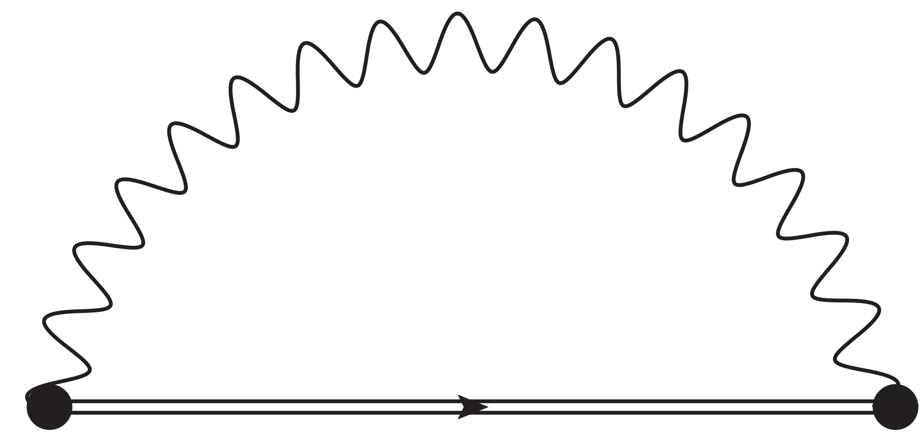
- Bare propagators

$$\langle \bar{\Phi}(\theta_1, p)\Phi(\theta_2, -p')\rangle = \frac{D^2_{\theta_1, p} - m_0}{p^2 + m_0^2}\delta^2(\theta_1 - \theta_2)(2\pi)^3\delta^3(p - p')$$

$$\langle \Gamma^-(\theta_1, p)\Gamma^-(\theta_2, -p')\rangle = -\frac{8\pi}{\kappa}\frac{\delta^2(\theta_1 - \theta_2)}{p_{--}}(2\pi)^3\delta^3(p - p')$$

Exact propagator to all loops

- Integral equation for self energy

$\Sigma(p, \theta_1, \theta_2) \; =$  $+$ 

- Both $P(r, \theta_1, \theta_2)$ and $\Sigma(r, \theta_1, \theta_2)$ satisfy off-shell susy ward identities and **are determined upto unknown functions of momenta**.

$$\Sigma(p, \theta_1, \theta_2) = 2\pi\lambda w \int \frac{d^3 r}{(2\pi)^3} \delta^2(\theta_1 - \theta_2) P(r, \theta_1, \theta_2)$$
$$- 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} D_-^{\theta_2, -p} D_-^{\theta_1, p} \left(\frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} P(r, \theta_1, \theta_2) \right)$$
$$+ 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} \frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} D_-^{\theta_1, r} D_-^{\theta_2, -r} P(r, \theta_1, \theta_2)$$

- **The integral equation determines these unknown functions**.

Exact propagator to all loops

- Solution to the **exact propagator** is extremely simple

$$P(p,\theta_1,\theta_2) = \frac{D^2 - m}{p^2+m^2}\delta^2(\theta_1-\theta_2)$$

- Same structure as bare propagator with m_0 replaced by m

$$m = \frac{2m_0}{2+(-1+w)\lambda\ \mathrm{Sgn(m)}}$$

- m is the pole mass. It is **invariant under the bosonization duality**.

- For the $\mathcal{N}=2$ theory, $w=1$, **there is no mass renormalization**.

Onshell supersymmetry: Characterizing the supersymmetric S matrix

Scattering a superfield

- **Asymptotic in/out states satisfy free field equation**

$$\left(D^2 + m\right)\Phi = 0$$

- Solution

$$\Phi(x,\theta) = \int \frac{d^2p}{\sqrt{2p^0}(2\pi)^2}\left[\left(a(\mathbf{p})(1+m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\right)e^{ip.x} \right.$$
$$\left. + \left(a^{c\dagger}(\mathbf{p})(1+m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})\right)e^{-ip.x}\right]$$

- Action of off-shell susy operator on onshell superfields

$$[Q_\alpha^{off}, \Phi] = Q_\alpha^{off}\Phi = i\left(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\partial_{\beta\alpha}\right)\Phi$$

- **Onshell representation of the supercharge Q**

$$-iQ_\alpha^{on} = u_\alpha(\mathbf{p}_i)\left(\alpha\partial_a + \alpha^c\partial_{a^c}\right) + u_\alpha^*(\mathbf{p}_i)\left(a\partial_\alpha + a^c\partial_{\alpha^c}\right)$$
$$- v_\alpha^*(\mathbf{p}_i)\left(a^\dagger\partial_{\alpha^\dagger} + (a^c)^\dagger\partial_{(\alpha^c)^\dagger}\right) + v_\alpha(\mathbf{p}_i)\left(\alpha^\dagger\partial_{a^\dagger} + (\alpha^c)^\dagger\partial_{(a^c)^\dagger}\right)$$

Scattering a superfield

- Introduce creation and annihilation operator superfields

$$A_i(\mathbf{p}) = a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i \ ,$$
$$A_i^\dagger(\mathbf{p}) = a_i^\dagger(\mathbf{p}) + \theta_i\alpha_i^\dagger(\mathbf{p}) \ .$$

- Action of onshell susy operator on these

$$[Q_\alpha^{on}, A_i(\mathbf{p}_i, \theta_i)] = Q_\alpha^1 A_i(\mathbf{p}_i, \theta_i)$$
$$[Q_\alpha^{on}, A_i^\dagger(\mathbf{p}_i, \theta_i)] = Q_\alpha^2 A_i^\dagger(\mathbf{p}_i, \theta_i)$$

$$Q_\beta^1 = i\left(-u_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p})\theta\right)$$
$$Q_\beta^2 = i\left(v_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p})\theta\right) \ .$$

Supersymmetric Ward identity

- $2 \to 2$ S matrix $p_1 + p_2 \to p_3 + p_4$

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\sqrt{(2p_1^0)(2p_2^0)(2p_3^0)(2p_4^0)} =$$
$$\langle 0|A_4(\mathbf{p}_4, \theta_4)A_3(\mathbf{p}_3, \theta_3)UA_2^\dagger(\mathbf{p}_2, \theta_2)A_1^\dagger(\mathbf{p}_1, \theta_1)|0\rangle$$

- Supersymmetric ward identity for the superspace S matrix

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1, \theta_1) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_2, \theta_2)$$
$$+ \overrightarrow{Q}^2_\alpha(\mathbf{p}_3, \theta_3) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_4, \theta_4)\Big)S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0$$

Supersymmetric scattering

- The solution satisfying the supersymmetric ward identity for the $\mathcal{N}=1$ theory is given in terms of two independent functions $\mathcal{S}_B$ and $\mathcal{S}_F$.

$$
\begin{aligned}
S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) &= \mathcal{S}_B + \mathcal{S}_F\ \theta_1\theta_2\theta_3\theta_4 + \\
&\left(\frac{1}{2}C_{12}\mathcal{S}_B - \frac{1}{2}C^*_{34}\mathcal{S}_F\right)\ \theta_1\theta_2 + \left(\frac{1}{2}C_{13}\mathcal{S}_B - \frac{1}{2}C^*_{24}\mathcal{S}_F\right)\ \theta_1\theta_3 \\
&+ \left(\frac{1}{2}C_{14}\mathcal{S}_B + \frac{1}{2}C^*_{23}\mathcal{S}_F\right)\ \theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{S}_B + \frac{1}{2}C^*_{14}\mathcal{S}_F\right)\ \theta_2\theta_3 \\
&+ \left(\frac{1}{2}C_{24}\mathcal{S}_B - \frac{1}{2}C^*_{13}\mathcal{S}_F\right)\ \theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{S}_B - \frac{1}{2}C^*_{12}\mathcal{S}_F\right)\ \theta_3\theta_4
\end{aligned}
$$

- no $\boldsymbol{\theta}$ term is four boson scattering, the four $\boldsymbol{\theta}$ term is four fermion scattering. C_{ij} are functions of $u_\alpha(p), v_\alpha(p)$.
- Thus $\mathcal{N}=1$ susy determines 6 of 8 processes in terms of 2 independent functions.
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Supersymmetric scattering

- The $\mathcal{N}=2$ S matrix is already $\mathcal{N}=1$ supersymmetric, but obeys additional constraint from $\mathcal{N}=2$ susy.
- This can also be formulated in $\mathcal{N}=1$ onshell superspace

$$\left(\sum_{i=1}^{4} Q_\alpha^i(\mathbf{p}_i,\theta_i)+\bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i)=0$$

$$\left(\sum_{i=1}^{4} Q_\alpha^i(\mathbf{p}_i,\theta_i)-\bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i)=0$$

- Additional constraint relates $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{13}u_\alpha(\mathbf{p}_3)+C_{14}u_\alpha(\mathbf{p}_4)+C_{12}v_\alpha(\mathbf{p}_2)+v_\alpha^*(\mathbf{p}_1)\right)$$
$$=\mathcal{S}_F(C_{24}^*u_\alpha(\mathbf{p}_3)-C_{23}^*u_\alpha(\mathbf{p}_4)+C_{34}^*v_\alpha(\mathbf{p}_2))$$

- The $\mathcal{N}=2$ S matrix is completely specified by one function. Eg:

$$p_1=p+q, p_2=-k-q, p_3=p, p_4=-k$$

$$\mathcal{S}_B=\mathcal{S}_F\frac{-2m(k-p)_-+iq_3(k+p)_-}{2m(k-p)_-+iq_3(k+p)_-}\ .$$