Amplitudes and hidden symmetries in N=2 Chern-Simons Matter theory

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Dec 19, 2017

Newe Shalom Joint Seminar Series

Based on

K.I, S.Jain, P.Nayak, V.Umesh, **arXiv :1710.04227** **(BCFW)**

K.I, S.Jain, S.Majumdar, P.Nayak, T.Neogi, T.Sharma, R.Sinha, V.Umesh, **arXiv : 1711.02672** **(Dual Superconformal symmetry)**

K.I, S.Jain, P.Nayak, T.Sharma, V.Umesh, **arXiv : 1801.nnppq** **(Yangian)**

References:

K.I, S.Jain, S.Mazumdar, S.Minwalla, V.Umesh, S.Yokoyama, **arXiv: 1505.06571**, JHEP 1510 (2015) 176.

S.Jain, M.Mandlik, S.Minwalla, T.Takimi, S.Wadia, S.Yokoyama, **arXiv: 1404.6373,** JHEP 1504 (2015) 129.

Part I

Introduction

Introduction

- When we say that a theory is **integrable**, what do we really mean?
- Classically $\text{symmetries} = \text{degrees of freedom}$
- **Several examples**: n - dimensional harmonic oscillator, Central force motion, Heisenberg spin chain system, Sine-Gordon equation.
- At **quantum** level, there is **no universal notion of integrability.**
- One indicator of integrability is the presence of **infinite dimensional symmetry structures**.
- Some of the best known quantum field theories that arise in the low energy limits of string theory are **conjectured to be integrable**.

$$d = 4,\ \mathcal{N} = 4 \text{ Super Yang Mills}$$
$$d = 3,\ \mathcal{N} = 6 \text{ ABJM}$$

1012.3982, Beisert et.al

- The infinite dimensional symmetry structures appear in **planar tree level scattering amplitudes**!

Introduction

- The **tree level superamplitudes** in both SYM and ABJM possess hidden internal symmetries known as **dual superconformal symmetry**.
- The **superconformal symmetry and the dual superconformal symmetry overlap** to generate an **infinite dimensional Yangian symmetry** of the superamplitude.
- This **seems to violate the Coleman-Mandula theorem** that states that **spacetime and internal symmetries** of the S matrix may only be combined by **trivial direct product**!
- However the Yangian generators do not obey a basic assumption of the theorem, **they do not act on multiparticle states via a tensor product generalization of the action of single product states**!
- They have a **non-trivial co-product structure**! In fact the Yangian algebra is not a Lie algebra, but a Hopf algebra. (more on that later!)

Motivation

- It will be interesting to look for such infinite dimensional symmetry structures in **theories with less or no supersymmetry**.
- $d=3$, $\mathcal{N}=2$ superconformal Chern-Simons theory coupled to matter in fundamental representation of U(N)

$$\mathcal{S}^{L}_{\mathcal{N}=2} = \int d^3x \Big[-\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + \bar{\psi}i\not{D}\psi - \mathcal{D}^\mu\bar{\phi}\mathcal{D}_\mu\phi + \frac{4\pi^2}{\kappa^2}(\bar{\phi}\phi)^3 + \frac{4\pi}{\kappa}(\bar{\phi}\phi)(\bar{\psi}\psi) + \frac{2\pi}{\kappa}(\bar{\psi}\phi)(\bar{\phi}\psi)\Big]$$

- The theory exhibits a **strong-weak self duality** under the duality map

$$\kappa' = -\kappa \ , \ N' = |\kappa| - N + 1 \ , \ \lambda' = \lambda - \mathrm{Sgn}(\lambda)$$

$$\lambda = \frac{N}{\kappa} \ , N \to \infty, \kappa \to \infty$$

- **K.I, Jain, Mazumdar, Minwalla, Umesh, Yokoyama :** $2 \to 2$ scattering amplitudes to all orders in the 't Hooft coupling. **(summing planar diagrams)**

Motivation

- Channels of scattering

$$\text{Fundamental} \otimes \text{Fundamental} \to \text{Symm}(U_d) \oplus \text{Asymm}(U_e)$$
$$\text{Fundamental} \otimes \text{Antifundamental} \to \text{Adjoint}(T) \oplus \text{Singlet}(S)$$

- In the symmetric, anti-symmetric and adjoint channels of scattering the **amplitude is tree-level exact to all orders in λ**.

$$T_{symm}^{all\ loop} = T_{Asymm}^{all\ loop} = T_{Adj}^{all\ loop} = T_{tree}$$

- In the singlet channel the coupling dependence is **extremely simple**.

$$T_{singlet}^{all\ loop} = N\frac{\sin\pi\lambda}{\pi\lambda}T_{tree}$$

- Tree level super amplitude

$$T_{\text{tree}} = \frac{4\pi}{\kappa}\frac{\langle 12\rangle}{\langle 23\rangle}\delta(\sum_{i=1}^{4}p_i)\delta^2\left(\mathcal{Q}\right)$$

$$\delta^2(Q) = \sum_{i<j=1}^{n}\langle ij\rangle\eta_i\eta_j$$

Motivation

- Why is the 2 → 2 particle scattering in the Symm/Anti-symm/Adjoint channels **tree level exact**? and why does it have a very **simple coupling dependence in Singlet channel**?
- Maybe some **powerful symmetry** that protects the amplitude from renormalization.
- Is it possible to compute **all loop m → n scattering amplitudes** in the N=2 theory at least in the planar limit?
- Does the **non-renormalization** results of the 2 → 2 scattering continue to persist for arbitrary higher point amplitudes?
- These computations would **test the duality** in regions un-probed by large N perturbation theory yet.

Our work

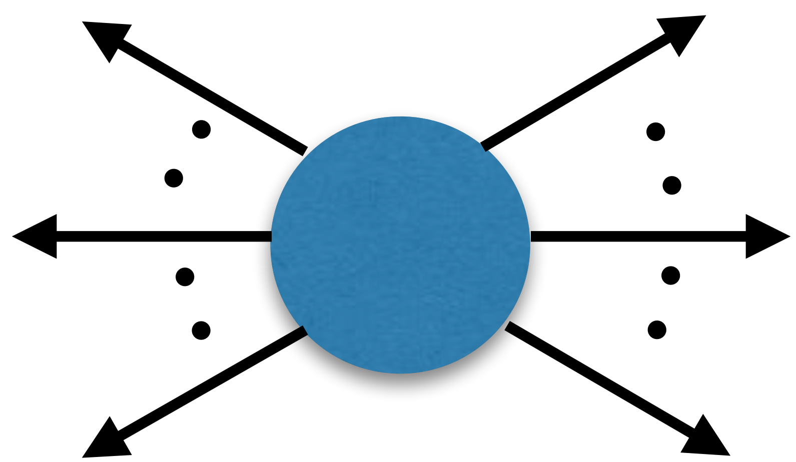
- As a first step towards the all loop m→n scattering, is it possible to write down **arbitrary m → n tree level amplitudes** ?
- We are able to achieve this via **BCFW recursions** **K.I, Jain, Nayak, Umesh**
- As a first step towards thinking about higher point amplitudes we identify a **hidden symmetry** in the 2 → 2 amplitude computed earlier that might explain the non-renormalization.
- This symmetry is known as **dual superconformal symmetry**.

  K.I, Jain, Majumdar, Nayak, Neogi, Sharma, Sinha, Umesh
- Answer analysis tells us this symmetry is **exact to all loops**!
- The superconformal symmetry and dual superconformal symmetry together generate the infinite dimensional **Yangian Symmetry**.

  K.I, Jain, Nayak, Sharma, Umesh, to appear
- If this persists for higher point amplitudes, this suggests that the theory we are dealing with may be **integrable!**

Part II

All tree level amplitudes



- K.I, S.Jain, P.Nayak, V.Umesh, **arXiv :1710.04227**

BCFW recursions in 2+1 dimensions

- Recursion relations enable to construct **n point tree level scattering amplitudes from lower point tree level amplitudes.**

Britto, Cachazo, Feng, Witten

- Central idea: **Dixon**
    - Tree level amplitudes are **continuously deformable** analytic functions of momenta.
    - Only type of singularities that can appear at tree level are **simple poles.**
    - One can **reconstruct amplitudes** for generic scattering kinematics knowing its behavior in **singular kinematics**.
    - In these singular regions **amplitudes factorize** into causally disconnected amplitudes with fewer legs, connected by an i**ntermediate onshell state**.

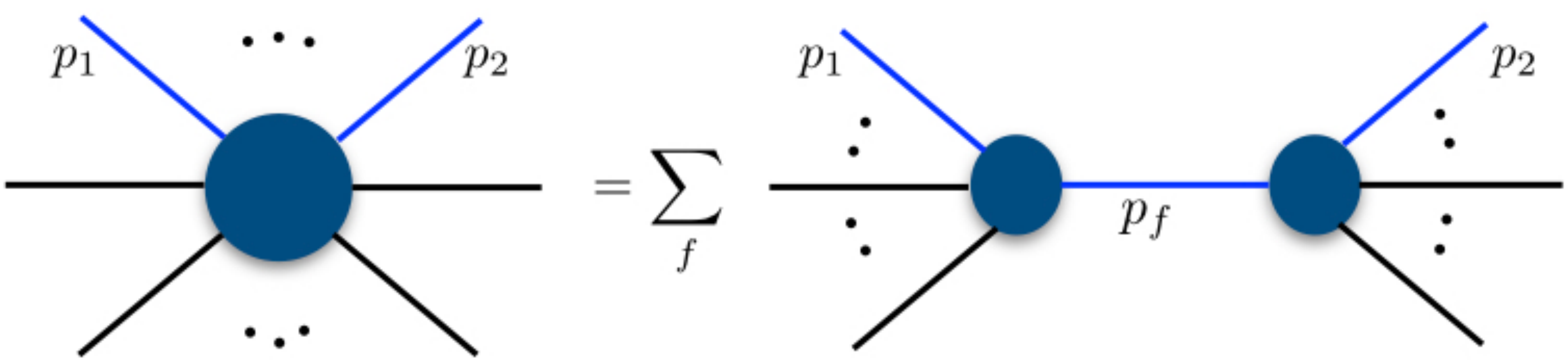
- We will focus on situation where all the external particles are massless.

BCFW recursions in 2+1 dimensions

- Promote the amplitude into a one complex parameter family of amplitudes

$$A(z) = A(p_1, \ldots p_i(z), p_{i+1}, \ldots p_l(z), \ldots, p_{2n})$$

- The necessary and sufficient conditions are:
    - **The momentum deformation should preserve on-shell conditions and momentum conservation.**
    - **The amplitude should be asymptotically well behaved under the deformation.**



- A higher point amplitude factorizes into lower point amplitudes!

The recursion formula for an arbitrary 2n point amplitude

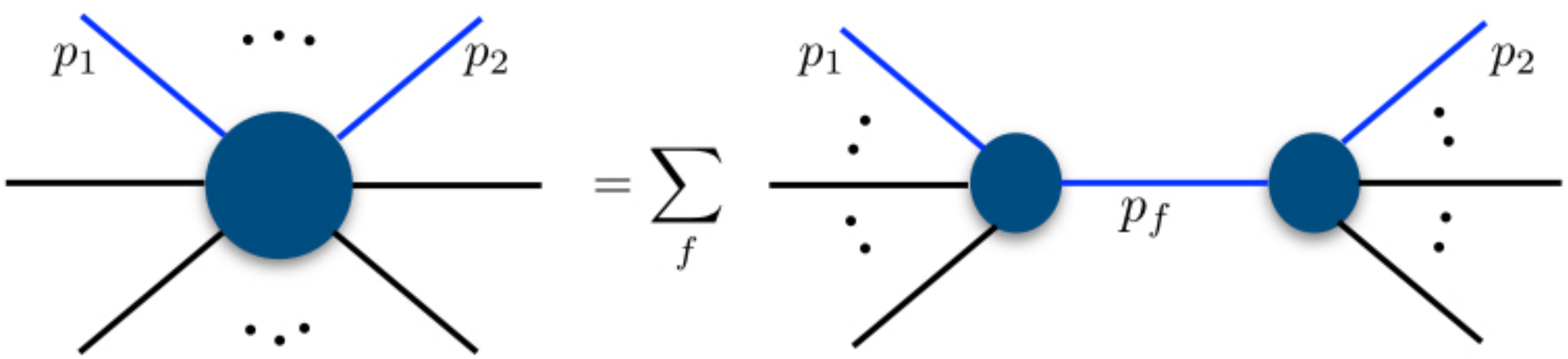
- Write a contour integral representation for the amplitude

$$\frac{1}{2\pi i}\oint_{C_{z=1}} \frac{dz}{z-1} A(z)$$

- Deform the contour to **z→∞**, If A(z) has no poles, the integral vanishes

$$A(z=1) = -\sum_{\mathrm{poles}:z^i} \mathrm{Res}_{z=z^i} \frac{A(z)}{z-1}$$

- remember that all the deformed momenta satisfy the onshell conditions!



$$A(z=1) = -\sum_f \sum_{\mathrm{poles}:z_f^i} \mathrm{Res}_{z=z_f^i} \frac{1}{z-1} \frac{A_L(p_1 \ldots p_i(z), \ldots p_n) A_R(p_{n+1} \ldots p_j(z), \ldots p_{2n})}{\hat{p}_f^2(z)}$$

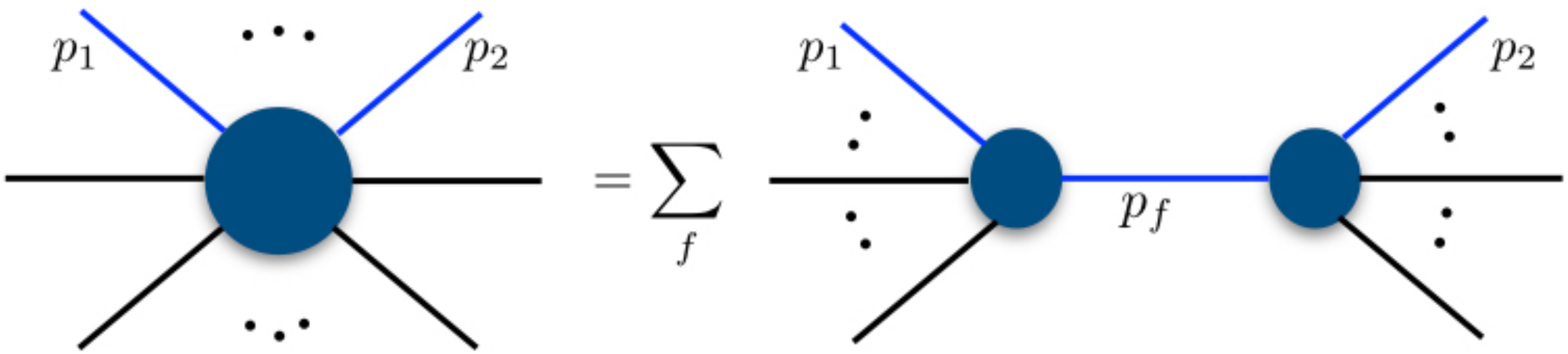
- We have used the fact that at **tree level the only possible singularities** are **simple poles**!

Asymptotic behavior

- Onshell susy methods, encode the **component amplitudes into a superamplitude**.
- **Susy ward identities** relate various component amplitudes and reduce the number of independent amplitudes.
- Susy also ensures that **if the independent component amplitudes are well behaved then the entire superamplitude is well behaved**.
- Using two independent methods we showed that the superamplitude is well behaved
  - **Background field expansion.** **Arkani-Hamed, Kaplan**
  - **Explicit Feynman diagram computation** of component amplitudes. (tedious, but provides explicit check for any given higher point amplitude).
- The recursion formula then follows from **Cauchy residue theorem**.

The recursion formula for arbitrary 2n point superamplitude

$$A_{2n}(z=1) = \sum_f \int \frac{d\theta}{p_f^2} \Bigg(z_{a;f} \frac{z_{b;f}^2 - 1}{z_{a;f}^2 - z_{b;f}^2} A_L(z_{a;f}, \theta) A_R(z_{a;f}, i\theta) + (z_{a;f} \leftrightarrow z_{b;f}) \Bigg)$$



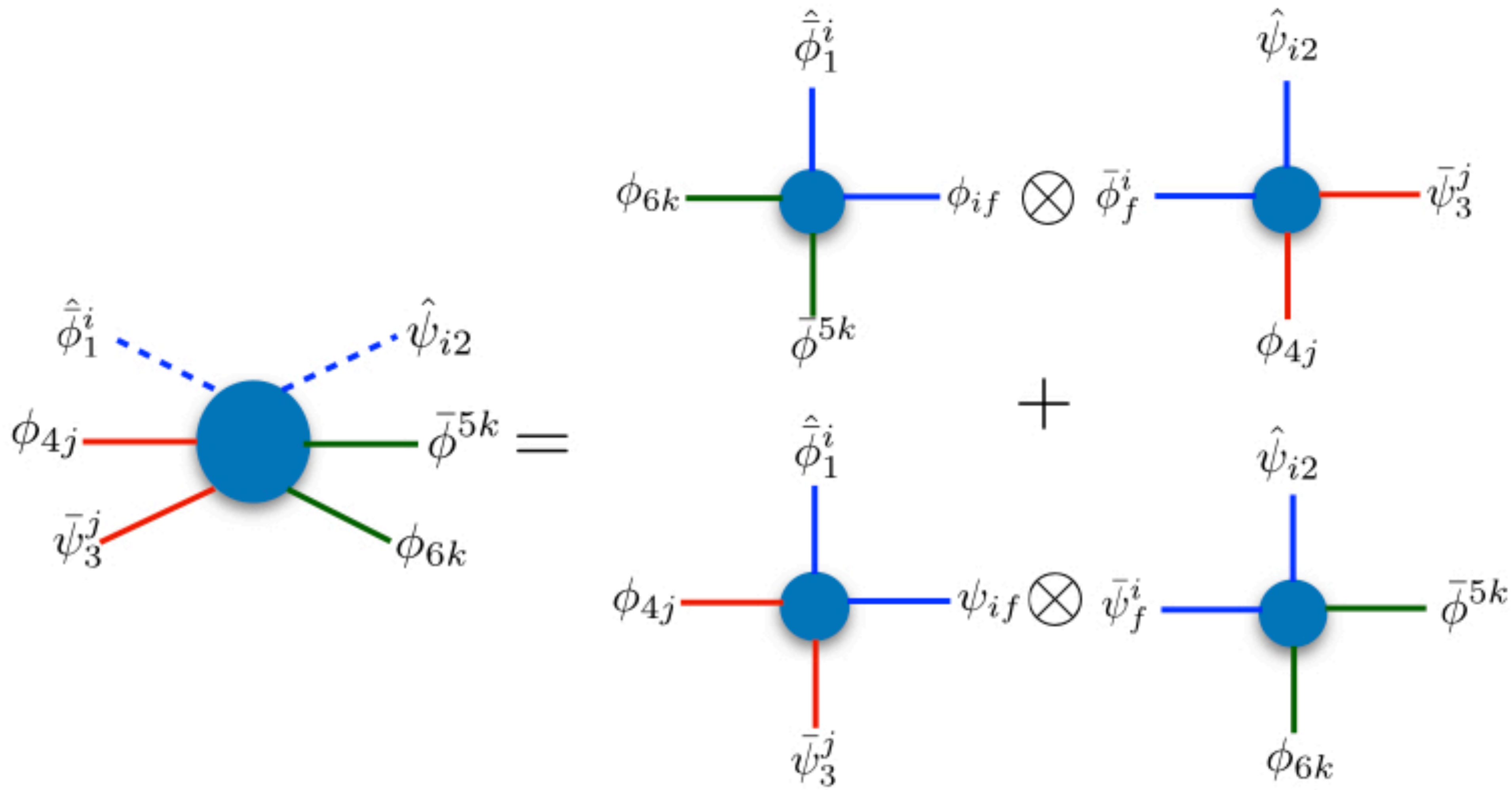
- $z_{a;f}, z_{b;f}$ are zeroes of $p_f^2(z) = 0$
- The formula can be recursively applied to write down any **higher point superamplitude in terms of products of the four point superamplitude**.

Eg: Six point amplitude as product of four point amplitudes

$$\langle\bar{\phi}_1\psi_2\bar{\psi}_3\phi_4\bar{\phi}_5\phi_6\rangle =$$

$$\left(z_{a;f}\,\frac{z_{b;f}^2-1}{z_{a;f}^2-z_{b;f}^2}\langle\hat{\bar{\phi}}_1\hat{\phi}_f\bar{\phi}_5\phi_6\rangle_{z_{a;f}}\langle\hat{\bar{\phi}}_{(-f)}\hat{\psi}_2\bar{\psi}_3\phi_4\rangle_{z_{a;f}} + (z_{a;f}\leftrightarrow z_{b;f})\right)\frac{i}{p_f^2}\Bigg|_{p_f=p_{234}}$$

$$+\left(z_{a;f}\,\frac{z_{b;f}^2-1}{z_{a;f}^2-z_{b;f}^2}\langle\hat{\bar{\phi}}_1\hat{\psi}_f\bar{\psi}_3\phi_4\rangle_{z_{a;f}}\langle\hat{\bar{\psi}}_{(-f)}\hat{\psi}_2\bar{\phi}_5\phi_6\rangle_{z_{a;f}} + (z_{a;f}\leftrightarrow z_{b;f})\right)\frac{i}{p_f^2}\Bigg|_{p_f=p_{256}}$$



Recursion relations for non-supersymmetric theories!

- BCFW does not apply to the **non-susy CS coupled to fermions/bosons** since the amplitudes **do not have good asymptotic behavior**.

- It is possible to extract the recursion relations for non-susy fermionic/bosonic CS matter theories from the N=2 results!! Eg:

    - At **tree level,** the Feynman diagrams for an **all fermion amplitude are same** for susy/non-susy theory.

    - **Susy ward identity**: The four point super amplitude is completely specified by one function, choose it to be the four fermion amplitude.

    - Use this information recursively in the BCFW formula!

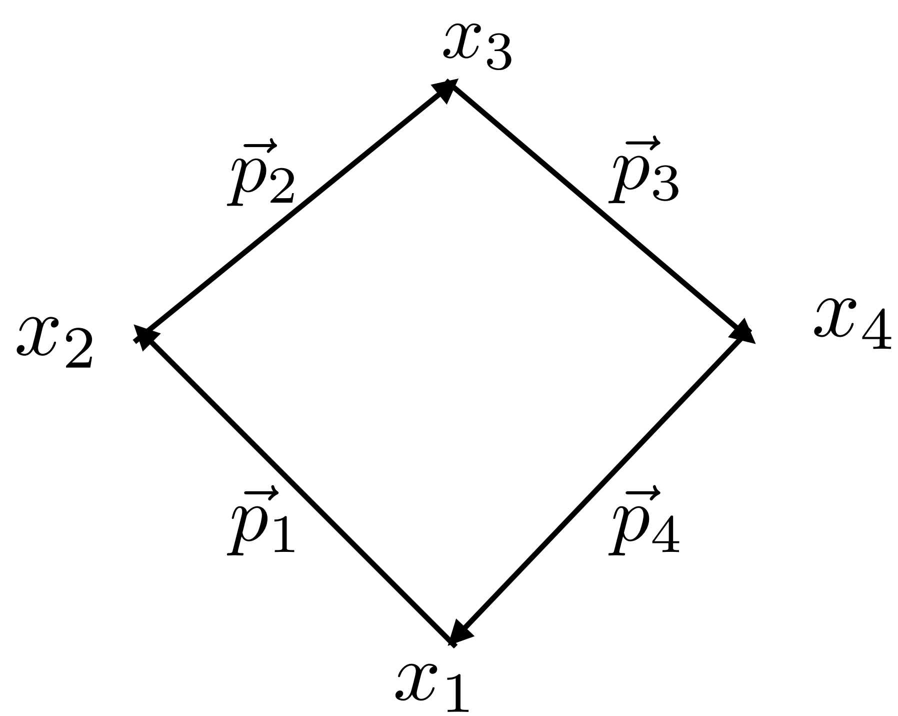
- An arbitrary higher point tree level amplitude in the fermionic CS matter theory can be entirely written in terms of **4 fermion amplitude**.

Recursion relations for non-supersymmetric theories!

$$\begin{aligned}
&\langle \bar\psi_1 \psi_2 \bar\psi_3 \psi_4 \bar\psi_5 \psi_6 \rangle = \\
&\Bigg(z_{a;f} \frac{z_{b;f}^2 - 1}{z_{a;f}^2 - z_{b;f}^2} \Bigg[-\frac{z_{a;f}^2 + 1}{2 z_{a;f}} + i\, \frac{z_{a;f}^2 - 1}{2 z_{a;f}} \; \frac{\langle \hat{1} 4 \rangle}{i \langle \hat{f} 4 \rangle} \, \frac{\langle \hat{f} 6 \rangle}{\langle \hat{2} 6 \rangle} \Bigg] \\
&\qquad\qquad\qquad \times \langle \hat{\bar\psi}_1 \hat\psi_f \bar\psi_3 \psi_4 \rangle \langle \hat{\bar\psi}_{(-f)} \hat\psi_2 \bar\psi_5 \psi_6 \rangle_{z_{a;f}} \\
&+ (z_{a;f} \leftrightarrow z_{b;f}) \Bigg) \frac{i}{p_f^2} \Bigg|_{p_f = p_{234}} \\
&- \Bigg(z_{a;f} \frac{z_{b;f}^2 - 1}{z_{a;f}^2 - z_{b;f}^2} \Bigg[-\frac{z_{a;f}^2 + 1}{2 z_{a;f}} + i\, \frac{z_{a;f}^2 - 1}{2 z_{a;f}} \; \frac{\langle \hat{1} 6 \rangle}{i \langle \hat{f} 6 \rangle} \, \frac{\langle \hat{f} 4 \rangle}{\langle \hat{2} 4 \rangle} \Bigg] \\
&\qquad\qquad\qquad \times \langle \hat{\bar\psi}_1 \hat\psi_f \bar\psi_5 \psi_6 \rangle \langle \hat{\bar\psi}_{(-f)} \hat\psi_2 \bar\psi_3 \psi_4 \rangle_{z_{a;f}} \\
&+ (z_{a;f} \leftrightarrow z_{b;f}) \Bigg) \frac{i}{p_f^2} \Bigg|_{p_f = p_{256}}
\end{aligned}$$

Part III

Hidden symmetry: Dual superconformal invariance



- K.I, S.Jain, S.Majumdar, P.Nayak, T.Neogi, T.Sharma, R.Sinha, V.Umesh, **arXiv : 1711.02672**

Dual variables

- The dual variables realize momentum conservation linearly in the x variables

$$x_{i,i+1}^{\alpha\beta} = x_i^{\alpha\beta} - x_{i+1}^{\alpha\beta} = p_i^{\alpha\beta} = \lambda_i^\alpha \lambda_i^\beta$$
$$\theta_{i,i+1}^{\alpha} = \theta_i^{\alpha} - \theta_{i+1}^{\alpha} = q_i^{\alpha} = \lambda_i^\alpha \eta_i$$

- momentum and supermomentum conservation imply

$$P^{\alpha\beta} = \sum_i p_i^{\alpha\beta} = x_{n+1}^{\alpha\beta} - x_1^{\alpha\beta} = 0,$$
$$\mathcal{Q}^{\alpha} = \sum_i q_i^{\alpha} = \theta_{n+1}^{\alpha} - \theta_1^{\alpha} = 0.$$

- The **four point super amplitude in dual space**

$$\mathcal{A}_4 = \frac{\langle 12\rangle}{\langle 23\rangle}\delta(\sum_{i=1}^{4} p_i)\delta^2(\mathcal{Q}) \xrightarrow[\textbf{dual space}]{} \mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}}\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)$$

- Goal is to show that this is **invariant under the superconformal symmetry in the dual variables**.

Superconformal algebra in dual space

- The N=2 superconformal algebra in **dual space** is generated by

$$\{P_{\alpha\beta},\ M_{\alpha\beta},\ D,\ K_{\alpha\beta},\ R,\ Q_\alpha,\ \bar{Q}_\alpha,\ S_\alpha,\ \bar{S}_\alpha\}$$

$$P_{\alpha\beta} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\alpha\beta}}, \quad D = -\sum_{i=1}^{n}\left(x_i^{\alpha\beta}\frac{\partial}{\partial x_i^{\alpha\beta}} + \frac{1}{2}\theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha}\right),$$

$$Q_\alpha = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i^\alpha}, \quad \bar{Q}_\alpha = \sum_{i=1}^{n} \theta_i^\beta \frac{\partial}{\partial x_i^{\beta\alpha}},$$

$$M_{\alpha\beta} = \sum_{i=1}^{n}\left(x_{i\alpha}{}^{\gamma}\frac{\partial}{\partial x_i^{\gamma\beta}} + \frac{1}{2}\theta_{i\alpha}\frac{\partial}{\partial \theta_i^\beta}\right), \quad R = \sum_{i=1}^{n}\theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha}$$

- The remaining generators can be expressed using the inversion operator

$$I\left[x_i^{\alpha\beta}\right] = \frac{x_i^{\alpha\beta}}{x_i^2}, \quad I\left[\theta_i^\alpha\right] = \frac{x_i^{\alpha\beta}\theta_{i\beta}}{x_i^2}$$

$$K_{\alpha\beta} = I P_{\alpha\beta} I, \quad S_\alpha = I Q_\alpha I, \quad \bar{S}_\alpha = I \bar{Q}_\alpha I.$$

Dual superconformal invariance of ABJM superamplitude

- **ABJM superamplitude** — Gang, Huang, Koh, Lee, Lipstein

$$A^{(4)}_{ABJM} = \frac{1}{\sqrt{x^2_{1,3}x^2_{2,4}}}\delta^{(3)}(x_1 - x_5)\delta^{(6)}(\theta_1 - \theta_5)$$

- The inversion operator leaves the delta functions invariant

$$I[\delta^{(3)}(x_1 - x_5)\delta^{(6)}(\theta_1 - \theta_5)] = \delta^{(3)}(x_1 - x_5)\delta^{(6)}(\theta_1 - \theta_5)$$

- Conformal transformations therefore act only on the pre factors.

$$I\left[A^{(4)}_{ABJM}\right] = \sqrt{\prod_{i=1}^{4} x_i^2 A^{(4)}_{ABJM}}$$

$$P_{\alpha\beta} I\left[A^{(4)}_{ABJM}\right] = -\frac{1}{2}\left(\sum_{i=1}^{4}\frac{x^i_{\alpha\beta}}{x_i^2}\right) I[A^{(4)}_{ABJM}]$$

$$\tilde{K}^{\alpha\beta}\mathcal{A}^{(4)}_{ABJM} = \left(K^{\alpha\beta} + \frac{1}{2}\sum_{j=1}^{4}\Delta_j x_j^{\alpha\beta}\right)\mathcal{A}^{(4)}_{ABJM} = 0\ ,\ \ \Delta_i = \{1,1,1,1\}$$

Dual superconformal invariance of N=2 superamplitude

$$\mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}}\,\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)$$

- In this form the **translation, Lorentz invariance and supersymmetry invariance of the amplitude is manifest.**
- Under Dilatations and R symmetry it transforms as a eigenfunction of weight 4 and weight 2 respectively.
- To show the dual superconformal invariance it is sufficient to show the invariance under $K_{\alpha\beta}, S_\alpha, \bar{S}_\alpha$
- Note that the **delta functions for N=2 transform under the inversion** as

$$I\left[\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)\right] = x_1^4\;\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)$$

- So **it was expected that the superamplitude in the N=2 theory would not have any dual superconformal invariance** at all.
- However, in the N=2 case, **dual superconformal invariance, still works but the dual conformal weights become non-homogeneous.**

Dual superconformal invariance of the superamplitude

- Under the action of $K_{\alpha\beta}, S_\alpha, \bar{S}_\alpha$

$$\tilde{K}^{\alpha\beta}\mathcal{A}^{(4)} = \left(K^{\alpha\beta} + \sum_{j=1}^{4}\Delta_j x_j^{\alpha\beta}\right)\mathcal{A}^{(4)} = 0$$

$$\tilde{\bar{S}}^{\alpha\beta}\mathcal{A}^{(4)} = \left(\bar{S}^{\alpha\beta} + \sum_{j=1}^{4}\Delta_j x_j^{\alpha\beta}\right)\mathcal{A}^{(4)} = 0$$

$$S_\alpha\mathcal{A}^{(4)} = 0$$

$$\Delta_i = \frac{1}{2}\{4-1, 1, -1, 1\}$$

- The factor of 4 is due to momentum+supermomentum conservation and can be removed. $I\left[\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)\right] = x_1^4\ \delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)$

- Thus the superamplitude is dual superconformal invariant with non-homogeneous weights

$$\tilde{\Delta}_i = \frac{1}{2}\{-1, 1, -1, 1\}$$

Dual superconformal invariance at all loops

- We showed that the function A4 is dual superconformal invariant!

$$\mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}}\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)$$

- The **tree level superamplitude is dual superconformal invariant**.

$$T_{tree} = \frac{4\pi}{\kappa}\mathcal{A}_4$$

- Thus the **all loop results** computed in **K.I, Jain, Mazumdar, Minwalla, Umesh, Yokoyama** are **also dual superconformal invariant.**

$$T^{all\ loop}_{sym} = T^{all\ loop}_{Asym} = T^{all\ loop}_{Adj} = T_{tree}$$

$$T^{all\ loop}_{sing} = N\frac{\sin\pi\lambda}{\pi\lambda}T_{tree}$$

- Unlike the SYM and the ABJM case, here the **symmetry is all loop exact in the planar limit.**
- Now that we know this symmetry exists, **can we reverse the argument and do an S matrix bootstrap to fix the general structure of the amplitude**?

Constraining amplitudes from dual superconformal symmetry

- The **four point amplitude in momentum space** can be interpreted as a **four point correlator in dual space**, then dual conformal invariance fixes

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$$
$$= \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_1-\Delta_2} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} f(u,v,\kappa,\lambda)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

- Since $x_{ij}^2 = p_i^2 = 0$, the correlator is understood in the limit

$$\frac{u}{v}\bigg|_{onshell} = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\bigg|_{onshell} = \frac{p_1^2 p_3^2}{p_2^2 p_4^2}\bigg|_{onshell} = constant$$

- **If dual superconformal symmetry is exact it fixes the momentum (x) dependence completely***

$$f(u,v,\kappa,\lambda) = g(\kappa,\lambda)$$

Constraining amplitudes from dual superconformal symmetry

- In general the S matrix could get complicated functions with poles and branch cuts.

- If dual conformal invariance is an exact symmetry at loop level then no such behavior appears.

- Non trivial momentum dependence could still appear from $x_{i,j}^{\Delta}$ when Δ gets correction from loops.

- This can give rise to log dependence for instance, However these do not appear if there are no IR divergences.

- **If we assume that there are no IR divergences (none seen in the calculation), and that the dual conformal invariance is an exact symmetry, then the momentum dependence is fixed to all loops**.

4 point amplitude as a free field correlator in dual space

- the operator dimensions are

$$\Delta_1 = \Delta_3 = -\frac{1}{2}$$
$$\Delta_2 = \Delta_4 = \frac{1}{2}$$

- The four point correlator in dual space gets fixed to (cancellations in limiting sense)

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = g(\kappa,\lambda)\sqrt{\frac{x_{13}^2}{x_{24}^2}}$$

- Same as the amplitude without the delta functions!

$$\mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}}\delta^{(3)}(x_1 - x_5)\delta^{(2)}(\theta_1 - \theta_5)$$

- However the function g cannot be completely fixed from general principles: Unitarity, Parity and Duality.

Part III
Yangian

K.I, S.Jain, P.Nayak, T.Sharma, V.Umesh, **arXiv : 1801.nnppq**

Yangian Symmetry

- The presence of the **superconformal and dual superconformal symmetries indicate a Yangian symmetry** in the amplitude.

- A Yangian algebra is an **associative Hopf Algebra** generated by

$$[J^A, J^B] = f_C^{AB} J^C \ , \ [J^A, Q^B] = f_C^{AB} Q^C$$

- J^A take values in a Lie group G, both J^A and Q^A are constrained to obey the **Serre relations in addition to Jacobi Identity**.

$$[Q^A,[Q^B,J^C]] + [Q^B,[Q^C,J^A]] + [Q^C,[Q^A,J^B]] = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\}$$

$$[[Q^A,Q^B],[J^C,Q^D]] + [[Q^C,Q^D],[J^A,Q^B]] = \frac{1}{24}(f^{AGL} f^{BEM} f_K^{CD} + f^{CGL} f^{DEM} f_K^{AB}) \times f^{KFN} f_{LMN} \{J_G, J_E, J_F\}$$

- Under repeated commutations the **Q's generate an infinite dimensional Symmetry algebra**.

- The **necessary and sufficient conditions for the Yangian to exist is that Q satisfies the Serre relations**.

Yangian Symmetry in N=2 Chern-Simons theory

- The Yangian has a **basis (labelled by levels)**

$$\mathcal{J}_0^A = J^A \ , \ \mathcal{J}_1^A = Q^A \ , \ldots , \ \mathcal{J}_n^A$$

$$[\mathcal{J}^A_{(0)}, \mathcal{J}^B_{(0)}\} = f^{AB}{}_C \mathcal{J}^C_{(0)} \qquad [\mathcal{J}^A_{(0)}, \mathcal{J}^B_{(1)}\} = f_C^{AB} \mathcal{J}^C_{(1)}$$

- The generators $\mathcal{J}_n^A$ are **"n local" operators**. The infinite dimensional symmetry is generated by commutators of $\mathcal{J}_n^A$.

- Multi-locality arises due to the **co-product structure** of the Hopf algebra

$$\Delta(\mathcal{J}^A_{(0)}) = \mathcal{J}^A_{(0)} \otimes 1 + 1 \otimes \mathcal{J}^A_{(0)}$$

$$\Delta(\mathcal{J}^A_{(1)}) = \mathcal{J}^A_{(1)} \otimes 1 + 1 \otimes \mathcal{J}^A_{(1)} + f^{ABC} \mathcal{J}_B^{(0)} \otimes \mathcal{J}_C^{(0)}$$

- This causes the bilocal part to act **non-trivially** on multiparticle states. In particular, it will not act like a trivial tensor product of single particle action.

- This allows the overlap between spacetime symmetries and internal symmetries, and allows for a non-trivial symmetry structure.

- on a product of fields, the bilocals act on different sites at the same time

$$(\mathcal{J}^A_{(1)})_{bi\ local}(\phi^1\phi^2\phi^3 ... \phi^n) = f^{ABC} \sum_{i<j=1}^{n} \phi^1\phi^2 \ldots \mathcal{J}_B^{(0)} \phi^i \ldots \mathcal{J}_C^{(0)} \phi^j \ldots \phi^n$$

Yangian Symmetry in N=2 Chern-Simons theory

- For the N=2 theory the spacetime superconformal symmetry is $Osp(2|4)$

$$\mathcal{J}^A_{(0)} = \{p_{\alpha\beta}, m_{\alpha\beta}, d, k_{\alpha\beta}, r, q_\alpha, \bar{q}_\alpha, s_\alpha, \bar{s}_\alpha\}$$

- The superconformal generators can be expressed in terms of the **on-shell variables**

$$p_{\alpha\beta} = \sum_i \lambda_{i\alpha}\lambda_{i\beta}, \quad k_{\alpha\beta} = \sum_i \frac{\partial}{\partial\lambda_i^\alpha}\frac{\partial}{\partial\lambda_i^\beta},$$

$$m^\alpha{}_\beta = \sum_i \lambda_i^\alpha \frac{\partial}{\partial\lambda_i^\beta} - \frac{1}{2}\delta^\alpha_\beta \lambda_i^\gamma \frac{\partial}{\partial\lambda_i^\gamma}, \quad d = \sum_i \left(\frac{1}{2}\lambda_i^\alpha \frac{\partial}{\partial\lambda_i^\beta} + \frac{1}{2}\right),$$

$$r = \sum_i \left(\eta_i \frac{\partial}{\partial\eta_i} - \frac{1}{2}\right), \quad q_\alpha = \sum_i \lambda_{i\alpha}\eta_i, \quad \bar{q}_\alpha = \sum_i \lambda_{i\alpha}\frac{\partial}{\partial\eta_i},$$

$$s_\alpha = \sum_i \eta_i \frac{\partial}{\partial\lambda_i^\alpha}, \quad \bar{s}_\alpha = \sum_i \frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\lambda_i^\alpha}.$$

- **Superconformal invariance** of the amplitude is the statement

$$\mathcal{J}^A_{(0)}\mathcal{A}_4 = 0$$

Superconformal algebra in dual space

- The N=2 superconformal algebra in **dual space** is generated by

$$\{P_{\alpha\beta},\ M_{\alpha\beta},\ D,\ K_{\alpha\beta},\ R,\ Q_\alpha,\ \bar{Q}_\alpha,\ S_\alpha,\ \bar{S}_\alpha\}$$

$$P_{\alpha\beta} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\alpha\beta}}, \quad D = -\sum_{i=1}^{n}\left(x_i^{\alpha\beta}\frac{\partial}{\partial x_i^{\alpha\beta}} + \frac{1}{2}\theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha}\right),$$

$$Q_\alpha = \sum_{i=1}^{n}\frac{\partial}{\partial \theta_i^\alpha}, \quad \bar{Q}_\alpha = \sum_{i=1}^{n}\theta_i^\beta \frac{\partial}{\partial x_i^{\beta\alpha}},$$

$$M_{\alpha\beta} = \sum_{i=1}^{n}\left(x_{i\alpha}{}^{\gamma}\frac{\partial}{\partial x_i^{\gamma\beta}} + \frac{1}{2}\theta_{i\alpha}\frac{\partial}{\partial \theta_i^\beta}\right), \quad R = \sum_{i=1}^{n}\theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha}$$

- The remaining generators can be expressed using the inversion operator

$$I\left[x_i^{\alpha\beta}\right] = \frac{x_i^{\alpha\beta}}{x_i^2}, \quad I\left[\theta_i^\alpha\right] = \frac{x_i^{\alpha\beta}\theta_{i\beta}}{x_i^2}$$

$$K_{\alpha\beta} = I P_{\alpha\beta} I, \quad S_\alpha = I Q_\alpha I, \quad \bar{S}_\alpha = I\bar{Q}_\alpha I.$$

Emergence of Yangian

(x, θ)

Dual Superconformal algebra

$(\lambda, \eta, x, \theta)$

Extended onshell superspace

$$x_{i,i+1}^{\alpha\beta} = x_i^{\alpha\beta} - x_{i+1}^{\alpha\beta} = p_i^{\alpha\beta} = \lambda_i^\alpha \lambda_i^\beta$$
$$\theta_{i,i+1}^{\alpha} = \theta_i^{\alpha} - \theta_{i+1}^{\alpha} = q_i^{\alpha} = \lambda_i^\alpha \eta_i$$

(λ, η)

Superconformal algebra

(λ, η)

Dual superconformal algebra in restricted superspace

$\mathcal{J}_A^{(0)}$

$\bar{q}_\alpha$, d, s_α, $m_{\alpha\beta}$, r

Automorphism

S_α, D, $\bar{Q}_\alpha$, $M_{\alpha\beta}$, R

$(P_{\alpha\beta}, Q_\alpha) \rightarrow \text{trivial}$

- The $K_{\alpha\beta}$ and $\bar{S}_\alpha$ do not map to any of the Level 0 generators.

Emergence of Yangian: Level one generators

- Dual conformal generator in extended superspace $(\lambda, \eta, x, \theta)$

$$\tilde{K}_{\alpha\beta} = \sum_{i=1}^{4}\Big[x_{i\alpha}{}^{m} x_{i\beta}{}^{n}\frac{\partial}{\partial x_i^{mn}} + \frac{1}{2}x_{i(\alpha}{}^{m}\theta_{i\beta)}\frac{\partial}{\partial\theta_i^m} + \frac{1}{4}(x_i + x_{i+1})_{(\alpha}{}^{\gamma}\lambda_{i\beta)}\frac{\partial}{\partial\lambda_i^\gamma}$$
$$+\frac{1}{4}(\theta_i + \theta_{i+1})_{(\alpha}\lambda_{i\beta)}\frac{\partial}{\partial\eta_i}\Big] + \frac{1}{2}\sum_{i=1}^{4}\Delta_i\ x_{i\alpha\beta}$$

- Dual conformal generator restricted to onshell superspace (λ, η)

$$\tilde{K}_{\alpha\beta} = \frac{1}{2}\sum_{i=1}^{4}\Big[x_{1(\alpha}{}^{\gamma}\lambda_{i\beta)}\frac{\partial}{\partial\lambda_i^\gamma} + \theta_{1(\alpha}\lambda_{i\beta)}\frac{\partial}{\partial\eta_i}\Big] - \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{i-1}\Big[\lambda_{j(\alpha}\lambda_{i\beta)}\lambda_j^\gamma\frac{\partial}{\partial\lambda_i^\gamma} + \lambda_{j(\alpha}\lambda_{i\beta)}\eta_j\frac{\partial}{\partial\eta_i}\Big]$$
$$-\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{i}\Big[\lambda_{j(\alpha}\lambda_{i\beta)}\lambda_j^\gamma\frac{\partial}{\partial\lambda_i^\gamma} + \lambda_{j(\alpha}\lambda_{i\beta)}\eta_j\frac{\partial}{\partial\eta_i}\Big] - \frac{1}{2}\sum_{i=1}^{4}\Delta_i\left(x_1 - \sum_{j=1}^{i-1}\lambda_{i\alpha}\lambda_{j\beta}\right)$$

- Acting the generator on the amplitude written in onshell space, it can be checked that the (x_1, θ_1) dependence cancels out.

Emergence of Yangian: Level one generators

- The **bilocal form of the level one generator** is visible when we rewrite it in terms of the level 0 generators.

$$\tilde{K}_{\alpha\beta} = \frac{1}{4}\sum_{i>j}\left[p_{(i\alpha}{}^{\gamma}d_{j\beta)\gamma} + q_{i(\alpha}\bar{q}_{j\beta)} - (i \leftrightarrow j)\right] - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{i-1}(\Delta_i - 1)p_{i\alpha\beta}$$

- Similarly

$$\tilde{\bar{S}}_{\alpha} = -\frac{1}{4}\sum_{i>j}\left[\bar{q}_j^{\beta}(m_{i\alpha\beta} + \epsilon_{\alpha\beta}d_i) - \bar{q}_{j\alpha}r_i + p_{j\alpha}{}^{\beta}s_{i\beta} - (i \leftrightarrow j)\right] - \frac{1}{2}\sum_{i>j}^{4}(\Delta_i - 1)q_{j\alpha}$$

- This is **consistent with the general form of level one generators**

$$\mathcal{J}^A_{(1)} = \frac{1}{2}f^A_{BC}\sum_{j<k}\mathcal{J}^C_{j,(0)}\mathcal{J}^B_{k,(0)} + \sum_k v^l\,\mathcal{J}^A_{l,(0)}$$

Kazakov et al.

- The precise identification with the **level one generators of superconformal algebra**

$$\tilde{K}_{\alpha\beta} \equiv p^{(1)}_{\alpha\beta}$$

$$\tilde{\bar{S}}_{\alpha} \equiv q^{(1)}_{\alpha}$$

- Remaining level one generators can be obtained from **adjoint condition**

$$[\mathcal{J}^A_{(0)}, \mathcal{J}^B_{(1)}\} = f^{AB}_C\,\mathcal{J}^C_{(1)}$$

Summary: Emergence of Yangian

(λ, η)
Superconformal algebra

(λ, η)
Dual superconformal algebra in restricted superspace

$\mathcal{J}_A^{(0)}$: $\bar{q}_\alpha$, d, s_α, $m_{\alpha\beta}$, r $\xleftarrow{\text{Automorphism}}$ S_α, D, $\bar{Q}_\alpha$, $M_{\alpha\beta}$, R

$(P_{\alpha\beta}, Q_\alpha) \rightarrow \text{trivial}$

$\mathcal{J}_A^{(1)}$: $p^{(1)}_{\alpha\beta}$, $q^{(1)}_\alpha$ $\xleftarrow{\text{map}}$ $\tilde{K}_{\alpha\beta}$, $\tilde{\bar{S}}_\alpha$

$$[\mathcal{J}^A_{(0)}, \mathcal{J}^B_{(1)}\} = f_C^{AB}\mathcal{J}^C_{(1)}$$

$\vdots$

$\mathcal{J}_A^{(2)}$: $[\mathcal{J}^A_{(1)}, \mathcal{J}^B_{(1)}\} = f_C^{AB}\mathcal{J}^C_{(2)}$ $\cdots\cdots\cdots$ Yangian superalgebra

$Y[Osp(2|4)]$

Summary: Yangian Invariance of the four point amplitude

- The four point amplitude is invariant under the Level one generators of the Yangian $K_{\alpha\beta}$ and $\bar{S}_{\alpha}$

- The invariance under the rest of the Level one generators follows from the adjoint condition since the amplitude is already invariant under the level 0 superconformal symmetry,

- Invariance under Level 2 and higher generators invariance follows through successive commutations

- Thus **superconformal and dual superconformal symmetries generate a Yangian symmetry!**

$$\mathcal{J}^{A}_{(0)}\mathcal{A}_4 = 0 \ , \ \mathcal{J}^{A}_{(1)}\mathcal{A}_4 = 0 \ , \implies \ \mathcal{Y}\mathcal{A}_4 = 0,$$

- **This is all loop exact for the four point amplitude in the planar limit.**

Part III
Summary

Summary

- We started with a goal of computing **arbitrary m→n tree level scattering amplitudes** in U(N) $\mathcal{N} = 2$ Chern-Simons matter theories with fundamental matter.
- We achieved this via **BCFW recursion relations**, this enabled us to express arbitrary n point amplitudes as products of four point amplitudes!
- We saw an explicit example where the six point amplitude is expressed as a product of two four point amplitudes via two factorization channels.
- We can also extract the **recursion relations for non-supersymmetric theories** eventhough BCFW does not directly apply.
- We saw an explicit example for the six fermion amplitude in the fermion coupled Chern-Simons theory.

Summary

- We showed that the **all loop 2→2 scattering amplitude** computed in **K.I, Jain, Mazumdar, Minwalla, Umesh, Yokoyama** is **dual superconformal invariant**.

- Thus **dual superconformal symmetry is all loop exact**, at least for the 4 point amplitude.

- The superconformal and dual superconformal symmetries together generate an **infinite dimensional Yangian symmetry**!

- **It follows that Yangian symmetry is all loop exact for the four point amplitude in the planar limit.**

- This is in contrast to SYM and ABJM where the Yangian symmetry is anomalous due to IR divergences.

Things to do

	$\mathcal{N}=4$ SYM	ABJM	$\mathcal{N}=2$ CSM
BCFW recursions for all tree level amplitudes	✓	✓	✓
Dual superconformal symmetry	✓	✓	$2\to 2$
Superconformal x Dual Superconformal symmetry = Yangian	✓	✓	$2\to 2$
Manifestly Yangian invariant representation (Orthogonal Grassmanian)	✓	✓	🔍
Symmetries at loop level	✗	✗	$2\to 2$
Yangian symmetry of the classical action	🔍	🔍	🔍

Even more things to do…

- **Amplitude-Wilson loop duality, CHY formulation, where is the higher spin symmetry?**

- What are the **crossing rules for anyonic channels in arbitrary higher point amplitudes**?

- Does some anomalous form of dual conformal invariance survive for non-supersymmetric amplitudes, if so can it constrain the form of the amplitude? (In the massless limit, these amplitudes are also quite simple)

- What is the anyonic phase structure for an **n-particle Aharonov-Bohm scattering**? (non-relativistic limit of the whole story)

תודה רבה!!

Aside: Spinor helicity basis

- Spinor helicity representation

$$p_i^{\alpha\beta} = p_i^{\mu}\sigma_{\mu}^{\alpha\beta} = \lambda_i^{\alpha}\lambda_i^{\beta}$$
$$(p_i + p_j)^2 = 2p_i.p_j = -\langle\lambda_i^{\alpha}\lambda_{j,\alpha}\rangle^2 = \langle ij\rangle^2$$

- Onshell Supersymmetry generators

$$\mathcal{Q} = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \lambda_i\eta_i,$$
$$\bar{\mathcal{Q}} = \sum_{i=1}^{n} \bar{q}_i = \sum_{i=1}^{n} \lambda_i\partial_{\eta_i}$$

Preserving the onshell conditions

- In 3d the momentum shift is non-linear in z

$$\begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix} = R \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \qquad R = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2\ i} \\ \frac{z-z^{-1}}{2\ i} & \frac{z+z^{-1}}{2} \end{pmatrix}$$

$$p_i \rightarrow \frac{p_{ij}}{2} + q z^2 + \tilde{q} z^{-2} \qquad q^{\alpha\beta} = \frac{1}{4}(\lambda_2 + i\lambda_1)^\alpha (\lambda_2 + i\lambda_1)^\beta$$

$$p_j \rightarrow \frac{p_{ij}}{2} - q z^2 - \tilde{q} z^{-2} \qquad \tilde{q}^{\alpha\beta} = \frac{1}{4}(\lambda_2 - i\lambda_1)^\alpha (\lambda_2 - i\lambda_1)^\beta\ .$$

- The momentum deformations **preserve the onshell conditions**

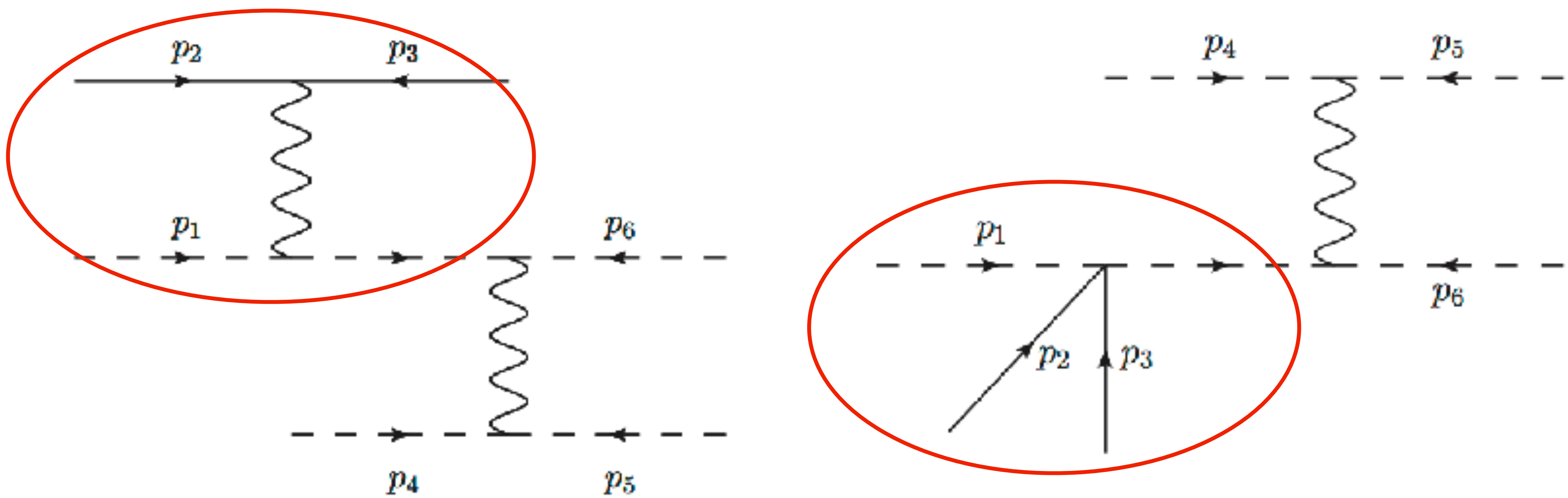
$$p_i^2 = 0\ , p_j^2 = 0$$

$$q.\tilde{q} = -\frac{1}{4} p_i.p_j\ ,\ q + \tilde{q} = \frac{1}{2}(p_i - p_j)\ , q.p_{ij} = 0\ ,\ \tilde{q}.p_{ij} = 0$$

Gang, Huang, Koh, Lee, Lipstein

Eg: Six point amplitude: Asymptotic behavior

- The **Asymptotic behavior involves very precise cancellations of divergences** in the Feynman diagram approach.
- For eg, the process $\langle\bar{\psi}_1\phi_2\bar{\phi}_3\psi_4\bar{\phi}_5\phi_6\rangle$ gets contribution from 15 diagrams.
- 5 of them are well behaved, the remaining 10 are **individually divergent**, However the **divergences cancel pair wise**.
- Typical cancellations are between



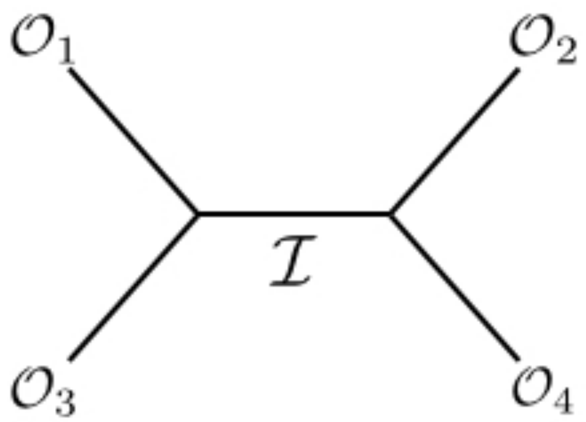
$$\sim -\frac{8\pi^2 iz}{\kappa^2}\frac{\langle q3\rangle\langle 45\rangle\langle 56\rangle\langle 46\rangle}{p_{45}^2 p_{123}^2} + \mathcal{O}(\frac{1}{z}) \ , \ \sim \frac{8\pi^2 iz}{\kappa^2}\frac{\langle q3\rangle\langle 45\rangle\langle 56\rangle\langle 46\rangle}{p_{45}^2 p_{123}^2} + \mathcal{O}(\frac{1}{z})$$

4 point amplitude as a free field correlator in dual space

- In a general CFT, in the **double light cone limit, only Identity operators are expected to contribute**!

- In the channel where $(\mathcal{O}_1, \mathcal{O}_3)$ and $(\mathcal{O}_2, \mathcal{O}_4)$ are brought together

$$\Delta_1 = \Delta_3 = -\frac{1}{2}$$
$$\Delta_2 = \Delta_4 = \frac{1}{2}$$



- The four point amplitude can be accounted for by an **identity exchange**.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \langle \mathcal{O}_1(x_1)\mathcal{O}_3(x_3)\rangle\langle \mathcal{O}_2(x_2)\mathcal{O}_4(x_4)\rangle$$
$$= c_1 c_2 \sqrt{\frac{x_{13}^2}{x_{24}^2}}$$

- This suggests $g(\kappa, \lambda) = c_1 c_2.$

- It would be interesting to understand the CFT interpretation of these operators, and also to see what happens in the cross channel.

Scattering in U(N) Chern-Simons matter theories

- Consider $2 \to 2$ scattering of particles in representations R_1 and R_2 of U(N)

$$R_1 \times R_2 = \sum_m R_m$$

- The S matrix takes the schematic form

$$S = \sum_m P_m S_m$$

$$P_m : \text{ projector in } m^{th} \text{ rep, } S_m \text{ is scattering in } m^{th} \text{ channel.}$$

- The Aharanov-Bohm phase of the particle R_1 as it circles around particle R_2 is $2\pi\nu_m$ where

$$\nu_m = \frac{4\pi}{\kappa} T_1^a T_2^a = \frac{2\pi}{\kappa} \left(C_2(R_1) + C_2(R_2) - C_2(R_m) \right)$$

- Scattering amplitude in the m'th exchange channel: Aharanov-Bohm scattering of a unit charge particle off a flux tube of flux $2\pi\nu_m$

Scattering in U(N) Chern-Simons matter theories

- Channels of scattering

$$\text{Fundamental} \otimes \text{Fundamental} \rightarrow \text{Symm}(U_d) \oplus \text{Asymm}(U_e)$$
$$\text{Fundamental} \otimes \text{Antifundamental} \rightarrow \text{Adjoint}(T) \oplus \text{Singlet}(S)$$

- The quadratic Casimirs

$$C_2(F) = C_2(A) = \frac{N^2-1}{2N}\ ,\ C_2(Sym) = \frac{N^2+N-2}{N}$$
$$C_2(ASym) = \frac{N^2-N-2}{N}\ ,\ C_2(Adj) = N\ ,\ C_2(Sing) = 0$$

- Anyonic phase

$$\nu_{Sym} = \frac{1}{\kappa} - \frac{1}{N\kappa}\ ,\ \nu_{Asym} = -\frac{1}{\kappa} - \frac{1}{N\kappa}$$
$$\nu_{Adj} = \frac{1}{N\kappa}\ ,\ \nu_{Sin} = -\frac{N}{\kappa} + \frac{1}{N\kappa}$$

- In the large N, large κ limit, define 't Hooft coupling $\lambda = \frac{N}{\kappa}$

$$\nu_{Asym} \sim \nu_{Sym} \sim \nu_{Adj} \sim O\left(\frac{1}{N}\right)\ ,\nu_{Sing} \sim O(\lambda)$$

Scattering in U(N) Chern-Simons matter theories

- Anyonic phases in the large N, large κ limit

$$\nu_{Asym} \sim \nu_{Sym} \sim \nu_{Adj} \sim O\left(\frac{1}{N}\right) \,, \nu_{Sing} \sim O(\lambda)$$

- The T matrices themselves have the large N behavior

$$T_{Asym} \sim T_{Sym} \sim T_{Adj} \sim O\left(\frac{1}{N}\right) \,, \, T_{sing} \sim O(1)$$

- The singlet channel is effectively anyonic in the large N ,large κ limit.

- Unitarity $i(T^\dagger - T) = TT^\dagger$ is a non-trivial check only for the singlet channel. In other channels it follows from hermiticity.

- Observation: Naive crossing symmetry rules from any of the non-anyonic channels to the singlet channel leads to a non unitary S matrix.

S.Jain, M. Mandlik, S.Minwalla,T.Takimi S.Wadia, S.Yokoyama

- Conjecture: Singlet channel S matrices have the form

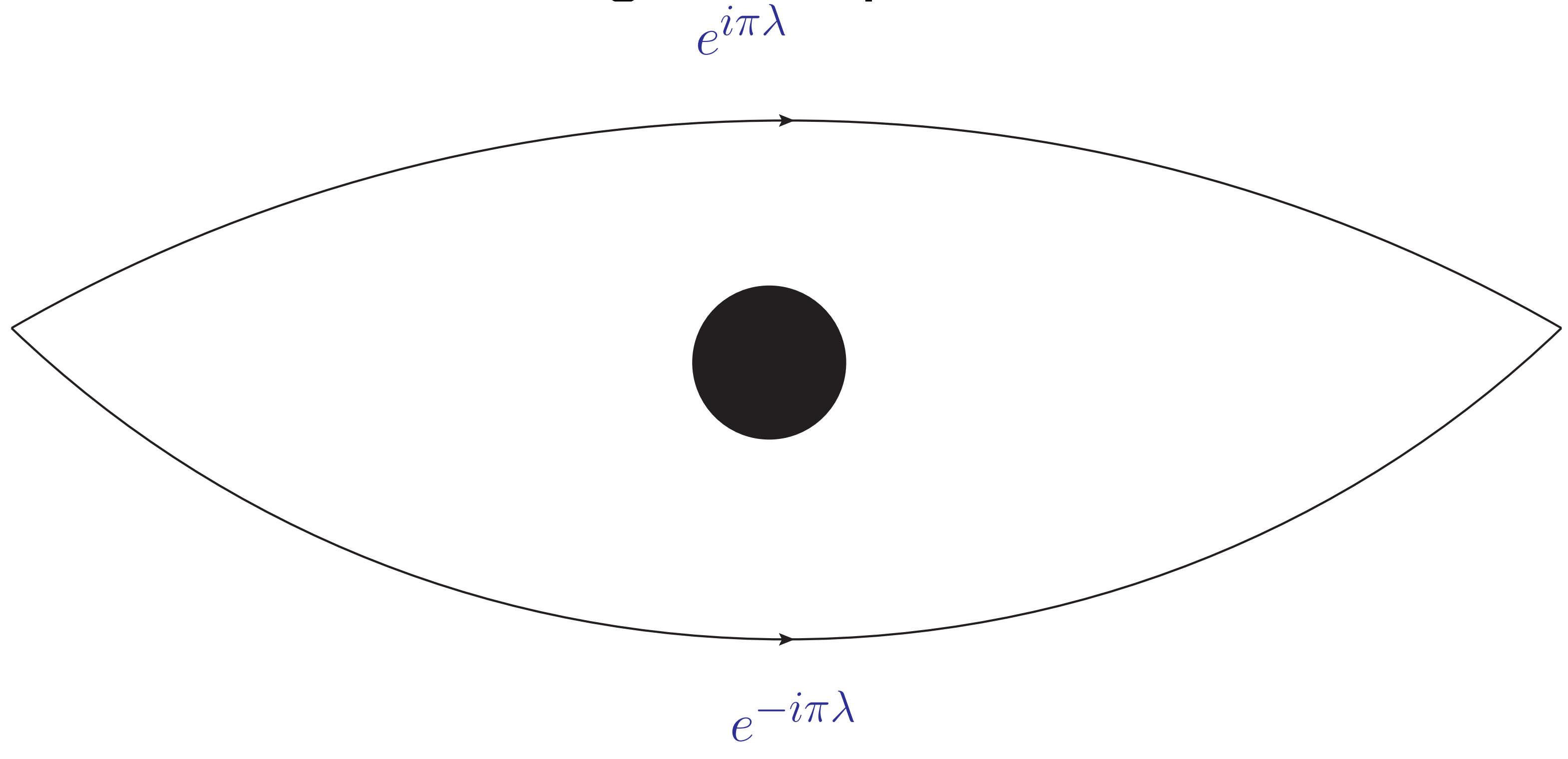
$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\mathrm{naive}}(s,\theta)$$

- $\mathcal{T}^{S;\mathrm{naive}}$ is the matrix obtained from naive analytic continuation of particle-particle scattering.

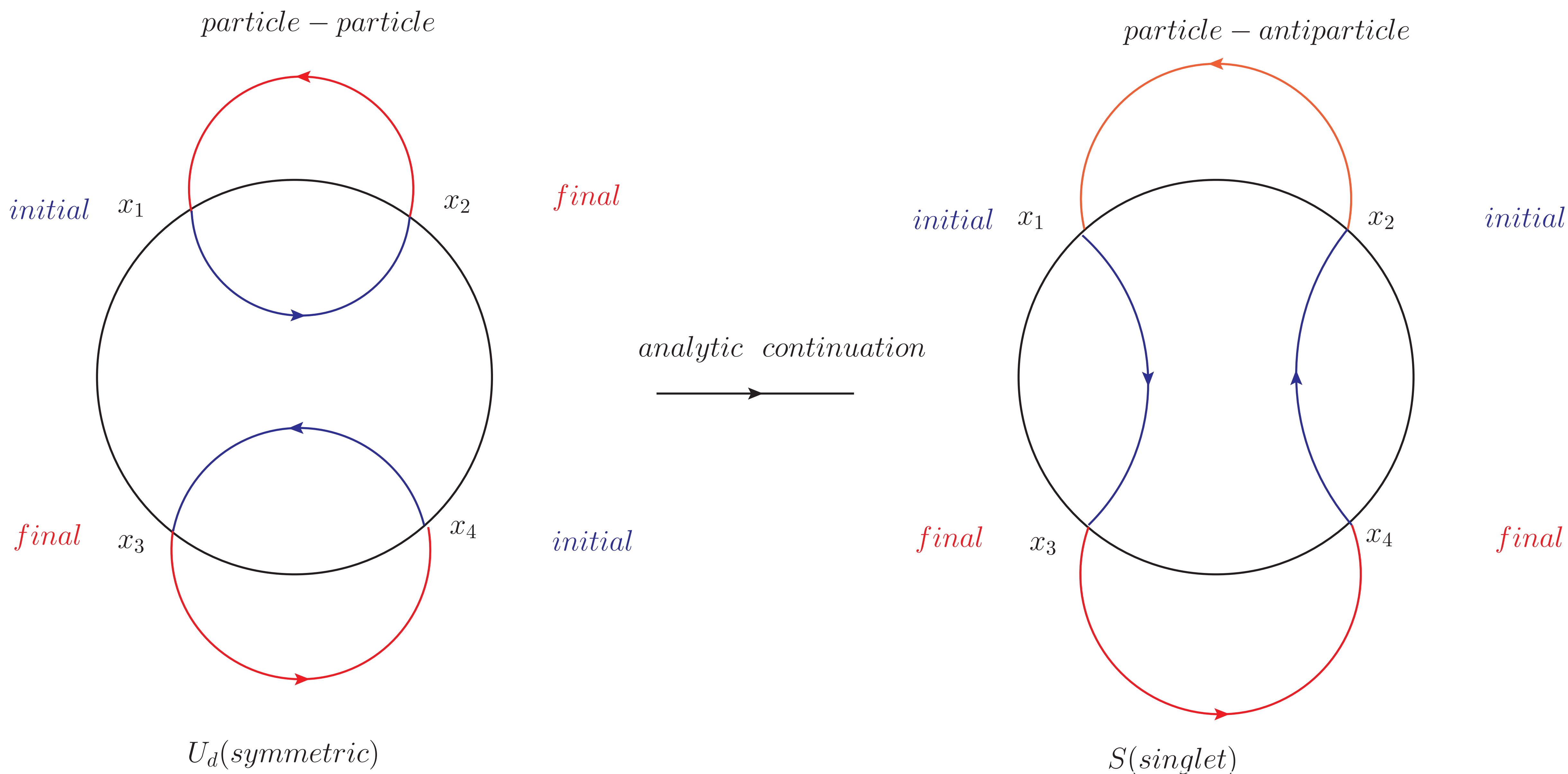
Nature of the conjecture: Delta function

$$\mathcal{S} = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}^{S;\mathrm{naive}}(s,\theta)$$

- The conjectured S matrix has a non-analytic delta function piece.
- The delta function modulated by anyonic phase is already known to be necessary to unitarize Aharonov-Bohm scattering.
- $\cos\pi\lambda$ in the identity term is due to the interference of the Aharanov-Bohm phases of the incoming wave packets.



Modified crossing rules: Heuristic explanation



- Attach Wilson lines to make correlators gauge invariant

$$T_{U_d} W_{U_d} \to T_S W_S$$

$$\frac{W_{U_d}}{W_S} = \frac{\oint \text{with 2 circular Wilson lines}}{\oint \text{with 1 circular Wilson line}} = \frac{\sin(\pi\lambda)}{\pi\lambda}$$

Witten