

Amplitudes and hidden symmetries in Supersymmetric Chern-Simons matter theories

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Based on

K.I, S.Jain, P.Nayak, V.Umesh, [arXiv :1710.04227](#), Accepted in Phys.

Rev. Letters.

K.I, S.Jain, S.Majumdar, P.Nayak, T.Neogi, T.Sharma, R.Sinha, V.Umesh,

[arXiv : 1711.02672](#) Submitted to Phys. Rev. Letters

K.I, S.Jain, P.Nayak, T.Sharma, [arXiv : 18ij.nppq](#) In preparation

References:

K.I, S.Jain, S.Mazumdar, S.Minwalla, V.Umesh, S.Yokoyama, [arXiv: 1505.06571](#), JHEP 1510 (2015) 176.

S.Jain, M.Mandlik, S.Minwalla, T.Takimi, S.Wadia, S.Yokoyama,
[arXiv:1404.6373](#), JHEP 1504 (2015) 129.

Part I

Introduction

Introduction

- When we say that a theory is **integrable**, what do we really mean?
- Classically symmetries = degrees of freedom
- **Several examples:** n - dimensional harmonic oscillator, Central force motion, Heisenberg spin chain system, Sine-Gordon equation.
- At **quantum** level, there is **no universal notion of integrability**.
- One indicator of integrability is the presence of **infinite dimensional symmetry structures**.
- Some of the best known quantum field theories that arise in the low energy limits of string theory are **conjectured to be integrable**.

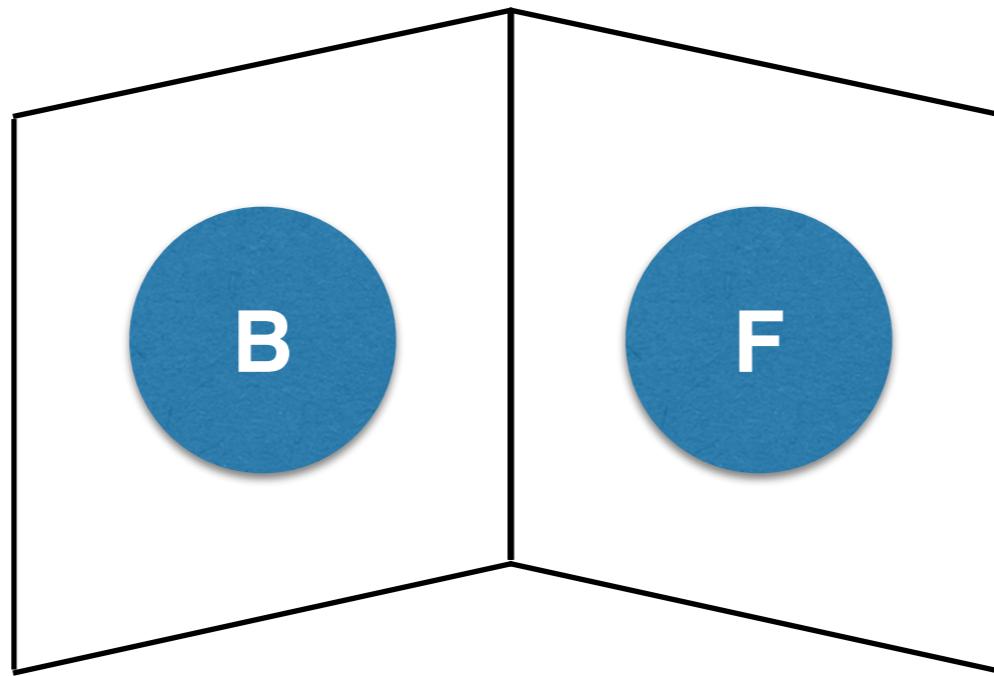
$d = 4, \mathcal{N} = 4$ Super Yang Mills
 $d = 3, \mathcal{N} = 6$ ABJM

1012.3982, Beisert et.al
- The infinite dimensional symmetry structures appear in **planar tree level scattering amplitudes!**

Introduction

- The **tree level superamplitudes** in both SYM and ABJM possess hidden internal symmetries known as **dual superconformal symmetry**.
- The **superconformal symmetry and the dual superconformal symmetry overlap** to generate an **infinite dimensional Yangian symmetry** of the superamplitude.
- This **seems to violate the Coleman-Mandula theorem** that states that **spacetime and internal symmetries** of the S matrix may only be combined by **trivial direct product!**
- However the Yangian generators do not obey a basic assumption of the theorem, **they do not act on multiparticle states via a tensor product generalization of the action of single product states!**
- Besides, the action of the Yangian symmetry is realized non-linearly on the fields of the theory and thus acts as a powerful constraint.

Motivation



Bosonization duality

Aharony, Bardeen, Benini, Chang, Frishman, Giombi, Giveon, Gur-Ari, Gurucharan, K.I.,
Karch, Kutasov, Jain, Maldacena, Mandlik, Minwalla, Moshe, Sharma, Prakash,
Radicevic, Takimi, Trivedi, Seiberg, Sonnenschein, Tong, Yacoby, Yin, Yokoyama, Wadia,
Witten, Zhiboedov

Large N Bosonization duality

- $U(N_F)$ Chern-Simons coupled to fundamental fermions (regular fermion)

$$S = \int d^3x \left(i\epsilon^{\mu\nu\rho} \frac{\kappa_F}{4\pi} \text{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \bar{\psi}^i \gamma^\mu D_\mu \psi_i + m_f \bar{\psi}^i \psi_i \right)$$

- $U(N_B)$ Chern-Simons coupled to fundamental bosons

$$\begin{aligned} S = \int d^3x & \left(i\epsilon^{\mu\nu\rho} \frac{\kappa_B}{4\pi} \text{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + D_\mu \bar{\phi} D^\mu \phi \right. \\ & \left. + \sigma \bar{\phi} \phi + N_B \frac{m_B^2}{b_4} \sigma - N_B \frac{\sigma^2}{2b_4} \right) \end{aligned}$$

- Wilson-Fisher limit (critical boson)

$$b_4 \rightarrow \infty, m_B \rightarrow \infty, 4\pi \frac{m_B^2}{b_4} = \text{fixed}$$

Large N Bosonization duality

t' Hooft large N limit

$$\lambda_B = \lim_{\substack{N_B \rightarrow \infty \\ \kappa_B \rightarrow \infty}} \frac{N_B}{\kappa_B} \quad \lambda_F = \lim_{\substack{N_F \rightarrow \infty \\ \kappa_F \rightarrow \infty}} \frac{N_F}{\kappa_F}$$

- $U(N_B)$ Chern-Simons coupled to fundamental bosons at Wilson-Fisher limit
dual
- $U(N_F)$ Chern-Simons coupled to fundamental Fermions

duality map

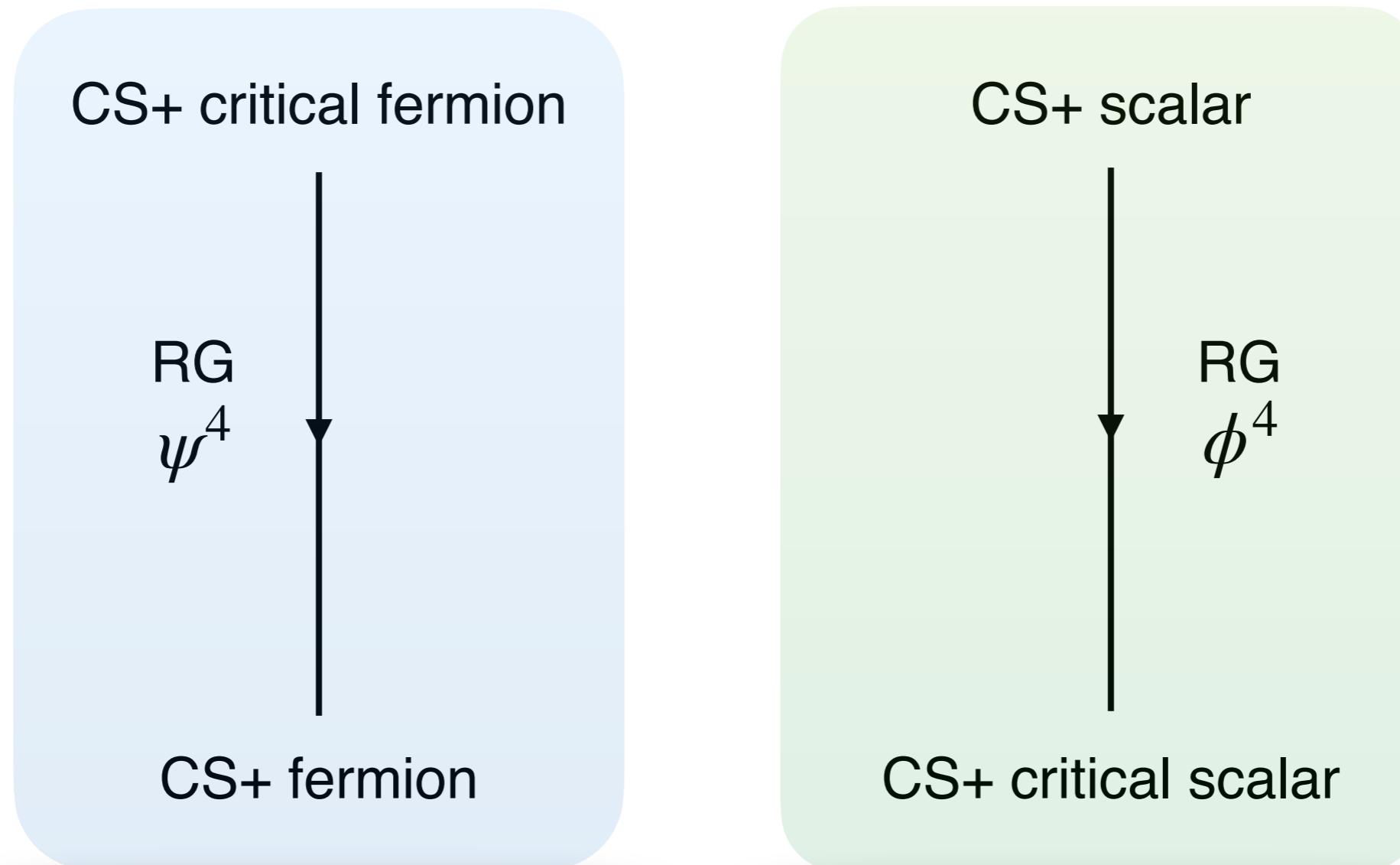
$$\left\{ \begin{array}{l} \kappa_F = -\kappa_B \\ N_F = |\kappa_B| - N_B \\ \lambda_B = \lambda_F - \text{Sign}(\lambda_F) \\ m_F = -m_B^{Crit} \lambda_B \end{array} \right.$$

- **Physical observables such as spectrum of single trace primaries, thermal partition functions, correlation functions of higher spin currents, S matrices, match on both sides under the map!**

Large N Bosonization duality

- The duality has been shown to hold along the RG flow fixed points labelled by the 't Hooft coupling and N.

UV



IR

Large N duality - Supersymmetric version

- $d = 3, \mathcal{N} = 2$ superconformal Chern-Simons theory coupled to matter in fundamental representation of $U(N)$

$$\begin{aligned} S_{\mathcal{N}=2}^L = & \int d^3x \left[-\frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) \right. \\ & \left. + \bar{\psi} i \not{D} \psi - \mathcal{D}^\mu \bar{\phi} \mathcal{D}_\mu \phi + \frac{4\pi^2}{\kappa^2} (\bar{\phi} \phi)^3 + \frac{4\pi}{\kappa} (\bar{\phi} \phi)(\bar{\psi} \psi) + \frac{2\pi}{\kappa} (\bar{\psi} \phi)(\bar{\phi} \psi) \right] \end{aligned}$$

- The theory exhibits a **strong-weak self duality** under the duality map
$$\begin{aligned}\kappa' &= -\kappa, \quad N' = |\kappa| - N + 1, \quad \lambda' = \lambda - \text{Sgn}(\lambda) \\ \lambda &= \frac{N}{\kappa}, \quad N \rightarrow \infty, \kappa \rightarrow \infty\end{aligned}$$
- **K.I, Jain, Mazumdar, Minwalla, Umesh, Yokoyama** : $2 \rightarrow 2$ scattering amplitudes to all orders in the 't Hooft coupling. (**summing planar diagrams**)
- The results confirm the duality, however there were lot of additional surprises. The amplitude had remarkable non-renormalization properties.

Exact S matrices computed to all loops in the planar limit

- Channels of scattering

$$\text{Fundamental} \otimes \text{Fundamental} \rightarrow \text{Symm}(U_d) \oplus \text{Asymm}(U_e)$$

$$\text{Fundamental} \otimes \text{Antifundamental} \rightarrow \text{Adjoint}(T) \oplus \text{Singlet}(S)$$

- In the symmetric, anti-symmetric and adjoint channels of scattering the **amplitude is tree-level exact to all orders in λ .**

$$T_{symm}^{all\ loop} = T_{Asymm}^{all\ loop} = T_{Adj}^{all\ loop} = T_{tree}$$

- In the singlet channel the coupling dependence is **extremely simple.**

$$T_{singlet}^{all\ loop} = N \frac{\sin \pi \lambda}{\pi \lambda} T_{tree}$$

- Tree level super amplitude

$$T_{tree} = \frac{4\pi}{\kappa} \frac{\langle 12 \rangle}{\langle 23 \rangle} \delta\left(\sum_{i=1}^4 p_i\right) \delta^2(Q)$$

$$\delta^2(Q) = \sum_{i < j=1}^n \langle ij \rangle \eta_i \eta_j$$

$$A_i = a_i + \eta_i \alpha_i$$
$$A_i^\dagger = a_i^\dagger \eta_i + \alpha_i^\dagger$$

$$S = \langle 0 | A_4 A_3 A_2^\dagger A_1^\dagger | 0 \rangle$$

Motivation

- It is incredibly rare in any quantum field theory that the amplitude has vanishing loop corrections.
- Maybe some **powerful symmetry** that protects the amplitude from renormalization.
- Is it possible to compute **all loop $m \rightarrow n$ scattering amplitudes** in the $N=2$ theory at least in the planar limit?
- Does the **non-renormalization** results of the $2 \rightarrow 2$ scattering continue to persist for arbitrary higher point amplitudes?
- These computations would **test the duality** in regions un-probed by large N perturbation theory yet.
- If the non-renormalizations do persist for higher point amplitudes, it is a sign of something special about this theory.

Our work

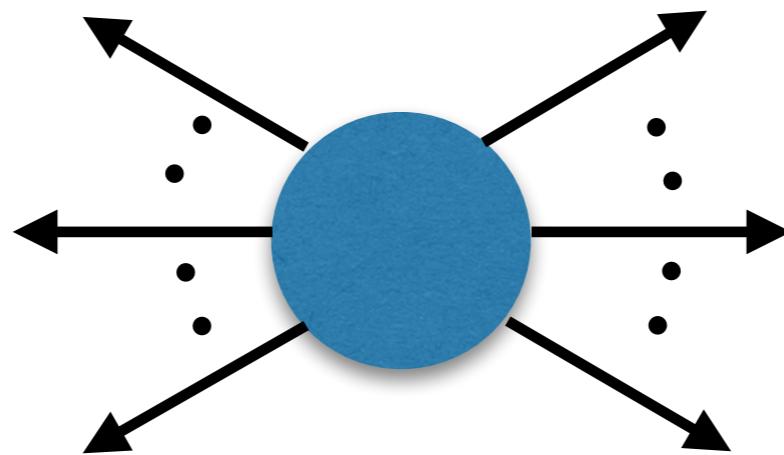
- As a first step towards the all loop $m \rightarrow n$ scattering, is it possible to write down **arbitrary $m \rightarrow n$ tree level amplitudes** ?
- We are able to achieve this via **BCFW recursions** **K.I, Jain, Nayak, Umesh**
- As a first step towards thinking about higher point amplitudes we identify a **hidden symmetry** in the $2 \rightarrow 2$ amplitude computed earlier that might explain the non-renormalization.
- This symmetry is known as **dual superconformal symmetry**.
K.I, Jain, Majumdar, Nayak, Neogi, Sharma, Sinha, Umesh
- Answer analysis tells us this symmetry is **exact to all loops!**
- The superconformal symmetry and dual superconformal symmetry together generate the infinite dimensional **Yangian Symmetry**.
K.I, Jain, Nayak, Sharma, Umesh, to appear
- If this persists for higher point amplitudes, this suggests that the theory we are dealing with may be **integrable!**

Where to?

- Recently there is a conjectured version of **bosonization duality at finite N and k**. Aharony
- It is very less understood. So far **not a single computation** has been done beyond the leading order in large N.
- The hindrances are computational complexity, **non-planar diagrams**, mixing of single trace and double trace operators in applying CFT methods.
- If one can make an **integrability conjecture** for the susy theory it could be the first arena for computations at finite N and k
- At **large N, integrability is not useful** since the anomalous dimensions of gauge invariant operators are protected by higher spin symmetries.
- However, these **symmetries are weakly broken beyond the planar limit** and the operaors acquire anomalous dimensions.
- It is **ok to dream big**, but first let us do the **baby steps**.

Part II

All tree level amplitudes



- K.I, S.Jain, P.Nayak, V.Umesh, [arXiv :1710.04227](#)

BCFW recursions in 2+1 dimensions

- Recursion relations enable to construct **n point tree level scattering amplitudes from lower point tree level amplitudes.**

Britto, Cachazo, Feng, Witten

- Central idea:
 - Tree level amplitudes are **continuously deformable** analytic functions of momenta.
 - Only type of singularities that can appear at tree level are **simple poles**.
 - One can **reconstruct amplitudes** for generic scattering kinematics knowing its behavior in **singular kinematics**.
 - In these singular regions **amplitudes factorize** into causally disconnected amplitudes with fewer legs, connected by an **intermediate onshell state**.
- We will focus on situation where all the external particles are massless.

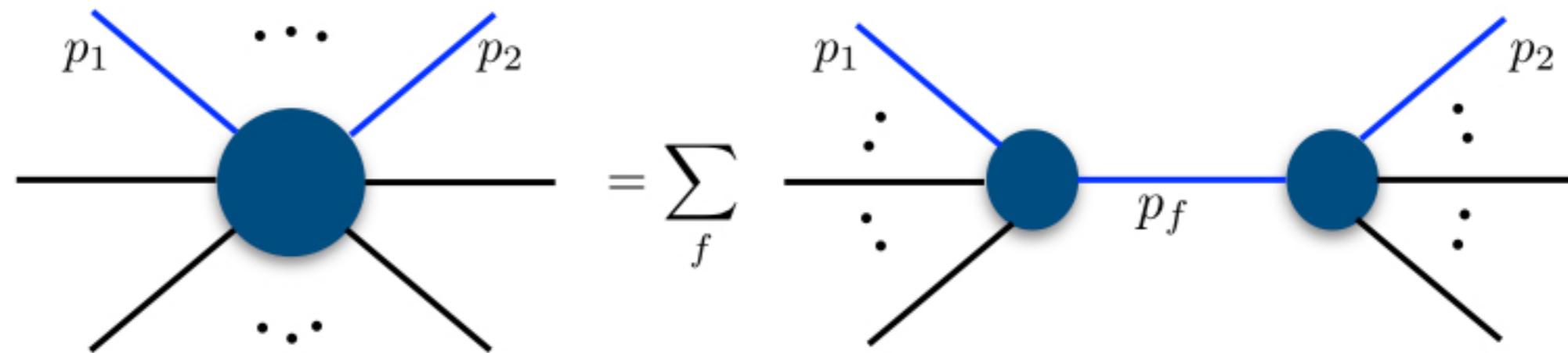
BCFW recursions in 2+1 dimensions

- Promote the amplitude into a one complex parameter family of amplitudes

$$A(z) = A(p_1, \dots, p_i(z), p_{i+1}, \dots, p_l(z), \dots, p_{2n})$$

- The necessary and sufficient conditions are:

- The momentum deformation should preserve on-shell conditions and momentum conservation.**
- The amplitude should be asymptotically well behaved under the deformation.**



- A higher point amplitude factorizes into lower point amplitudes!

The recursion formula for an arbitrary $2n$ point amplitude

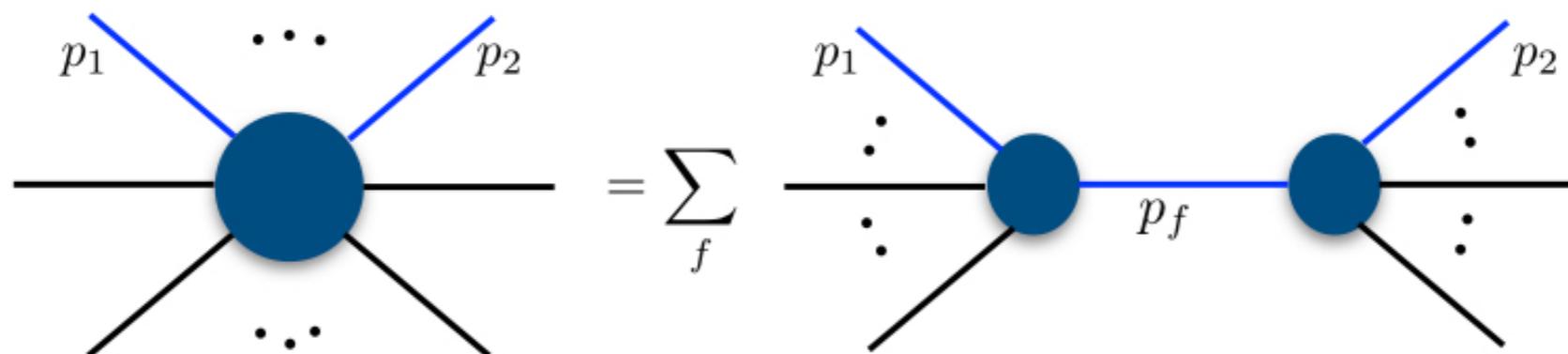
- Write a contour integral representation for the amplitude

$$\frac{1}{2\pi i} \oint_{C_{z=1}} \frac{dz}{z-1} A(z)$$

- Deform the contour to $\mathbf{z \rightarrow \infty}$, If $A(z)$ has no poles, the integral vanishes

$$A(z=1) = - \sum_{\text{poles: } z^i} \text{Res}_{z=z^i} \frac{A(z)}{z-1}$$

- remember that all the deformed momenta satisfy the onshell conditions!



$$A(z=1) = - \sum_f \sum_{\text{poles: } z_f^i} \text{Res}_{z=z_f^i} \frac{1}{z-1} \frac{A_L(p_1 \dots p_i(z), \dots p_n) A_R(p_{n+1} \dots p_j(z), \dots p_{2n})}{\hat{p}_f^2(z)}$$

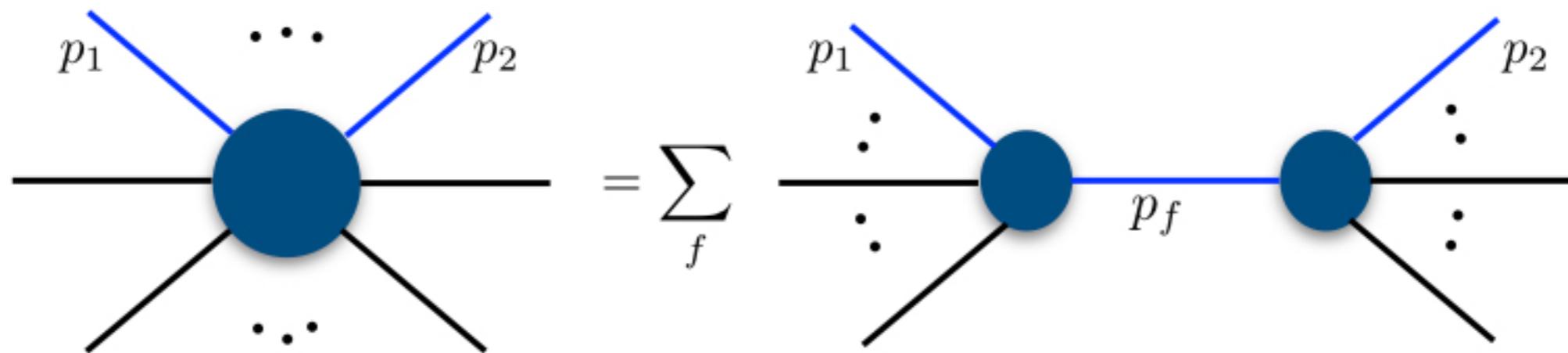
- We have used the fact that at **tree level the only possible singularities** are **simple poles**!

Asymptotic behavior

- Onshell susy methods, encode the **component amplitudes into a superamplitude**.
- **Susy ward identities** relate various component amplitudes and reduce the number of independent amplitudes.
- Susy also ensures that **if the independent component amplitudes are well behaved then the entire superamplitude is well behaved**.
- Using two independent methods we showed that the superamplitude is well behaved
 - **Background field expansion.** Arkani-Hamed, Kaplan
 - **Explicit Feynman diagram computation** of component amplitudes. (tedious, but provides explicit check for any given higher point amplitude).
 - The recursion formula then follows from **Cauchy residue theorem**.

The recursion formula for arbitrary $2n$ point superamplitude

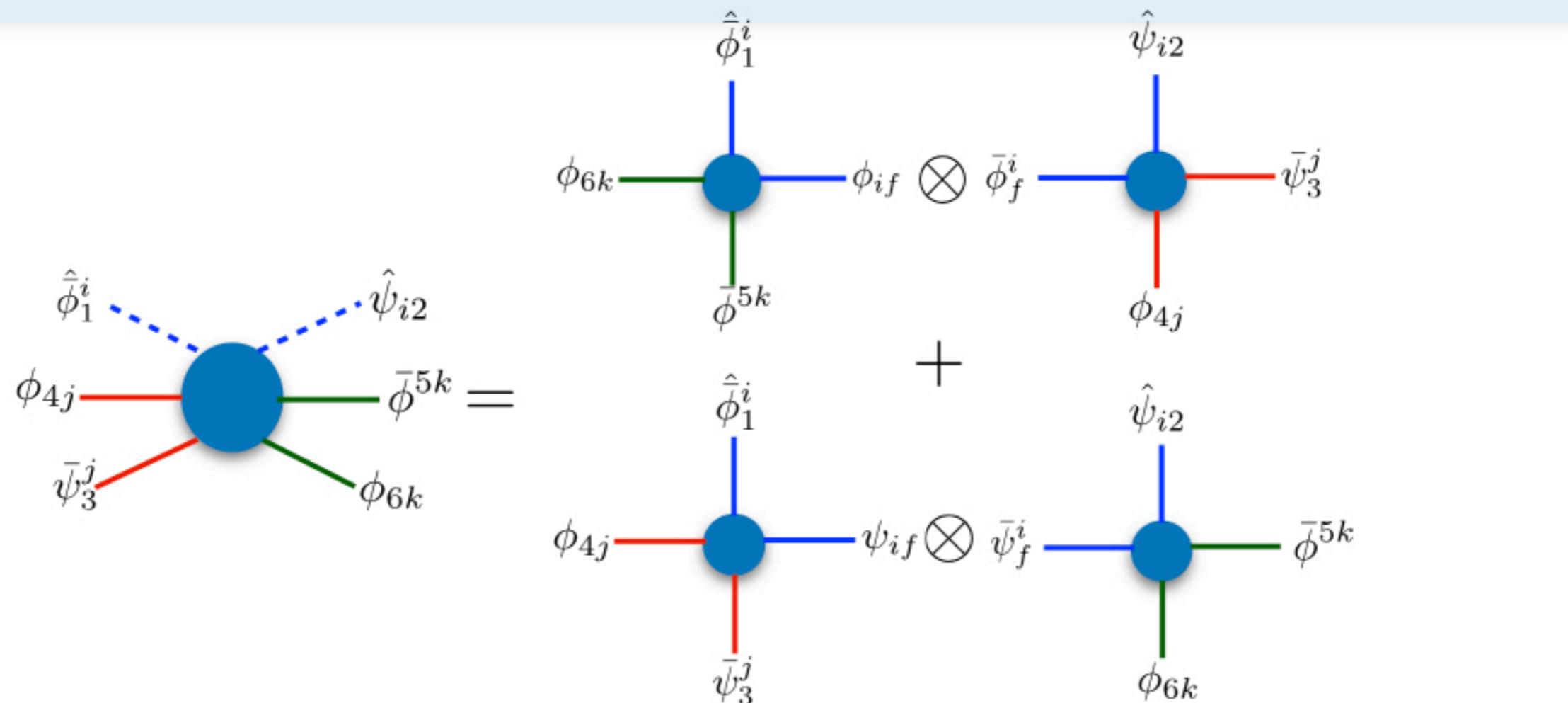
$$A_{2n}(z = 1) = \sum_f \int \frac{d\theta}{p_f^2} \left(z_{a;f} \frac{z_{b;f}^2 - 1}{z_{a;f}^2 - z_{b;f}^2} A_L(z_{a;f}, \theta) A_R(z_{a;f}, i\theta) + (z_{a;f} \leftrightarrow z_{b;f}) \right)$$



- $z_{a;f}, z_{b;f}$ are zeroes of $p_f^2(z) = 0$
- The formula can be recursively applied to write down any **higher point superamplitude in terms of products of the four point superamplitude**.

Eg: Six point amplitude as product of four point amplitudes

$$\begin{aligned} \langle \bar{\phi}_1 \psi_2 \bar{\psi}_3 \phi_4 \bar{\phi}_5 \phi_6 \rangle = & \left(\frac{32\pi^2 i}{\kappa^2} \right) \left[\frac{\langle 2 | p_4 | 3 \rangle p_{12} \cdot p_{56} - \langle 2 | p_1 | 3 \rangle p_{34} \cdot p_{56}}{p_{256}^2 p_{124}^2} \right. \\ & + \left(\langle 3 | p_{12} | 5 \rangle (\langle 2 | p_1 | 5 \rangle p_{34} \cdot p_{56} - \langle 2 | p_6 | 5 \rangle p_{34} \cdot p_{12}) \right. \\ & \left. \left. - \langle 34 \rangle \langle 12 \rangle (\langle 1 | p_6 | 4 \rangle p_{12} \cdot p_{56} - \langle 1 | p_2 | 5 \rangle \langle 4 | p_6 | 5 \rangle) \right) \frac{1}{p_{234}^2 p_{123}^2 p_{126}^2} \right] \end{aligned}$$



Recursion relations for non-supersymmetric theories!

- BCFW does not apply to the **non-susy CS coupled to fermions/bosons** since the amplitudes **do not have good asymptotic behavior**.
- It is possible to extract the recursion relations for non-susy fermionic/bosonic CS matter theories from the N=2 results!! Eg:
 - At **tree level**, the Feynman diagrams for an **all fermion amplitude are same** for susy/non-susy theory.
 - **Susy ward identity**: The four point super amplitude is completely specified by one function, choose it to be the four fermion amplitude.
 - Use this information recursively in the BCFW formula!
- An arbitrary higher point tree level amplitude in the fermionic CS matter theory can be entirely written in terms of **4 fermion amplitude**.

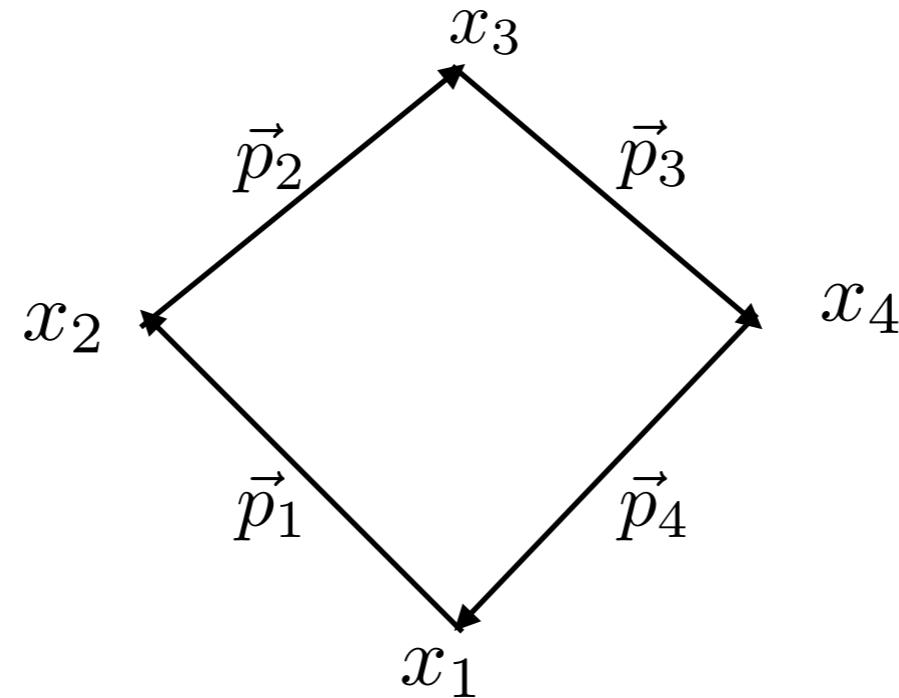
Recursion relations for non-supersymmetric theories!

$$\begin{aligned}\langle \bar{\psi}_1 \psi_2 \bar{\psi}_3 \psi_4 \bar{\psi}_5 \psi_6 \rangle = & \left(\frac{16\pi^2 i}{\kappa^2} \right) \left(\frac{1}{p_{124}^2 p_{125}^2 p_{256}^2} - \frac{1}{p_{123}^2 p_{126}^2 p_{234}^2} \right) \\ & \left[- \langle 1 | p_{34} | 2 \rangle \langle 3 | p_{56} | 4 \rangle \langle 5 | p_{12} | 6 \rangle + \langle 12 \rangle \langle 34 \rangle \langle 56 \rangle \right. \\ & \left. \times (\langle 12 \rangle \langle 1 | p_{34} | 2 \rangle + \langle 34 \rangle \langle 3 | p_{56} | 4 \rangle + \langle 5 | p_{12} | 6 \rangle \langle 56 \rangle) \right]\end{aligned}$$

- The six fermion amplitude is remarkably simple and is manifestly symmetric under permutations of particle pairs (12),(34),(56).

Part III

Hidden symmetry: Dual superconformal invariance



- K.I, S.Jain, S.Majumdar, P.Nayak, T.Neogi, T.Sharma, R.Sinha, V.Umesh,
[arXiv : 1711.02672](https://arxiv.org/abs/1711.02672)

Dual variables

- The dual variables realize momentum conservation linearly in the x variables

$$x_{i,i+1}^{\alpha\beta} = x_i^{\alpha\beta} - x_{i+1}^{\alpha\beta} = p_i^{\alpha\beta} = \lambda_i^\alpha \lambda_i^\beta$$
$$\theta_{i,i+1}^\alpha = \theta_i^\alpha - \theta_{i+1}^\alpha = q_i^\alpha = \lambda_i^\alpha \eta_i$$

- momentum and supermomentum conservation imply

$$P^{\alpha\beta} = \sum_i p_i^{\alpha\beta} = x_{n+1}^{\alpha\beta} - x_1^{\alpha\beta} = 0,$$

$$\mathcal{Q}^\alpha = \sum_i q_i^\alpha = \theta_{n+1}^\alpha - \theta_1^\alpha = 0.$$

- The four point super amplitude in dual space

$$\mathcal{A}_4 = \frac{\langle 12 \rangle}{\langle 23 \rangle} \delta\left(\sum_{i=1}^4 p_i\right) \delta^2(\mathcal{Q}) \xrightarrow[\text{dual space}]{} \mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}} \delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5)$$

- Goal is to show that this is invariant under the superconformal symmetry in the dual variables.

Superconformal algebra in dual space

- The N=2 superconformal algebra in **dual space** is generated by

$$\{P_{\alpha\beta}, M_{\alpha\beta}, D, K_{\alpha\beta}, R, Q_\alpha, \bar{Q}_\alpha, S_\alpha, \bar{S}_\alpha\}$$

$$P_{\alpha\beta} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\alpha\beta}}, \quad D = - \sum_{i=1}^n \left(x_i^{\alpha\beta} \frac{\partial}{\partial x_i^{\alpha\beta}} + \frac{1}{2} \theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha} \right),$$

$$Q_\alpha = \sum_{i=1}^n \frac{\partial}{\partial \theta_i^\alpha}, \quad \bar{Q}_\alpha = \sum_{i=1}^n \theta_i^\beta \frac{\partial}{\partial x_i^{\beta\alpha}},$$

$$M_{\alpha\beta} = \sum_{i=1}^n \left(x_{i\alpha}{}^\gamma \frac{\partial}{\partial x_i^{\gamma\beta}} + \frac{1}{2} \theta_{i\alpha} \frac{\partial}{\partial \theta_i^\beta} \right), \quad R = \sum_{i=1}^n \theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha}$$

- The remaining generators can be expressed using the inversion operator

$$I[x_i^{\alpha\beta}] = \frac{x_i^{\alpha\beta}}{x_i^2}, \quad I[\theta_i^\alpha] = \frac{x_i^{\alpha\beta} \theta_{i\beta}}{x_i^2}$$

$$K_{\alpha\beta} = IP_{\alpha\beta}I, \quad S_\alpha = IQ_\alpha I, \quad \bar{S}_\alpha = I\bar{Q}_\alpha I.$$

Dual superconformal invariance of ABJM superamplitude

- **ABJM superamplitude**

Gang, Huang, Koh, Lee, Lipstein

$$A_{ABJM}^{(4)} = \frac{1}{\sqrt{x_{1,3}^2 x_{2,4}^2}} \delta^{(3)}(x_1 - x_5) \delta^{(6)}(\theta_1 - \theta_5)$$

- The inversion operator leaves the delta functions invariant

$$I[\delta^{(3)}(x_1 - x_5) \delta^{(6)}(\theta_1 - \theta_5)] = \delta^{(3)}(x_1 - x_5) \delta^{(6)}(\theta_1 - \theta_5)$$

- Conformal transformations therefore act only on the pre factors.

$$I[A_{ABJM}^{(4)}] = \sqrt{\prod_{i=1}^4 x_i^2} A_{ABJM}^{(4)}$$

$$P_{\alpha\beta} I[A_{ABJM}^{(4)}] = -\frac{1}{2} \left(\sum_{i=1}^4 \frac{x_{\alpha\beta}^i}{x_i^2} \right) I[A_{ABJM}^{(4)}]$$

$$\tilde{K}^{\alpha\beta} \mathcal{A}_{ABJM}^{(4)} = \left(K^{\alpha\beta} + \frac{1}{2} \sum_{j=1}^4 \Delta_j x_j^{\alpha\beta} \right) \mathcal{A}_{ABJM}^{(4)} = 0 , \quad \Delta_i = \{1, 1, 1, 1\}$$

Dual superconformal invariance of N=2 superamplitude

$$\mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}} \delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5)$$

- In this form the **translation, Lorentz invariance and supersymmetry invariance of the amplitude is manifest.**
- Under Dilatations and R symmetry it transforms as a eigenfunction of weight 4 and weight 2 respectively.
- To show the dual superconformal invariance it is sufficient to show the invariance under $K_{\alpha\beta}, S_\alpha, \bar{S}_\alpha$
- Note that the **delta functions for N=2 transform under the inversion** as
$$I \left[\delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5) \right] = x_1^4 \delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5)$$
- So **it was expected that the superamplitude in the N=2 theory would not have any dual superconformal invariance at all.**
- However, in the N=2 case, **dual superconformal invariance, still works but the dual conformal weights become non-homogeneous.**

Dual superconformal invariance of the superamplitude

- Under the action of $K_{\alpha\beta}, S_\alpha, \bar{S}_\alpha$

$$\tilde{K}^{\alpha\beta} \mathcal{A}^{(4)} = \left(K^{\alpha\beta} + \sum_{j=1}^4 \Delta_j x_j^{\alpha\beta} \right) \mathcal{A}^{(4)} = 0$$

$$\tilde{\bar{S}}^{\alpha\beta} \mathcal{A}^{(4)} = \left(\bar{S}^{\alpha\beta} + \sum_{j=1}^4 \Delta_j x_j^{\alpha\beta} \right) \mathcal{A}^{(4)} = 0$$

$$S_\alpha \mathcal{A}^{(4)} = 0$$

$$\Delta_i = \frac{1}{2} \{4 - 1, 1, -1, 1\}$$

- The factor of 4 is due to momentum+supermomentum conservation and can be removed. $I \left[\delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5) \right] = x_1^4 \delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5)$
- Thus the superamplitude is dual superconformal invariant with non-homogeneous weights

$$\tilde{\Delta}_i = \frac{1}{2} \{-1, 1, -1, 1\}$$

Dual superconformal invariance at all loops

- We showed that the function A_4 is dual superconformal invariant!

$$\mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}} \delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5)$$

- The tree level superamplitude is dual superconformal invariant.

$$T_{tree} = \frac{4\pi}{\kappa} \mathcal{A}_4$$

- Thus the **all loop results** computed in **K.I, Jain, Mazumdar, Minwalla, Umesh, Yokoyama** are **also dual superconformal invariant**.

$$T_{sym}^{all \ loop} = T_{Asym}^{all \ loop} = T_{Adj}^{all \ loop} = T_{tree}$$

$$T_{sing}^{all \ loop} = N \frac{\sin \pi \lambda}{\pi \lambda} T_{tree}$$

- Unlike the SYM and the ABJM case, here the **symmetry is all loop exact in the planar limit**.
- Now that we know this symmetry exists, **can we reverse the argument and do an S matrix bootstrap to fix the general structure of the amplitude?**

Constraining amplitudes from dual superconformal symmetry

- The **four point amplitude in momentum space** can be interpreted as a **four point correlator in dual space**, then dual conformal invariance fixes

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle$$

$$= \frac{1}{x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}} \left(\frac{x_{24}}{x_{14}} \right)^{\Delta_1 - \Delta_2} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_3 - \Delta_4} f(u, v, \kappa, \lambda)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

- Since $x_{ij}^2 = p_i^2 = 0$, the correlator is understood in the limit

$$\frac{u}{v} \Big|_{onshell} = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2} \Big|_{onshell} = \frac{p_1^2 p_3^2}{p_2^2 p_4^2} \Big|_{onshell} = \text{constant}$$

- If dual superconformal symmetry is exact it fixes the momentum (x) dependence completely

$$f(u, v, \kappa, \lambda) = g(\kappa, \lambda)$$

Constraining amplitudes from dual superconformal symmetry

- the operator dimensions are

$$\begin{aligned}\Delta_1 &= \Delta_3 = -\frac{1}{2} \\ \Delta_2 &= \Delta_4 = \frac{1}{2}\end{aligned}$$

- The four point correlator in dual space gets fixed to (cancellations in limiting sense)

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle = g(\kappa, \lambda) \sqrt{\frac{x_{13}^2}{x_{24}^2}}$$

- Same as the prefactor in the amplitude! The delta functions are fixed by momentum and supermomentum conservation.

$$\mathcal{A}_4 = \sqrt{\frac{x_{1,3}^2}{x_{2,4}^2}} \delta^{(3)}(x_1 - x_5) \delta^{(2)}(\theta_1 - \theta_5)$$

- However it is much harder to fix the complete coupling dependence of g from general principles: Unitarity, Parity and Duality.

Constraining amplitudes from dual superconformal symmetry

- In general the S matrix could get complicated functions with poles and branch cuts.
- If dual conformal invariance is an exact symmetry at loop level then no such behavior appears.
- Non trivial momentum dependence could still appear from $x_{i,j}^{\Delta}$ when Δ gets correction from loops.
- This can give rise to log dependence for instance, However these do not appear if there are no IR divergences.
- **If we assume that there are no IR divergences (none seen in the calculation), and that the dual conformal invariance is an exact symmetry, then the momentum dependence is fixed to all loops.**
- This is a rather unique feature, and suggests more symmetry structures in play.

Part IV

Hidden symmetry: Yangian

K.I, S.Jain, P.Nayak, T.Sharma, V.Umesh, [arXiv : 1801.nnppq](#)

Yangian Symmetry

- The presence of the **superconformal and dual superconformal symmetries indicate a Yangian symmetry** in the amplitude.
- A Yangian algebra is an **associative Hopf Algebra** generated by
$$[J^A, J^B] = f_C^{AB} J^C, \quad [J^A, Q^B] = f_C^{AB} Q^C$$
- J^A take values in a Lie group G, both J^A and Q^A are constrained to obey the **Serre relations in addition to Jacobi Identity**.

$$[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\}$$

$$[[Q^A, Q^B], [J^C, Q^D]] + [[Q^C, Q^D], [J^A, Q^B]] = \frac{1}{24} (f^{AGL} f^{BEM} f_K^{CD} + f^{CGL} f^{DEM} f_K^{AB}) \\ \times f^{KFN} f_{LMN} \{J_G, J_E, J_F\}$$

- Under repeated commutations the **Q's generate an infinite dimensional Symmetry algebra**.
- The **necessary and sufficient conditions for the Yangian to exist is that Q satisfies the Serre relations**.

Yangian Symmetry in N=2 Chern-Simons theory

- The Yangian has a **basis (labelled by levels)**

$$\mathcal{J}_0^A = J^A, \quad \mathcal{J}_1^A = Q^A, \dots, \quad \mathcal{J}_n^A$$

$$[\mathcal{J}_{(0)}^A, \mathcal{J}_{(0)}^B] = f_{C}^{AB} \mathcal{J}_{(0)}^C \quad [\mathcal{J}_{(0)}^A, \mathcal{J}_{(1)}^B] = f_C^{AB} \mathcal{J}_{(1)}^C$$

- The generators \mathcal{J}_n^A are **“n local” operators**. The infinite dimensional symmetry is generated by commutators of \mathcal{J}_n^A .
- Multi-locality arises due to the **co-product structure** of the Hopf algebra

$$\Delta(\mathcal{J}_{(0)}^A) = \mathcal{J}_{(0)}^A \otimes 1 + 1 \otimes \mathcal{J}_{(0)}^A$$

$$\Delta(\mathcal{J}_{(1)}^A) = \mathcal{J}_{(1)}^A \otimes 1 + 1 \otimes \mathcal{J}_{(1)}^A + f^{ABC} \mathcal{J}_B^{(0)} \otimes \mathcal{J}_C^{(0)}$$

- This causes the bilocal part to act **non-trivially** on multiparticle states. In particular, it will not act like a trivial tensor product of single particle action.
- This allows the overlap between spacetime symmetries and internal symmetries, and allows for a non-trivial symmetry structure.
- on a product of fields, the bilocals act on different sites at the same time

$$(\mathcal{J}_{(1)}^A)_{bi \text{ local}} (\phi^1 \phi^2 \phi^3 \dots \phi^n) = f^{ABC} \sum_{i < j=1}^n \phi^1 \phi^2 \dots \mathcal{J}_B^{(0)} \phi^i \dots \mathcal{J}_C^{(0)} \phi^j \dots \phi^n$$

Yangian Symmetry in N=2 Chern-Simons theory

- For the N=2 theory the spacetime superconformal symmetry is $Osp(2|4)$

$$\mathcal{J}_{(0)}^A = \{p_{\alpha\beta}, m_{\alpha\beta}, d, k_{\alpha\beta}, r, q_\alpha, \bar{q}_\alpha, s_\alpha, \bar{s}_\alpha\}$$

- The superconformal generators can be expressed in terms of the **on-shell variables**

$$\begin{aligned} p_{\alpha\beta} &= \sum_i \lambda_{i\alpha} \lambda_{i\beta}, & k_{\alpha\beta} &= \sum_i \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \lambda_i^\beta}, \\ m^\alpha{}_\beta &= \sum_i \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\beta} - \frac{1}{2} \delta_\beta^\alpha \lambda_i^\gamma \frac{\partial}{\partial \lambda_i^\gamma}, & d &= \sum_i \left(\frac{1}{2} \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\beta} + \frac{1}{2} \right), \\ r &= \sum_i \left(\eta_i \frac{\partial}{\partial \eta_i} - \frac{1}{2} \right), & q_\alpha &= \sum_i \lambda_{i\alpha} \eta_i, & \bar{q}_\alpha &= \sum_i \lambda_{i\alpha} \frac{\partial}{\partial \eta_i}, \\ s_\alpha &= \sum_i \eta_i \frac{\partial}{\partial \lambda_i^\alpha}, & \bar{s}_\alpha &= \sum_i \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \lambda_i^\alpha}. \end{aligned}$$

- Superconformal invariance** of the amplitude is the statement

$$\mathcal{J}_{(0)}^A \mathcal{A}_4 = 0$$

Superconformal algebra in dual space

- The N=2 superconformal algebra in **dual space** is generated by

$$\{P_{\alpha\beta}, M_{\alpha\beta}, D, K_{\alpha\beta}, R, Q_\alpha, \bar{Q}_\alpha, S_\alpha, \bar{S}_\alpha\}$$

$$P_{\alpha\beta} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\alpha\beta}}, \quad D = - \sum_{i=1}^n \left(x_i^{\alpha\beta} \frac{\partial}{\partial x_i^{\alpha\beta}} + \frac{1}{2} \theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha} \right),$$

$$Q_\alpha = \sum_{i=1}^n \frac{\partial}{\partial \theta_i^\alpha}, \quad \bar{Q}_\alpha = \sum_{i=1}^n \theta_i^\beta \frac{\partial}{\partial x_i^{\beta\alpha}},$$

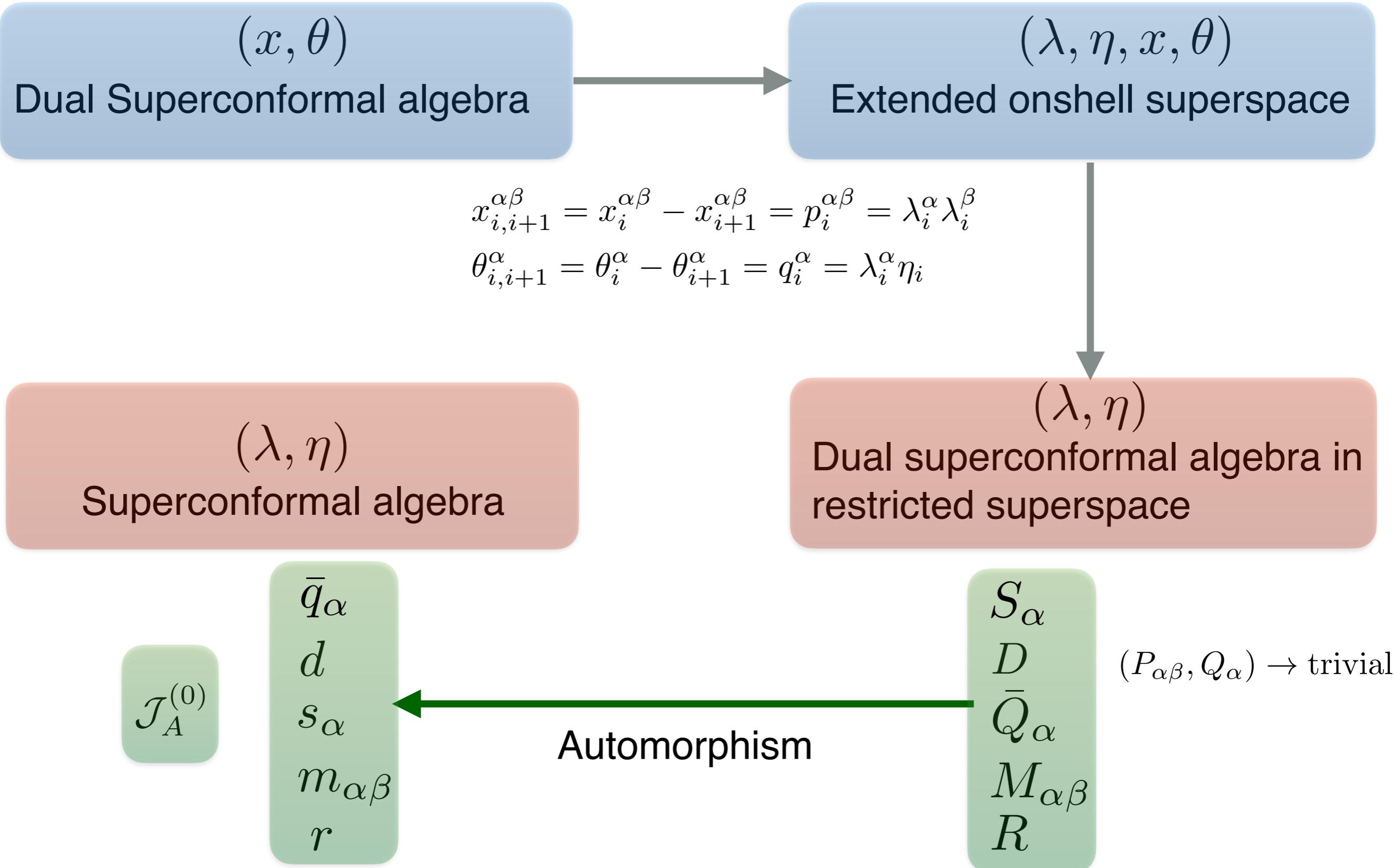
$$M_{\alpha\beta} = \sum_{i=1}^n \left(x_{i\alpha}{}^\gamma \frac{\partial}{\partial x_i^{\gamma\beta}} + \frac{1}{2} \theta_{i\alpha} \frac{\partial}{\partial \theta_i^\beta} \right), \quad R = \sum_{i=1}^n \theta_i^\alpha \frac{\partial}{\partial \theta_i^\alpha}$$

- The remaining generators can be expressed using the inversion operator

$$I[x_i^{\alpha\beta}] = \frac{x_i^{\alpha\beta}}{x_i^2}, \quad I[\theta_i^\alpha] = \frac{x_i^{\alpha\beta} \theta_{i\beta}}{x_i^2}$$

$$K_{\alpha\beta} = IP_{\alpha\beta}I, \quad S_\alpha = IQ_\alpha I, \quad \bar{S}_\alpha = I\bar{Q}_\alpha I.$$

Emergence of Yangian



- The $K_{\alpha\beta}$ and \bar{S}_α do not map to any of the Level 0 generators.

Emergence of Yangian: Level one generators

- Dual conformal generator in extended superspace $(\lambda, \eta, x, \theta)$

$$\begin{aligned}\tilde{K}_{\alpha\beta} = \sum_{i=1}^4 & \left[x_{i\alpha}{}^m x_{i\beta}{}^n \frac{\partial}{\partial x_i^{mn}} + \frac{1}{2} x_{i(\alpha}{}^m \theta_{i\beta)} \frac{\partial}{\partial \theta_i^m} + \frac{1}{4} (x_i + x_{i+1})_{(\alpha}{}^\gamma \lambda_{i\beta)} \frac{\partial}{\partial \lambda_i^\gamma} \right. \\ & \left. + \frac{1}{4} (\theta_i + \theta_{i+1})_{(\alpha} \lambda_{i\beta)} \frac{\partial}{\partial \eta_i} \right] + \frac{1}{2} \sum_{i=1}^4 \Delta_i x_{i\alpha\beta}\end{aligned}$$

- Dual conformal generator restricted to onshell superspace (λ, η)

$$\begin{aligned}\tilde{K}_{\alpha\beta} = \frac{1}{2} \sum_{i=1}^4 & \left[x_{1(\alpha}{}^\gamma \lambda_{i\beta)} \frac{\partial}{\partial \lambda_i^\gamma} + \theta_{1(\alpha} \lambda_{i\beta)} \frac{\partial}{\partial \eta_i} \right] - \frac{1}{4} \sum_{i=1}^4 \sum_{j=1}^{i-1} \left[\lambda_{j(\alpha} \lambda_{i\beta)} \lambda_j^\gamma \frac{\partial}{\partial \lambda_i^\gamma} + \lambda_{j(\alpha} \lambda_{i\beta)} \eta_j \frac{\partial}{\partial \eta_i} \right] \\ & - \frac{1}{4} \sum_{i=1}^4 \sum_{j=1}^i \left[\lambda_{j(\alpha} \lambda_{i\beta)} \lambda_j^\gamma \frac{\partial}{\partial \lambda_i^\gamma} + \lambda_{j(\alpha} \lambda_{i\beta)} \eta_j \frac{\partial}{\partial \eta_i} \right] - \frac{1}{2} \sum_{i=1}^4 \Delta_i \left(x_1 - \sum_{j=1}^{i-1} \lambda_{i\alpha} \lambda_{j\beta} \right)\end{aligned}$$

- Acting the generator on the amplitude written in onshell space, it can be checked that the (x_1, θ_1) dependence cancels out.

Emergence of Yangian: Level one generators

- The **bilocal form of the level one generator** is visible when we rewrite it in terms of the level 0 generators.

$$\tilde{K}_{\alpha\beta} = \frac{1}{4} \sum_{i>j} \left[p_{(i\alpha}{}^\gamma d_{j\beta)\gamma} + q_{i(\alpha} \bar{q}_{j\beta)} - (i \leftrightarrow j) \right] - \frac{1}{2} \sum_{i=1}^4 \sum_{j=1}^{i-1} (\Delta_i - 1) p_{i\alpha\beta}$$

- Similarly

$$\tilde{\bar{S}}_\alpha = -\frac{1}{4} \sum_{i>j} \left[\bar{q}_j^\beta (m_{i\alpha\beta} + \epsilon_{\alpha\beta} d_i) - \bar{q}_{j\alpha} r_i + p_{j\alpha}{}^\beta s_{i\beta} - (i \leftrightarrow j) \right] - \frac{1}{2} \sum_{i>j}^4 (\Delta_i - 1) q_{j\alpha}$$

- This is **consistent with the general form of level one generators**

$$\mathcal{J}_{(1)}^A = \frac{1}{2} f_{BC}^A \sum_{j<k} \mathcal{J}_{j,(0)}^C \mathcal{J}_{k,(0)}^B + \sum_k v^l \mathcal{J}_{l,(0)}^A$$

Kazakov et al.

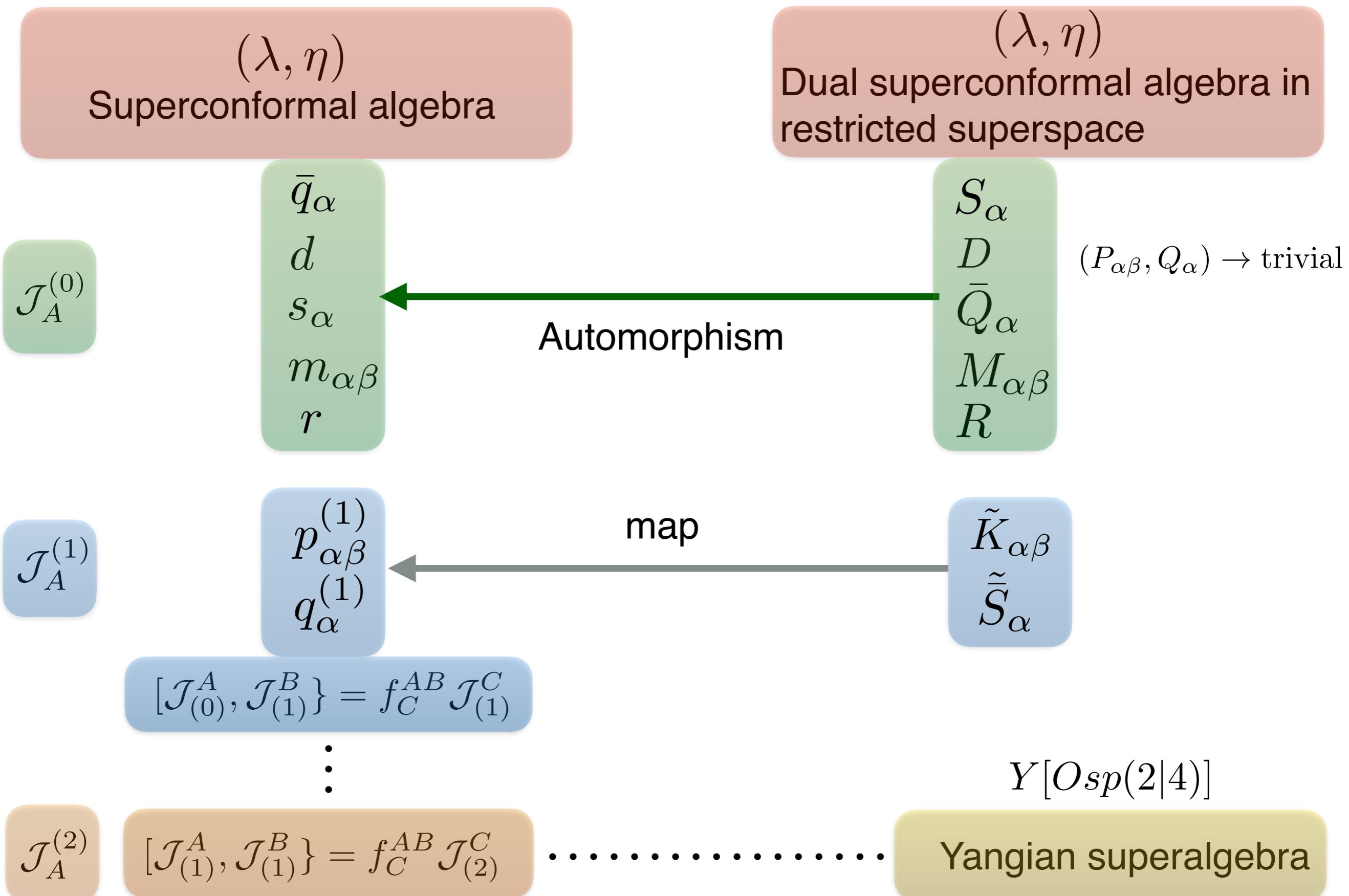
- The precise identification with the **level one generators of superconformal algebra**

$$\begin{aligned} \tilde{K}_{\alpha\beta} &\equiv p_{\alpha\beta}^{(1)} \\ \tilde{\bar{S}}_\alpha &\equiv q_\alpha^{(1)} \end{aligned}$$

- Remaining level one generators can be obtained from **adjoint condition**

$$[\mathcal{J}_{(0)}^A, \mathcal{J}_{(1)}^B] = f_C^{AB} \mathcal{J}_{(1)}^C$$

Summary: Emergence of Yangian



Summary: Yangian Invariance of the four point amplitude

- The four point amplitude is invariant under the Level one generators of the Yangian $K_{\alpha\beta}$ and \bar{S}_α
- The invariance under the rest of the Level one generators follows from the adjoint condition since the amplitude is already invariant under the level 0 superconformal symmetry,
- Invariance under Level 2 and higher generators invariance follows through successive commutations
- Thus **superconformal and dual superconformal symmetries generate a Yangian symmetry!**

$$\mathcal{J}_{(0)}^A \mathcal{A}_4 = 0, \quad \mathcal{J}_{(1)}^A \mathcal{A}_4 = 0, \implies \gamma \mathcal{A}_4 = 0,$$

- **This is all loop exact for the four point amplitude in the planar limit.**

Part V

Summary

Summary

- We started with a goal of computing **arbitrary $m \rightarrow n$ tree level scattering amplitudes** in $U(N)$ $\mathcal{N} = 2$ Chern-Simons matter theories with fundamental matter.
- We achieved this via **BCFW recursion relations**, this enabled us to express arbitrary n point amplitudes as products of four point amplitudes!
- We saw an explicit example where the six point amplitude is expressed as a product of two four point amplitudes via two factorization channels.
- We can also extract the **recursion relations for non-supersymmetric theories** even though BCFW does not directly apply.
- We saw an explicit example for the six fermion amplitude in the fermion coupled Chern-Simons theory.

Summary

- We showed that the **all loop $2 \rightarrow 2$ scattering amplitude** computed in **K.I, Jain, Mazumdar, Minwalla, Umesh, Yokoyama** is **dual superconformal invariant**.
- Thus **dual superconformal symmetry is all loop exact**, at least for the 4 point amplitude.
- The superconformal and dual superconformal symmetries together generate an **infinite dimensional Yangian symmetry**!
- **It follows that Yangian symmetry is all loop exact for the four point amplitude in the planar limit.**
- This is in contrast to SYM and ABJM where the Yangian symmetry is anomalous due to IR divergences.

$$\mathcal{N} = 4 \text{ SYM} \quad \text{ABJM} \quad \mathcal{N} = 2 \text{ CSM}$$

	$\mathcal{N} = 4 \text{ SYM}$	ABJM	$\mathcal{N} = 2 \text{ CSM}$
BCFW recursions for all tree level amplitudes	✓	✓	✓
Dual superconformal symmetry for tree level amplitudes	✓	✓	$2 \rightarrow 2$
Superconformal x Dual Superconformal symmetry = Yangian	✓	✓	$2 \rightarrow 2$
Orthogonal Grassmannian	✓	✓	??
Exact Symmetries at loop level	✗	✗	$2 \rightarrow 2$
Amplitude-Wilson loop duality	✓	$2 \rightarrow 2$??
Integrability	✓	✓	

Thank you!