Linear Algebra

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1. Matrices and Gaussian Eliminations

1.1. Geometry of linear equations

Matrix representation: $Ax = b$

- *Row picture*: Lines in a ($\mathbb{R}^2$) or planes ($\mathbb{R}^3$).
- *Column picture*: b is linear combination of the columns of A. $\therefore$ b is in the column space of A.

Singular case: No solution or infinite solution.

1.2. Gaussian Elimination

- Apply row operations and row exchange to convert the system of linear equations to triangular form.
- Use back substitution to solve the system.

Row operations preserve the nullspace and rowspace of A,
but this normally alters the eigenvalues.

Gaussian elimination in matrix form

- Each row operation on A can be represented as a matrix multiplication on A. If a triangular form U is obtained by three row operations on A, each of them can be represented by matrices, say G, F and E.
  Then, $G \times F \times E \times A = U$.
- The product of G, F and E would be a lower triangular matrix (L). The matrix A can be written as $PA = LU$. L can be found by keeping track of the row operations. P is a permutation matrix.
- LU can also be written as LDU by splitting U to DU. D is a diagonal matrix containing the pivot elements.

1.3. Matrix multiplication

5 ways to multiply matrices

Given, $A \times B = C$

- Standard.
- Each column of C is a linear combination of columns of A.
  $\begin{bmatrix} Ab_1 & \cdots & Ab_n \end{bmatrix}$, where b_is are columns of B.
- Each row of C is a linear combination of rows of B.
  $\begin{bmatrix} a_1B \\ \vdots \\ a_nB \end{bmatrix}$, where a_is are rows of A.
- $\sum_{i=1}^{n} [ca_i] \times [rb_i]$
  Where, ca_i and rb_i are i-th column and row of A and B, respectively.
- Multiplying Blocks of matrix. Eg:
  $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \times \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$
  A_i, B_i and C_1 are blocks of the matrices A, B and C. The blocks are treated as a single entity and multiplied like a normal matrix. Eg: $C_1 = A_1B_1 + A_2B_3$
  Last approach would be useful for proving matrix theorems by inductions.

Properties

- Associative: $(AB)C = A(BC)$.
- Distributive: $A(B + C) = AB + AC$; $(B + C)D = BD + CD$.
- **NOT** commutative: Usually $EF \neq FE$.
- $AB = 0 \;\not\Rightarrow\; A = 0 (or) B = 0$.
- $AC = AD \;\not\Rightarrow\; C = D$
- Let A, B, C be $n \times n$ matrices. Then:
- If $rank(A) = n$ and $AB = AC$, then $B = C$.
- If $rank(A) = n$, then $AB = 0 \implies B = 0$. If $AB = 0$ but $A \neq 0$ and $B \neq 0$, then $rank(A) < n$ and $rank(B) < n$.

1.4. Inverse and Transpose of a Matrix

If the system of equations is non-singular, then it could be solved by using inverse of A.

- A^{-1} is inverse of $A \Leftrightarrow A^{-1}A = A^{-1}A = I$.
- If A^{-1} exist, then A is invertible or non-singular matrix.

Properties

- A square matrix is singular or not invertible if:
  - Its determinant is 0.
  - Dimension(Column Space) < Matrix dimension.
    Proof (by contradiction): $\exists x \neq 0, \ni Ax = 0$.
    Assuming A^{-1} exists $\Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$.
- If A is invertible, then the one and only solution to $Ax = b$ is $x = A^{-1}b$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(AB)^T = B^TA^T$
- $(A^{-1})^T = (A^T)^{-1}$
- For any rectangular matrix R, RR^T and R^TR is symmetric.
- A symmetric matrix could be decomposed to LDL^T.

Finding inverse by Gauss-Jordan elimination

Start with augmented matrix. Achieve upper triangular matrix on the left part and then identity matrix by row operations:

$$\boxed{[A|I] \longrightarrow [U|X] \longrightarrow [I|A^{-1}]}$$

2. Vector Spaces

2.1. Vector spaces and subspaces

Vector space Set of vectors with vector addition and scalar multiplication, satisfying the following properties.

- Associativity of vector addition.
- Commutativity of vector addition.
- Identity element of vector addition.
- Inverse elements of vector addition.
- Compatibility of scalar multiplication with field multiplication. i.e., $a(b\mathbf{v}) = ab(\mathbf{v})$
- Identity element of scalar multiplication.
- Distributivity of scalar multiplication w.r.t. vector addition. i.e., $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- Distributivity of scalar addition w.r.t. vector multiplication. i.e. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

If S and P are two vector spaces, $S \cup P$ need not be a vector space, but $S \cap P$ is a vector space.
Subspace $Q \subseteq P$ is a subspace of P, if it satisfies the following property in addition to all the properties of a vector space.

- $\mathbf{a} \in Q$ and $\mathbf{b} \in Q \implies \mathbf{a} + \mathbf{b} \in Q$.
- $\mathbf{a} \in Q \implies c\mathbf{a} \in Q$ for all scalar c.

2.2. Solving $Ax = b$[Revisited]

Augmented matrix $[A|b] \rightarrow$ Row operation leading to echelon form.
Pivot elements First non-zero element in a row with a column of zeroes below it.
Rank = number of pivot elements.
Free columns = Columns without pivot elements.
Null space: of a matrix A, is the set of all x that satisfies $Ax = 0$.
General solution: $x = x_p + x_n$, where x_n is a vector from the nullspace of A.
The existence and number of solutions depends on the rank of A and that of the augmented matrix $[A|b]$.

2.3. Linear independence, basis and dimension

Linear indepenedence: A set of vectors $v_1, v_2, \cdots, v_k$ are said to be linearly independent iff
$c_1v_1 + c_2v_2 + \cdots + c_kv_k \Rightarrow c_1 = c_2 = \cdots = c_k$.

$$\boxed{\text{Columns of } A \text{ are linearly independent iff } N(A) = \{\mathbf{0}\}}$$

Spanning a vector space: $W = \{w_1, w_2, \cdots, w_k\}$ is said to span a vector space V, if for all $v \in V$,, $\exists\, c_1, c_2, \cdots, c_k \in \mathbb{R}$ such that $v = c_1w_1 + c_2w_2 + \cdots + c_kw_k$
Basis: Basis of a vector space is a set of vectors with the following two properties.

- The vectors in the set are linearly independent.
- They span the vector space.

Dimension of a vector space: Number of basis vectors.

2.4. Four fundamental subspaces

For a matrix A of dimensions $m \times n$ and rank r.

- Column space of A, $C(A)$: Smallest subspace containing the columns of A.
  $C(A) \subseteq \mathbb{R}^m$.
  Dimension of $C(A)$ is r.
- Row space of A, $R(A)$: Smallest subspace containing the rows of A.
  $R(A) \subseteq \mathbb{R}^n$.
  Dimension of $R(A)$ is r.
- Null space of A, $N(A)$: Set of all x such that $Ax = 0$.
  $N(A) \subseteq \mathbb{R}^n$.
  Dimension of $N(A)$ is $n - r$.
- Null space of A^T, $N(A^T)$: Set of all x such that $A^T x = 0$.
  $N(A^T) \subseteq \mathbb{R}^m$.
  Dimension of $N(A^T)$ is $m - r$.

Relationship between the subspaces:

- $C(A) \perp N(A^T)$ and $C(A) \cup N(A^T) = \mathbb{R}^m$.
- $R(A) \perp N(A)$ and $R(A) \cup N(A) = \mathbb{R}^n$.

2.5. Linear Transformation

Linear Transformation: A transformation T that converts a vector u to $v(= T(x))$, is a linear transformation if it satisfies the following properties.

- $T(0) = 0$.
- $T(u + v) = T(u) + T(v)$
- $T(au) = aT(u)$

Important notes:

- Every matrix multiplication to a vector is a linear transformation.
- Every linear transformation can be represented by a matrix multiplication.
- NOTE: The transformation matrix need not be a square matrix.
- If Ax / $T(x)$ is known for all the basis of the vector space, then it is known for all the vectors in vector space.

Examples of linear transformation:
Scaling, Rotation, Reflection, Projection, Differentiation and Integration.

3. Orthogonality

3.1. Orthogonal vectors and subspaces

Orthogonal vectors: x and y are orthogonal iff $x^T y = 0$.
Length of a vector: $\|x\| = x^T x$.

If a set of vectors $v_1, v_2, \cdots, v_k$ of non-zero length are mutually perpendicular, then those vectors are linearly independent.
Proof:
Assume they are not linearly independent. $\therefore$ there exist non-zero constants $c_1, c_2, \cdots, c_k$ such that,
$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$
$\therefore v_1^T (c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = 0$
$\implies c_1 = 0$, a contradiction.

Orthogonal subspaces: Two subspaces V and W are said to be orthogonal if $\forall v \in V$ and $\forall w \in W$ $v \perp w$.

Fundamental theorem of linear algebra part-I:
Row space is orthogonal to nullspace (in $\mathbb{R}^n$) and column space if orthogonal to the left nullspace (in $\mathbb{R}^m$).
Proof-1:
If x is in $N(A)$, $Ax = 0$. This implies Product of every row in A to x is 0. Therefore $x \perp$ rowspace of A.
Proof-2:
Let $x \in N(A) \implies Ax = 0$ and,
$y \in$ Rowspace(A) $\implies \exists z \ni y = A^T z$.
$y^T x = (A^T z)^T x = z^T Ax = z^T \mathbf{0} = \mathbf{0}$. $\implies x \perp y$.

I think:
If V and W are subspaces of dimensions p and q in $\mathbb{R}^n$ and if $p + q > n$ then, V cannot be orthogonal to W.

Orthogonal complement: Given a subspace V in $\mathbb{R}^n$, the space of all vectors orthogonal to V is called orthogonal complement of V. It is denoted by $V^\perp$.

Fundamental theorem of linear algebra part-II:
Nullspace is orthogonal complement of row-space in $\mathbb{R}^n$.
Left nullspace is orthogonal complement of column-space in $\mathbb{R}^m$.

A deeper meaning of matrix multiplication

The matrix multiplication Ax transforms the rowspace componenet of x (x_r) to column-space of A and the nullspace component of x (x_n) to 0.
Here $x = x_r + x_n$.
The real action is between rowspace and column space.

Theorem: From rowspace to columnspace A is invertible.
Every b in the column space comes from one exactly one x_r in the rowspace.

3.2. Cosines and Projections onto lines

Cosine The cosine of the angle between any two non-zero vectors is defined as follows:

$$cos\theta = \frac{a^T b}{\|a\|\|b\|}$$

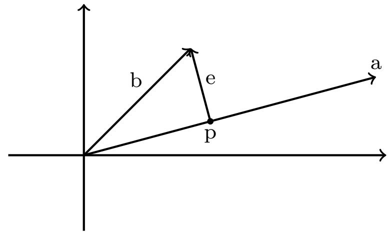
Law of cosines

$$\|b - a\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\|cos\theta$$

Schwarz inequality

$$\|a^T b\| \leq \|a\|\|b\|, \because cos\theta \leq 1$$

Projection onto a line (1D subspace)



p is projection of b on a. $\therefore p = xa$ for some x.

$$e = b - p \text{ and } e \perp a,$$
$$\therefore a^T(b - xa) = 0$$
$$xa^T a = a^T b$$
$$\implies x = \frac{a^T b}{a^T a}$$

$$\boxed{p = a\frac{a^T b}{a^T a}}$$

This can be written as,
$p = \frac{aa^T}{a^T a} b = Pb$, where $P = \frac{aa^T}{a^T a}$ is the projection matrix of a.
Properties of P

- $C(P) =$ line through a.
- Rank(P) = 1.
- P is symmetric: $P = P^T$
- $P^2 = P$

Projection onto a higher dimension subspace

Finding projection of b on A
Let p be the projection of b on subspace spanned by columns of A, then there exists $\hat{x}$ such that $A\hat{x} = p$
$b - p \perp A \implies A^T(b - A\hat{x}) = 0$
$\therefore \hat{x} = (A^T A)^{-1} A^T b$
$\therefore p = A(A^T A)^{-1} A^T b$
Here the projection matrix $P = A(A^T A)^{-1} A^T$
If A is invertible then $P = I$

$A^T A$ has the same nullspace of A
i.e., $A^T A$ is invertible if columns of A are independent.

Least squares problem Solve: $Ax = b$

- Normal equations: $A^TA\hat{x} = A^Tb$
- Best estimate: $\hat{x} = (A^TA)^{-1}A^Tb$
- Projection: $p = A\hat{x} = A(A^TA)^{-1}A^Tb$
- $b = p + e$
- p is in $C(A)$
- e is in N(A^T)
- If b is in C(A),
$p = A(A^TA)^{-1}A^Tb = A(A^TA)^{-1}A^TAx = Ax = b$
- If b is in $N(A^T)$, $p = A(A^TA)^{-1}A^Tb = A(A^TA)^{-1}0 = 0$
- If A is invertible, $p = A(A^TA)^{-1}A^Tb = b$
- (I think) $\|e\|^2 = \|b-p\|^2$ is the squared error.

3.3. Orthogonal basis and Gram-Schmidt

Orthogonal basis: A set of bases, $q_1, q_2, \cdots, q_k$, is said to be orthogonal bases, if for all $i \neq j$ $q_iTq_j = 0$.
Orthonormal basis: Orthonormal bases,$q_1, q_2, \cdots, q_k$, satisfies the following property:

$$q_i^Tq_j = \begin{cases} 1 \text{ if } i = j. \\ 0 \text{ otherwise} \end{cases}$$

Gram-Schmidt process

- Eg: A case with two bases a and b:
$a' = a$
$b' = b - \frac{aa^T}{a^Ta}b$, here, $b' = e$ and $b = p + e$
- Eg: Case with three bases a, b and c.
$a' = a$
$b' = b - \frac{aa^T}{a^Ta}b$
$c' = c - \frac{aa^T}{a^Ta}c - \frac{bb^T}{b^Tb}c$
From c subtract projection of c on a and projection of c on b.

4. Determinants

4.1. Properties of determinants

Three most basic properties of determinants

1. det I = 1.
2. Determinant changes sign when two rows are exchanged.
3. Determinant depend linearly on the first row.
Eg:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Derivative properties of determinants

4. $\mid cA \mid = c^n \mid A \mid$.
5. If two rows of A are equal, then $detA = 0$
Follows from rule 2.
6. Adding a multiple of one row to another does not alter the value of the matrix.
Follows from rule 3 and rule 4.
7. If A has a row of 0s, then $detA = 0$.
Follows form rule 4 and rule 5.
8. If A is triangular, $detA = \prod d_i$.
Proof: Convert A to an equivalent diagonal determinant by row operation. Now apply rule 3 and finally rule 1.
9. If A is singular, then $detA = 0$. If A is invertible, then $detA \neq 0$.
Proof:
If A is singular, elimination leads to zero row, $\therefore detA = 0$
If A is invertible, elimination leads to non-zero pivots.
$detA = \prod$ pivots $\neq 0$.
10. $det(AB) = det(A) \times det(B)$
Proof:
For a diagonal matrix D, $det(DB) = det(D)det(B)$
A can be converted to a diagonal matrix D by row operations.
Using the exact same row operation as above AB can be converted to DB
For another proof see book.
11. $det(A^T) = det(A)$.
Proof:
$PA = LDU$ and $det(PA) = det(LDU)$
$det(A^TP^T) = det(U^TD^TL^T) = det(LDU) = det(PA)$
$det(P^T) = det(P) \therefore det(A^T) = det(A)$.

4.2. Formulae for determinant

Based on LDU decomposition

If A is invertible $PA = LDU$.
$det(A) = \pm det(L)det(D)det(U) = \pm$(product of pivots)

Using linearity of determinants

Eg: 2×2 case.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = ad - bc$$

Expansion in cofactors

The determinant of A is any row i times its cofactors

That is, $det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$
Where , $C_{ij} = (-1)^{i+j}det(M_{ij})$
Proof:
Split the $n \times n$ matrix to sum of $n \times n$ matrices by linearity property of detetermінants. Each of the constituent matrices can be converted to LDU form.

Block matrices

- $det\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = det(A)det(D) = det\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$
- If A is invertible
$det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = det(A)det(D - CA^{-1}B)$
- If blocks are square matrices of the same size. And $AC = CA$.
$det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = det(AD - CB)$
- If blocks are square matrices of same size and if $A = D$ and $B = C$. Here A and B need not commute.
$det\begin{bmatrix} A & B \\ B & A \end{bmatrix} = det(A - B)det(A + B)$

4.3. Application of determinants

Definition 4.1 (Adjugate Matrix):
Transpose of cofactor matrix: $adj(A) = C^T$. NOTE: $\mid adj(A) \mid = \mid A \mid^{n-1}$.

Computation of A^{-1}

$$A^{-1} = \frac{C^T}{det(A)} = \frac{adj(A)}{det(A)}$$

C is cofactor matrix.
Proof:

$$A^{-1} = \frac{C^T}{det(A)} \implies det(A)I = AC^T$$

Now the diagonal of AC^T will be $det(A)$. To complete the proof we just need to show that the off-diagonal elements are all 0.
Each off-diagonal element is represented as:
$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$
where $i \neq j$
This is equal to the determinant of modified A where the j'th row is replaced by a copy of i'th row times ± 1. This determinant $= 0$

Cramer's rule; Solution to $Ax = b$

$$x_j = \frac{det(B_j)}{detA}$$

Where, B_j is A with it's j'th column replaced by b.
Proof: $x = A^{-1}b = \frac{1}{\det(A)}C^Tb$

Volume of a box

$det(A)$ represents the volume of a box(higher dimensional parellelogram) whose edges are represented by the row vectors of A (or the column vectors of A, I think).

5. Eigen values and eigen vectors

5.1. Introduction

$$\boxed{\text{Solve: } Ax = \lambda x}$$

This give: $(A - \lambda I)x = 0$
For a non-zero solution x must lie in the nullspace of $A - \lambda I$.
Therefore $det(A - \lambda I) = 0$
This equation gives rise to a n dimensional polynomial equation in λ(**Characteristic equation**). Solving this gives eigen values.
Eigen vectors corresponding to each Eigen value form a vector space.
Algebraic multiplicity(M_λ): Order of the eigenvalue in the characteristic equation.
Geometric multiplicity(m_λ): Dimension of the eigenspace corresponding to the eigenvalue λ.
$m_\lambda \leq M_\lambda$

Theorem:

- $\sum \lambda_i = tr(A)$
- $\prod \lambda_i = det(A)$
- For a triangular matrix the Eigen values are along the diagonal.
- If $Ax = \lambda x$, then $A^k x = \lambda^k x$.

5.2. Diagonalization

If a $n \times n$ matrix A has n independent eigen vectors then, it can be written as follows:

$$A = S\Lambda S^{-1}$$

where, each column of S is one of the eigenvectors of A (e_i)

$$S = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

and Λ is a diagonal matrix with corresponding eigen values in the diagonal.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Proof:

$AS = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = S\Lambda$

$\implies A = S\Lambda S^{-1}$

Corollary: Powers of A

$A^n = S\Lambda^n S^{-1}$

Caley-Hamilton's theorem.

Theorem 5.1:
A satisfies it's own characteristic equation.

Proof.
Substitute $A = S\Lambda S^{-1}$ in $(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$. □

5.3. Difference equations and powers A^k

Difference equations

$$u_{k+1} = Au_k$$

In some sense difference equations are analogous to differential equations.

Examples

Fibonacci numbers

$F_{k+2} = F_{k+1} + F_k$

Let, $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ then, difference representation of Fibonacci sequence would be:

$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k$

The solution to this difference equation, $u_{k+1} = Au_k$ is

$$\boxed{u_k = A^k u_0}$$

If A can be diagonalized then $u_k = S\Lambda^k S^{-1} u_0$.

Markov matrices

Markov matrix is similar to the above examples along with the following two properties:

1. Each column of Markov matrix adds to 1.
2. The numbers outside and inside can never become negative.

$u_{k+1} = Au_k$ and the solution is $u_k = A^k u_0$

If A can be diagonalized, then $u_k = S\Lambda^k S^{-1} u_0$. *Steady state*($u_\infty$): $Au_\infty = u_\infty$.

Properties of a Markov matrix A:

1. $\lambda_1 = 1$ is an eigenvalue of A.
2. Its eigenvector x_1 is nonnegative and $Ax_1 = x_1$.
3. The other eigenvalues satisfy $|\lambda_1| \leq 1$.
4. If A or any power of a A has all positive entries, these other $|\lambda_i|$ are below 1.
5. The solution $A^k u_0$ approaches a multiple of x_1 which is the steady state u_∞.

Stability of $u_{k+1} = Au_k$

The difference equation $u_{k+1} = Au_k$ is:

- *stable* if all eigenvalues satisfy $|\lambda_i| < 1$.
- *neutrally stable* if some $|\lambda_i| = 1$ and all other $|\lambda_i| < 1$.
- *unstable* if at least one $|\lambda_i| > 1$.

5.4. Differential equations and e^{At}

System of differential equations

Matrices are useful to solve a system of differential equations. Eg:

$$\frac{du_1}{dt} = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$
$$\frac{du_2}{dt} = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$
$$\vdots$$
$$\frac{du_n}{dt} = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

This system in matrix notation is $\dfrac{du}{dt} = Au$, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Solution:

$$u(t) = e^{At}u(0)$$

Is the above solution valid if A is not diagonalizable.

Proof: If A is diagonalizable then let $v = S^{-1}u$, where S is the eigenvector matrix of A. Using eigenvector matrix the system of linear equations is decoupled.

$$\begin{aligned} \therefore S\frac{dv}{dt} &= ASv \\ \implies \frac{dv}{dt} &= S^{-1}ASv = \Lambda v \\ v(t) &= e^{\Lambda t}v(0) \\ u(t) &= Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0) \end{aligned}$$

where e^{At} is the exponential of the matrix A.

Matrix exponential

Matrix exponential is defined as follows:

$$e^{At} = I + At + \frac{(At)^2}{2} + \cdots + \frac{(At)^n}{dt} + \cdots$$

This series **always converges** and has the following properties:

- $(e^{As})(e^{At}) = e^{A(s+t)}$
- $(e^{At})(e^{-At}) = I$
- $\dfrac{d}{dt}(e^{At}) = Ae^{At}$
- If A can be diagonalized, $A = S\Lambda S^{-1}$, then $du/dt = Au$ has the solution $u(t) = e^{At}u(0)$.
- e^{At} is never singular.
  Proof 1: If λ is an eigenvalue of A, then $e^{\lambda t}$ is the corresponding eigenvalue of e^{At}, which is never 0.
  Proof 2: $det(e^{At}) = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t} = e^{trace(At)} \neq 0$.

Stability of differential equations

Behaviour of $u(t)$ as $t \to \infty$: The differential equation $du/dt = Au$ is:

- **stable:** and $e^{At} \to 0$ whenever, all $Re(\lambda_i) < 0$.
- **neutrally stable:** when all $Re(\lambda_i) < 0$ and $Re(\lambda_1) = 0$.
- **unstable:** and e^{At} is unbounded if any eigenvalue has $Re(\lambda_i) > 0$.

These results are true even if A is not diagonalizable.

5.5. Complex matrices

Length of a vector: $\|x\| = |x_1| + |x_2| + \cdots + |x_n|$

Hermitian (and symmetric) matrices

Conjugate transpose: $\overline{A}^T = A^H$ (Read as A Hermitian) has entries $(A^H)_{ij} = \overline{A_{ji}}$
Hermitian matrices: A is Hermitian iff $A^H = A$.
Theorem: If $A = A^H$ then $\forall\ x$, $x^H Ax$ is real.
Proof: $(x^H Ax)^H = x^H A^H x = x^H Ax$.
Theorem: If $A = A^H$ then all the eignevalues are real.
Proof: $x^H Ax = \lambda x^H x$. LHS is real and $x^H x$ is real, therefore λ is real.
Theorem: Eigenvectors of Hermitian matrices that come from two different eigenvalues are orthogonal.
Proof:
Let, $Ax = \lambda x$ and $Ay = \mu y$
$\lambda x^H y = (Ax)^H y = x^H Ay = x^H \mu y \implies x^H y = 0, \because \lambda \neq \mu$
Orthonormal matrices: $QQ^T = I$.
Theorem (Spectral theorem): A real symmetric matrix can be factored into $\boxed{A = Q\Lambda Q^T}$. Its orthonormal eigenvectors are in the columns of Q.

Unitary matrices

$\boxed{U^H U = I}$
Multiplication by U has no effect on inner products, angles and lengths.
Property 1:

- $(Ux)^T(Uy) = x^T U^T Uy = x^T y$
- $\|Ux\|^2 = x^H U^H Ux = x^H x = \|x\|^2$

Property 2:
For every eigenvalue(λ) of U, $|\lambda| = 1$
Property 3:
Eigen vectors corresponding different eigenvalues are orthogonal.
Let, $Ux = \lambda x$ and $Uy = \mu y$.
$x^H y = (Ux)^H(Uy) = (\lambda x)^H(\mu y) = \overline{\lambda}\mu x^H y$
By property-2 $\overline{\lambda}\lambda = 1, \therefore \overline{\lambda}\mu \neq 1$
$\implies x^H y = 0$.

$\boxed{\text{If } A \text{ is Hermitian then } K = iA \text{ is skew-Hermitian.}}$
Theorem: Eigenvalues of K are purely imaginary.

Fourier matrix

THE FOLLOWING IS INCOMPLETE

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}$$

5.6. Similarity transformation

Definition: A and B are similar if $\exists M \ni B = M^{-1}AM$.
Going from one to another is known as **similarity transformations**.
In the case where $M = S$, $M^{-1}AM$ becomes the diagonal matrix Λ. This is the best case scenario. But other M' s are also useful. Usually M is chosen such that, $M^{-1}AM$ is easier to work with than A.

Theorem: Suppose $B = M^{-1}AM$, then A and B have the same eigenvalues.
Proof: $Ax = \lambda x \implies B(M^{-1}x) = \lambda(M^{-1}x)$
$M^{-1}x$ is the eigenvector of B corresponding to λ.

Change of Basis = Similarity transformation

Every linear transformation is represented by a matrix. Similar matrices represent the same transformation T with respect to different basis.

$[T]_{\text{V to V}}$	$= [I]_{\text{v to V}}$	$[T]_{\text{v to v}}$	$[I]_{\text{V to v}}$
B	$= M^{-1}$	A	M

Triangular forms with unitary M

Shcur's lemma:
There is a unitary matrix $M = U$ such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.

Diagonalizing Hermitian matrices

Theorem:
If $A = A^H$ This triangular form T, is a diagonal matrix.
Proof: $(U^{-1}AU)^H = U^H A^H (U^{-1})^H = U^{-1}AU \implies T = T^H$.

Spectral theorem:
Every real symmetric matrix can be diagonalized by an orthogonal matrix Q. Every Hermitian matrix can be diagonalized by a unitary matrix U:

$$\begin{array}{rlcl} \text{(real)} & Q^{-1}AQ = \Lambda & \text{or} & A = Q\Lambda Q^T \\ \text{(complex)} & U^{-1}AU = \Lambda & \text{or} & A = U\Lambda U^H \end{array}$$

Normal matrices
The matrix N is normal if it commutes with N^H: $NN^H = N^H N$.

Theorem:
A triangular matrix T that is normal must be diagonal
Proof:
Tip: Use induction and multiplication by blocks.
For details see: https://math.stackexchange.com/a/2538528/633346

Theorem:
$T = U^{-1}AU$ is diagonal if and only if A is a normal matrix.
Proof:
If A is normal,
$TT^H = U^{-1}AUU^H A^H U = U^{-1}AA^H U = U^{-1}A^H AU = U^H A^H UU^{-1}AU = T^H T \implies T$ is diagonal.

Jordan form

Theorem: If A has s independednt eigenvectors, it is simillar to a matrix with s blocks.

$$\textbf{Jordan form } J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$

Each Jordan block is a triangular matrix that has only a single eigenvalue (λ_i) along the diagonal corresponding to only one eigenvector. For each missing eigenvector there will be a 1 just above the diagonal.

$$\text{Jordan block } J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}$$

6. Positive definite matrices

Definition:
A $n \times n$ symmetric matrix A is positive-definite if $\forall\ x \neq 0, \in \mathbb{R}^n$ $x^T Ax > 0$.

Test for positive definiteness Each of the following test is a necessary and sufficient condition for a matrix to be positive definite.

1. $x^T Ax > 0$ for all nonzero vectors x.
   Definition.
2. All eigenvalues of A satisfy $\lambda_i > 0$.
   Proof: $x^T Ax = \lambda x^T x = \lambda\|x\|^2$
   Converse: Every $\forall y > 0, \in \mathbb{R}^n$, $y = a_1 x_1 + \cdots + a_n x_n$
   $Ay = a_1\lambda_1 + \cdots + a_n\lambda_1$
   $y^T Ay = a_1^2\lambda_1 + \cdots + a_n^2\lambda_n > 0$
3. All the upper left submatrices A_k have positive determinants.
   Clue: $x^T Ax = \begin{bmatrix} x_k^T & 0 \end{bmatrix} \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} \begin{bmatrix} x_k \\ 0 \end{bmatrix} = x_k^T A_k x_k$
4. All the pivots (without row exchanges) satisfy $d_k > 0$.
   Proof: Incomplete

Theorem: Another test for positive definiteness
The symmetric matrix A is positive definite if and only if:
$\exists\ R$ with independent columns $\ni A = R^T R$.

Positive semidefinite Matrices

The tests for semidefinite matrices will relax to allow zeros.
Definition:
$\forall\ x\ x^T Ax \geq 0$
Test for positive semi definiteness

- $\forall x \neq 0\ x^T Ax \geq 0$.
- All eigenvalues satisfy $\lambda_i \geq 0$.
- No principal submatrices of A have negative determinants.
- No pivots are negative.
- $\exists\ R$, possibly with dependent columns such that $A = R^T R$.

6.1. Singular value decomposition

Any $m \times n$ matrix A can be decomposed in to:

$$\boxed{A = U\Sigma V^T}$$

Σ: Diagonal matrix, $m \times n$. This diagonal matrix has eignevalues form $A^T A$, not from A.
The positive entries $\sigma_1, \sigma_2, \cdots, \sigma_s$, form the first r diagonal elements of Σ. These elements are called **singular values**. The remainder of entries in Σ is 0.

U: Orthogonal matrix, $m \times m$.
The columns of U are the eigenvectors of $A^T A$. V: Orthogonal

matrix, $n \times n$.
The columns of V are the eigenvectors of AA^T.

Remark 1
For positive definite matrices, Σ is Λ and $U\Sigma V^T$ is identical to $Q\Lambda Q^T$.

Remark 2
U and V give orthonormal bases for all four fundamental subspaces.

first	r	columns of U:	**column space** of A
last	$m-r$	columns of U:	**left nullspace** of A
first	r	columns of V:	**row space** of A
last	$n-r$	columns of V:	**nullspace** of A

Remark 3
$Av_j = \sigma_j u_j$ or $AV = U\Sigma$

Remark 4
Eigenvectors of AA^T and $A^T A$ goes into columns of U and V.
$AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$, and,
$A^T A = V\Sigma^T \Sigma V^T$.

7. Matrix Factorizations

LU factorizability

$$\boxed{A = LU}$$

Requirements:
An invertible matrix A is LU factorizable, if the determinant of all its principal minors are > 0. Or if A has rank k, A is LU factorizable, if all its first k principal minors are > 0.
Procedure:
Do Gaussian elimination without row exchanges.
LDU:
Pivots in U are divided out into D to have 1s in the diagonal of U.

PLU factorization

$$\boxed{PA = LU}$$

Always exists.
Procedure:
Do Gaussian elimination.
If A is invertible, then L are U are invertible. P is always invertible and $P^{-1} = P^T$.