

COPULA FUNCTIONS FOR REGULAR VINES USAGE

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INHALTSVERZEICHNIS

1	COPULAE	1
1.1	Independence Copula	1
1.2	Gauss Copula	1
1.3	Student t Copula	1
1.4	Clayton Copula	2
1.5	Frank Copula	2
1.6	Gumbel Copula	3
1.7	Farlie-Gumbel-Morgenstern Copula	3
1.8	Galambos Copula	4
A	SYMBOLS	5
	LITERATURVERZEICHNIS	7

1 Copulae

Copulae in general

Probability Function	$C(x, y)$
Density Function	$c(x, y) = \frac{\partial C(x, y)}{\partial x \partial y}$
Conditional Probability Function	$h(x, y) = \frac{\partial C(x, y)}{\partial y}$
Inverse h Function	$h^{-1}(x, y)$
Kendall's Tau Function	$\tau = \iint_{[0,1]^2} C(x, y) dC(x, y)$

1.1 Independence Copula

Probability Function	$C(x, y) = xy$
Density Function	$c(x, y) = 1$
Conditional Probability Function	$h(x, y) = x$
Inverse h Function	$h^{-1}(x, y) = x$
Kendall's Tau Function	$\tau = 0$

1.2 Gauss Copula

Let $p \in (-1, 1)$ be probability parameter.

Let $\tilde{x} = \Phi^{-1}(x)$ and $\tilde{y} = \Phi^{-1}(y)$.

Probability Function	$C(x, y) = \Phi_2(\tilde{x}, \tilde{y}, p)$ [1]
Density Function	$c(x, y) = \frac{1}{\sqrt{1-p^2}} \exp\left(-\frac{p^2(\tilde{x}^2 + \tilde{y}^2) - 2p\tilde{x}\tilde{y}}{2(1-p^2)}\right)$ [2]
Conditional Probability Function	$h(x, y) = \Phi\left(\frac{\tilde{x} - p\tilde{y}}{\sqrt{1-p^2}}\right)$ [2]
Inverse h Function	$h^{-1}(x, y) = \Phi(\tilde{x}\sqrt{1-p^2} - p\tilde{y}\sqrt{1-p^2})$ [2]
Kendall's Tau Function	$\tau = \frac{2}{\pi} \arcsin(p)$ [3]

1.3 Student t Copula

Let $p \in (-1, 1)$ be probability parameter and $v \in \mathbb{N} \setminus 0$ be degree of freedom.

Let $\tilde{x} = t_v^{-1}(x)$ and $\tilde{y} = t_v^{-1}(y)$.

Probability Function	$C(x, y) = t_{2v}(\tilde{x}, \tilde{y}, p)$ [1]
Density Function	$c(x, y) = \frac{1}{2\pi dt_v(\tilde{x}) dt_v(\tilde{y}) \sqrt{1-p^2}} \left(1 + \frac{\tilde{x}^2 + \tilde{y}^2 - 2p\tilde{x}\tilde{y}}{v(1-p^2)} \right)^{-\frac{v+1}{2}}$ [2]
Conditional Probability Function	$h(x, y) = t_{v+1} \left(\frac{\tilde{x} - p\tilde{y}}{\sqrt{\frac{(v+\tilde{y}^2)(1-p^2)}{v+1}}} \right)$ [2]
Inverse h Function	$h^{-1}(x, y) = t_v \left(t_{v+1}^{-1}(x) \sqrt{\frac{(v+\tilde{y}^2)(1-p^2)}{v+1}} + p\tilde{y} \right)$ [2]
Kendall's Tau Function	$\tau = \frac{2}{\pi} \arcsin(p)$ [3]

Archimedean Copulae

Induced by a function $\varphi : [0, 1] \rightarrow [0, \infty]$ continual and monotonically decreasing.

Probability function is defined by $C(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y))$.

Kendall's tau for archimedean copulae is given by $\tau = 1 + 4 \int_0^1 \frac{\phi(v)}{\phi'(v)} dv$. [4]

1.4 Clayton Copula

Let $\delta \in [-1, \infty)$.

Let $\varphi(x) = \frac{1}{\delta}(x^{-\delta} - 1)$. [5]

Probability Function	$C(x, y) = (x^{-\delta} + y^{-\delta} - 1)^{-\frac{1}{\delta}}$ [2]
Density Function	$c(x, y) = (1 + \delta)(xy)^{-1-\delta}(x^{-\delta} + y^{-\delta} - 1)^{-\frac{1}{\delta}-2}$ [2]
Conditional Probability Function	$h(x, y) = y^{-\delta-1}(x^{-\delta} + y^{-\delta} - 1)^{-1-\frac{1}{\delta}}$ [2]
Inverse h Function	$h^{-1}(x, y) = \left((x \cdot y^{\delta+1})^{-\frac{\delta}{\delta+1}} + 1 - y^{-\delta} \right)^{-\frac{1}{\delta}}$ [2]
Kendall's Tau Function	$\tau = \frac{\delta}{\delta+2}$ [6]

1.5 Frank Copula

Let $\delta \in (-\infty, \infty)$.

Let $\varphi(x) = -\log\left(\frac{e^{-\delta x} - 1}{e^{-\delta} - 1}\right)$. [5]

Probability Function	$C(x, y) = -\frac{1}{\delta} \log \left(1 + \frac{(e^{-\delta x} - 1)(e^{-\delta y} - 1)}{e^{-\delta} - 1} \right)$ [6]
Density Function	via Wavelet estimation[7] [TODO]
Conditional Probability Function	$h(x, y) = \frac{e^{-\delta y}}{\frac{1-e^{-\delta}}{1-e^{-\delta x}} + e^{-\delta y} - 1}$ [10]
Inverse h Function	$h^{-1}(x, y) = -\log \left(1 - \frac{1-e^{-\delta}}{(x^{-1}-1)e^{-\delta y}+1} \right) \frac{1}{\delta}$ [10]
Kendall's Tau Function	$\tau = 1 - \frac{4}{\delta} + 4 \frac{D_1(\delta)}{\delta}$ [6]

1.6 Gumbel Copula

Let $\delta \in [1, \infty)$.

Let $\varphi(x) = (-\log(x))^\delta$. [5]

Probability Function	$C(x, y) = e^{(-((-\log(u_1))^\delta + (-\log(u_2))^\delta)^{\frac{1}{\delta}})}$ [2]
Density Function	$c(x, y) = \frac{1}{C(x, y)} (-((-\log(x))^\delta + (-\log(y))^\delta)^{\frac{2}{\delta}-2}) \cdot (\log(x)\log(y))^{\delta-1} \cdot (1 + (\delta-1)((-\log(x))^\delta + (-\log(y))^\delta)^{\frac{1}{\delta}})$ [2]
Conditional Probability Function	$h(x, y) = C(x, y) \cdot \frac{1}{y} \cdot (-\log(y))^{\delta-1} \cdot \left((-\log(x))^\delta + (-\log(y))^\delta \right)^{\frac{1}{\delta-1}}$ [2]
Inverse h Function	via Gauss-Raphson method[2]
Kendall's Tau Function	$\tau = 1 - \frac{1}{\delta}$ [6]

1.7 Farlie-Gumbel-Morgenstern Copula

Let $\delta \in [-1, 1]$.

Probability Function	$C(x, y) = xy + \delta xy(1-x)(1-y)$ [5]
Density Function	$c(x, y) = 1 + \delta(1-2x)(1-2y)$ [5]
Conditional Probability Function	$h(x, y) = x(1 + \delta(1-x)(1-2y))$ [5]
Inverse h Function	via Gauss-Raphson method
Kendall's Tau Function	$\tau = \frac{2}{9}\delta$ [5]

Extreme-Value Copulae

Induced by a function $A : [0, 1] \rightarrow [1/2, 1]$, the Pickands dependence function, which is convex and $\max(t, 1-t) \leq A(t) \leq 1 \quad \forall t \in [0, 1]$.

Probability function is defined by $C(x, y) = \exp\left(\log(uv)A\left(\frac{\log(v)}{\log(uv)}\right)\right)$.

Kendall's tau for extreme-value copulae is given by $\tau = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t)$. [8]

1.8 Galambos Copula

Let $A(t) = 1 - (t^{-\delta} + (1-t)^{-\delta})^{-1/\delta}$ where $\delta \in [0, \infty)$. [9]

Let c_{Gu} be density of Gumbel Copula.

Probability Function	$C(x, y) = xy \exp(((-\log(x))^{-\delta} + (-\log(y))^{-\delta})^{-\frac{1}{\delta}})$ [10]
Density Function	$c(x, y) = c_{Gu}(1-x)(1-y)$ [11]
Conditional Probability Function	$h(x, y) = \frac{C(x, y)}{y} \left(1 - \left(1 + \left(\frac{-\log(x)}{-\log(y)}\right)^{\delta}\right)^{-1-\frac{1}{\delta}}\right)$ [10]
Inverse h Function	via Gauss-Raphson method
Kendall's Tau Function	Use numerical integration (e.g. Simpson's rule) [TODO]

A Symbols

τ	Kendall's tau function.
$C(x, y)$	Copula probability function.
$c(x, y)$	Copula density function.
$h(x, y)$	Copula conditional probability function.
$\Phi(x)$	Standard normal distribution.
$\Phi_2(x, y, \rho)$	Bivariate standard normal distribution.
$t_v(x)$	Student t distribution with v degrees of freedom.
$t_{2,v}(x, y, \rho)$	Bivariate student t distribution with v degrees of freedom.
$\varphi(x)$	Generator function for archimedean copulae.
$A(x)$	Generator function for extreme-value copulae.

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