COPULA FUNCTIONS FOR REGULAR VINES USAGE

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INHALTSVERZEICHNIS

1 Copulae

Copulae in general

Probability Function	$C(x,y)$
Density Function	$c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y}$
Conditional Probability Function	$h(x,y) = \frac{\partial C(x,y)}{\partial y}$
Inverse h Function	$h^{-1}(x,y)$
Kendall's Tau Function	$\tau = \iint\limits_{[0,1]^2} C(x,y) dC(x,y)$

1.1 Independence Copula

Probability Function	$C(x,y) = xy$
Density Function	$c(x,y) = 1$
Conditional Probability Function	$h(x,y) = x$
Inverse h Function	$h^{-1}(x,y) = x$
Kendall's Tau Function	$\tau = 0$

1.2 Gauss Copula

Let $p \in (-1,1)$ be probability parameter.
Let $\tilde{x} = \Phi^{-1}(x)$ and $\tilde{y} = \Phi^{-1}(y)$.

Probability Function	$C(x,y) = \Phi_2(\tilde{x}, \tilde{y}, p)$ [1]
Density Function	$c(x,y) = \frac{1}{\sqrt{1-p^2}} \exp\left(-\frac{p^2(\tilde{x}^2+\tilde{y}^2)-2p\tilde{x}\tilde{y}}{2(1-p^2)}\right)$ [2]
Conditional Probability Function	$h(x,y) = \Phi\left(\frac{\tilde{x}-p\tilde{y}}{\sqrt{1-p^2}}\right)$ [2]
Inverse h Function	$h^{-1}(x,y) = \Phi(\tilde{x}\sqrt{1-p^2} + p\tilde{y})$ [2]
Kendall's Tau Function	$\tau = \frac{2}{\pi} \arcsin(p)$ [3]

1.3 Student t Copula

Let $p \in (-1,1)$ be probability parameter and $\nu \in \mathbb{N} \backslash 0$ be degree of freedom.
Let $\tilde{x} = t_\nu^{-1}(x)$ and $\tilde{y} = t_\nu^{-1}(y)$.

Probability Function	$C(x,y) = t_{2_\nu}(\tilde{x}, \tilde{y}, p)$ [1]
Density Function	$c(x,y) = \frac{1}{2\pi dt_\nu(\tilde{x}) dt_\nu(\tilde{y})\sqrt{1-p^2}} \left(1 + \frac{\tilde{x}^2+\tilde{y}^2-2p\tilde{x}\tilde{y}}{\nu(1-p^2)}\right)^{-\frac{\nu+1}{2}}$ [2]
Conditional Probability Function	$h(x,y) = t_{\nu+1}\left(\frac{\tilde{x}-p\tilde{y}}{\sqrt{\frac{(\nu+\tilde{y}^2)(1-p^2)}{\nu+1}}}\right)$ [2]
Inverse h Function	$h^{-1}(x,y) = t_\nu\left(t_{\nu+1}^{-1}(x)\sqrt{\frac{(\nu+\tilde{y}^2)(1-p^2)}{\nu+1}} + p\tilde{y}\right)$ [2]
Kendall's Tau Function	$\tau = \frac{2}{\pi}\arcsin(p)$ [3]

Archimedean Copulae

Induced by a function $\varphi : [0,1] \to [0,\infty]$ continual and monotonically decreasing. Probability function is defined by $C(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y))$.

Kendall's tau for archimedean copulae is given by $\tau = 1 + 4\int_0^1 \frac{\phi(\nu)}{\phi'(\nu)} d\nu$. [4]

1.4 Clayton Copula

Let $\delta \in (0,\infty)$.
Let $\varphi(x) = \frac{1}{\delta}(x^{-\delta} - 1)$. [5]

Probability Function	$C(x,y) = (x^{-\delta} + y^{-\delta} - 1)^{-\frac{1}{\delta}}$[2]
Density Function	$c(x,y) = (1+\delta)(xy)^{-1-\delta}(x^{-\delta} + y^{-\delta} - 1)^{-\frac{1}{\delta}-2}$[2]
Conditional Probability Function	$h(x,y) = y^{-\delta-1}(x^{-\delta} + y^{-\delta} - 1)^{-1-\frac{1}{\delta}}$[2]
Inverse h Function	$h^{-1}(x,y) = \left(\left(x \cdot y^{\delta+1}\right)^{-\frac{\delta}{\delta+1}} + 1 - y^{-\delta}\right)^{-\frac{1}{\delta}}$ [2]
Kendall's Tau Function	$\tau = \frac{\delta}{\delta+2}$[6]

1.5 Frank Copula

Let $\delta \in (-\infty,\infty)$.
Let $\varphi(x) = -\log\left(\frac{e^{-\delta x}-1}{e^{-\delta}-1}\right)$.[5]

Probability Function	$C(x,y) = -\frac{1}{\delta}\log\left(1 + \frac{(e^{-\delta x}-1)(e^{-\delta y}-1)}{e^{-\delta}-1}\right)$[6]
Density Function	$c(x,y) = \frac{\delta e^{\delta(1+x+y)}(e^{\delta}-1)}{(e^{\delta}(-e^{\delta x}+e^{\delta(x+y-1)}-e^{\delta y}))^2}$
Conditional Probability Function	$h(x,y) = \frac{e^{-\delta y}}{\frac{1-e^{-\delta}}{1-e^{-\delta x}}+e^{-\delta y}-1}$[10]
Inverse h Function	$h^{-1}(x,y) = -\log\left(1 - \frac{1-e^{-\delta}}{(x^{-1}-1)e^{-\delta y}+1}\right)\frac{1}{\delta}$[10]
Kendall's Tau Function	$\tau = 1 - \frac{4}{\delta} + 4\frac{D_1(\delta)}{\delta}$ [6]

1.6 Gumbel Copula

Let $\delta \in [1, \infty)$.
Let $\varphi(x) = (-\log(x))^{\delta}$. [5]

Probability Function	$C(x,y) = e^{(-((-\log(u_1))^{\delta}+(-\log(u_2))^{\delta})^{\frac{1}{\delta}})}$[2]
Density Function	$c(x,y) = \frac{C(x,y)}{xy}(-((-\log(x))^{\delta} + (-\log(y))^{\delta})^{\frac{2}{\delta}-2})$ $\cdot(\log(x)\log(y))^{\delta-1}$ $\cdot(1 + (\delta-1)((-\log(x))^{\delta} + (-\log(y))^{\delta})^{-\frac{1}{\delta}})$ [2]
Conditional Probability Function	$h(x,y) = C(x,y) \cdot \frac{1}{y} \cdot (-\log(y))^{\delta-1}$ $\cdot\left((-\log(x))^{\delta} + (-\log(y))^{\delta}\right)^{\frac{1}{\delta}-1}$ [2]
Inverse h Function	via numerical approximation (Bisection Method)
Kendall's Tau Function	$\tau = 1 - \frac{1}{\delta}$[6]

1.7 Farlie-Gumbel-Morgenstern Copula

Let $\delta \in [-1, 1]$.

Probability Function	$C(x,y) = xy + \delta xy(1-x)(1-y)$[5]
Density Function	$c(x,y) = 1 + \delta(1-2x)(1-2y)$[5]
Conditional Probability Function	$h(x,y) = x(1 + \delta(1-x)(1-2y))$[5]
Inverse h Function	via numerical approximation (Bisection Method)
Kendall's Tau Function	$\tau = \frac{2}{9}\delta$[5]

Extreme-Value Copulae

Induced by a function $A : [0,1] \to [1/2, 1]$, the Pickands dependence function, which is convex and $\max(t, 1-t) \leqslant A(t) \leqslant 1 \quad \forall t \in [0,1]$.
Probability function is defined by $C(x,y) = \exp\left(\log(xy)A\left(\frac{\log(y)}{\log(xy)}\right)\right)$.

Kendall's tau for extreme-value copulae is given by $\tau = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t)$.

If $A'(t)$ is fully differentiable it holds $\tau = \int_0^1 \frac{t(1-t)}{A(t)} A''(t)dt$. [8]

1.8 Galambos Copula

Let $A(t) = 1 - (t^{-\delta} + (1-t)^{-\delta})^{-1/\delta}$ where $\delta \in [0, \infty)$.[9]
Let $\tilde{x} = -\log(x)$ and $\tilde{y} = -\log(y)$

Probability Function	$C(x,y) = xy \exp(\tilde{x}^{-\delta} + \tilde{y}^{-\delta})^{-\frac{1}{\delta}}$ [10]
Density Function	$c(x,y) = \frac{C(x,y)}{xy}\Big(1 - (\tilde{x}^{-\delta} + \tilde{y}^{-\delta})^{-1-\frac{1}{\delta}}(\tilde{x}^{-\delta-1} + \tilde{y}^{-\delta-1})$ $+(\tilde{x}^{-\delta} + \tilde{y}^{-\delta})^{-2-\frac{1}{\delta}}(\tilde{x}\tilde{y})^{-\delta-1}\big(1 + \delta + (\tilde{x}^{-\delta} + \tilde{y}^{-\delta})^{-\frac{1}{\delta}}\big)\Big)$ [10]
Conditional Probability Function	$h(x,y) = \frac{C(x,y)}{y}\Big(1 - \big(1 + (\frac{\tilde{y}}{\tilde{x}})^{\delta}\big)^{-1-\frac{1}{\delta}}\Big)$[10]
Inverse h Function	via numerical approximation (Bisection Method)
Kendall's Tau Function	Use numerical integration (using Simpson's rule, N = 1000)

A Symbols

τ	Kendall's tau function.
$C(x, y)$	Copula probability function.
$c(x, y)$	Copula density function.
$h(x, y)$	Copula conditional probability function.
$\Phi(x)$	Standard normal distribution.
$\Phi_2(x, y, p)$	Bivariate standard normal distribution.
$t_\nu(x)$	Student t distribution with v degrees of freedom.
$t_{2_\nu}(x, y, p)$	Bivariate student t distribution with v degrees of freedom.
$\varphi(x)$	Generator function for archimedean copulae.
$A(x)$	Generator function for extreme-value copulae.

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