COPULA FUNCTIONS FOR REGULAR VINES USAGE

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INHALTSVERZEICHNIS

1 Copulae

Copulae in general

Probability Function	$C(x, y)$
Density Function	$c(x, y) = \frac{\partial C(x,y)}{\partial x \partial y}$
Conditional Probability Function	$h(x, y) = \frac{\partial C(x,y)}{\partial y}$
Inverse h Function	$h^{-1}(x, y)$
Kendall's Tau Function	$\tau = \iint\limits_{[0,1]^2} C(x, y) dC(x, y)$

1.1 Independence Copula

Probability Function	$C(x, y) = xy$
Density Function	$c(x, y) = 1$
Conditional Probability Function	$h(x, y) = x$
Inverse h Function	$h^{-1}(x, y) = x$
Kendall's Tau Function	$\tau = 0$

1.2 Gauss Copula

Let $p \in (-1, 1)$ be probability parameter.
Let $\tilde{x} = \Phi^{-1}(x)$ and $\tilde{y} = \Phi^{-1}(y)$.

Probability Function	$C(x, y) = \Phi_2(\tilde{x}, \tilde{y}, p)$ [1]
Density Function	$c(x, y) = \frac{1}{\sqrt{1-p^2}} \exp\left(-\frac{p^2(\tilde{x}^2+\tilde{y}^2)-2p\tilde{x}\tilde{y}}{2(1-p^2)}\right)$ [2]
Conditional Probability Function	$h(x, y) = \Phi\left(\frac{\tilde{x}-p\tilde{y}}{\sqrt{1-p^2}}\right)$ [2]
Inverse h Function	$h^{-1}(x, y) = \Phi(\tilde{x}\sqrt{1-p^2} - p\tilde{y}\sqrt{1-p^2})$ [2]
Kendall's Tau Function	$\tau = \frac{2}{\pi} \arcsin(p)$ [3]

1.3 Student t Copula

Let $p \in (-1, 1)$ be probability parameter and $\nu \in \mathbb{N} \backslash 0$ be degree of freedom.
Let $\tilde{x} = t_\nu^{-1}(x)$ and $\tilde{y} = t_\nu^{-1}(y)$.

Probability Function	$C(x,y) = t_{2_\nu}(\tilde{x},\tilde{y},p)$ [1]
Density Function	$c(x,y) = \frac{1}{2\pi dt_\nu(\tilde{x})dt_\nu(\tilde{y})\sqrt{1-p^2}}\left(1+\frac{\tilde{x}^2+\tilde{y}^2-2p\tilde{x}\tilde{y}}{\nu(1-p^2)}\right)^{-\frac{\nu+1}{2}}$ [2]
Conditional Probability Function	$h(x,y) = t_{\nu+1}\left(\frac{\tilde{x}-p\tilde{y}}{\sqrt{\frac{(\nu+\tilde{y}^2)(1-p^2)}{\nu+1}}}\right)$ [2]
Inverse h Function	$h^{-1}(x,y) = t_\nu\left(t_{\nu+1}^{-1}(x)\sqrt{\frac{(\nu+\tilde{y}^2)(1-p^2)}{\nu+1}}+p\tilde{y}\right)$ [2]
Kendall's Tau Function	$\tau = \frac{2}{\pi}\arcsin(p)$ [3]

Archimedean Copulae

Induced by a function $\varphi : [0,1] \to [0,\infty]$ continual and monotonically decreasing.
Probability function is defined by $C(x,y) = \varphi^{-1}(\varphi(x)+\varphi(y))$.
Kendall's tau for archimedean copulae is given by $\tau = 1 + 4\int_0^1 \frac{\phi(v)}{\phi'(v)}dv$. [4]

1.4 Clayton Copula

Let $\delta \in [-1,\infty)$.
Let $\varphi(x) = \frac{1}{\delta}(x^{-\delta}-1)$. [5]

Probability Function	$C(x,y) = (x^{-\delta}+y^{-\delta}-1)^{-\frac{1}{\delta}}$[2]
Density Function	$c(x,y) = (1+\delta)(xy)^{-1-\delta}(x^{-\delta}+y^{-\delta}-1)^{-\frac{1}{\delta}-2}$[2]
Conditional Probability Function	$h(x,y) = y^{-\delta-1}(x^{-\delta}+y^{-\delta}-1)^{-1-\frac{1}{\delta}}$[2]
Inverse h Function	$h^{-1}(x,y) = \left(\left(x\cdot y^{\delta+1}\right)^{-\frac{\delta}{\delta+1}}+1-y^{-\delta}\right)^{-\frac{1}{\delta}}$ [2]
Kendall's Tau Function	$\tau = \frac{\delta}{\delta+2}$[6]

1.5 Frank Copula

Let $\delta \in (-\infty,\infty)$.
Let $\varphi(x) = -\log\left(\frac{e^{-\delta x}-1}{e^{-\delta}-1}\right)$.[5]

Probability Function	$C(x,y) = -\frac{1}{\delta}\log\left(1+\frac{(e^{-\delta x}-1)(e^{-\delta y}-1)}{e^{-\delta}-1}\right)$[6]
Density Function	via Wavelet estimation[7]
Conditional Probability Function	$h(x,y) = \frac{e^{-\delta y}}{\frac{1-e^{-\delta}}{1-e^{-\delta x}}+e^{-\delta y}-1}$
Inverse h Function	$h^{-1}(x,y) = -\log\left(1-\frac{1-e^{-\delta}}{(x^{-1}-1)e^{-\delta y}+1}\right)\frac{1}{\delta}$
Kendall's Tau Function	$\tau = 1-\frac{4}{\delta}+4\frac{D_1(\delta)}{\delta}$ [6]

1.6 Gumbel Copula

Let $\delta \in [1,\infty)$.
Let $\varphi(x) = (-\log(x))^{\delta}$. [5]

Probability Function	$C(x,y) = e^{(-((-\log(u_1))^{\delta}+(-\log(u_2))^{\delta})^{\frac{1}{\delta}})}$[2]
Density Function	$c(x,y) = \frac{1}{C(x,y)}(-((-\log(x))^{\delta}+(-\log(y))^{\delta})^{\frac{2}{\delta}-2})$ $\cdot(\log(x)\log(y))^{\delta-1}$ $\cdot(1+(\delta-1)((-\log(x))^{\delta}+(-\log(y))^{\delta})^{\frac{1}{\delta}})$ [2]
Conditional Probability Function	$h(x,y) = C(x,y)\cdot\frac{1}{y}\cdot(-\log(y))^{\delta-1}$ $\cdot\left((-\log(x))^{\delta}+(-\log(y))^{\delta}\right)^{\frac{1}{\delta-1}}$ [2]
Inverse h Function	via Gauss-Raphson method[2]
Kendall's Tau Function	$\tau = 1-\frac{1}{\delta}$[6]

1.7 Farlie-Gumbel-Morgenstern Copula

Let $\delta \in [-1,1]$.

Probability Function	$C(x,y) = xy+\delta xy(1-x)(1-y)$[5]
Density Function	$c(x,y) = 1+\delta(1-2x)(1-2y)$[5]
Conditional Probability Function	$h(x,y) = x(1+\delta(1-x)(1-2y))$[5]
Inverse h Function	via Gauss-Raphson method
Kendall's Tau Function	$\tau = \frac{2}{9}\delta$[5]

Extreme-Value Copulae

Induced by a function $A : [0,1] \to [1/2,1]$, the Pickands dependence function, which is convex and $\max(t,1-t) \leqslant A(t) \leqslant 1 \quad \forall t \in [0,1]$.

Probability function is defined by $C(x,y) = exp\left(\log(uv)A\left(\frac{\log(v)}{\log(uv)}\right)\right)$.

Kendall's tau for extreme-value copulae is given by $\tau = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t)$. [8]

1.8 Galambos Copula

Let $A(t) = 1-(t^{-\delta}+(1-t)^{-\delta})^{-1/\delta}$ where $\delta \in [0,\infty)$.[9]
Let c_{Gu} be density of Gumbel Copula.

Probability Function	$C(x,y) = xy\ exp(((-\log(x))^{-\delta} + (-\log(y))^{-\delta})^{-\frac{1}{\delta}})$ [10]
Density Function	$c(x,y) = c_{Gu}(1-x)(1-y)$ [11]
Conditional Probability Function	$h(x,y) = \frac{C(x,y)}{y}\left(1-\left(1+\left(\frac{-\log(x)}{-\log(y)}\right)^{\delta}\right)^{-1-\frac{1}{\delta}}\right)$[10]
Inverse h Function	via Gauss-Raphson method
Kendall's Tau Function	Use numerical integration (e.g. Simpson's rule)

A Symbols

τ	Kendall's tau function.
$C(x, y)$	Copula probability function.
$c(x, y)$	Copula density function.
$h(x, y)$	Copula conditional probability function.
$\Phi(x)$	Standard normal distribution.
$\Phi_2(x, y, p)$	Bivariate standard normal distribution.
$t_\nu(x)$	Student t distribution with v degrees of freedom.
$t_{2_\nu}(x, y, p)$	Bivariate student t distribution with v degrees of freedom.
$\varphi(x)$	Generator function for archimedean copulae.
$A(x)$	Generator function for extreme-value copulae.

[1] Song, P. X.-K. Multivariate dispersion models generated from gaussian copula. *Scandinavian Journal of Statistics*, 27(2):305–320, **2000**.

[2] Aas, K., Czado, C., Frigessi, A. and Bakken, H. Pair-copula constructions of multiple dependence. *Insurance Mathematics and Economics*, 44(2):182–198, **2009**.

[3] Fang, H. B., Fang, K. T., and Kotz, S. The meta-elliptical distributions with given marginals. *Journal of Multivariate Analysis*, 82(1):1–16, **2002**.

[4] Genest, C., and MacKay, J. The joy of copulas: bivariate distributions with uniform marginals. *The American Statistician*, 40(4):280–283, **1986**.

[5] Genest, C., and Favre, A. C. Everything you always wanted to know about copula modeling but were afraid to ask. *Journal of hydrologic engineering*, 12(4):347–368, **2007**.

[6] Mahfoud, M., and Michael, M. Bivariate archimedean copulas: an application to two stock market indices. *BMI Paper*, **2012**.

[7] Genest, C., Masiello, E., Tribouley, K. Estimating copula densities through wavelets. *Insurance: Mathematics and Economics*, 44(2):170–181, **2009**.

[8] Genest, C., Kojadinovic, I., NeÅąlehovÃą, J., and Yan, J. A goodness-of-fit test for bivariate extreme-value copulas. *Bernoulli*, 17(1):253–275, **2011**.

[9] HU Berlin. Multivariate Time Series. `http://fedc.wiwi.hu-berlin.de/xplore/tutorials/stfhtmlnode13.html`. [Online; accessed 05-March-2017].

[10] Schirmacher, D., and Schirmacher, E. Multivariate dependence modeling using pair-copulas. *Technical report*, pages 14–16, **2008**.

[11] Doyon, G. On densities of extreme value copulas. M.Sc. Thesis, ETH Zurich, **2013**.