Simulation Guide to "Marginal and Conditional Multiple Inference for Linear Mixed Model Predictors"

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The notation in this document correspond to the notation used in the main article and coincide with the standard notation of Rao and Molina (2015).

For the conditions below, the following identities and simplifications are used in the file balanced.jl.

i) $Z_i = \mathbf{1}_{n_i}$ (random intercept)

ii) $l_i = \bar{x}_i$

iii) $n_i = n_k$ for $i, k = 1, \ldots, m$ (balanced panel)

iv) $p = 2$

For unbalanced panels and the Fay-Herriot model, these simplifications can be straightforwardly adapted.

For i, ii, iv) the identities hold ($V_i = \sigma_v^2 \mathbf{1}_{n_i} \mathbf{1}_{n_i}^t + \sigma_e^2 I_{n_i}$):

$$
\begin{aligned}
b_i^t &= h_i^t G Z_i^t V_i^{-1} = \frac{\gamma_i}{n_i} \mathbf{1}_{n_i}^t = \frac{\sigma_v^2}{(\dots)} \mathbf{1}_{n_i}^t \\
d_i^t &= l_i^t - b_i^t X_i = (1-\gamma_i)\bar{x}_i^t
\end{aligned}
$$

$$
\begin{aligned}
x_i^t V_i^{-1} \mathbf{1}_{n_i} &= \frac{\gamma_i}{\sigma_v^2} \bar{x}_i \\
x_i^t V_i^{-1} x_i &= \frac{x_i^t x_i - n_i \gamma_i \bar{x}_i^2}{\sigma_e^2} \\
\mathbf{1}_{n_i}^t V_i^{-1} \mathbf{1}_{n_i} &= \frac{\gamma_i}{\sigma_v^2} \\
\mathbf{1}_{n_i}^t V_i^{-1} y_i &= \frac{\gamma_i \bar{y}_i}{\sigma_v^2} \\
x_i^t V_i^{-1} y_i &= (x_i^t y_i - n_i \gamma_i \bar{x}_i \bar{y}_i)/\sigma_e^2 \\
\mathbf{1}_{n_i}^t V_i \mathbf{1}_{n_i} &= n_i^2 \sigma_v^2 + n_i \sigma_e^2 = \frac{n_i^2 \sigma_v^2}{\gamma_i}
\end{aligned}
$$

This implies, if further iii) holds:

$$
X^t V^{-1} X = \begin{pmatrix} \frac{m\gamma_i}{\sigma_v^2} & \frac{\gamma_i}{\sigma_v^2}\bar{x} \\ \frac{\gamma_i}{\sigma_v^2}\bar{x} & (x^t x - n_i \gamma_i \sum_{k=1}^m \bar{x}_k^2)/\sigma_e^2 \end{pmatrix} \tag{1}
$$

Further,

$$
\begin{aligned}
\frac{\partial}{\partial \sigma_v^2} b_i &= \frac{\sigma_e^2}{(n_i\sigma_v^2 + \sigma_e^2)^2} \mathbf{1}_{n_i} = \frac{\sigma_e^2}{n_i^2} \left(\frac{\gamma_i}{\sigma_v^2}\right)^2 \mathbf{1}_{n_i} = \frac{\gamma_i^2}{n_i^2 \sigma_v^2} \cdot \frac{\sigma_e^2}{\sigma_v^2} \mathbf{1}_{n_i} = \frac{\sigma_e^2}{(n_i\sigma_v^2 + \sigma_e^2)^2} \mathbf{1}_{n_i} \\
\frac{\partial}{\partial \sigma_e^2} b_i &= -\frac{\sigma_v^2}{(n_i\sigma_v^2 + \sigma_e^2)^2} \mathbf{1}_{n_i} = -\frac{\gamma_i^2}{n_i^2 \sigma_v^2} \mathbf{1}_{n_i} = -\frac{\sigma_v^2}{(n_i\sigma_v^2 + \sigma_e^2)^2} \mathbf{1}_{n_i} \\
\frac{\partial^2}{\partial (\sigma_v^2)^2} b_i &= -\frac{2n_i\sigma_e^2}{(n_i\sigma_v^2 + \sigma_e^2)^3} \\
\frac{\partial^2}{\partial \sigma_v^2 \partial \sigma_e^2} b_i &= \frac{n_i\sigma_v^2 - \sigma_e^2}{(n_i\sigma_v^2 + \sigma_e^2)^3} \\
\frac{\partial^2}{\partial (\sigma_e^2)^2} b_i &= \frac{2\sigma_v^2}{(n_i\sigma_v^2 + \sigma_e^2)^3}
\end{aligned}
$$

Thus:

$$\begin{aligned}
\partial_v b_i^t V_i \partial_v b_i &= \frac{\gamma_i^3}{n_i^2 \sigma_v^2} \cdot \frac{\sigma_e^4}{\sigma_v^4} \\
\partial_e b_i^t V_i \partial_e b_i &= \frac{\gamma_i^3}{n_i^2 \sigma_v^2} \\
\partial_e b_i^t V_i \partial_v b_i &= -\frac{\gamma_i^3}{n_i^2 \sigma_v^2} \cdot \frac{\sigma_e^2}{\sigma_v^2} \\
\frac{\partial b_i^t}{\partial \delta} V_i \frac{\partial b_i}{\partial \delta} &= \frac{\gamma_i^3}{n_i^2 \sigma_v^2} \begin{pmatrix} \sigma_e^4/\sigma_v^4 & -\sigma_e^2/\sigma_v^2 \\ -\sigma_e^2/\sigma_v^2 & 1 \end{pmatrix}
\end{aligned}$$

And dropping i and adapting the simplified notation $\partial_v b_i = b_v$:

$$\begin{aligned}
\frac{1}{2} \nabla_v b^t R b &= b_v^t R b \\
\frac{1}{2} \nabla_e b^t R b &= b_e^t R b + b^t b/2 \\
\frac{1}{2} \nabla_{vv} b^t R b &= b_{vv}^t R b + b_v^t R b_v \\
\frac{1}{2} \nabla_{ee} b^t R b &= b_{ee}^t R b + b_e^t R b_e + 2 b_e^t b \\
\frac{1}{2} \nabla_{ev} b^t R b &= b_{ev}^t R b + b_v^t R b_e + b_v^t b
\end{aligned}$$

$$
\begin{aligned}
b_v^t R b_v &= \frac{n_i \sigma_e^6}{(\dots)^4} \\
b_{vv}^t R b &= -\frac{2n_i^2 \sigma_e^4 \sigma_v^2}{(\dots)^4} \\
\nabla_{vv} L_1 &= \frac{n_i \sigma_e^4}{(\dots)^4} (\sigma_e^2 - 2n_i \sigma_v^2) \\
b_{ee}^t R b &= \frac{2n_i \sigma_v^4 \sigma_e^2}{(\dots)^4} \\
b_e^t R b_e &= \frac{n_i \sigma_v^4 \sigma_e^2}{(\dots)^4} \\
b_e^t b &= -\frac{n_i \sigma_v^4}{(\dots)^4} \\
\nabla_{ee} L_1 &= \frac{(n_i \sigma_v^4)}{(\dots)^4} (\sigma_e^2 - 2n_i \sigma_v^2) \\
b_{ev}^t R b &= \frac{n_i \sigma_e^2 \sigma_v^2 (n_i \sigma_v^2 - \sigma_e^2)}{(\dots)^4} \\
b_v^t R b_e &= -\frac{n_i \sigma_e^4 \sigma_v^2}{(\dots)^4} \\
b_v^t b &= \frac{n_i \sigma_e^2 \sigma_v^2}{(\dots)^3} \\
\nabla_{ev} L_1 &= \frac{n_i \sigma_v^2 \sigma_e^2}{(\dots)^4} (2n_i \sigma_v^2 - \sigma_e^2)
\end{aligned}
$$

The information matrix under i, iii) is given as

$$
\begin{aligned}
a &= n_i \sigma_v^2 + \sigma_e^2 \\
I_{ve} &= \frac{n}{2a^2} \\
I_{vv} &= \frac{n n_i}{2a^2} \\
I_{ee} &= \frac{m((n_i - 1)\sigma_e^{-4} + a^{-2})}{2} = \frac{n}{2n_i a^2} + \frac{n - m}{2\sigma_e^4} \\
I &= \frac{n}{2a^2} \begin{pmatrix} n_i & 1 \\ 1 & 1/n_i + (n_i - 1)/n_i a^2 / \sigma_e^4 \end{pmatrix}
\end{aligned}
$$

Further, if $n_i \neq 1$,

$$\frac{\partial}{\partial \sigma_v^2} I = -\frac{n}{a^3} \begin{pmatrix} n_i^2 & n_i \\ n_i & 1 \end{pmatrix} \qquad \frac{\partial}{\partial \sigma_e^2} I = -\frac{n}{a^3} \begin{pmatrix} n_i & 1 \\ 1 & \frac{n_i-1}{ni}\frac{a^3}{\sigma_e^6} - \frac{1}{ni} \end{pmatrix}$$

Eventually,

$$\begin{aligned} \bar{V} &= \frac{2a^2}{n} \cdot \frac{1}{(n_i-1)a^2/\sigma_e^4} \begin{pmatrix} 1/n_i + (n_i-1)/n_i a^2/\sigma_e^4 & -1 \\ -1 & n_i \end{pmatrix} \\ &= \frac{2\sigma_e^4}{n(n_i-1)} \begin{pmatrix} 1/n_i + (n_i-1)/n_i a^2/\sigma_e^4 & -1 \\ -1 & n_i \end{pmatrix} \end{aligned}$$

Thus, noting that $a^2/\sigma_e^4 = n_i^2\sigma_v^4/\sigma_e^4/\gamma_i^2$ and $\sigma_e^2/n_i/\sigma_v^2 + 1 = 1/\ \gamma_i$,

$$\begin{aligned} tr\left(\frac{\partial b_i^t}{\partial \delta} V_i \frac{\partial b_i}{\partial \delta} \bar{V}\right) &= \frac{2\sigma_e^4}{n(n_i-1)} \cdot \frac{\gamma_i^3}{n_i^2\sigma_v^2}\Big((1+(n_i-1)a^2/\sigma_e^4)/n_i \cdot \sigma_e^4/\sigma_v^4 + 2\sigma_e^2/\sigma_v^2 + n_i\Big) \\ &= \frac{2\sigma_e^4}{n(n_i-1)} \cdot \frac{\gamma_i^3}{n_i^2\sigma_v^2}\left(\frac{(n_i-1)n_i}{\gamma_i^2} + \frac{1}{n_i}\sigma_e^4/\sigma_v^4 + 2\sigma_e^2/\sigma_v^2 + n_i\right) \\ &= \frac{2\sigma_e^4}{n(n_i-1)} \cdot \frac{\gamma_i^3}{n_i\sigma_v^2}\left(\frac{n_i-1}{\gamma_i^2} + \sigma_e^4/n_i^2/\sigma_v^4 + 2\sigma_e^2/n_i/\sigma_v^2 + 1\right) \\ &= \frac{2\sigma_e^4}{n(n_i-1)} \cdot \frac{\gamma_i^3}{n_i\sigma_v^2}\left(\frac{n_i-1}{\gamma_i^2} + 1/\gamma_i^2\right) \\ &= 2\frac{\sigma_e^4}{\sigma_v^2}\frac{\gamma_i}{n(n_i-1)} \end{aligned}$$

For term L_5,

$$tr\left\{\nabla^2 L_1 \bar{V}\right\} = \frac{1}{n}\frac{n_i}{n_i-1}\frac{2\sigma_e^4(2n_i\sigma_v^2 - \sigma_e^2)}{a^2}$$

For the calculation of the Woodbury formula it is helpful to know that, setting $a =$

diagonal and $D = (d_1^t, \ldots, d_m^t)$,

$$X^t V X - DD'/a = \frac{(1-\gamma_i)^2}{a} \begin{pmatrix} \frac{m\gamma_i}{\sigma_v^2} & \frac{\gamma_i}{\sigma_v^2}\bar{x} \\ \frac{\gamma_i}{\sigma_v^2}\bar{x} & (x^t x - n_i \gamma_i \sum_{k=1}^m \bar{x}_k^2)/\sigma_e^2 \end{pmatrix} \begin{pmatrix} m & \bar{x} \\ \bar{x} & \sum_{i=1}^m \bar{x}_i^2 \end{pmatrix}$$

Regarding the conditional case,

$$b_i^t R_i b_i = \sigma_e^2 \frac{\gamma_i^2}{n_i}$$

Inversion with Woodbury: For an estimate Σ with leading diagonal entries α,

$$\Sigma^{-1} = (I_m - D(\alpha X^t V^{-1} X + D^t D)^{-1} D^t)/\alpha$$