UCDAVIS
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Uniformly Valid Inference Based on the Lasso in Linear Mixed Models

joint work with U. Schneider and T. Krivobokova

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Model Selection and Inference

General Linear Model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}_0 + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \sim \mathcal{N}_n\{\mathbf{0}_n, \mathbf{V}(\boldsymbol{\theta}_0)\}$$

for $\boldsymbol{\beta}_0 \in \mathbb{R}^p$, $\boldsymbol{\theta}_0 \in \Theta \subset \mathbb{R}^r$ unknown, $n, p, r \in \mathbb{N}$, $n > p$

Goal: Inference on $\boldsymbol{\beta}_0$ after model selection.

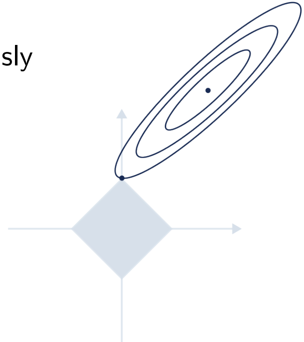
- Model selection is data driven
- Selection and estimation are two sources of uncertainty
- Naive inference after model selection is not valid

Least Absolute Shrinkage and Selection Operator (Lasso)

for $\lambda_j > 0$, $j = 1, \ldots, p$ and REML estimator $\widehat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}_0$,

$$\widehat{\boldsymbol{\beta}}_L = \underset{\boldsymbol{\beta} \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \ell(\boldsymbol{\beta}, \widehat{\boldsymbol{\theta}}) + 2 \sum_{j=1}^{p} \lambda_j \, |\beta_j| \right\}$$

- Selection and estimation simultaneously
- Lasso not uniformly consistent
  Pötscher and Leeb [2009]



Related Work

In mixed models:

- Penalization of $\boldsymbol{\theta}_0$: Bondell et al. [2010], Ibrahim et al. [2011]
- Penalization of the random effects: Peng and Lu [2012]
- Many instances of penalized $\boldsymbol{\beta}_0$ in mixed models.

In linear models:

- Uniformly valid inference based on the Lasso if $n > p$
  Ewald and Schneider [2018]

$$\inf_{\boldsymbol{\beta}_0} \mathrm{P}_{\boldsymbol{\beta}_0}(\boldsymbol{\beta}_0 \in S) = \min_{\mathbf{d} \in \{-1,1\}^p} \mathrm{P}(\boldsymbol{\beta}_0 \in S_{\mathbf{d}}) = 1 - \alpha$$

Lasso

for $\lambda_j > 0$, $j = 1, \ldots, p$ and REML estimator $\widehat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}_0$,

$$\widehat{\boldsymbol{\beta}}_L = \underset{\boldsymbol{\beta} \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \ell(\boldsymbol{\beta}, \widehat{\boldsymbol{\theta}}) + 2 \sum_{j=1}^{p} \lambda_j \, |\beta_j| \right\}$$

Goal: Uniformly valid inference on $\boldsymbol{\beta}_0$ based on the Lasso.

- Find $S \subset \mathbb{R}^p$ such that

$$\inf_{\boldsymbol{\beta}_0, \boldsymbol{\theta}_0} \mathrm{P}_{\boldsymbol{\beta}_0, \boldsymbol{\theta}_0}(\widehat{\boldsymbol{\beta}}_L \in S) \approx 1 - \alpha$$

- Idea: Separate estimation for $\boldsymbol{\theta}_0$ and $\boldsymbol{\beta}_0$
- Requires that $\widehat{\boldsymbol{\theta}}$ is uniformly bounded

Restricted Maximum Likelihood (REML) Methodology

Estimate $\boldsymbol{\theta}_0$ from $\mathbf{A}^t\mathbf{y}$ instead of $\mathbf{y}$, for $\mathbf{A} \in \mathbb{R}^{n\times(n-p)}$ such that $\mathbf{A}^t\mathbf{X} = \mathbf{0}_{n-p}$. Then,

$$\widehat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\Theta}{\operatorname{argmax}}\, \ell_R(\boldsymbol{\theta}),$$

where $\mathbf{P}(\boldsymbol{\theta}) = \mathbf{A}\{\mathbf{A}^t\mathbf{V}(\boldsymbol{\theta})\mathbf{A}\}^{-1}\mathbf{A}^t$ and

$$\ell_R(\boldsymbol{\theta}) = \underset{\boldsymbol{\theta}\in\Theta}{\operatorname{argmax}} -\frac{1}{2}\ln|\mathbf{V}(\boldsymbol{\theta})| - \frac{1}{2}\ln|\mathbf{X}^t\mathbf{V}(\boldsymbol{\theta})^{-1}\mathbf{X}| - \frac{1}{2}\mathbf{y}^t\mathbf{P}(\boldsymbol{\theta})\mathbf{y}$$

- Requires low-dimensional setting $n > p$
- $\widehat{\boldsymbol{\theta}}$ is independent of $\boldsymbol{\beta}_0$
- *Uniform* consistency has to be proved
- Maximum not unique, hence consider *Cramér* consistency

Theorem 1: Cramér Consistency of REML Estimator

Under some regularity conditions, set

$$\nu_k(\boldsymbol{\theta}_0) = \frac{\mathrm{tr}\left\{\mathbf{V}(\boldsymbol{\theta}_0)^{-1}\frac{\partial \mathbf{V}}{\partial \theta_k}\right\}}{\sqrt{\mathrm{rk}\left(\frac{\partial \mathbf{V}}{\partial \theta_k}\right)}}, \qquad k = 1, \ldots, r.$$

There exists a sequence $\widehat{\boldsymbol{\theta}}$ *of local maximizers of* ℓ_R*, such that*

$$\nu_k(\boldsymbol{\theta}_0)\left|\widehat{\theta}_k - \theta_{0,k}\right| = O_P(1)$$

uniformly over $\boldsymbol{\theta}_0 \in \Theta$*.*

- If $\boldsymbol{\theta}_0$ were fixed, $\nu_k(\boldsymbol{\theta}_0) \equiv \sqrt{n}$
- Note that $-\mathrm{E}_{\boldsymbol{\theta}_0}\{\partial^2 \ell_R(\boldsymbol{\theta})/\partial \theta_k^2|_{\boldsymbol{\theta}_0}\} = O\{\nu_k(\boldsymbol{\theta}_0)^2\}$

Theorem 2: Uniformly Valid Confidence Sets

There exists a sequence $\widehat{\boldsymbol{\theta}}$ of local maximizers of ℓ_R such that for $E(\widehat{\mathbf{C}},\widehat{\tau}) = \{\mathbf{z} \in \mathbb{R}^p \mid \mathbf{z}^t\widehat{\mathbf{C}}\mathbf{z} \leq \widehat{\tau}\}$, with $\widehat{\mathbf{C}} = n^{-1}\mathbf{X}^t\mathbf{V}(\widehat{\theta})^{-1}\mathbf{X}$ and

$$\widehat{\tau} = \max_{\mathbf{d}\in\{-1,1\}^p} \chi^2_{p,1-\alpha}\left\{n^{-1}\left\|\widehat{\mathbf{C}}^{-1/2} diag(\lambda_1,\ldots,\lambda_p)\mathbf{d}\right\|^2\right\},$$

a quantile of a non-central χ^2_p-distribution, it holds that

$$\inf_{\substack{\boldsymbol{\beta}_0\in\mathbb{R}^p\\ \boldsymbol{\theta}_0\in\Theta}} \mathrm{P}_{\boldsymbol{\beta}_0,\boldsymbol{\theta}_0}\left\{\sqrt{n}\left(\widehat{\boldsymbol{\beta}}_L - \boldsymbol{\beta}_0\right) \in E\left(\widehat{\mathbf{C}},\widehat{\tau}\right)\right\} = 1-\alpha+O\left(\frac{1}{\sqrt{n}}\right).$$

- Idea:

$$\inf_{\boldsymbol{\beta}_0,\boldsymbol{\theta}_0} \mathrm{P}_{\boldsymbol{\beta}_0,\boldsymbol{\theta}_0}(\boldsymbol{\beta}_0 \in S) = \inf_{\boldsymbol{\theta}_0}\min_{\mathbf{d}\in\{-1,1\}^p} \mathrm{P}_{\boldsymbol{\beta}_0,\boldsymbol{\theta}_0}(\boldsymbol{\beta}_0 \in S_{\mathbf{d}})$$

- Confidence set: $M_L = \left\{\boldsymbol{\beta} \in \mathbb{R}^p\colon \quad n\left\|\widehat{\mathbf{C}}^{1/2}\left(\widehat{\boldsymbol{\beta}}_L - \boldsymbol{\beta}\right)\right\|^2 \leq \widehat{\tau}\right\}$

Simulation Design

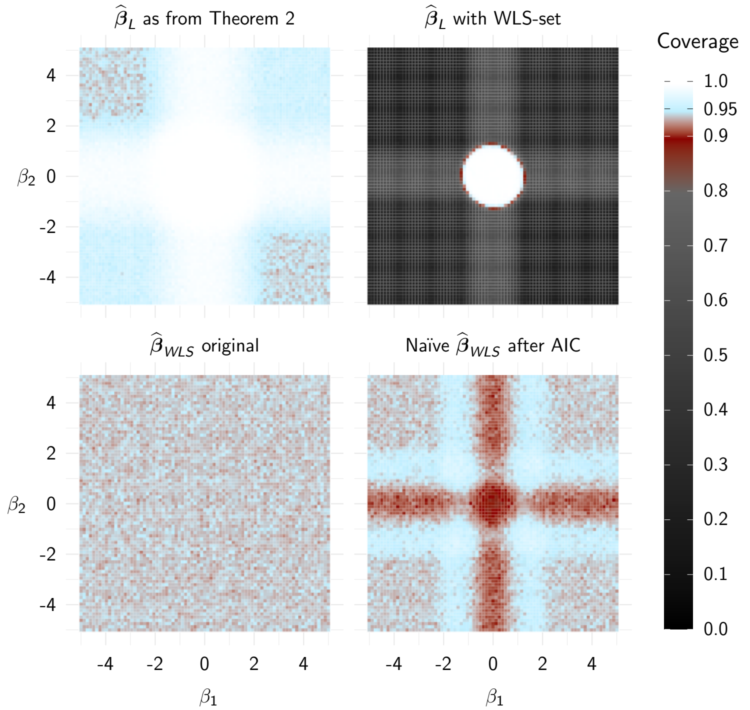
Random Intercept Model

For $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$, let

$$y_{ij} = \mathbf{x}_{ij}^t \boldsymbol{\beta}_0 + v_i + u_{ij}, \qquad u_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_u^2), \quad v_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_v^2)$$

and $m = 20$, $n = 400$, $n_i = 20$, $\sigma_u = \sigma_v = 4$, $\lambda_i = n^{1/2}/2$, $p = 2$.

- For each $\boldsymbol{\beta}_0 \in [-4, 4]^2$, perform $3,000$ replications
- For each replications, check if $\boldsymbol{\beta}_0 \in M_L$



$\widehat{\beta}_L$ as from Theorem 2
$\widehat{\beta}_L$ with WLS-set
$\widehat{\beta}_{WLS}$ original
Naïve $\widehat{\beta}_{WLS}$ after AIC
Coverage
1.0
0.95
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
β_2
β_1
4
2
0
-2
-4

Real Data Example

Acidity in US Lakes Opsomer et al. [2009]

$$\mathbf{y} = \mathbf{1}_n\beta_1 + \mathbf{x}\beta_2 + \mathbf{Z}\mathbf{v} + \mathbf{D}\mathbf{u} + \boldsymbol{\epsilon},$$

$$\mathbf{v} \sim \mathcal{N}(\mathbf{0}_m, \sigma_v^2\mathbf{I}_m), \quad \mathbf{u} \sim \mathcal{N}(\mathbf{0}_K, \sigma_u^2\mathbf{I}_K) \quad \text{and} \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}_n, \sigma_e^2\mathbf{I}_n)$$

- WLS: $\widehat{\boldsymbol{\beta}}_{WLS} = (483, -1.20)^t$; $sd(\widehat{\boldsymbol{\beta}}_{WLS}) = (1718, 0.14)^t$
- Lasso: $\widehat{\boldsymbol{\beta}}_L = (0, -1.12)^t$
- Lasso-based confidence set $\sim$ 200 times larger, but

$$M_L(\beta_2) = [-1.49, -0.75] \qquad M_{WLS}(\beta_2) = [-1.57, -0.83]$$

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Conclusions

- REML estimator for $\boldsymbol{\theta}_0$ is uniformly Cramér consistent
- Inference on $\boldsymbol{\beta}_0$ based on $\widehat{\boldsymbol{\beta}}_L$ uniformly valid over $\mathbb{R}^p \times \Theta$
- Resulting coverage probability depends on $\boldsymbol{\beta}_0$

- If interest in inference only, do no selection
- Lasso-based confidence sets are large
- Valid inference (for $n > p$) if nuisance and target parameters can be separated

Regularity Conditions

(A) $\boldsymbol{\theta}_0 \in \Theta = \left\{\boldsymbol{\theta} \in \mathbb{R}^r_{>0} \mid \max(\boldsymbol{\theta})/\min(\boldsymbol{\theta}) \le c\right\}$; $\infty > c$ const.

(B) $\mathbf{V}(\boldsymbol{\theta}_0) = \sum_{k=1}^r \theta_{0,k}\mathbf{H}_k$; $\mathbf{H}_k \ge 0$, $k = 1\ldots, r-1$ and $\mathbf{H}_r > 0$

(C) $\mathrm{rk}(\mathbf{X}) = p < n$

(D) For constants $0 < \underline{\omega} \le \overline{\omega} < \infty$ it holds

$$\underline{\omega} \le \frac{\mathrm{tr}\left\{\mathbf{V}(\mathbf{1}_r)^{-1}\mathbf{H}_k\right\}}{\mathrm{rk}(\mathbf{H}_k)} \le \overline{\omega},$$

where $\mathbf{1}_r = (1, \ldots, 1)' \in \mathbb{R}^r$

(E) $m = \min\{\mathrm{rk}(\mathbf{H}_1), \ldots, \mathrm{rk}(\mathbf{H}_r)\} \to \infty$