Differential Calculus

UNIT 1 DIFFERENTIAL CALCULUS

Structure

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1.0 INTRODUCTION

In this Unit, we shall define the concept of limit, continuity and differentiability.

1.1 OBJECTIVES

After studying this unit, you should be able to:

- define limit of a function;
- define continuity of a function; and
- define derivative of a function.

1.2 LIMITS AND CONTINUITY

We start by defining a function. Let A and B be two non empty sets. A function f from the set A to the set B is a rule that assigns to each element x of A a unique element y of B.

We call y the image of x under f and denote it by f(x). The domain of f is the set A, and the co-domain of f is the set B. The range of f consists of all images of elements in A. We shall only work with functions whose domains and co-domains are subsets of real numbers.

Given functions f and g, their sum f + g, difference f - g, product f. g and quotient f/g are defined by

$$(f + g)(x) = f(x) + g(x)$$

 $(f - g)(x) = f(x) - g(x)$

$$(f. g)(x) = f(x) g(x)$$

and
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$





For the functions f + g, f - g, f. g, the domain is defined to be intersections of the domains of f and g, and for f / g the domain is the intersection excluding the points where g(x) = 0.

The composition of the function f with function g, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set of all x in the domain of g such that g(x) is in the domain of f.

Limit of a Function

We now discuss intuitively what we mean by the limit of a function. Suppose a function f is defined on an open interval (α, β) except possibly at the point $a \in (\alpha, \beta)$ we say that

$$f(x) \rightarrow L$$
 as $x \rightarrow a$

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(read f(x) approaches L as x approaches a), if f(x) takes values very, very close to L, as x takes values very, very close to a, and if the difference between f(x) and L can be made as small as we wish by taking x sufficiently close to but different from a.

As a mathematical short hand for $f(x) \to L$ as $x \to a$, we write

$$\lim_{x \to a} f(x) = L.$$

Example 1: Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$

Solution: Let $f(x) = \frac{x^2 - 9}{x - 3}$. This function is defined for each x except for x = 3. This function is defined for each x except for x = 3. Let us calculate the value of f at x = 3 + h, where $h \ne 0$. We have

$$f(3+h) = \frac{(3+h)^2 - 9}{3+h-3} = \frac{9+6h+h^2-9}{h} = \frac{h(6+h)}{h} = 6+h$$

We now note that as x takes values which are very close to 3, that is, h takes values very close to 0, f(3 + h) takes values which are very close to 6. Also, the difference between f(3 + h) and 6 (which is equal to h) can be made as small as we wish by taking h sufficiently close to zero.

Thus,

$$\lim_{x \to 3} f(x) = 6$$

Properties of Limits

We now state some properties of limit (without proof) and use them to evaluate limits.





Theorem 1 : Let a be a real number and let f(x) = g(x) for all $x \ne a$ in an open interval containing a. If the limit g(x) as $x \to a$ exists, then the limit of f(x) also exists, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

Theorem 2: If c and x are two real numbers and n is a positive integer, then the following properties are true:

$$(1) \quad \lim_{x \to a} c = c$$

$$(2) \quad \lim_{x \to a} x = a$$

$$(3) \quad \lim_{x \to a} x^n = a^n$$

Theorem 3: Let c and a be two real numbers, n a positive integer, and let f and g be two functions whose limit exist as $x \to a$. Then the following results hold:

1.
$$\lim_{x \to a} [c f(x)] = c \left[\lim_{x \to a} f(x) \right]$$

$$2 \quad \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

3.
$$\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)] [\lim_{x \to a} g(x)]$$

4.
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \text{provided } \lim_{x \to a} g(x) \neq 0,$$

5.
$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$$

6. If
$$\lim_{x \to a} f(x) = f(a)$$
, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$

Example 2: Evaluate $\lim_{x\to 3} (4x^2 + 7)$

Solution:
$$\lim_{x \to 3} (4x^2 + 7) = \lim_{x \to 3} 4x^2 + \lim_{x \to 3} 7$$

= $4\lim_{x \to 3} x^2 + \lim_{x \to 2} 7$
= $4(3)^2 + 7 = 4 \times 9 + 7$
= 43

Note: If p(x) is a polynomial, then $\lim_{x \to a} p(x) = p(a)$. If q(x) is also a polynomial and $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$









Example 3: Evaluate the following limits:



(i)
$$\lim_{x \to 2} [(x-1)2 + 6]$$

(i)
$$\lim_{x \to 2} [(x-1)^2 + 6]$$
 (ii) $\lim_{x \to 0} \frac{ax + b}{cx + d} (d \neq 0)$

(iii)
$$\lim_{x \to 2} \frac{x^2 + 5x + 7}{x^2 + 8}$$

(iv)
$$\lim_{x \to -1} \sqrt{x + 17}$$

Solution: (i) $\lim_{x\to 2} [(x-1)^2+6] = (2-1)^2+6 = 1+6=7$

(ii) Since $\lim_{x\to 0} cx + d = d \neq 0$,

$$\lim_{x \to 0} \frac{ax + b}{cx + d} = \frac{a(0) + b}{c(0) + d} = \frac{b}{d}$$

 $\lim_{x \to 0} \frac{ax + b}{cx + d} = \frac{a(0) + b}{c(0) + d} = \frac{b}{d}$ (iii) Since $\lim_{x \to 3} (x^2 + 8) = 3^2 + 8 = 17 \neq 0$,

$$\therefore \lim_{x \to 3} \frac{x^2 + 5x + 7}{x^2 + 8} = \frac{3^2 + 5(3) + 7}{3^2 + 8} = \frac{31}{17}$$

(iv) Since $\lim_{x\to -1} x + 17 = -1 + 17 = 16$, we have $\lim_{x \to -1} \sqrt{x + 17} = \sqrt{16} = 4$

Example 4: Evaluate the following limits.



(i)
$$\lim_{x \to 5} \frac{x^2 - 7x + 10}{x - 5}$$

(ii)
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

(iii)
$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$$

(i)
$$\lim_{x \to 5} \frac{x^2 - 7x + 10}{x - 5}$$
 (ii) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ (iv) $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$

Solution: (i) Here, $\lim_{x\to 5} (x-5) = 0$. So direct substitution will not work.

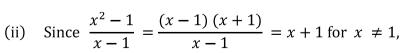
We can proceed by cancelling the common factor (x - 5) in numerator and denominator and using theorem 1, as shown below:

$$\frac{\lim_{x \to 5} \frac{x^{2} - 1}{x}}{1}$$
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$$\lim_{x \to 5} \frac{x^2 - 7x + 10}{x - 5} = \lim_{x \to 5} \frac{(x - 2)(x - 5)}{(x - 5)}$$

$$= \lim_{x \to 5} (x - 2), \text{ for } x \neq 5$$

$$= 5 - 2 = 3$$



therefore by theorem 1, we have

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2.$$







$$\frac{\sqrt{x+2}-\sqrt{2}}{x} = \left(\frac{\sqrt{x+2}-\sqrt{2}}{x}\right) \left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)$$
$$= \left(\frac{x+2-2}{\sqrt{x+2}+\sqrt{2}}\right) = \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$$
$$= \frac{1}{\sqrt{x+2}+\sqrt{2}}$$

Therefore, by Theorem 1, we have

$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \lim_{x \to 0} \frac{1}{\sqrt{0+2} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(iv) For $x \neq 0$, we have

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right) \left(\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$$
$$= \frac{2x}{x\sqrt{1+x} - \sqrt{1-x}} = \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

 \therefore by theorem 1, we have

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = \frac{2}{2} = 1$$

An important limit

Example 5: Prove that $\lim_{x\to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$ where *n* is positive integer

Solution: We know that

$$x^{n} - a^{n} = (x - a) (x^{n-1} + x^{n-2}a + x^{n-3}a^{2} + \dots xa^{n-2} + a^{n-1})$$

Therefore, for
$$x \neq a$$
, we get
$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots xa^{n-2} + a^{n-1}$$

Hence by Theorem 1, we get

$$\lim_{x \to a} \frac{x^{n} - a^{n}}{x - a} = \lim_{x \to a} \left(x^{n-1} + x^{n-2}a + x^{n-3}a^{2} + \dots x a^{n-2} + a^{n-1} \right)$$
$$= a^{n-1} + a^{n-2}a + a^{n-3}a^{2} + \dots a a^{n-2} + a^{n-1}$$





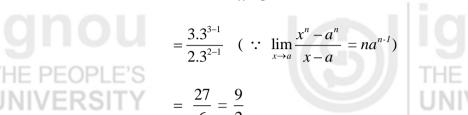


Note: The above limit is valid for negative integer n, and in general for any rational index n provided a > 0. The above formula can be directly used to evaluate limits.

Example 6: Evaluate
$$\lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9}$$

Solution:
$$\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9} = \lim_{x \to 3} \frac{x^3 - 3^3}{x^2 - 3^2}$$

$$= \lim_{x \to 3} \frac{\frac{x^3 - 3^3}{x - 3}}{\frac{x^2 - 3^2}{x - 3}}$$



$$=\frac{27}{6}=\frac{9}{2}$$

One-sided Limits

Definition : Let f be a function defiend on an open interval (a-h, a+h) (h>0). A number L is said to be the **Left Hand Limit** (**L.H.L.**) of f at a if f(x) takes values very close to L as x takes values very close to a on the left of a ($x \neq a$). We then write

$$\lim_{x \to a} f(x) = L$$

We similarly define L to be the **Right Hand Limit** if f(x) takes values close to L as x takes values close to a on the right of a and write $\lim_{x \to a+} f(x) = L$

Note that $\lim_{x\to a} f(x)$ exists and is equal to L if and only if $\lim_{x\to a^{-}} f(x)$ and $\lim_{x\to a+} f(x)$ both exist and are equal to L.

$$\lim_{x \to a^{-}} f(x) = L = \lim_{x \to a^{+}} f(x) = \lim_{x \to a} f(x)$$

Example 7: Show that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution: Let
$$f(x) = \frac{|x|}{x}$$
, $x \neq 0$.

Since
$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\therefore f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

So,
$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} (1) = 1$$
 and

$$\lim_{x \to 0^{-}} f(x) = \lim_{x \to 0} (-1) = -1$$

Thus $\lim_{x\to 0} f(x)$ does not exist.







Definition: A function f is said to be **continuous** at x = a if the following three conditions are met:

- (1) f(a) is defined
- (2) $\lim_{x \to a} f(x)$ exists
- $(3) \lim_{x \to a} f(x) = f(a)$

Example 8: Show that f(x) = |x| is continuous at x = 0

Solution: Recall that

$$f(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

To show that f is continuous at x = 0, it is sufficient to show that

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} (x) = f(0) \text{ and}$$

We have

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0+} f(0-h) = \lim_{h \to 0+} f(-h)$$

$$=\lim_{h\to 0+}-\left(-h\right)$$

$$= \lim_{h \to 0+} h = 0$$

and
$$\lim_{x\to 0+} f(x) = \lim_{h\to 0+} f(0+h) = \lim_{h\to 0+} f(h)$$

$$=\lim_{h\to 0+}(h)=0.$$

Thus,
$$\lim_{h \to 0^{-}} f(x) = \lim_{h \to 0^{+}} f(x) = 0$$

Also,
$$f(0) = 0$$

Therefore,
$$\lim_{x\to 0+} f(x) = 0 = f(0)$$

Hence, f is continuous at x = 0.

Example 9: Check the continuity of f at the indicated point

(i)
$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$
 at $x = 0$





(ii)
$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$
 at $x = 1$

Solution: (i) We have already seen in Example 7 that $\lim \frac{|x|}{|x|}$ Hence, f is not continous at x = 0

(ii) Here,
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} (x+1) \angle$$

$$= 2$$
Also, $f(1) = 2$

$$\lim_{x \to 1} f(x) = f(1)$$

Hence, f is continuous at x = 1.

Definition: A function is said to be **continuous on an open interval** (a,b) if it is continuous at each point of the interval. A function which is continuous on the entire real line $(-\infty,\infty)$ is said to be **everywhere continuous.**

Algebra of Continuous Functions

Theorum : Let c be a real number and let f and g be continuous at x = a. Then the functions cf, f+g, f-g, fg are also continuous at x=a. The functions $\frac{1}{a}$ and $\frac{f}{a}$ are continuous provided $g(a) \neq 0$.

Remark: It must be noted that polynomial functions, rational functions, trigonometric functions, exponential and logarithmic function are continuous in their domains.

Example 10: Find the points of discontinuity of the following functions:

(i)
$$f(x) = \begin{cases} x^2 & \text{if } x > 0\\ x + 3 & x \le 0 \end{cases}$$
(ii)
$$f(x) = \begin{cases} x & \text{if } x \ne 0\\ 1 & x = 0 \end{cases}$$

(ii)
$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & x = 0 \end{cases}$$

Solution: (i) Since x^2 and x + 3 are polynomial functions, and polynomial functions are continuous at each point in R, f is continuous at each $x \in$ R except possibly at x = 0. For x = 0, we have

$$\lim_{x \to 0^{-}} f(x) = \lim_{h \to 0^{+}} f(0 - h) = \lim_{h \to 0^{+}} (-h + 3) = 0 + 3 = 3$$

and
$$\lim_{x \to 0+} f(x) = \lim_{h \to 0+} f(0+h) = \lim_{h \to 0+} f(h) = \lim_{h \to 0+} h^2 = 0$$
.

Therefore, since $\lim_{x\to 0^-} f(x) f \lim_{x\to 0^+} f(x)$, f is not continuous at x = 0

Since, polynomial functions are continuous at each point of Differential Calculus (ii) R, f is also continuous at each $x \in R$ except possibly at x = 0.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \neq f(0).$$

Thus,
$$f$$
 is not continuous at $x = 0$

Check Your Progress – 1

1. Evaluate the following limits:

(i)
$$\lim_{x \to 2} (3x^3 + 2x + 1)$$

(ii)
$$\lim_{x \to 2} \frac{x-2}{x+2}$$

(iii)
$$\lim_{x \to 2} \frac{x^2 - 5x + 2}{x - 1}$$
 (iv) $\lim_{x \to 2} \sqrt[3]{3x^2 - 19}$

(iv)
$$\lim_{x \to 2} \sqrt[3]{3x^2 - 19}$$

2. Evaluate the following limits:

(i)
$$\lim_{x \to 2} \frac{x^2 - 4}{x + 2}$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x + 2} \quad \text{(ii)} \qquad \lim_{x \to 5} \frac{\sqrt{x - 1} - 2}{x - 5}$$

3. Evaluate the following limits:

(i)
$$\lim_{x \to a} \frac{x^{7/6} - a^{7/6}}{x^{3/5} - a^{3/5}} \quad (a > 0)$$

(ii)
$$\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n}$$
 (*m*, *n* are rational numbers, $a > 0$)

Check the continuity of f at the indicated point where

$$f(x) = \begin{cases} 2 - x & \text{if } x < 0 \\ x + x & \text{if } x \ge 0 \end{cases} \text{ at } x = 0$$

5. For what value of constant k the function f is continuous at x = 5?

$$f(x) = \begin{cases} \frac{x^2 - 25}{x - 5} & \text{if } x \neq 5\\ k & \text{if } x = 5 \end{cases}$$

1.3 DERIVATIVE OF A FUNCTION

Definition: A function f is said to be **differentiable** at x if and only if

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists. If this limit exists, it is called the derivative of f at x and is denoted by

$$f^{1}(x)$$
 or $\frac{dy}{dx}$.

i.e.,
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f^{I}(x)$$

A function is said to be **differentiable on an open interval I** if it is differentiable at each point of I.

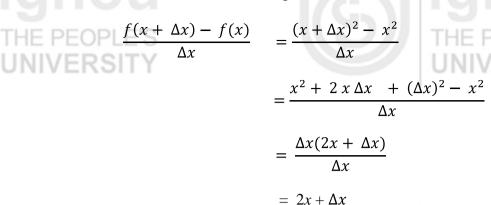






Example 11: Differentiate $f(x) = x^2$ by using the definition.

Solution: We first find the difference quotient as follows:



It follows that

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Remark: It can be easily proved that if f is differentiable at a point x, then f is continuous at x. Thus, if f is not continuous at x, then f is not differentiable at x.

Some differentiation Rules

We now develop several "rules" that allow us to calculate derivatives without the direct use of limit definition.

Theorem 1 (Constant Rule). The derivative of a constant is zero. That is,

$$\frac{d}{dx}[c] = 0$$

where c is a real number.

Proof: Let f(x) = c then

$$\frac{d}{dx}[c] = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

Theorem 2: (Scalar Multiple Rule). If f is differentiable function and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof: By definiton

$$\frac{d}{dx}[cf(x)] = \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} c \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] = cf'(x)$$





Theorem 3 : (Sum and Difference Rule). If f and g are two differentiable **Differential Calculus** functions, then

Sum Rule
$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

Difference Rule
$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Proof: We have

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) + g(x + \Delta x) - [f(x) + g(x)]}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

We can similarly prove the difference rule.

Theorum 4 : (**Product Rule**). If *f* and *g* are two differentiable functions, then

$$\frac{d}{dx}[f(x) g(x)] = f(x) + g'(x) + f'(x) + g(x)$$

Proof: We have
$$\frac{d}{dx}[f(x) g(x)] = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) g(x + \Delta x) - f(x) g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) (g(x + \Delta x) - f(x + \Delta x) g(x) + f(x + \Delta x) g(x) - f(x) g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) (g(x + \Delta x) - g(x))}{\Delta x} + g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= \left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x)}{\Delta x}\right] \left[\lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] + g(x) \left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

(using the product and scalar multiple rules of limits). Now, since f is differentiable at x, it is also continuous at x.

$$\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$$

Thus
$$\frac{d}{dx}[f(x) g(x)] = f(x)g'(x) + g(x)f'(x)$$



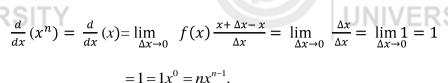


Theorem 5 : (Power Rule) If n is a positive integer, then



$$\frac{d}{dx}(x^n) = nx^{n-1}$$

For n = 1, we have



If n > 1, then the binomial expansion produces

$$\frac{d}{dx}(x^{n}) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^{n} - x^{n}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{C_{0 x^{n} +} C_{1 x^{n-1} +} C_{2 x^{n-2} (\Delta x)^{2} + \dots } C_{n (\Delta x)^{n} - x^{n}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \dots + (\Delta x)^{n-1}]$$

$$= nx^{n-1}.$$

Theorem 6: (Reciprocal Rule). If f is differentiable function such that $f(x) \neq 0$, then

$$\frac{d}{dx} \left[\frac{1}{f(x)} \right] = \frac{-f'(x)}{[f(x)]^2}$$

$$\mathbf{Proof} \quad \frac{d}{dx} \left[\frac{1}{f(x)} \right] = \lim_{\Delta x \to 0} \quad \frac{1}{\Delta x} \quad \left[\frac{1}{f(x + \Delta x)} - \frac{1}{f(x)} \right]$$

$$= \lim_{\Delta x \to 0} \quad \left[\frac{f(x) - f(x + \Delta x)}{f(x + \Delta x)f(x)} \right]$$

$$= \lim_{\Delta x \to 0} \left[-\left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right) \right] \left[\left(\frac{1}{f(x + \Delta x)f(x)} \right) \right]$$

$$= -f'(x) \cdot \frac{1}{f(x)f(x)} \quad (\because \lim_{\Delta x \to 0} (f(x + \Delta x)) = f(x)$$
as f being diff. at x is continuous at x)

$$=\frac{-f'(x)}{[f(x)]^2}$$

Theorem 7 : (Quotient Rule) : If f and g are two differentiable function such that $g(x) \neq 0$, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$$









Proof:
$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{d}{dx} \left[f(x) \frac{1}{g(x)} \right]$$

$$= \frac{1}{g(x)} \frac{d}{dx} [f(x)] + f(x) \frac{d}{dx} \left[\frac{1}{g(x)} \right] \text{ [Product Rule]}$$

$$= \frac{1}{g(x)} + f'(x) + f(x) \left[\frac{-g'(x)}{[g(x)]^2} \right]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$



Remark : The power rule can be extended for any integer. Indeed, if n = 0, we have

$$\frac{d}{dx}(x^{n)} = \frac{d}{dx}(1) = 0 = 0x^{-1}$$
 $x \neq 0$,

and if n is a negative integer, then by using reciprocal rule we can prove

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Thus we have

$$\frac{d}{dx}(x^n) = nx^{n-1}$$
, for any integer n .

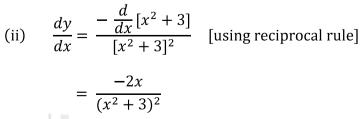
Example 2: Find the derivatives of the following function.

(i)
$$y = 2x^5 - 3x$$
 (ii) $y = \frac{1}{x^2 + 3}$

(iii)
$$y = \frac{x}{x+2}$$
 (iv) $y = \frac{x^2}{x^2-5}$

Solution: (i) $\frac{dy}{dx} = \frac{d}{dx} (2x^5 - 3x)$

$$= 2\frac{d}{dx}(x^5) - 3\frac{d}{dx}(x)$$
$$= 2.(5x^4) - 3.1$$
$$= 10x^4 - 3$$







Calculus
$$(iii) \frac{dy}{dx} = \frac{(x+2)\frac{d}{dx}(x) - x\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2).1 - x.1}{(x+2)^2}$$

$$= \frac{2}{(x+2)^2}$$

(iv) $\frac{dy}{dx} = \frac{(x^2 - 5)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x^2 - 5)}{(x^2 - 5)^2}$ (Quotient Rule)

$$= \frac{(x^2 - 5)(2x^{-}) - x^2(2x)}{(x^2 - 5)^2}$$

$$= \frac{2x^3 - 10x - 2x^3}{(x^2 - 5)^2} = \frac{-10x}{(x^2 - 5)^2}$$

Derivative of Exponential and Logarithmic Functions

To find the derivatives of the natural exponential function e^x and the natural logarithmic function lnx, we need the following limits.

(1)
$$\lim_{\Delta x \to 0} \frac{e^{x} - 1}{x} = 1$$
(2)
$$\lim_{\Delta x \to 0} \frac{\ln(1 + x)}{x} = 1$$



Theorem 8: The derivative of the natural exponential function is given by

$$\frac{d}{dx}(e^x) = e^x \qquad (x \in \mathbb{R})$$

Proof: By definition

$$\frac{d}{dx}(e^{x}) = \lim_{\Delta x \to 0} \frac{e^{x + \Delta x} - e^{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x}(e^{\Delta x} - 1)}{\Delta x}$$

$$= e^{x} \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$
$$= e^{x} (1)$$
$$= e^{x}$$









$$\frac{d}{dx}(lnx) = \frac{1}{x} \ (x > 0)$$

Proof: By definition

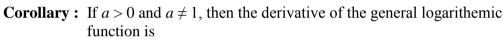
$$\frac{d}{dx}(lnx) = \lim_{\Delta x \to 0} \frac{ln(x + \Delta x) - lnx}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} ln \frac{(x + \Delta x)}{x} (\because lna - lnb) = ln \frac{a}{b})$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} ln \left(1 + \frac{\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x/x}$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x/x} = \frac{1}{x}(1) = \frac{1}{x}$$



$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$$

Proof: We know that

$$\log_a x = (\ln x)(\log_a e)$$

$$\Rightarrow \frac{d}{dx}(\log_a x) = \frac{d}{dx}[(\log x)(\log_a e)]$$

$$= \log_a e \frac{d}{dx}(\log x)$$

$$= \frac{1}{x}(\log_a e)$$

Remark : Similar to the proof of theorem, we can prove that if a > 0, and $a \ne 1$, then the derivative of the general exponential function is

$$\frac{d}{dx}\left(a^{x}\right) = a^{x} lna \quad (x \in R)$$

Example 13: Find the derivative of the following functions.

(i)
$$x^2e^x$$

(ii)
$$\frac{ln}{r}$$

(iii)
$$\frac{e^x}{x^2+3}$$

(iv)
$$5^x lnx$$







Solution: (i) Using the product rule



$$\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x)$$

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$$= 2x e^x + x^2 e^x = (2x + x^2) e^x$$



(i) Using the quotient rule, we have

$$\frac{d}{dx}\frac{lnx}{x} = \frac{x\frac{d}{dx}(lnx) - lnx\frac{d}{dx}(x)}{x^2}$$
$$= \frac{x \cdot \frac{1}{x} - (lnx)(1)}{x^2}$$

 $= \frac{1 - \ln x}{x^2}$

 $= \frac{1}{x^2}$ Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{e^x}{x^2+3}\right) = \frac{(x^2+3)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$=\frac{(x^2+3)(e^x)-e^x(2x)}{(x^2+3)^2}$$

 $=\frac{(x^2-2x+3)e^x}{(x^2+3)^2}$

(iii) Using the product rule, we have

$$\frac{d}{dx}(5^x lnx) = \frac{d}{dx}(5^x) lnx + 5^x \frac{d}{dx}(lnx)$$
$$= (5^x lnx5) + 5^x \left(\frac{1}{x}\right)$$

 $= 5^x (ln5) lnx + \left(\frac{5^x}{x}\right).$



Check Your Progress – 2

1. Find the derivative of each of the following functions.

(i)
$$y = x^5 - 3x^4 + 2x - 1$$
 (ii) $y = \frac{2x - 1}{\pi^2}$

(iii)
$$\frac{3x+5}{2x+7}$$

(iv)
$$y = \frac{x^3 - 4}{x^3}$$



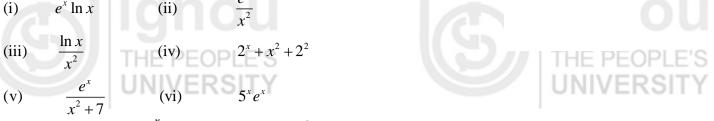
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2. Find the derivative of each of the following functions.

(i)
$$e^x \ln x$$
 (ii) $\frac{e^x}{x^2}$
(iii) $\frac{\ln x}{x^2}$ (iv) $2^x + x^2 + 2^2$
(v) $\frac{e^x}{x^2 + 7}$ (vi) $5^x e^x$



3. Using the limit $\lim_{x\to 0} \frac{a^{x}-1}{x} = \ln a$, prove that $\frac{d}{dx}(a^{x}) = a^{x} \ln a$, where a > 0 and $a \neq 1$.

1.4 THE CHAIN RULE

We now discuss one of the most powerful rules in differential calculus, the chain rule, which deals with composite of functions.

Theorem 10: If y = f(u) is differentiable function of u and u = g(x) is a differentiable function of x, then y = f(g(x)) is differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or, equivalently,
$$\frac{d}{dx}[f(gx)] = f'(g(x))g'(x)$$
.

Proof: Let (F(x) = f(g(x))). We have to show that for x = c,

$$F'(c) = f'(g\bigl((c)\bigr)g'(c).$$

An important consideration in this proof is the behaviour of g as x approaches c. A problem occurs if there are values of x other than c such that g(x) = g(c). However, in this proof we shall assume that $g(x) \neq g(c)$ for values of x other than c. Thus, we can multiply and divide by the same (non–zero) quantity g(x) - g(c). Note that as g is differentiable, it is continuous and it follows that $g(x) \rightarrow g(c)$ as $x \rightarrow c$.

$$F'(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c}$$

$$= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \frac{g(x) - g(c)}{x - c} \right] \quad [\because g(x) \neq g(c)]$$

$$= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \lim_{x \to c} \frac{g(x) - g(c)}{x \to c}$$

$$= f'(g(c)g'(c))$$

Remark: We can extend the chain rule for more than two functions. For example, if F(x) = f[g(h(x))], then

$$F'(c) = f'[g(h(c))]g'(h(c))h'(c).$$



In other words

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$$

Example 14: Find the derivatives of the following functions.

(i)
$$y = (x^2 + 1)^3$$

(ii)
$$y = e^{x^2}$$

(iii)
$$y = ln (2x^2 + e^x)$$

(iv)
$$y = (x + lnx)^2$$

Solution : (i) Put $x^2 + 1 = u$

Then
$$y = u^3$$
 where $u = x^2 + 1$

$$\therefore \frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 2x$$

Then by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2)(2x)$$

$$=6x(x^2+1)^2$$

(ii) In this case we take
$$x^2 = u$$
, so that $y = e^u$

Then by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)(2x) = 2xe^{x^2}.$$

(iii) Take
$$u = 2x^2 + e^x$$
, so that $y = lnu$.

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{u}(4x + e^x)$$

$$= \frac{4x + e^x}{2x^2 + e^x}$$

(iv) Take
$$u = x + \ln x$$
, so that $y = u^2$

Then
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2u)\left(1 + \frac{1}{x}\right)$$

$$= 2(x + \ln x)\left(1 + \frac{1}{x}\right)$$

We will now extend the power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

to real exponents. We will do this in two stages – first to rational exponents and then to real exponents. We shall use the chain rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof: Let $n = \frac{p}{q}$, where p, q are integers and q > 0. Then nq = p is an integer. Let $u = x^n$ and consider the equation.

$$(x^n)^q = x^{nq} = x^p \text{ or } u^q = x^p \dots (1)$$

Now differentiate (1) using the chain rule on the left and the power rule (for integers) on the right to obtain

$$qu^{q-1} \frac{du}{dx} = p x^{p-1}$$

$$\Rightarrow \frac{d}{dx} (x^n) = \frac{px^{p-1}}{qu^{q-1}}$$
But $u^{q-1} = \frac{u^q}{u} = \frac{x^p}{x^n}$, because $u = x^n$. Thus

$$\frac{d}{dx}(x^n) = \frac{px^{p-1}}{q^{x^p}/x^n} = nx^{p-1+n-p} = nx^{n-1}$$

Theorem 12: For a real number n

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Proof: Recall that if n is real, then by definition

$$x^n = e^{nlnx}$$

Now put u = nlnx, so that $x^n = e^u$. Then by the chain rule

$$\frac{d}{dx}(x^n) = \frac{d}{du}(e^u)\frac{du}{dx} = (e^u)\frac{d}{dx}(nlnx) = (e^{nlnx})\left(\frac{n}{x}\right)$$
$$= \frac{nx^n}{x} = nx^{n-1}$$

Example 15: Find the derivative of each of the following functions:

(i)
$$y = (x^2 + 2)^{2/3}$$

(ii)
$$y = e^{\sqrt{x}}$$

(iii)
$$y = \ln(1 + \sqrt{1 + x^2})$$

$$(iv) y = x^2 e^{x^2}$$

Solution : (i) Putting $u = x^2 + 2$, we have

$$\frac{dy}{dx} = \frac{2}{3}(x^2 + 2)^{\frac{2}{3}-1} \frac{d}{dx}(x^2 + 2)$$
$$= \frac{2}{3}(x^2 + 2)^{-1/3}(2x)$$
$$= \frac{4x}{3(x^2 + 1)^{1/3}}$$



(ii) Putting
$$u = \sqrt{x}$$
, we have



$$\frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx} (\sqrt{x}) = e^{\sqrt{x}} \frac{1}{\sqrt[2]{x}} = \frac{e^{\sqrt{x}}}{\sqrt[2]{x}}$$

$$\frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx} (\sqrt{x}) = e^{\sqrt{x}} \frac{1}{\sqrt[2]{x}} = \frac{e^{\sqrt{x}}}{\sqrt[2]{x}}$$
(iii)
$$\frac{dy}{dx} = \frac{1}{1 + \sqrt{1 + x^2}} \frac{d}{dx} \left(1 + \sqrt{1 + x^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{1+\sqrt{1+x^2}} \frac{d}{dx} \left(1+\sqrt{1+x^2}\right)$$

$$=\frac{1}{1+\sqrt{1+x^2}}\frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2)$$

$$= \left(\frac{1}{1+\sqrt{1+x^2}}\right) \left(\frac{1}{2\sqrt{1+x^2}}\right) (2x)$$

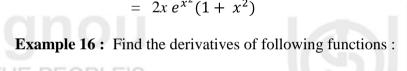
$$= \left(\frac{x}{(1+\sqrt{1+x^2})\sqrt{1+x^2}}\right)$$
(iv) $\frac{dy}{dx} = \frac{d}{dx}(x^2)e^{x^2} + x^2\frac{d}{dx}(e^{x^2})$

(iv)
$$\frac{dy}{dx} = \frac{d}{dx} (x^2) e^{x^2} + x^2 \frac{d}{dx} (e^{x^2})$$

$$= 2x e^{x^2} + x^2 e^{x^2} \frac{d}{dx} (x^2)$$

$$= 2x e^{x^2} + x^2 e^{x^2} (2x)$$

$$= 2x e^{x^2} (1 + x^2)$$



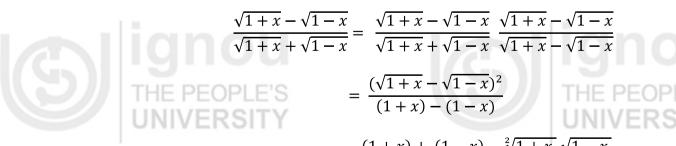


(i)
$$y = \ln\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$$
 (ii) $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$$\frac{e^x + e^{-x}}{e^x - e^{-x}}$$

(iii)
$$y = \sqrt[3]{x(x+1)(x+2)}$$

Solution: (i) Rewriting the argument of the log, we have



$$=\frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(1+x)-(1-x)}$$

$$=\frac{(1+x)+(1-x)-\sqrt[2]{1+x}\sqrt{1-x}}{2x}$$

$$= \frac{2 - \sqrt[2]{1 - x^2}}{2x} = \frac{1 - \sqrt{1 - x^2}}{x}$$









$$y = \ln \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$$

$$= \ln\left(\frac{1 - \sqrt{1 - x^2}}{x}\right)$$
$$= \ln\left(1 - \sqrt{1 - x^2}\right) - \ln x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - \sqrt{1 - x^2}} \frac{d}{dx} \left(1 - (1 - x^2)^{-1/2} \right) - \frac{1}{x}$$

$$= \left[\frac{1}{1 - \sqrt{1 - x^2}} \left\{ \frac{d}{dx} 0 - \frac{1}{2} (1 - x^2)^{-1/2} (-2x) \right\} - \frac{1}{x} \right]$$

$$= \frac{1}{1 - \sqrt{1 - x^2}} \frac{x}{\sqrt{1 - x^2}} - \frac{1}{x}$$

$$=\frac{x^2-[\sqrt{1-x^2}(1-\sqrt{1-x^2})]}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$=\frac{x^2-\sqrt{1-x^2}+(1-x^2)}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$=\frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}=\frac{1}{x\sqrt{1-x^2}}$$

(i) One can apply the quotient rule in this case. However, we will avoid it by rewriting the given expression.

$$Y = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \frac{e^{x} + \frac{1}{e^{x}}}{e^{x} - \frac{1}{e^{x}}} = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{e^{2x} - 1 + 2}{e^{2x} - 1}$$

$$=1+\frac{2}{e^{2x}-1}=1+2(e^{2x}-1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 0 + 2(-1)(e^{2x} - 1)^{-2} \frac{d}{dx}(e^{2x} - 1)$$

$$= \frac{-2}{(e^{2x} - 1)^2} (2e^{2x}) = \frac{-4e^{2x}}{(e^{2x} - 1)^2}$$

(ii) We have
$$y = [x(x+1)(x+2)]^{1/3}$$

So,
$$\frac{dy}{dx} = \frac{1}{3} [x(x+1)(x+2)]^{\frac{1}{3}-1} \frac{d}{dx} [x(x+1)(x+2)]$$
 (chain Rule)

$$= \frac{1}{3} [x(x+1)(x+2)]^{-\frac{2}{3}} \frac{d}{dx} [x(x+1)(x+2)]$$
 (product rule)

$$= \frac{1}{3} [x(x+1)(x+2)]^{-\frac{2}{3}} \frac{d}{dx} [(x+1)(x+2) + x(x+2) + x(x+1)]$$



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$$= \frac{1}{3} [x(x+1)(x+2)]^{-\frac{2}{3}} x(x+1)(x+2) \left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} \right]$$
$$= \frac{1}{3} [x(x+1)(x+2)]^{1/3} \left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} \right]$$

Check Your Progress 3

Find the derivatives of each of the following functions:

(i)
$$y = (x^3 + x)^{3/2}$$

(ii)
$$y = ln\left(\frac{x^2}{2}\right)$$

(iv) $y = ln\left(x + \sqrt{x}\right)$

(iii)
$$y = e^{(x^2 + 2x)}$$

(iv)
$$y = ln(x + \sqrt{x})$$

2. Find
$$\frac{dy}{dx}$$
 where

2. Find
$$\frac{dy}{dx}$$
 where

(i) $y = \frac{1 - e^x}{e^{2x}}$

(ii)
$$y = \frac{x}{\sqrt{x^2 - 1}}$$

(iii)
$$y = 2^{x/lnx}$$

Differentiate each of the following functions:

(i)
$$y = ln \left[e^x \left(\frac{x-2}{x+2} \right)^{3/4} \right]$$
 (ii) $y = \sqrt{\frac{1-x}{1+x}}$

(ii)
$$y = \sqrt{\frac{1-x}{1+x}}$$

(iii)
$$y = \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1 - \sqrt{x^2 - 1}}}$$

DIFFERENTIATION OF PARAMETRIC FORMS

Suppose x and y are given as functions of another variable t. We call t, the variable in which x and y are expressed as parameter. In this case, we find as follows:

Let x = f(t) and y = g(t), where f and g are differentiable functions of t and $f'(t) \neq 0 \ \forall t$. Let Δx and Δy be the increments and x and y respectively, corresponding to the increment Δt in t. That is $\Delta x = f(t + \Delta t) - f(t)$ and $\Delta y = g(t + \Delta t) - g(t)$

Since
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

and $\Delta x \to 0$ as $\Delta t \to 0$, we can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{g(t + \Delta t) - g(t)}{f(t + \Delta t) - f(t)}$$

Dividing both the numerator and denominator by Δt , we can use the differentiability of f and g to conclude that



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\left[\frac{g(t + \Delta t) - g(t)}{\Delta t}\right]}{\left[\frac{f(t + \Delta t) - f(t)}{\Delta t}\right]}$$
$$= \frac{g'(t)}{f'(t)} = \frac{dy/dt}{dx/dt}$$



Example 17: Find $\frac{dy}{dx}$ when

(a)
$$x = at^2$$
, $y = 2at$

(b) =
$$ct$$
, y = $\frac{c}{t}$

(c)
$$x = lnt, y = \frac{1}{t}$$

(b) =
$$ct$$
, $y = {c \over t}$
(d) $y = {3at \over 1 + t^2}$

Solution: (a) We have

(a) We have
$$\frac{dy}{dx} = \frac{d}{dt}[at^2] = 2at$$
 and
$$\frac{dy}{dt} = \frac{d}{dt}[2at] = 2a$$

$$so, \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{d}{dt}[ct] = c, and \frac{dy}{dx} = \frac{d}{dt}\left[\frac{c}{t}\right] = \frac{d}{dt}[ct^{-1}]$$
$$= [c(-1)t^{-2}] = \frac{c}{t^2}$$

since,
$$\frac{dy}{dx} = \frac{dy}{dx/dt}$$
, we get

$$\frac{dy}{dx} = \frac{-c/t^2}{c} = -\frac{1}{t^2}$$

(c) We have
$$\frac{dx}{dt} = \frac{1}{t}$$
 and $\frac{dy}{dt} = -\frac{1}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dx/dt} = (-1) \frac{1/t}{1/t^2} = -\frac{1}{t}$$

(d) We have

$$\frac{dx}{dt} = \frac{d}{dt} \left[\frac{3at}{1+t^2} \right]$$









alus
$$= 3a \frac{(1+t^2)\frac{dx}{dt} - t \frac{d}{dt} - (1+t^2)}{(1+t^2)^2}$$

$$= 3a \frac{(1+t^2)(1) - t (2t)}{(1+t^2)^2}$$

$$= 3a \frac{(1-t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{d}{dt} \left[\frac{3at^2}{(1+t^2)} \right] \text{ and}$$

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 $= 3a \frac{(1+t^2)\frac{d}{dt}(t^2) - (t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$

$$= 3a \frac{(1+t^2)(2t) - (t^2)(2t)}{(1+t^2)^2}$$

$$=\frac{6at}{(1+t^2)^2}$$

Since,
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

we get

$$\frac{dy}{dx} = \frac{\frac{6at}{(1+t^2)^2}}{3a(1-t^2)/(1+t^2)^2}$$
$$= \frac{2t}{1-t^2}$$

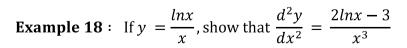
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Second Order Derivatives

Let y = f(x) be a function. If f is a differentiable function, then its derivative is a function. If the derivative is itself differentiable we can differentiate it and get another function called the second derivative. The second derivative is denoted

by y or
$$f(x)$$
 or $\frac{d^2y}{dx^2}$

Thus
$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$



Solution: we have

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{lnx}{x} \right] = \frac{d}{dx} \left[x^{-1} lnx \right]$$









$$= \frac{d}{dx}(x^{-1})lnx + x^{-1}\frac{d}{dx}(lnx)$$

$$= (-1)x^{-2}lnx + x^{-1}\frac{1}{x}$$

$$= x^{-2}[1-lnx]$$
Differentiating both sides with respect to x, we get

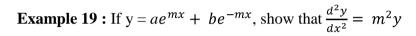
$$\frac{d^2y}{dx^2} = \frac{d}{dx} [x^{-2}][1 - \ln x] + x^{-2} \frac{d}{dx} [1 - \ln x]$$

$$= (-2) x^{-3} (1 - \ln x) + x^{-2} \left(0 - \frac{1}{x}\right)$$

$$= -2x^{-3} (1 - \ln x) + x^{-3}$$

$$= -x^{-3} (2 - 2\ln x + 1)$$

$$= \frac{2\ln x - 3}{x^3}$$



Solution : We have $y = ae^{mx} + be^{-mx}$

Differentiating both sides with respect to x, we get $\frac{dy}{dx} = \frac{d}{dx} (ae^{mx} + be^{-mx})$ $= ame^{mx} - bme^{-mx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(ame^{mx} - bme^{-mx})$$

$$= am^2e^{mx} - bm(-m)e^{-mx}$$

$$= am^2e^{mx} + bm^2e^{-mx}$$

$$= m^2(ae^{mx} + be^{-mx})$$

$$= m^2y$$

Example 20 : If $y = ln (x + \sqrt{x^2 + 1})$, prove that

$$(x^2+1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$$

Solution : We have $y = ln (x + \sqrt{x^2 + 1})$

Differentiating both sides, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx} \left[x + (x^2 + 1)^{\frac{1}{2}} \right] \quad \text{(chain rule)}$$



(product rule)









$$= \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} (2x) \right]$$

$$= \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{x}{\sqrt{x^2 + 1}} \right]$$

$$= \frac{1}{x + \sqrt{x^2 + 1}} \left[\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right]$$

$$=\frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

Now,
$$(x^2 + 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx}$$

$$= (x^2 + 1) \left[\frac{-x}{(x^2 + 1)^{\frac{3}{2}}} \right] + x \frac{1}{\sqrt{x^2 + 1}}$$

$$= -\frac{x}{\sqrt{x^2 + 1}} + \frac{x}{\sqrt{x^2 + 1}} = 0$$

Thus,
$$(x^2 + 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$$



- 1. Find $\frac{dy}{dx}$ when
- Find $\frac{dy}{dx}$ when

(a)
$$x = \frac{1}{2} (e^{\theta} - e^{-\theta})$$

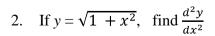
and
$$y = \frac{1}{2}(e^{\theta} - e^{-\theta})$$

(a)
$$x = \frac{1}{2}(e^{\theta} - e^{-\theta})$$
 and $y = \frac{1}{2}(e^{\theta} - e^{-\theta})$
(b) $x = a\left(t - \frac{1}{t}\right)$ and $y = a\left(t + \frac{1}{t}\right)$

and
$$y = a \left(t + \frac{1}{t} \right)$$

(c)
$$x = \frac{a(1-t^2)}{(1+t^2)}$$

and
$$y = \frac{2bt}{1+t^2}$$



3. If
$$y = ln (\sqrt{x-1} + \sqrt{x+1})$$
, prove that

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$















4. If
$$y = ax + \frac{b}{x}$$
, show that $\frac{xd^2y}{dx^2} + \frac{xdy}{dx} - y = 0$

1.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress - 1

1. (i)
$$\lim_{x \to 3} (3x^3 + 2x + 1) = 3.(2)^3 + 2(2) + 1 = 29$$

(ii)
$$\lim_{x \to 2} \frac{x-2}{x+2} = \frac{2-2}{2+2} = \frac{0}{4} = 0$$

(iii)
$$\lim_{x \to 2} \frac{x^2 - 5x + 2}{x - 1} = \frac{2^2 - 5(2) + 2}{2 - 1} = -2$$

(iv)
$$\lim_{x \to 3} \sqrt[3]{3x^2 - 19} = \sqrt[3]{3(3)^2 - 19} = \sqrt[3]{27 - 19} = \sqrt[3]{8} = 2$$

2. (i)
$$\lim_{x \to 2} \frac{x^2 - 4}{x + 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x + 2} = \lim_{x \to -2} (x - 2) = -2 - 2 = -4$$

(ii)
$$\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5} = \lim_{x \to 5} \left[\left(\frac{\sqrt{x-1}-2}{x-5} \right) \left(\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2} \right) \right]$$

$$x \to 5 \qquad x \to$$

$$= \lim_{x \to 5} \frac{(x-5)}{(x-5)(\sqrt{x-1}+2)}$$

$$= \lim_{x \to 5} \frac{1}{(\sqrt{x-1} + 2)} = \lim_{x \to 5} \frac{1}{(\sqrt{5-1} + 2)} = \frac{1}{4}$$

3. (i)
$$\lim_{x \to a} \frac{x^{7/6} - a^{7/6}}{x^{3/5} - a^{3/5}} = \lim_{x \to a} \frac{\frac{x^{7/6} - a^{7/6}}{x - a}}{\frac{x^{3/5} - a^{3/5}}{x - a}}$$

$$= \lim_{\substack{x \to a \\ \lim_{x \to a}}} \frac{x^{7/6} - a^{7/6}}{\frac{x - a}{x - a}} = \frac{(7/6)}{(3/5)} \frac{a^{7/6 - 1}}{a^{3/5 - 1}} \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n - 1} \right)$$

$$= \frac{35}{18} \ \frac{a^{1/6}}{a^{-2/5}} = \frac{35}{18} \ a^{\frac{17}{30}}$$







(ii)
$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to a} \frac{(x^m - a^m)/(x - a)}{(x^n - a^n)/(x - a)}$$

$$= \frac{\lim_{x \to a} \frac{x^m - a^m}{x - a}}{\lim_{x \to a} \frac{x^n - a^n}{x - a}}$$

$$=\frac{ma^{m-1}}{na^{n-1}}=\frac{m}{n}a^{m-n}$$

4. We have

$$\lim_{x \to 0^{-}} f(x) = \lim_{h \to 0^{+}} f(0 - h) = \lim_{h \to 0^{+}} f(-h)$$

$$= \lim_{h \to 0^{+}} [2 - (-h)] = \lim_{h \to 0^{+}} (2 + h) = 2$$

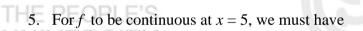
and
$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0^+} f(0+h) = \lim_{h \to 0^+} (f(h)) = \lim_{h \to 0^+} (2+h) = 2$$

Thus,
$$\lim_{x \to 0^{-}} f(x) = \lim_{x \to 0^{+}} f(x) = 2 \implies \lim_{x \to 0^{+}} f(x) = 2$$

Also,
$$f(0) = 2 + 0 = 2$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Hence, f is continuous at x = 0.



$$f(5) = \lim_{x \to 5} f(x)$$

$$\Rightarrow k = \lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x - 5)(x + 5)}{x - 5}$$

So,
$$k = \lim_{x \to 5} (x + 5) = 5 + 5 = 10$$

Thus, k = 10

Check Your Progress 2

1. (i)
$$\frac{dy}{dx} = \frac{d}{dx}(x^5 - 3x^4 + 2x - 1) = 5x^4 - 12x^3 + 2$$

(ii)
$$\frac{dy}{dx} = \frac{d}{dx} \frac{2x-1}{\pi^2} = \frac{1}{\pi^2} \frac{d}{dx} (2x-1) = \frac{2}{\pi^2}$$

(iii)
$$\frac{dy}{dx} = \frac{(2x+7)\frac{d}{dx}(3x+5) - (3x+5)\frac{d}{dx}(2x+7)}{(2x+7)^2}$$
 (Quotient Rule)

$$= \frac{(2x+7) \cdot 3 - (3x+5) \cdot 2}{(2x+7)^2}$$
$$= \frac{11}{(2x+7)^2}$$

(iv)
$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^3 - 4}{x^3} \right) = \frac{(x^3) \frac{d}{dx} (x^3 - 4) - (x^3 - 4) \frac{d}{dx} (x^3)}{(x^3)^2}$$
$$= \frac{x^3 (3x^2) - (x^3 - 4)(3x^2)}{x^6}$$

$$= \frac{4x^2}{x^6} = \frac{4}{x^4}$$
2. (i)
$$\frac{d}{dx}(e^x \ln x) = \frac{d}{dx}(e^x) \ln x + e^x \frac{d}{dx} \ln x$$

$$= (e^x \ln x) + \frac{e^x}{x} = e^x (\ln x + \frac{1}{x})$$

(ii)
$$\frac{d}{dx} \left(\frac{e^x}{x^2} \right) = \frac{x^2 \frac{d}{dx} (e^x) - (e^x) \frac{d}{dx} (x^2)}{x^4} = \frac{e^x (x - 2)}{x^3}$$

(iii)
$$\frac{d}{dx} \left(\frac{\ln x}{x^3} \right) = \frac{x^3 \frac{d}{dx} (\ln x) - (\ln x) \frac{d}{dx} (x^3)}{(x^3)^2}$$
$$= \frac{x^3 \frac{1}{x} - (\ln x) \frac{d}{dx} (3x^2)}{x^6}$$

$$= \frac{x^2(1 - 3lnx)}{x^6} = \frac{1 - 3lnx}{x^4}$$

(iv)
$$\frac{d}{dx} (2^x + x^2 + 2^2) = \frac{d}{dx} (2^x) + \frac{d}{dx} (x^2) + \frac{d}{dx} (2^2)$$

$$= 2^{x} ln 2 + 2 x + 0$$
$$= 2^{x} ln 2 + 2 x$$

$$(v)\frac{d}{dx}\left(\frac{e^x}{x^2+7}\right) = \frac{(x^2+7)\frac{d}{dx}(e^x) - (e^x)\frac{d}{dx}(x^2+7)}{(x^2+7)^2}$$
$$= \frac{(x^2+7)e^x - e^x(2x)}{(x^2+7)^2}$$
$$= \frac{e^x[x^2-2x+7]}{(x^2+7)^2}$$



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(vi)
$$\frac{d}{dx} (5^x e^x) = \frac{d}{dx} (5^x) e^x + 5^x \frac{d}{dx} (e^x)$$

$$= 5^x \ln 5 e^x + 5^x e^x$$

$$= 5^x e^x (\ln 5 + 1)$$

3.
$$\frac{d}{dx} (a^{x}) = \lim_{\Delta x \to 0} \frac{a^{x + \Delta x} - a^{x}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{a^{x} (a^{\Delta x} - 1)}{\Delta x}$$

$$= a^{x} \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^{x} \ln a$$
Check Your Progress – 3

(using the given limit)

1. (i)
$$\frac{dy}{dx} = \frac{3}{2}(x^3 + x)^{\frac{3}{2}-1}\frac{d}{dx}(x^3 + x)$$

$$= \frac{3}{2}(x^3 + x)^{1/2} (3x^2 + 1)$$

(ii)
$$\frac{dy}{dx} = \frac{1}{(x^2/2)} \frac{d}{dx} \left(\frac{x^2}{2}\right) = \frac{2}{x^2} \left(\frac{2x}{2}\right) = \frac{2}{x}$$

(iii)
$$\frac{dy}{dx} = e^{(x^2 + 2x)} \frac{d}{dx} (x^2 + 2x) = e^{(x^2 + 2x)} (2x + 2)$$
$$= 2(x + 1)e^{(x^2 + 2x)}$$

(iv)
$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x}} \frac{d}{dx} (x + \sqrt{x}) = \frac{1}{x + \sqrt{x}} \left(1 + \frac{1}{\sqrt[2]{x}} \right) = \frac{\sqrt[2]{x} + 1}{\sqrt[2]{x} (x + \sqrt{x})}$$

$$2. (i) \qquad \frac{dy}{dx} = \frac{\frac{d}{dx} (1 - e^x) e^{2x} - (1 - e^x) \frac{d}{dx} (e^{2x})}{(e^{2x})^2}$$

2. (i)
$$\frac{dy}{dx} = \frac{\frac{d}{dx}(1 - e^x)e^{2x} - (1 - e^x)\frac{d}{dx}(e^{2x})}{(e^{2x})^2}$$

$$=\frac{e^{2x}(-e^x)-(1-e^x)(2e^{2x})}{e^{4x}} = \frac{e^x-2}{e^{2x}}$$





(ii)
$$\frac{dy}{dx} = \frac{\sqrt{(x^2 - 1)} \frac{d}{dx}(x) - x \frac{d}{dx} \sqrt{(x^2 - 1)}}{(\sqrt{(x^2 - 1)})^2}$$
$$= \frac{\sqrt{(x^2 - 1)} - x \left(\frac{1}{\sqrt[2]{(x^2 - 1)}}\right) 2x}{x^2 - 1}$$
$$= \frac{(x^2 - 1) - x^2}{(x^2 - 1)\sqrt{x^2 - 1}} = \frac{-1}{(x^2 - 1)^{3/2}}$$

(iii)
$$\frac{dy}{dx} = 2x^{x/\ln 2} \ln 2 \frac{d}{dx} \left(\frac{x}{\ln x}\right)$$
$$= 2x^{x/\ln x} \left[\frac{1 \cdot \ln x - x \cdot \frac{1}{x}}{(\ln x)^2}\right]$$
$$= \frac{2x^{x/\ln x} \ln 2(\ln x - 1)}{(\ln x)^2}$$

3. (i) Rewriting the given expression, we have

$$y = \ln \left[e^x \left(\frac{x-2}{x+2} \right)^{3/4} \right]$$

$$= \ln e^x + \ln \left(\frac{x-2}{x+2} \right)^{\frac{3}{4}} \qquad [\ln(ab) = \ln a + \ln b]$$

$$= x \ln e^{\frac{3}{4}} + \ln \left(\frac{x-2}{x+2} \right) \qquad [\ln a^x = x \ln a]$$

$$= x + \frac{3}{4} [\ln(x-2) - \ln(x+2)] \qquad [\ln(e) = 1 \text{ and } \ln(a/b) = \ln a - \ln b]$$

$$\Rightarrow \frac{dy}{dx} = 1' + \frac{3}{4} \left[\frac{1}{x - 2} - \frac{1}{x + 2} \right]$$

$$= 1 + \frac{3}{4} \left[\frac{(x + 2) - (x - 2)}{(x - 2)(x + 2)} \right]$$

$$= 1 + \frac{3}{4} \left[\frac{x + 2 - x + 2}{x^2 - 4} \right]$$

$$= 1 + \frac{3}{x^2 - 4}$$

$$= \frac{x^2 - 4 + 3}{x^2 - 4} = \frac{x^2 - 1}{x^2 - 4}$$



(ii)
$$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{1-x}{1+x} \right) \quad \text{(Chain Rule)}$$

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$$= \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{-1/2} \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

(Quotient Rule)

$$= \frac{1}{2} \sqrt{\frac{1+x}{1-x}} \frac{-2}{(1+x)^2}$$
$$= \frac{-1}{(1+x)^2} \sqrt{\frac{1+x}{1-x}}$$

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(iii) Rewriting the given expression, we have

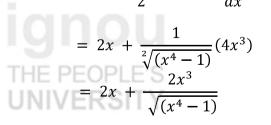
$$y = \frac{\sqrt{(x^2 + 1)} + \sqrt{(x^2 - 1)}}{\sqrt{(x^2 + 1)} - \sqrt{(x^2 - 1)}} = \frac{\sqrt{(x^2 + 1)} + \sqrt{(x^2 - 1)}}{\sqrt{(x^2 + 1)} - \sqrt{(x^2 - 1)}} \frac{\sqrt{(x^2 + 1)} + \sqrt{(x^2 - 1)}}{\sqrt{(x^2 + 1)} - \sqrt{(x^2 - 1)}}$$

$$= \frac{\left(\sqrt{(x^2+1)} + \sqrt{(x^2-1)}\right)^2}{(x^2+1) - (x^2-1)} = \frac{(x^2+1) + (x^2-1) + \sqrt[2]{(x^2+1)(x^2-1)}}{2}$$

$$= \frac{2x^2 + \sqrt[2]{x^4-1}}{2} = x^2 + (x^4-1)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}[((x^4-1)^{\frac{1}{2}}]$$

$$= 2x + \frac{1}{2}(x^4-1)^{-\frac{1}{2}}\frac{d}{dx}(x^4-1)$$





Check Your Progress 4

1. (a)
$$\frac{dx}{d\theta} = (e^{\theta} + e^{-\theta})/2$$
$$\frac{dx}{d\theta} = (e^{\theta} - e^{-\theta})/2$$



$$\therefore \frac{dx}{dy} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{1}{2}(e^{\theta} - e^{-\theta})}{\frac{1}{2}(e^{\theta} + e^{-\theta})} = \frac{x}{y}$$

(b)
$$\frac{dx}{dt} = a\left(1 + \frac{1}{t^2}\right), \quad \frac{dy}{dt} = b\left(1 - \frac{1}{t^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b(1 - \frac{1}{t^2})}{a(1 + \frac{1}{t^2})} = \frac{b(t^2 - 1)}{a(t^2 + 1)}$$

(c)
$$\frac{dx}{dt} = \frac{a(1+t^2)(-2t) - a(1-t^2)(2t)}{(1+t^2)^2}$$
 (Quotient Rule)

$$=\frac{a[-2t-2t^3-2t+2t^3]}{(1+t^2)^2}$$

$$= \frac{-4at}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b \frac{(1-t^2)(1) - t(2t)}{(1+t^2)^2}$$

$$=\frac{2b(1-t^2)}{-4at}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(1-t^2)}{-4at} = \frac{-b(t^2-1)}{(1+t^2)^2}$$

2.
$$\frac{dy}{dx} = \frac{d}{dx}[(1+x^2)] = \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}(2x)$$

$$= x(1+x^2)^{-1/2}$$

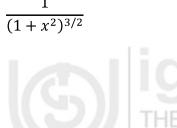
$$= x(1+x^2)^{-1/2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[x(1+x^2)^{-\frac{1}{2}} \right] = \frac{d}{dx} (x)(1+x^2)^{-\frac{1}{2}} + x\frac{d}{dx} (1+x^2)^{-\frac{1}{2}}$$

= 1.
$$(1 + x^2)^{-\frac{1}{2}} + x \left[-\frac{1}{2} (1 + x^2)^{-\frac{1}{2}-1} (2x) \right]$$

$$= (1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-3/2}$$

$$= (1+x^2)^{-\frac{1}{2}} \left[1 - \frac{x^2}{1+x^2} \right] = \frac{(1+x^2)^{-\frac{1}{2}}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$



3. We have
$$y = \ln (\sqrt{x-1} + \sqrt{x+1})$$



$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x-1} + \sqrt{x+1}} \frac{d}{dx} (\sqrt{x-1} + \sqrt{x+1})$$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x+1}} \frac{1}{dx} (\sqrt{x-1} + \sqrt{x+1})$$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x+1}} \left(\frac{1}{\sqrt[2]{x-1}} + \frac{1}{\sqrt[2]{x-1}} \right)$$

$$= \frac{(\sqrt{x-1} + \sqrt{x+1})}{2(\sqrt{x-1} + \sqrt{x+1})\sqrt{x-1}\sqrt{x+1}}$$



$$= \frac{1}{\sqrt[2]{x^2 - 1}} = \frac{1}{2}(x^2 - 1)^{-1/2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\frac{d}{dx}\left[(x^2 - 1)^{-\frac{1}{2}}\right]$$

$$= -\frac{1}{4}[(x^2 - 1)^{-\frac{1}{2}-1}(2x)]$$
 (Chain Rule)

$$= -\frac{1}{2}x(x^2 - 1)^{-3/2}$$

4. When have
$$y = ax + \frac{b}{x}$$

$$\therefore \frac{dy}{dx} = a - \frac{b}{x^2} \text{ and } \frac{d^2y}{dx^2} = \frac{d}{dx} (a - bx^{-2}) = 2bx^{-3} = \frac{2b}{x^3}$$

$$\therefore x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2 \left(\frac{2b}{x^3}\right) + x \left(a - \frac{b}{x^2}\right) - \left(ax + \frac{b}{x}\right)$$

$$= \frac{2b}{x} + ax - \frac{b}{x} - ax - \frac{b}{x}$$









1.7 SUMMARY

In **section 1.2** of the unit, to begin with, the concept of limit of a function is defined. Then, some properties of limits are stated. Next, the concept of one-sided limit is defined. Then, the concept of continuity of a function is defined. Each of these concepts is illustrated with a number of examples.

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In **section 1.3**, the concepts of differentiability of a function at a point and in an open interval are defined. Then, a number of rules for finding derivatives of simple functions are derived. In **section 1.4**, chain rule of differentiation is derived and is explained with a number of examples. In **section 1.5**, the concept of differentiation of parametric forms is defined followed by the definition of the concept of second order derivative. Each of these concepts is explained with a number of suitable examples.

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Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.6**.











