UNIT 1   SEQUENCE AND SERIES

Structure

1.0   INTRODUCTION

We begin by looking at some examples which exhibit some pattern.

1. Arrangement of seats in a conference hall.   Each row (except the first) contains, one seat more than the number of seats in the row ahead of it. See the following figure.

2. The number of dots used to draw the following triangles :

3. The money in your account in different years when you deposit Rs. 10,000 and at the rate of 10% per annum compounded annually.

10000	11000	12100	13310
$n = 0$	$n = 1$	$n = 2$	$n = 3$

In this unit, we shall study sequences exhibiting some patterns as they grow.

5

1.1 OBJECTIVES

After studying this unit, you will be able to :

- define an arithmetic progression, geometric progression and harmonic progression;
- find the nth terms of an A.P., G.P., and H.P.;
- find the sum to n terms of an A.P. and G.P.;
- find sum of an infinite G.P.; and
- obtain sum of first n natural numbers, their squares and cubes.

1.2 ARITHMETIC PROGRESSION (A.P.)

An **arithmetic progression** is a sequence of terms such that the difference between any term and the one immediately preceding it is a constant. This difference is called the common difference.

For example, the sequences

(i)   3, 7, 11, 15, 19…………..

(ii)   7, 5, 3, 1, -1…………………..

(iii)   1, $\dfrac{3}{4}$, $\dfrac{1}{2}$, $\dfrac{1}{4}$, 0, $-\dfrac{1}{4}$ ……………….

(iv)   2, 2, 2, 2.Y……………..

are arithmetic progressions.

In (i) common difference is   4,

In (ii) common difference is   –2,

In (iii) common difference is $-\dfrac{1}{4}$, and

In (iv) common difference is 0,

In general, an arithmetic progression (A.P.) is given by

$a,\ \ a+d,\ \ a+2d,\ \ a+3d$……………….

We call a as **first term** and d as the **common difference.**

The nth term of the above A.P. is denoted by a_n and is given by

$$a_n = a + (n-1)d$$

Example 1 : Find the first term and the common difference of each of the
following arithmetic progressions.

(i)   7, 11, 15, 19, 23....................

(ii) $\dfrac{1}{6}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{5}{6}$ 1 ... ...

(iii) $a + 2b, a + b, a, a - b, a - 2b, ...$ ... ...

Solution :

		First term	**Common difference**
	(i)	7	4
	(ii)	$\dfrac{1}{6}$	$\dfrac{1}{3}$
	(iii)	$a + 2b$	$-b$

Example 2 : Find the 18th, 23rd and n^{th} terms of the arithmetic progression.

$$-11, -9, -7, -5....................$$

Solution: Here $a = -11,$ and $d = -9 + 11 = 2$

Thus,

$$a_{18} = a + (18 - 1)\, d$$
$$= -11 + (17)\,(2)$$
$$= -11 + 34 = 23;$$
$$a_{23} = a + (23 - 1)d$$
$$= -11 + (22)\,(2)$$
$$= -11 + 44 = 33; \text{ and}$$
$$a_n = a + (n - 1)d$$
$$= -11(n - 1)\,(2)$$
$$= -11 + 2n - 2 = 2n - 13$$

Difference of two Terms of an A.P.

Let the A.P. be

$a, a + d, a + 2d, a + 3d$............

we have

$$a_r - a_s = [a + (r - 1)\, d] - [a + (s - 1)\, d]$$
$$= a + rd - d - [a + sd - d]$$
$$= a + rd - d - a - sd + d$$
$$= (r - s)d$$
$$\therefore a_{r - s} = (r - s)d$$

Example 3 : Which term of the A.P. 3, 15, 27, 39, ………..will be 132 more than its 54th term ?

Solution: Common difference of the given A.P. is 12.

If *nth* term is the required term, then

$$a_n - a_{54} = 132$$

$$\Rightarrow \quad (n - 54)(12) = 132$$

$$\Rightarrow \quad n - 54 = \frac{132}{12} = 11$$

$$\Rightarrow \quad n = 54 + 11 = 65$$

Thus, the 65th term of the A.P. is 132 more than the 54th term.

Example 4 : If *pth* term of an A.P. is *q* and its *qth* term is *p*, show that its *rth* term is $p+q-r$. What is its $(p + q)th$ term ?

Solution : If *d* is the common difference of the A.P., then

$$a_p - a_q = (p - q)\ d$$

$$\Rightarrow q - p = (p - q)d$$

$$\Rightarrow d = \frac{q - p}{p - q} = -1$$

Now,

$$a_r - a_p = (r - p)\ d = (r - p)(-1)$$

$$\Rightarrow a_r = a_p - r + p$$

$$= q - r + p = p + q - r$$

$$\therefore\ a_{p+q} = p + q - (p + q) = 0 \qquad [\text{ put } r = p + q]$$

Example 5 : If *m* times the *mth* term of an A.P. is *n* times its *nth* term, show that the $(m+n)th$ term of the A.P. is 0.

Solution : We are given that $m\ a_m = n\ a_n$

$$\Rightarrow \quad m\ [a + (m - 1)\ d] = n\ [a + (n - 1)d]$$

$$\Rightarrow \quad m\ [a + md - d] = n\ [a + nd - d]$$

$$\Rightarrow \quad m\ [\alpha + md] = n[\alpha + nd]\ where\ \alpha = a - d$$

$$\Rightarrow \quad m\ \alpha + m^2 d = n\ \alpha + n^2 d \Rightarrow (m - n)\ \alpha + (m^2 - n^2)d = 0$$

$$\Rightarrow \quad (m - n)[\ \alpha + (m + n)d] = 0 \quad \Rightarrow \alpha + (m + n)d = 0$$

$$\Rightarrow \quad a + (m + n - 1)d = 0 \qquad\qquad [\because \alpha = a - d]$$

Left hand side is nothing but the $(m + n)th$ term of the A.P.

Check Your Progress 1

Find the common difference and write next four terms of the A.P. (1 – 5)

1. 16, 11, 6, 1………………

2. 2, 5, 8, 11, 14………………

3. $\dfrac{1}{n}, \dfrac{2n+1}{n}, \dfrac{4n+1}{n}, \dfrac{6n+1}{n}$ ....

4. $m-1, m-2, m-3, m-4$………………………

5. $\sqrt{3}, \sqrt{27}, \sqrt{48},$ ……………………

6. If a_r denotes the *rth* term of an AP., show that

$$a_{p+q} + a_{p-q} = 2a_p$$

7. Find k so that $\dfrac{2}{3} - k, k, \dfrac{5}{8} - k$ consecutive are three terms of an A.P.

8. Which term of the sequence

$17, 16\dfrac{1}{5}, 15\dfrac{2}{5}, 14\dfrac{3}{5}$ … … … … … is the first negative term.

9. The fourth term of an arithmetic progression is equal to 3 times the first term and the seventh term exceeds twice the third term by 1. Find its first term and the common difference.

10. If $(p + 1)th$ term of an arithmetic progression is twice the $(q+1)th$ term, show that $(3p+1)th$ term is twice the $(p+q+1)th$ term.

11. If 7 times the 7th term of an arithmetic progression is equal to 11 times its 11th term, show that the 18th term of an arithmetic progression is zero.

1.3 FORMULA FOR SUM TO *N* TERMS OF AN AP

Let

$$S = a+(a+d) +(a+2d) +………..+ (a+ \overline{n-2}\ d) +(a+(\overline{n-1}\ d) \qquad (1)$$

Writing the expression in the reverse order, we get

$$S = (a + \overline{n-1}\ d) +(a + \overline{n-2}d)+ \quad .... +(a + d) + a \qquad (2)$$

Adding (1) and (2) vertically, we get

$$2S = \underbrace{[2a + (n-1)\ d] +[2a + (n-1)d] + ………+[2a + (n-1)d]}$$

n expressions

9

$$2S = n[2a+(n-1)d]$$

$$\Rightarrow \quad S = \frac{n}{2}\,2a+(n-1)d$$

Alternative form for the Sum Formula

$$S = \frac{n}{2}\{a+a+(n-1)d\}$$

$$S = \frac{n}{2}\{a + l\}$$

where $l = a + (n-1)d$ is the last term of the AP

Solved Examples

Example 6 : Find the sum of first 100 natural numbers.

Solution : Here $a = 1$, $d = 1$ and $n = 100$.

Using

$$S_n = \frac{n}{2}\,[2a + (n-1)d]$$

we get

$$S_{100} = \frac{100}{2}\,[2(1) + (100 - 1)(1)] = 50(101) = 5050$$

Example 7 : Find the sum of 23 terms and n terms of the A.P.

16, 11, 6, 1……………

Solution : Here $a = 16$, $a + d = 11$ Therefore $d = -5$

Since $S_n = \frac{n}{2}\,[2a + a + (n-1)d]$ we get

$$S_{23} = \frac{23}{2}\,[2(16) + (23 - 1)(-5)] = \frac{23}{2}\,(32 - 110)$$

$$= \frac{23}{2}\,(-78) = -23 \times 39 = -897.$$

Also $S_n = \dfrac{n}{2}\,[2\,(16) + (n-1)(-5)] = \dfrac{n}{2}\,[32 - 5n + 5] = \dfrac{n}{2}\,[37 - 5n]$

Example 9 : How many terms of the A.P. 1,4,7,…….. must be taken so that the sum may be 715 ?

Solution : Here $a = 1$, $d = 3$ and $S_n = 715$.

Using $S_n = \dfrac{n}{2}\,[2a + (n-1)d]$, we get $715 = \dfrac{n}{2}\,[2(1) + (n-1)(3)]$

$\Rightarrow 715 = \dfrac{n}{2}\,(3n + 1) \Rightarrow 3n^2 + n - 1430 = 0.$

This is a quadratic equation in n. Its discriminant is positive. We can use the quadratic formula to obtain

$$n = \dfrac{1 \pm \sqrt{1 + (4)\,(3)(1430)}}{6} = \dfrac{1 \pm \sqrt{17161}}{6} = \dfrac{1 \pm 131}{6} = 22,\ -\dfrac{65}{3}.$$

As $-65/3$ is a negative fraction, the number of terms cannot be equal to $-65/3$
Thus $n = 22$.

Example 10: If in an A.P. $a = 2$ and the sum of first five terms is one–fourth of the sum of the next five terms, show that $a_{20} = -112$.

Solution :   Here $a = 2$

We are given that sum of the first five terms is one–fourth of the next five terms.

Think of the A.P. as
$$a,\ a + d,\ a + 2d,\ a + 3d, …………$$

The sum of the first five terms is
$$a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) = S_5$$
and the sum of the next five terms is
$$(a + 5d) + (a + 6d)\ (a + 7d) + (a + 8d) + (a + 9d).$$
Note that this expression equals $S_{10} - S_5.$

According to the give condition

$$S_{10} = \dfrac{1}{4}(S_{10} - S_5) \Rightarrow 4S_{10} - S_5 \ or \ 5S_5 = S_{10}$$

$$(S_{10} - S_5) = \ or \ 5\,S_5 = S_{10}$$

$$= 5\left[\dfrac{10}{2}(2a + 9d)\right]$$

$$= 5\left[\frac{10}{2}(2a+9d)\right]$$

$$\Rightarrow 5a + 10d = 2a + 9d \Rightarrow d = -3a$$

Now, $a_{20} = a + (20-1)d = a + 19d = a + 19(-3a) = a - 57a = -56a$
$$= (-56)(2) = -112.$$

Example 11 : If sum of the pth, q^{th} and r^{th} terms of an AP are a, b, c respectively, show that

$$(q-r)\,\frac{a}{p} + (r-p)\,\frac{b}{q} + (p-q)\,\frac{c}{r} = 0 \qquad (1)$$

Solution : Let the first term of the AP be A and the common difference be D.
We are given :

$$a = S_p = \frac{p}{2}\,[2A + (p-1)D] \qquad (2)$$

$$b = S_q = \frac{q}{2}\,[2A + (q-1)D] \qquad (3)$$

$$c = S_r = \frac{r}{2}\,[2A + (r-1)D] \qquad (4)$$

Form (2), (3) and (4), we get

$$\frac{2a}{p} = (2A - D) + pD \qquad (4)$$

$$\frac{2b}{q} = (2A - D) + qD \qquad (5)$$

$$\frac{2c}{r} = (2A - D) + rD \qquad (6)$$

Multiplying (4) by $q-r$, (5) $r-p$ and (6) by $p-q$, we get

$$2(q-r)\,\frac{a}{p} + 2(r-p)\,\frac{b}{q} + 2(p-q)\,\frac{c}{r}$$

$$= (2A - D)(q - r) + p(q - r)D$$

$$+ (2A - D)(r - p) + q(r - p)D$$

$$+ (2A - D)(p - q) + r(p - q)D$$

$$= (2A - D)\{q - r + r - p + p - q\}$$

$$+ (pq - pr + qr - qp + rp - rq)D$$

$$= (2A - D)(0) + (0)D = 0$$

Dividing both the sides by 2 we get (1).

Example 12: If the sum of the first n terms of an A.P. is given by $S_n = 2n^2 + 5n$, Find the nth term of the A.P.

Solution : We shall use the formula

$$a_n = S_n - S_{n-1} \qquad \forall\, n \geq 1,$$

Where $S_o = 0$

$$\therefore a_n = 2n^2 + 5n - [2(n-1)^2 + 5(n-1)]$$

$$= 2n^2 + 5n - [2(n^2 - 2n + 1) + 5n - 5]$$

$$= 2n^2 + 5n - 2n^2 + 4n - 2 - 5n + 5$$

$$= 4n + 3$$

Thus, nth term of the A.P. is $4n + 3$

Example 13 : Find the sum of all integers between 100 and 1000 which are divisible by 9.

Solution : The first integer greater than 100 and divisible by 9 is 108 and the integer just smaller than 1000 and divisible by 9 is 999. Thus, we have to find the sum of the series.

$$108 + 117 + 126 + \ldots\ldots\ldots + 999.$$

Here $t_1 = a = 108$, $d = 9$ and $l = 999$

Let n be the total number of terms in the series be n. Then

$$999 = 108 + 9\,(n-1) \Rightarrow 111 = 12 + (n-1) \Rightarrow n = 100$$

Hence, the required sum $= \dfrac{n}{2}\,(a + l) = \dfrac{100}{2}\,(108 + 999)$

$$= 50\,(1107) = 55350.$$

Example 14 : The interior angles of a convex polygon are in A.P. If the smallest angle is 120^0 and the common difference is 5^0, show that there are 9 sides in the polygon.

Solution : We know that the sum of the interior angles of a convex polygon is $(n-2)(180^0)$ We are given that $a = 120^0$ and $d = 5^0$.

$$\therefore \quad \dfrac{n}{2}\,[2(120) + (n-1)(5)] = (n-2)(180)$$

13

$$\Rightarrow n[48 + (n - 1] = (n - 2)(72) \Rightarrow n^2 - 25n + 144 = 0$$

$$\Rightarrow (n - 9)(n - 16) = 0 \Rightarrow n = 9, 16.$$

But when $n = 16$, the greatest angle is equal to

$a + (16 - 1)d = 120^0 + 15 \times 5^0 = 195^0$

Which is not possible as in this case one of the interior angle becomes 180^0.

[interior angles are 120^0, 125^0,............ 175^0, 180^0, 185^0, 190^0, 195^0]

Thus $n = 9$.

Check Your Progress – 2

In questions 1 to 3, find the sum of indicated number of terms.

1. 1,3,5,7, ...;   100 terms, 200 terms
2. 0.9, 091, 0.92, 0.93..........., 20 terms n terms.

3. $\dfrac{m - 1}{m}, \dfrac{m - 2}{m}, \dfrac{m - 3}{m}$ ....; 10 terms, n terms

4. If the first term of an A.P. is 22, the common difference is -4, and the sum to n terms is 64, Find n.   Explain the double answer.

5. If sum of p terms of an A.P. is $3p^2 + 4p$, find its nth term.

6. Find the sum of all three digit numbers which leaves remainder (1) when divided by 4.

1.4   GEOMETRIC PROGRESSION (G.P.)

Suppose a ball always rebounds exactly 70 per cent of the distance it falls. For instance, if this ball falls from a height of 100 units, then it will rebound exactly 70 units.   As a result the second fall will be from a height of 70 units. See Fig. 1

Now the ball will rebound exactly $(0.7)(70) = 49$ units

And so, on.

100 units

70 units

49 units

Figure 1
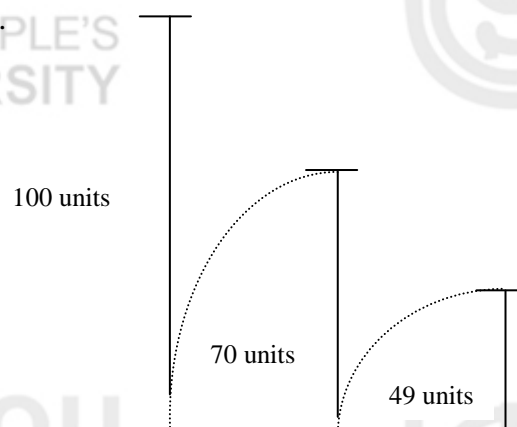
The following table gives the distance through which the ball bounces in the first 5 falls.

Number of fall	Ist fall	2nd fall	3rd fall	4th fall	5th fall
Distance of fall	100	$100\,(0.7)$	$100\,(0.7)^2$	$100\,(0.7)^3$	$100\,(0.7)^4$

If you look carefully you will find that each term in sequence

$100,\ 100\,(0.7),\ 100\,(0.7)^2,\ 100\,(0.7)^3,\ \ldots\ldots\ldots$.(except the first) is obtained by multiplying the previous term by a fixed constant 0.7.

Such a sequence is called **geometric sequence or geometric progression** or briefly, G.P. In other words, a geometric sequence or geometric progression is a sequence in which each term, except the first, is obtained by multiplying the term immediately preceding it by a fixed, non-zero number. The fixed number is called the **common ratio.**

Definition :   A sequence $a_1,\ a_2,\ \ldots\ldots\ldots a_n,\ \ldots\ldots$ is called a geometric progression (G.P.).   If there exists a constant r, such that

$$a_{k+1} = ra_k \qquad \forall\,k \in \mathbf{N.}$$

Thus, a geometric progression (G.P.) looks as follows :

$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ldots\ldots$

where a is called the **first term** of G.P. and r is called as the **common ratio** of the G.P.

The nth term of the G.P., is given by

$$a_n = ar^{n-1}$$

Some other examples of GPs are

1. $0.1,\ 0.01,\ 0.001,\ 0.0001\ldots(0.1)^n,\ldots\ldots\ldots\ldots$

2. $2,\ 4,\ 6,\ 8,\ 16,\ldots\ldots\ldots\ldots,\ 2^n,\ldots\ldots$

3. $-\dfrac{1}{3},\ -\dfrac{1}{9}\ -\dfrac{1}{27}\ -\dfrac{1}{81}\ \ldots\ldots\ldots\ldots\left(-\dfrac{1}{3}\right)^n\ldots$

4. $\frac{1}{\sqrt{3}}, 1, \sqrt{3}, 3, 3\sqrt{3}, \ldots \ldots 3^{(n-2)/2}, \ldots \ldots \ldots$

Ratio of two terms of G.P.

Let the G.P. be

$a, ar, ar^2, \ldots \ldots \ldots$

Now,

$$\frac{a_n}{a_m} = \frac{ar^{n-1}}{ar^{m-1}} = r^{n-1-m+1} = r^{n-m}$$

Thus, $\frac{a_n}{a_m} = r^{n-m}$

Solved Examples

Example 15: Find r and the next four terms of the G.P.

$$-3, 1, -\frac{1}{3}, \frac{1}{9}, \ldots \ldots \ldots \ldots$$

Solution: Here $a = -3$, $ar = 1$

So that

$$r = \frac{ar}{a} = \frac{1}{-3} = -\frac{1}{3}$$

The next four terms of G.P. are

$$a_5 = a_4 r = \left(\frac{1}{9}\right)\left(-\frac{1}{3}\right) = -\frac{1}{27}$$

$$a_6 = a_5 r = -\frac{1}{81}$$

$$a_7 = a_6 r = \left(\frac{1}{81}\right)\left(-\frac{1}{3}\right) = -\frac{1}{243}$$

and $a_8 = a_7 r = \left(-\frac{1}{243}\right)\left(-\frac{1}{3}\right) = -\frac{1}{729}$

Example 16 : Determine the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

Solution : We have

$$\frac{a_{12}}{a_8} = r^{12-8} = r^4 = 2^4$$

$$\Rightarrow \quad a_{12} = a_8(2^4) = 192 \times 16 = 3072$$

Example 17 : Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. find the numbers.

Solution : Let the three numbers in AP be $a - d$, a, $a + d$. we are given

$$(a - d) + a + (a + d) = 15 \Rightarrow 3a = 15 \ or \ a = 5$$

According to the given condition $a - d + 1$, $a + 3$ *and* $a + d + 9$ are in GP

$$\Rightarrow \quad \frac{a + 3}{a - d + 1} = \frac{a + d + 9}{a + 3}$$

$$\Rightarrow (a + 3)^2 = (a - d + 1)(a + d + 9) \Rightarrow (5 + 3)^2 = (5 - d + 1)(5 + d + 9)$$

$$\Rightarrow 64 = (6 - d)(14 + d) \Rightarrow 64 = 84 - 8d - d^2$$

$$\Rightarrow d^2 + 8d - 20 = 0 \Rightarrow (d + 10)(d - 2) = 0 \Rightarrow d = -10 \ or \ d = 2$$

If $d = -10$, the numbers are 15, 5, –5.

If $d = 2$, the numbers are 3, 5, 7.

Thus, the numbers are 15, 5, –5, or 3, 5, 7.

Check Your Progress - 3

For question 1 to 3 find the common ratio of each of the following G.P.

1. $2, 1, \dfrac{1}{2}, \dfrac{1}{4} \dots \dots \dots \dots \dots$

2. $-3, 1, -\dfrac{1}{3}, \dfrac{1}{9} \dots \dots \dots \dots$

3.   3, 6, 12, 24…………..

For questions 4 to 6, find the nth term of the GP.

4.   128, –96, 72………

5.   100, –110, 121…………….

6.   3, 3.3, 3.63……………

7.   Determine the 18th term of the GP whose 5th term is 1 and common ratio is 2/3.

8.   The 5[th,] 8th and 11th terms of a GP are a, b, c respectively. Show that a, b, c are in GP.

1.5   SUM TO n TERMS OF a G.P.

We wish to find the sum of first n terms of the GP whose first term is a and the common ratio r.

Let us denote the sum of first n terms by S_n, that is,

$$S_n = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1}$$

Let us multiply both sides of this equation by r. We write the results of this operation below the original equation and line up vertically the terms of the same exponent (to prepare for a subtraction).

$$S_n = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1} \qquad (1)$$

$$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots ar^{n-2} + ar^{n-1} + ar^n \qquad (2)$$

Now subtract both the sides of lower equation [equation (2)] from the upper equation [equation (1)].

Most of the summands on the right side cancel. All that is left is the equation.

$$S_n = a - ar^n \Rightarrow (1 - r)\, S_n = a(1 - r^n)$$

Now, if $r \neq 1$ then $S_n = \dfrac{a(1 - r^n)}{1 - r}$

Thus, the sum to n terms of a GP is $S_n = \dfrac{a(1 - r^n)}{1 - r}, \; r \neq 1$

Clearly $S_n = na$ where $r = 1$.

Example 18 : Find the sum to 20 terms of a GP 128, -96, 72, -54,………

Solution

Here $a = 128$, $ar = -96$, therefore, $r = -96/128 = -3/4$.

Now that $r \neq 1$. Therefore,

$$S_n \frac{128[1 - (-3/4)^{20}]}{1 - (-3/4)} = 128\left[1 - \left(-\frac{3}{4}\right)^{20}\right]\left(\frac{4}{7}\right)$$

$$= \frac{128(4^{20} - 3^{20})}{4^{20}}\left(\frac{4}{7}\right) = \frac{2^6(4^{20} - 3^{20})(2^2)}{2^{40}(7)} = \frac{4^{20} - 3^{20}}{(2^{32})(7)}$$

Example 19 : How many terms of the GP $\sqrt{3}$, 3, 3$\sqrt{3}$, … Add upto $39 + 13\sqrt{3}$.

Solution : Here $a = \sqrt{3}$, $ar = 3$, so that $r = \sqrt{3}$.

Let $39 + 13\sqrt{3}$ be the sum to n terms of the given GP, that is,

$$S_n = 39 + 13\sqrt{3}$$

$$\Rightarrow \frac{a(r^n - 1)}{r - 1} = 39 + 13\sqrt{3} \quad \Rightarrow \quad \frac{(\sqrt{3})\left[(\sqrt{3})^n - 1\right]}{\sqrt{3} - 1}$$

$$= 13\sqrt{3}\,(\sqrt{3} + 1)$$

$\Rightarrow$ $\qquad$ $3^{n/2} = 1 + 13(3–1) = 1 + 26 = 27 = 3^3 \Rightarrow n/2 = 3$ or $n = 6$.

Thus, 6 terms of $\sqrt{3}, 3, 3\sqrt{3} ... ...$

are required to obtain a sum of $39 + 13\sqrt{13}$.

Example 20 : Find the sum to n terms of the series $9 + 99 + 999 + ...$

Solution : Note that we can write

$9 = 10 – 1, 99 = 100 – 1 = 10^2 – 1, 999 = 10^3 – 1$, etc.

The sum to n terms of the series can be written as

$S = (10 – 1) + (10^2 – 1) + (10^3 – 1) + ...+(10^n – 1)$

$= (10 + 10^2 + \cdots . + 10^n) - n = \dfrac{10(10^n – 1)}{10 – 1} - n = \dfrac{10}{9}(10^n – 1) - n$

Three Terms in GP

Three terms in GP whose product is given are taken as

$$\dfrac{a}{r}, a, ar$$

Example 21 : If sum of three numbers in GP is 38 and their product is 1728, find the numbers.

Solution : Let the three number be $a/r, a, ar$

Then, $\qquad \dfrac{a}{r} + a + ar = 38$ $\qquad\qquad\qquad$ (1)

and $\left(\dfrac{a}{r}\right)(a)(ar) = 1728$ $\qquad\qquad\qquad$ (2)

We can write (2) as $a^3 = 1728 = 12^3 \Rightarrow a = 12$.

Putting $a = 12$ in (1), we get $\dfrac{12}{r} + 12 + 12r = 38$

$\Rightarrow \dfrac{12}{r} + 12r = 26$ or $\dfrac{1}{r} + r = \dfrac{26}{12} = \dfrac{13}{6}$

$\Rightarrow 6(r^2 + 1) = 13r \Rightarrow 6r^2 – 13r + 6 = 0 \Rightarrow 6r^2 – 9r – 4r + 6 = 0$

$\Rightarrow 3r(2r – 3) – 2(2r – 3) = 0 \Rightarrow (3r – 2)(2r – 3) = 0 \Rightarrow r = 2/3$ or $3/2$.

When $a = 12$ and $r = \dfrac{2}{3}, \qquad \dfrac{a}{r} = \dfrac{12}{2/3} = 18, a = 12, ar = 12\left(\dfrac{2}{3}\right) = 8.$

When $a = 12$ and $r = \dfrac{3}{2}$, $\dfrac{a}{r} = \dfrac{12}{3/2} = 8, a = 12, ar = 12\left(\dfrac{3}{2}\right) = 8.$

Hence, the numbers are either 18, 12, 8 or 8, 12, 18.

Sum of the Infinite Number of a G.P.

If $-1 < r < 1$, then sum of the infinite G.P.

$$a + ar + ar^2 + \cdots$$

is $\dfrac{a}{1 - r}$

Example 22: Find the sum of the infinite G.P.

$$1, -1/3, 1/9, -1/27\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution: Here $a = 1$, $r = -1/3$

Thus, sum of the infinite G.P. is

$$\frac{a}{1 - r} = \frac{a}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{4}$$

Example 23 : The common ratio of a GP is –4/5 and the sum to infinity is 80/9. Find the first term of the GP.

Solution : Here $r = -4/5$

We are given

$$\frac{a}{1 - r} = \frac{80}{9}$$

$$\Rightarrow a = \frac{80}{9}(1 - r) = \frac{80}{9}\left[1 - \left(\frac{4}{5}\right)\right]$$

$$= \frac{80}{9} \times \frac{9}{5} = 16$$

Thus, first term of the G.P. is 16.

Check Your Progress – 4

1. Find the sum of 10 terms and n terms of the G.P.

$$1, \frac{2}{3}, \frac{4}{9}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.,$$

2. Find the sum of 12 terms and n terms of the G.P.

$$2, -\frac{1}{2}, \frac{1}{8}$$

3. Find the sum to *n* terms of the series

   $5 + 55 + 555 + \ldots\ldots\ldots$

4. Find the sum to *n* terms of the series

   $0.6 + 0.66 + 0.666 + \ldots\ldots\ldots\ldots\ldots$

5. Show that

   $\underbrace{111\ldots\ldots\ldots\ldots1}_{}$

   91 times

   is not a prime number.

6. The sum of three numbers in G.P. is 31 and sum of their squares is 651. Find the numbers.

7. Find the of the infinite G.P.

   $7, \ -1, \ \dfrac{1}{7}, \ -\dfrac{1}{49}\ldots\ldots\ldots\ldots\ldots\ldots,$

8. Find the sum of an infinite G.P. whose first term is 28 and the fourth term is 4/99.

1.6   ARITHEMETIC – GEOMETRIC PROGRESSION (A.G.P.)

A sequence is said to be arithmetic geometric sequence if the *nth* term of the sequence is obtained by multiplying the *nth* term of A.P. and a G.P.

Thus, A.G.P. is given by

ab, (a + d) br, (a +2d) br² …………….

*n*th term of the A. G. P. is given by $a_n = (a + \overline{n - 1})br^{n-1}$

Sum to *n* terms of the A.G.P. is given by

$$S_n = \frac{ab}{1 - r} + \frac{bdr(1 - r^{n-1})}{(1 - r)^2} - \frac{[a + (n - 1)d]br^n}{1 - r}, \qquad r \neq 1$$

If $r = 1$, then $Sn = b\dfrac{n}{2}[2a + (n - 1)\,d]$

Sum of Infinite terms of A.G.P

If $|r| < 1, \ r^{n-1} \to 0, r^n \to 0, nr^n \to 0$

21

As $n \to \infty$ the sum of infinite A.G.P. is

$$S = \frac{ab}{1-r} + \frac{bdr}{(1-r)^2}$$

Example 23: Find the sum to n terms of the series.

$$1 + \frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots\ldots\ldots\ldots\ldots$$

Solution : Here $a = 1$, $d = 3$, $b = 1$, $r = 1/5$

$$S_n = \frac{1}{1 - 1/5} + \frac{3\left(\frac{1}{5}\right)\left[1 - \left(\frac{1}{5}\right)^{n-1}\right]}{(1 - \frac{1}{5})^2} - \frac{[1 + (n-1)(3)]\left(\frac{1}{5}\right)^n}{1 - 1/5}$$

$$= \frac{5}{4} + \frac{15}{16}\left[1 - \left(\frac{1}{5}\right)^{n-1}\right] - \frac{5}{4}[3n - 2]\left(\frac{1}{5}\right)^n$$

$$= \frac{5}{4} + \frac{15}{16} - \left[\frac{15}{16} + \frac{3n - 2}{4}\right]\left(\frac{1}{5}\right)^{n-1}$$

$$= \frac{35}{16} - \frac{12n + 7}{16}\left(\frac{1}{5}\right)^{n-1}$$

Example 24: Find the sum to infinite number of terms of A.G. P.

$$3 + 5\,(1/4) + 7\left(\frac{1}{4}\right)^2 + 9\left(\frac{1}{4}\right)^3 + \ldots\ldots\ldots\ldots$$

Solution : Here $a = 3$, $d = 2$, $b = 1$, $r = 1/4$

$$\text{Thus,} \quad S = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2}$$

$$= \frac{3}{1 - 1/4} + \frac{2\left[\frac{1}{4}\right]}{\left(1 - \frac{1}{4}\right)^2}$$

$$= \frac{3(4)}{3} + \frac{2(4)}{9} = \frac{44}{9}$$

Check Your Progress 5

For Question 1– 2 : Find the sum to n terms of the A.G.P.

1. $3 + \frac{5}{4} + \frac{7}{4^2} + \frac{9}{4^3} + \ldots\ldots\ldots\ldots$

2. $1 + 3x + 5x^2 + 7\,x^3 + \ldots\ldots\ldots\ldots\ , x \neq 1$

3. Find the sum of the infinite number of terms of the A.G. P.

$$1 + 5x + 9x^2 + 13\,x^3 + \ldots\ldots\ldots\ldots, \quad (|x| < 1)$$

1.7 HARMONIC PROGRESSION (H.P.)

A sequence a_1, a_2, a_3……………of non-zero numbers is said to a harmonic progression (H.P.) if

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \dots \dots \dots \dots$$

forms an A.P.

Example 25 : Find the 10th term of the H.P.

$$\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \dots \dots \dots \dots \dots \dots$$

Solution : We find the 10th term of the A.P. 3, 7, 11, 15…………

$$a_{10} = 3 + (10 - 1)(4) = 39$$

Thus, tenth term of the H. P. is $\dfrac{1}{39}$

1.8 SUM OF FIRST n NATRUAL NUMBERS THEIR SQUARES AND CUBES

The sequence of natural numbers

1,2,3,…………………..n,……………

is an AP with first term as well as the common difference equal to 1. Thus

$$1 + 2 + 3 + \dots \dots \dots + n = \frac{n}{2}(1 + n)$$

Therefore, $S_n = 1 + 2 + \dots \dots \dots n = \frac{n}{2}(n+1)$

We now consider an alternative way of obtaining the sum of first n natural numbers.

Sum of Squares of first n natural Numbers*

$$(r + 1)^2 - (r - 1)^2 = 4r$$

By letting $r = 1, 2, 3, \dots \dots \dots, (n - 2), (n - 1), n$ successively, we obtain

$2^2 - 0^2 = 4(1);$

$3^2 - 1^2 = 4(2);$

$4^2 - 2^2 = 4(3);$

$5^2 - 3^2 = 4(4);$

………………

---

* We could also have started with identity $(r + 1)^2 - r^2 = 2r + 1$

$(n-1)^2 - (n-3)^2 = 4(n-2)$;

$n^2 - (n-2)^2 = 4(n-1)$;

$(n+1)^2 - (n-1)^2 = 4(n)$;

When we add the above identities, we find that all terms except $(n+1)^2, n^2, -1^2$ and -0^2 cancel out from the LHS and we obtain

$(n+1)^2 + n^2 - 1 - 0 = 4(1 + 2 + 3 + \ldots\ldots\ldots + n)$

But $(n+1)^2 + n^2 - 1 = n^2 + 2n + 1 + n^2 - 1 = 2n^2 + 2n = 2n(2n+1)$

Therefore, $2n(n+1) = 4(1 + 2 + 3 + \ldots\ldots.. + n)$

$\Rightarrow 1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = \dfrac{1}{2}n(n+1)$

Sum of Square of First n Natural Numbers

In this case we begin with the identity

$(r+1)^3 - (r-1)^3 = 6r^2 + 2$

Letting $r = 1, 2, 3, 4, \ldots\ldots..(n-2), (n-1), n$ we obtain

$2^3 - 0^2 = 6(1^2) + 2$

$3^2 - 1^3 = 6(2^2) + 2$

$4^3 - 2^3 = 6(3^2) + 2$

$5^3 - 3^3 = 6(4^2) + 2$

$\ldots\ldots\ldots\ldots\ldots$

$(n-1)^3 - (n-3)^3 = 6(n-2)^2 + 2$

$n^3 - (n-2)^3 = 6(n-1)^2 + 2$

$(n+1)^3 - (n-1)^3 = 6(n)^2 + 2$

Adding, we obtain

$(n+1)^3 + n^3 - 1^3 - 0^3 = 6(1^2 + 2^2 + \ldots.. + n^2) + 2n$

$\Rightarrow 6(1^2 + 2^2 + \ldots.. + n^2) = (n+1)^3 + n^3 - 1 - 2n$

But $(n+1)^3 + n^3 - 1 - 2n = n^3 + 3n^2 + 3n + 1 + n^3 - 1 - 2n$

$= 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1) = n(n+1)(2n+1)$.

Thus, $6(1^2 + 2^2 + \ldots\ldots\ldots + n^2) = n(n+1)(2n+1)$

$\Rightarrow 1^2 + 2^2 + \ldots\ldots\ldots.. + n^2 = \dfrac{1}{6}n(n+1)(2n+1)$

Sum of Cubes of First n Natural Numbers

In this case, we begin with the identity

$$(r + 1)^4 - (r - 1)^4 = 8 r^3 + 8$$

Letting $r = 1, 2, 3, 4, \ldots\ldots(n{-}2), (n{-}1), n$, successively, we obtain

$$2^4 - 0^4 = 8 (1^3) + 8(1)$$
$$3^4 - 1^4 = 8 (2^3) + 8(2)$$
$$4^4 - 2^4 = 8 (3^3) + 8(3)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$(n - 1)^4 - (n - 3)^4 = 8 (n - 2)^3 + 8(n - 2)$$
$$n^4 - (n - 2)^4 = 8(n - 1)^3 + 8(n - 1)$$
$$(n + 1)^4 - (n - 1)^4 = 8(n)^3 + 8(n)$$

Adding, we obtain

$$(n + 1)^4 + n^4 - 1^4 - 0^4 = 8(1^3 + 2^3 + 3^3 \ldots\ldots + n^3) + 8(1 + 2 + 3 + \ldots + n)$$

But $(1 + 2 + 3 + \ldots\ldots + n = \frac{1}{2}n(n{+}1)$.

Thus $(n + 1)^4 + n^4 - 1 = 8(1^3 + 2^3 + 3^3 \ldots\ldots + n^3) + 4n(n + 1)$

$$= 8(1^3 + 2^3 + 3^3 + \ldots n^3) = (n + 1)^4 + n^4 - 1 - 4n (n + 1)$$

But $(n + 1)^4 + n^4 - 1 - 4n(n + 1)$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1 + n^4 - 1 + 4n^2 - 4n$$

$$= 2 n^4 + 4n^3 + 2n^2 = 2n^2 (n^2 + 2n + 1) = 2n^2(n + 1)^2$$

Thus, $8(1^3 + 2^3 + 2^3 + \ldots + n^3) = 2n^2(n + 1)^2$

$$\Rightarrow 1^3 + 2^3 + \ldots\ldots + n^3 = \frac{1}{4}n^2 (n + 1)^2$$

In sigma notation the above three identities read as

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n + 1),$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

and $$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n + 1)^2$$

Example 26 : Find the sum of the following series

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2$$

Solution : Let t_r denote the rth term of $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2$, then

$$t_r = (2r-1)^2 = 4r^2 - 4r + 1$$

Thus

$$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} (4r^2 - 4r + 1) = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

But $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\displaystyle\sum_{r=1}^{n} 1 = n$.

Therefore, $\displaystyle\sum_{r=1}^{n} t_r = 4\{\frac{1}{6}n(n+1)(2n+1)\} - 4\{\frac{1}{2}n(n+1)\} + n$

$$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n.$$

We now take $\frac{1}{3}n$ common from each on the right side, so that

$$\sum_{r=1}^{n} t_r = \frac{1}{3}n[2(n+1)(2n+1) - 6(n+1) + 3)]$$

$$= \frac{1}{3}n[2(2n^2 + 2n + n + 1) - (6n+6) + 3]$$

$$= \frac{1}{3}n[(4n^2 + 6n + 2 - 6n - 6 + 3) = \frac{1}{3}n(4n^2 - 1)$$

Example 27 : Find the sum of the series

$$(1)(2^2) + (2)(3^2) + (3)(4^2) + \dots\dots\text{upto } n \text{ terms}$$

Solution : Let t_r denote the rth term of the given series, then

$$t_r = (r)(r+1)^2 = r(r^2 + 2r + 1) = r^3 + 2r^2 + r$$

Thus, $\displaystyle\sum t_r = \sum_{r=1}^{n} (r^3 + 2r^2 + r) = \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = 1$

$$= \frac{1}{4}n^2(n+1)^2 + 2\{\frac{1}{6}n(n+1)(2n+1)\} + \frac{1}{2}n(n+1)$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{12}n(n+1) \quad [3n(n+1) + 4(2n+1) + 6]$$

$$= \frac{1}{12}n(n+1) \quad [3n^2 + 3n + 8n + 4 + 6] = \frac{1}{12}n(n+1)(3n^2 + 11n + 10)$$

$$= \frac{1}{12}n(n+1)(3n^2 + 6n + 5n + 10) = \frac{1}{12}n(n+1)[3n(n+2) + 5(n+2)]$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+5).$$

27

Check Your Progress 6

1. Find the nth term of the H.P. $\dfrac{1}{2}, \dfrac{1}{7}, \dfrac{1}{12}, \dfrac{1}{17}, \ldots \ldots \ldots \ldots$

2. $2^2 + 4^2 + 6^2 + \ldots \ldots + (2n)^2$

3. $1.2.3 + 2.3.4 + 3.4.5 + \ldots \ldots$ upto n terms

4. $1.3.5 + 3.5.7 + 5.7.9 + \ldots \ldots$ upto n terms

1.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. $d = 11 - 16 = -5$

   Next four terms are

   $-4, -9, -14, -19$

2. $d = 5 - 2 = 3$

   Next four terms are

   $17, 26, 23, 26$

3. $d = \dfrac{2n+1}{n} - \dfrac{1}{n} = 2 + \dfrac{1}{n} - \dfrac{1}{n} = 2$

   Next four terms are

   $\dfrac{6n+1}{n} + 2, \quad \dfrac{6n+1}{n} + 2 + 2, \quad \dfrac{6n+1}{n} + 3\,(2), \quad \dfrac{6n+1}{n} + 4\,(2)$

   or $\dfrac{8n+1}{n}, \quad \dfrac{10n+1}{n}, \quad \dfrac{12n+1}{n}, \quad \dfrac{14n+1}{n}$

4. $d = (m-2) - (m-1) = -1$

   next four terms are

   $m-5, m-6, m-7, m-8$

5. The given A.P, is

   $\sqrt{3}, \sqrt[2]{3}, \sqrt[3]{3}, \sqrt[4]{3}$

   $\therefore d = \sqrt{3}$

   Next four terms are

   $\sqrt[5]{3}, \sqrt[6]{3}, \sqrt[7]{3}, \sqrt[8]{3},$

6. We have $a_r = a(r-1)d$, so that

$$a_{p+q} + a_{p-q} = a + (p+q-1) + a + (p-q-1)d$$
$$= 2a + (p+q-1+p-q-1)d$$
$$= 2a + 2(p-1)d = 2ap$$

7. $k - \left(\dfrac{2}{3} - k\right) = \left(\dfrac{5}{8} - k\right) - k$

$\Rightarrow 2k - 2/3 = 5/8 - 2k$

$\Rightarrow 4k = 2/3 + 5/8 = 31/24$

$\Rightarrow k = 31/96$

8. $d = 16\dfrac{1}{5} - 17 = -\dfrac{4}{5}$

Let nth term be the first negative term we have

$$a_n = a + (n-1)d = 17(n-1)\left(-\dfrac{4}{5}\right) < 0$$

$$\Rightarrow 17 < (n-1)\left(\dfrac{4}{5}\right) \Rightarrow n - 1 > \dfrac{85}{4}$$

$$\Rightarrow n > \left(\dfrac{89}{4}\right) = 22\dfrac{1}{4}$$

Thus, first negative is the 23rd term.

9. $a_4 = 3a_1$

$\Rightarrow \quad a + 3d = 3a$

$\Rightarrow \quad 3d = 2a$ \hfill (i)

Also, $a_7 - 2a_3 = 1$

$\Rightarrow \quad a + 6d - 2[a + 2d] = 1$

$\Rightarrow \quad a + 6d - 2a - 4d = 1$

$\Rightarrow \quad -a + 2d = 1$ \hfill (ii)

Putting $d = 2a/3$ (from (i)) in (ii), we get

$$-a + \dfrac{4a}{3} = 1 \Rightarrow \dfrac{a}{3} = 1 \Rightarrow a = 3$$

From (i), $3d = 2(3) \Rightarrow d = 2$

10. $a_{p+1} = 2a_{q+1}$

$\Rightarrow \quad a + (p+1-1)d = 2[a + ((q+1) - 1)d]$

$\Rightarrow \quad a + pd = 2[a + qd]$

$$\Rightarrow \quad a = (p - 2q)d$$

Now, $a_{3p+1} = a + (3p + 1 - 1)d$

$$= a + 3pd = (p - 2q)d + 3pd$$

$$= (4p - 2q)d = 2(2p - q)d$$

and $a_{p+q+1} = a + (p + q + 1 - 1)d$

$$= (p - 2q)d + (p + q)d$$

$$= (p - 2q + p + q)d$$

$$= (2p - q)d$$

Thus, $a_{3p+1} = 2a_{p+q+1}$

11. $7a_7 = 11a_{11}$

$$\Rightarrow \quad 7[a + 6d] = 11[a + 10d]$$

$$\Rightarrow \quad 7a + 42d = 11a + 110d$$

$$\Rightarrow \quad 4a + 68d = 0 \Rightarrow a + 17d = 0$$

$$\Rightarrow \quad a_{18} = 0$$

Check Your Progress 2

1. Here $a = 1$, $d = 2$

$$S_{100} = \frac{100}{2}[2a + (100 - 1)d]$$

$$= 50[2(1) + (99)2] = (50)(200)$$

$$= 10,000$$

$$S_{200} = \frac{200}{2}[2a + (200 - 1)d]$$

$$= 100[2(1) + (200 - 1)2] = (100)(400)$$

$$= 40,000$$

2. $a = 0.9$, $d = 0.01$

$$S_{20} = \frac{20}{2}[2a + (20 - 1)d]$$

$$= 10[2(0.9) + (19)0.01] = 19.9$$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$= \frac{n}{2}[2(0.9) + (n - 1)(0.01)]$$

$$= n(n + 179)/200$$

3. $a = \dfrac{m-1}{m}, d = \dfrac{m-2}{m} - \dfrac{m-1}{m} = -\dfrac{1}{m}$

$$S_{10} = \frac{10}{2} [2a + (10 - 1)d]$$

$$= 5 \left[2 \frac{(m-1)}{m} + 9 \left(\frac{-1}{m} \right) \right]$$

$$= 5 (2m - 11)/m$$

$$S_n = \frac{n}{2} [2a + (n - 1)]d = \frac{n}{2} \left[2 \frac{(m-1)}{m} + \frac{(n-1)(-1)}{m} \right]$$

$$= \frac{n}{2m} [2m - 2 - n + 1] = \frac{n}{2m} (2m - n - 1)$$

4. $a = 22, d = -4$

$$S_n = \frac{n}{2} [2a + (n - 1)d] = \frac{n}{2} [2(22) + (n-1)(-4)].$$

$$= \frac{n(4)}{2} [11 - n + 1] = 2n(12 - n)$$

Now, $2n (12 - n) = 64$

$\Rightarrow$ $n^2 - 12n + 32 = 0$

$\Rightarrow$ $(n - 4)(n - 8) = 0$

$\Rightarrow$ $n = 4, n = 8$

Double answer occurs as

$a_5 + a_6 + a_7 + a_8$

$= 6 + 2 - 2 - 6 = 0$

5. $S_p = 3p^2 + 4p$

$$a_n = S_n - S_{n-1}$$

$$= 3n^2 + 4n - [3(n - 1)^2 + 4(n - 1)]$$

$$= 3n^2 + 4n - [3n^2 - 2n + 1 + 4n - 4]$$

$$= 3n^2 + 4n - [3n^2 - 6n + 3 + 4n - 4]$$

$$= 6n + 1$$

6.   The smallest 3 digit number which leaves remainder 1 when divided by 4 is 101 and the last 3 digit number with this property is 997. Let 997 be the nth term of the A.P. then

$$997 = 101 + (n - 1)(4) \Rightarrow n = 225$$

Required sum $= \frac{n}{2} [a + l] = \frac{225}{2} (101 + 997) = 123525$

Check Your Progress – 3

1. $r = \dfrac{1}{2}$

2. $r = \dfrac{1}{(-3)} = -\dfrac{1}{3}$

3. $r = \dfrac{6}{3} = 2$

4. $r = \dfrac{-96}{128} = -\dfrac{3}{4}$

   $a_n = a\ r^{n-1} = 128\left(-\dfrac{3}{4}\right)^{n-1}$

5. $r = \dfrac{-110}{100} = -1.1$

   $a_n = a\ r^{n-1} = 100(-1.1)^{n-1}$

6. $r = \dfrac{3.3}{3} = 1.1.$

   $a_n = r^{n-1} = 3(1.1)^{n-1}$

7. $a_{18} = a\ r^{18-1} = a\ r^{17}$

   $a_5 = a\ r^{5-1} = a\ r^4 = 1$

   Now, $\dfrac{a_{18}}{a_5} = \dfrac{a\ r^{17}}{a\ r^4} = r^{13}$

   $\Rightarrow a_{18} = a_5 r^{13} = 1\left(\dfrac{2}{3}\right)^{13}$

8. $a = a_5 = AR^4$

   $b = a_8 = AR^7$

   and $c = a_{11} = AR^{10}$

   We have $\dfrac{b}{a} = R^3 = \dfrac{c}{b}$

   $\Rightarrow a, b, c$ *are in G.P.*

Check Your Progress – 4

1. Here $a = 1,\ r = 2/3$

$$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{1-\left(\dfrac{2}{3}\right)^{10}}{1-\left(\dfrac{2}{3}\right)} = 3\left[1-\left(\dfrac{2}{3}\right)^{10}\right]$$

and $s_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{1-\left(\dfrac{2}{3}\right)^n}{1-2/3} = 3\left[1-\left(\dfrac{2}{3}\right)^n\right]$

2. $a = 2, \quad r = \dfrac{-1/2}{2} = -\dfrac{1}{4}$

$$S_{12} = \frac{a(1-r^{12})}{1-r} = \frac{2\left(1-\left(-\dfrac{1}{4}\right)^{12}\right)}{1-\left(\dfrac{-1}{4}\right)} = \frac{8}{5}\left[1-\left(\dfrac{1}{4}\right)^{12}\right]$$

and $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{2\left(1-\left(-\dfrac{1}{4}\right)^{n}\right)}{1-\left(\dfrac{-1}{4}\right)} = \dfrac{8}{5}\left[1-\left(-\dfrac{1}{4}\right)^{n}\right]$

3. Let $s_n = 5 + 55 + 555 + \ldots\ldots + $ upto n terms

$= 5\,[1 + 11 + 111 + \ldots\ldots\ldots + $ upto n terms $]$

$= \dfrac{5}{9}\,[9 + 99 + 999 + \ldots\ldots\ldots. + $ upto n terms $]$

$= \dfrac{5}{9}\,[(10-1) + (10_2 - 1) + (10_3 - 1) + \ldots\ldots\ldots. + (10n - 1)]$

$= \dfrac{5}{9}\,[(10 + 10_2 + \cdots\ldots\ldots\ldots\ldots.. + 10_n\,]$

$= \dfrac{5}{9}\left[\dfrac{10(10^n - 1)}{10-1} - n\right] = \dfrac{5}{9}\left\{\dfrac{10(10^n - 1)}{10-1}\right\} - n]$

4. Let $S_n = 0.6 + 0.66 + 0.666 + \ldots\ldots\ldots\ldots + $ upto n terms

$= 6[0.1 + 0.11 + 0.111 + \ldots\ldots\ldots. + $ upto n terms $]$

$= \dfrac{6}{9}\,[0.9 + 0.99 + 0.999 + \ldots\ldots\ldots. + $ upto n terms $]$

$= \dfrac{2}{3}\,[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + \ldots\ldots\ldots. (1 - (0.1)^n)]$

$= \dfrac{2}{3}\,[\,n - \{0.1 + (0.1)^2 + \ldots\ldots\ldots.. + (0.1)^n\}\,]$

$= \dfrac{2}{3}\,[\,n - \{0.1 + (0.1)^2 + \ldots\ldots\ldots.. + (0.1)^n\}]$

$= \dfrac{2}{3}\left[n - \dfrac{(0.1)(1 - (0.1)^n)}{1 - 0.1}\right]$

$= \dfrac{2}{3}\,[n - \dfrac{1}{9}(1 - (0.1)^n)$

5. We have

$$A = \underbrace{111\ldots\ldots\ldots\ldots1}_{91 \text{ times}} = \frac{1}{9}\underbrace{(99\ldots\ldots\ldots9)}_{91 \text{ times}}$$

$$= \frac{1}{9}\,(10^{91} - 1)$$

$$= \frac{10^{91} - 1}{10^{13} - 1} \cdot \frac{10^{13} - 1}{10 - 1}$$

$$= [1 + 10^{13} + 10^{26} + \ldots\ldots\ldots\ldots 10^{78}]$$

$$[1 + 10 \ldots\ldots\ldots 10^{12}]$$

$\Rightarrow$ a is not a prime number.

6. Let three numbers in G. P. be $\dfrac{a}{r}, a, ar$

We have

$$\frac{a}{r} + a + ar = 31$$

and $\dfrac{a^2}{r^2} + a^2 + a^2\,r^2 = 651$

$$\Rightarrow a\left(r + \frac{1}{r} + 1\right) = 31$$

and $a^2\left(r^2 + \dfrac{1}{r^2} + 2 - 1\right) = 651$

$$\Rightarrow a\left(r + \frac{1}{r} - 1\right) = 31$$

and $a^2\left[\left(r + \dfrac{1}{r}\right)^2 - 1\right] = 651$

$$\Rightarrow a\left(r + \frac{1}{r} + 1\right) = 31$$

and $a\left(r + \dfrac{1}{r} + 1\right) = \dfrac{651}{31} = 21$

Subtracting we get $2a = 10 \Rightarrow a = 5$.

Also, $5\left(r + \dfrac{1}{r} + 1\right) = 31$

$$\Rightarrow r + \frac{1}{r} = \frac{26}{5} \Rightarrow r = 5, \frac{1}{5}$$

Thus, numbers are

1, 5, 25 or 25, 5, 1

7. $S = \dfrac{a}{1-r} = \dfrac{7}{1-\left(-\dfrac{1}{7}\right)} = \dfrac{49}{8}$

8. $a = 28, \qquad ar^3 = \dfrac{4}{49}$

$\Rightarrow r^3 = \dfrac{4}{49} \times \dfrac{1}{28} = \dfrac{1}{7^3}$

$\Rightarrow r = 1/7$

Thus, $S = \dfrac{a}{1-r} = \dfrac{28}{1-1/7} = \dfrac{28 \times 7}{6} = \dfrac{98}{3}$

Check Your Progress – 5

1. Here, $a = 3$, $d = 2$, $r = 1/4$

   Thus

   $$S_n = \dfrac{3}{1-1/4} + \dfrac{(2)\left(\dfrac{1}{4}\right)\left[1-\left(\dfrac{1}{4}\right)^{n-1}\right]}{\left(1-\dfrac{1}{4}\right)^2} - \dfrac{3+(n-1)(2)\ (1/4)^n}{1-\dfrac{1}{4}}$$

   $$= 4 + \dfrac{8}{9}\left[1-\left(\dfrac{1}{4}\right)^{n-1}\right] - \dfrac{1}{3}(2n-1)\left(\dfrac{1}{4}\right)^{n-1}$$

   $$= 4 + \dfrac{8}{9}\left[\dfrac{8}{9} + \dfrac{2n+1}{3}\right]\left(\dfrac{1}{4}\right)^{n-1}$$

   $$= \dfrac{44}{9} - \dfrac{6n+11}{9}\left(\dfrac{1}{4}\right)^{n-1}$$

2. Here, $a = 1$, $d = 2$, $r = x$ Thus,

   $$S_n = \dfrac{1}{1-x} - \dfrac{(2)(x)(1-x^{n-1})}{(1-x)^2} - \dfrac{(1+2n-1)x^n}{1-x}$$

   $$= \dfrac{1}{1-x} + \dfrac{2x}{(1-x)^2} + \dfrac{2x^n}{(1-x)^2} - \dfrac{(2n-1)x^n}{1-x}$$

   $$= \dfrac{1-3x}{(1-x)^2} + \dfrac{2x^n}{(1-x)^2} - \dfrac{(2n-1)x^n}{1-x}$$

3. Here $a = 1$, $d = 4$, $r = x$

   Thus,

   $$S = \dfrac{1}{1-x} + \dfrac{4x}{(1-x)^2} = \dfrac{1-x+4x}{(1-x)^2}$$

35

$$= \frac{1 - 3x}{(1 - x)^2}$$

Check Your Progress 6

1. 2,7,12,17 ………. are in A.P.

    $$a_n = a + (n - 1)d = 2 + (n - 1)(5) = 5n - 3$$

    Thus, nth term of the H. P. is $\dfrac{1}{5n - 3}$

2. We have

    $$2^4 + 2^4 \dots \dots \dots \dots \dots \dots + (n^2)^2$$

    $$= 2^2 [1^2 + 2^2 + \dots \dots \dots . 2^n]$$

    $$= 4 \frac{n(n + 1)(2n + 1)}{6} = \frac{2}{3} n(n + 1)(2n + 1)$$

3. The rth term of $1.2.3. + 2\,3.4 + 3.4.5 + \dots$ is given by

    $$t_r = r(r + 1)(r + 2) = r(r^2 + r + 2r + 2) = r^3 + 3r^2 + 2r$$

    Thus $\displaystyle\sum_{r=1}^{n} t_r = \sum_{r=1}^{n}(r^3 + 3r^2 + 2r) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

    $$= \frac{1}{4} n^2 (n + 1)^2 + 3\left\{\left\{\frac{1}{6} n(n + 1)(2n + 1)\right\} + 2\left\{\frac{1}{2} n(n + 1)\right\}\right\}$$

    $$= \frac{1}{4} n^2 (n + 1)^2 + \frac{1}{2} n(n + 1)(2n + 1) + n(n + 1)$$

    $$= \frac{1}{4} n(n + 1)[n(n + 1) + 2(2n + 1) + 4] = \frac{1}{2} n(n + 1)[n^2 + n + 4n + 2 + 4]$$

    $$= \frac{1}{4} n(n + 1)(n^2 + 5n + 6) = \frac{1}{4} n(n + 1)(n + 2)(n + 3)$$

4. The rth term of the series is given by

    $$t_r = (2r - 1)(2r + 1)(2r + 3) = 8\,r^3 + 12r^3 - 2\,r - 3$$

    Therefore, S_n the sum to n terms of the series is given by

    $$S_n = 8\sum_{r=1}^{n} r^3 + 12\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r - 3n$$

    $$= 8\left[\frac{n(n + 1)}{2}\right]^2 + 12\,\frac{n(n + 1)(2n + 1)}{6} - 2\,\frac{n(n + 1)}{2} - 3n$$

$$= 2n^2(n + 1)^2 + 2n(n + 1)(2n + 1) - n(n + 1) - 3n$$

$$= n(2n^3 + 4n^2 + 2n + 4n^2 + 6n + 2 - n - 1 - 3)$$

$$= n(2n^3 + 8n^2 + 7n - 2)$$

1.10 SUMMARY

In this unit, the well-known concepts of arithmetic progression (A.P.), geometric progression (G.P.), arithmetico-geometric progression (A.G.P.) and harmonic progression as special cases of sequences of numbers, are discussed. First of all, each of these concepts is defined. Then for each concept, through suitable examples, methods for finding nth term and sum upto n terms are explained. A.P is discussed in **sections 1.2 and 1.3,** G.P is discussed in **sections 1.4 and 1.5,** A.G.P is discussed in **section 1.6** and H.P. is discussed in **section 1.7**. The derivation of formulae for the sum of natural numbers, their squares and cubes are discussed in **section 1.8.**

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.9**.