Naïve Tsunami Generation via Shallow Water Theory

MATH484 Final Project Report

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"The Great Wave off Kanagawa", Katsushika Hokusai est. 1829

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1 Introduction

A tsunami is a single or series of ocean waves which are generated by sudden displacements in the sea floor, landslides, or volcanic activity [1]. These potentially catastrophic ocean waves frequently occur in a basin of the Pacfic Ocean known as the "Ring of Fire" for its historically intense volcanic activity [1]. Further background into the geological mechanisms resposible for tsunamis can be found in Zirker's (2013) introductory text *The Science of Ocean Waves: Ripples, Tsunamis, and Stormy Seas* [1]. An example of tsunami generation can be seen in figure 1, which was taken from chapter 3 of Marghany's (2018) *Advanced Geoscience Remote Sensing* [2].

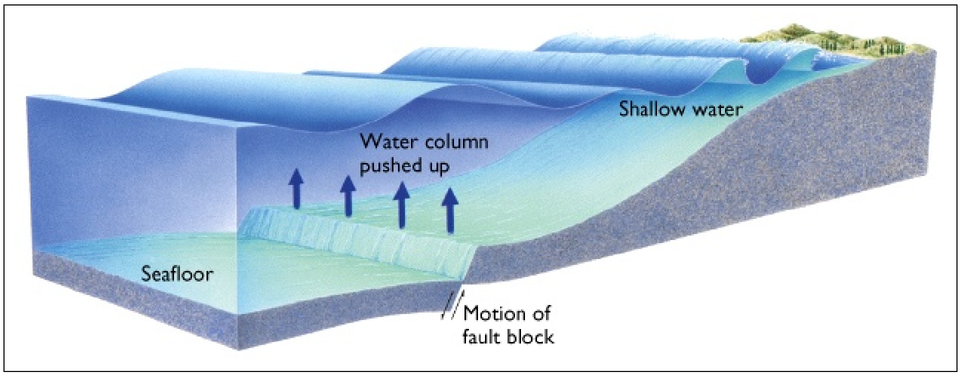


Figure 1: Tsnumai generation due to horizontal seafloor displacement [2].

Transient horizontal displacements resulting from submarine earthquakes and their role in tsunami generation were the focus of Tanioka' and Satake's (1996) study *Tsunami generation by horizontal displacement of ocean bottom* published in Geophysical Research Letters [3]. In this study, they utilize the Satake's (1995) earlier work *Linear and nonlinear computations of the 1992 Nicaragua earthquake tsunami*, which compared and contrasted the success of linear and nonlinear models that governed the generation and propagaion of tsunamis. The linear model utilized in Satake's case study of the Nicaragua earthquake, known as the *shallow-water equations*, and its numerical simulation via a finite-difference method will be the focus of this project.

2 Shallow-Water Theory

Shallow-Water Theory is the canonical mathematical model for waves propagating non-dispersively in a medium which is much wider than it is deep. The *shallow-water equations* are a system of linear partial-differential equations at the center of Shallow-Water Theory. In this section they will be stated, for a full derivation, see appendix **??**. The linear shallow-water equations for a single spatial dimension x and seafloor shape $H = H(x, y)$ are

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial x}\left(gH\frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(gH\frac{\partial \eta}{\partial y}\right) \tag{1}$$

3 The Conservative Form of a Hyperbolic Equation

3.1 Introduction

Before we discuss our process of selection for the scheme used to numerically solve the linear shallow-water equations, it is important to understand the idea of a *conservative form* of a hyperbolic partial-differential equation. First we state the conservative form:

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{F}(u) = 0, \tag{2}$$

where u is the *density of the conserved quantity*, and $\vec{F}(u)$ is the *density flux*. Note that the actual form of $\vec{F}(u)$ is dependent on the original hyperbolic equation that has been converted into this form. Since u is a density, we will denote its corresponding quantity $\mathbf{u}$ defined by:

$$\mathbf{u} = \int_V u \, dV, \tag{3}$$

where V denotes the spatial domain. In physics, a *conservation law* requires that some measurable property of a system does not change with respect to time. For example, the conservation of mass (ignoring energy) requires that any system with perfectly balanced mass transfers, or net-zero mass flux along its boundaries, cannot change its total mass over time. To illustrate the connection between such a conservation law and the conservative form of a hyperbolic equation, we present a short derivation proving the conservation of the quantity $\mathbf{u}$. To start, integrate both sides of the conservative form 2 term-by-term over the spatial domain V:

$$\int_V \frac{\partial u}{\partial t} dV + \int_V \nabla \cdot \vec{F}(u) \, dV = 0. \tag{4}$$

Now, we apply Leibniz's rule to change the order of differentiation in the first integral:

$$\int_V \frac{\partial u}{\partial t} dV = \frac{d}{dt} \int_V u \, dV. \tag{5}$$

Then, we use the divergence theorem on the second integral, which states that integrating the divergence of the vector field, $\nabla \cdot \vec{F}(u)$, over the spatial domain V is equal to integrating the vector field over the spatial domain's boundary A:

$$\int_V \nabla \cdot F(u) \, dV = \int_A \vec{F}(u) \cdot \vec{n} \, dA, \tag{6}$$

where $\vec{n}$ is the outward-pointing unit vector along the boundary A (see figure 2). By substituting 5 and 6 into 4, one obtains

$$\frac{d}{dt} \int_V u \, dV = - \int_A \vec{F}(u) \cdot \vec{n} \, dA. \tag{7}$$

Thus, the time derivative of $\mathbf{u}$ (see 3) in V is equal to the net flux across the boundary A (the second integral). Note that the unit vectors $\vec{n}$ are outward-pointing along A, meaning that $-\vec{n}$ must be inward-pointing. Thus, this equation does not necessarily say that $\mathbf{m}$ is conserved: it is a more general statement that $\mathbf{m}$ can only change if there is non-zero net flux along the boundary A.

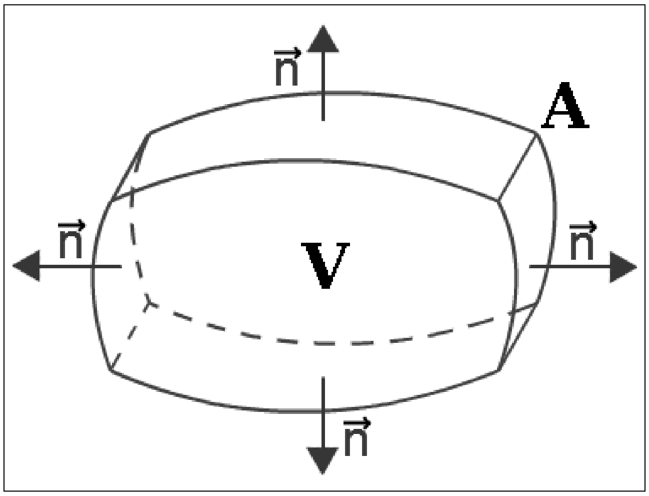


Figure 2: A spatial domain Ω and the unit vectors $\vec{n}$ along its boundary $\partial\Omega$.

Now we return to the law of conservation of mass to provide practical intuition regarding 7. Assume that the physical system in question is a mass distribution m over some spatial domain V, and some mass transfer $\vec{F}(m)$ within V as well as along A. We may state the law of conservation of mass in this case using our previous result 7:

$$\frac{d}{dt}\int_V m\, dV = -\int_A \vec{F}(m)\cdot\vec{n}\, dA. \tag{8}$$

This offers a clearer picture of the significance of the conserved quantity, as in this case it is the total mass $\mathbf{m}$ of the system:

$$\mathbf{m} = \int_V m\, dV,$$

In the case where the system is isolated with respect to mass transfers, we know that the flux A is zero everywhere along the boundary:

$$\vec{F}(m) = 0, \quad \text{along } A.$$

This implies that

$$\int_A \vec{F}(m)\cdot\vec{n}\, dA = 0, \quad \text{as } \vec{F}(m) = 0 \text{ along } A.$$

Thus, by the statement 8, we have

$$\frac{d}{dt}\int_V m\, dV = \frac{d}{dt}\mathbf{m} = 0,$$

that is: the total mass of our system does not change with respect to time. If a system cannot gain or lose mass, the total mass does not change. However, our statement of 8 allows for a more general result: if the net flux along the boundary is zero, we have:

$$\int_A \vec{F}(m)\cdot\vec{n}\, dA = 0.$$

This implies by 8 that the total mass, yet again, cannot change in time. If a system is given some mass but the exactly same amount of mass is removed from the system, the total mass does not change even though the system is not closed with respect to mass transfers along its boundary. Further, if we know the net flux, then we know exactly how the total mass $\mathbf{m}$ will change in time.

3.2 The Conservative Form of The Shallow-Water Equations

First we derive the conservative form of the 1D Shallow-Water Equations. The 1D form of the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(gH\frac{\partial u}{\partial x}\right) \tag{9}$$

We convert to flux-conservative form by defining two substitution variables:

$$\left.\begin{aligned}\alpha &= \frac{\partial u}{\partial x}\\ \gamma &= \frac{\partial u}{\partial t}\end{aligned}\right\} \Rightarrow \mathbf{U} = \begin{bmatrix}\alpha\\ \gamma\end{bmatrix}.$$

Our goal is to decouple the equation 9 into a system of two equations. This is accomplished by taking derivatives of the above defined substitution variables with respect to time, and applying the law of the equivalence of mixed partials:

$$\begin{aligned}\frac{\partial \alpha}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial \gamma}{\partial x},\\ \frac{\partial \gamma}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial \gamma}{\partial t} = \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(gH\alpha\right).\end{aligned}$$

This leaves us with the system:

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha\\ \gamma\end{bmatrix} + \begin{bmatrix}\partial_x & 0\\ 0 & \partial_x\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ -gH & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}\right) = \mathbf{0},$$

which is written succinctly as:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla\cdot\mathbf{F}(\mathbf{U}) = \mathbf{0}.$$

4 Numerical Methodolgy

4.1 A Test Problem

In this section we derive and evaluate a number of common numerical schemes for the solution of a test problem: the wave equation in one dimensions

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } (x) \in V, \tag{10}$$

where c denotes the propagation speed, and Ω denotes the interior of our spatial domain. For this equation, we consider homogenous Dirchlet boundary conditions

$$u(t,x) = 0 \qquad \text{for } x \in A, \tag{11}$$

where $\partial\Omega$ denotes the exterior of our spatial domain. Finally, we also consider some initial condition

$$u(t,x,y) = f(x,y) \qquad \text{for } t = t_0. \tag{12}$$

The equation 1 that we plan to numerically solve is very similar to this equation, as both are linear second-order hyperbolic partial differential equations. Further, we will be using the same boundary and initial conditions. Because of this, we select the scheme that we find to be optimal in terms of both time and accuracy.
However, we first state in conservative form. To do this we define variables α and γ such that:

$$\left.\begin{aligned} \gamma &= \frac{\partial u}{\partial t}, \\ \alpha &= c\frac{\partial u}{\partial x}. \end{aligned}\right\} \Rightarrow \mathbf{U} = \begin{bmatrix} \gamma \\ \alpha \end{bmatrix}$$

Now, we take derivatives with respect to time obtain a coupled system of equations:

$$\begin{aligned} \frac{\partial \gamma}{\partial t} &= \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2 \frac{\partial u}{\partial x}\right) = c^2 \frac{\partial \alpha}{\partial x} \\ \frac{\partial \alpha}{\partial t} &= \frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial \gamma}{\partial x} \end{aligned}$$

We may write this in matrix form by:

$$\frac{\partial}{\partial t}\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} + \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x \end{bmatrix}\left(\begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}\right) = \mathbf{0},$$

4.2 The Lax-Wendroff Scheme

The Lax-Wendroff scheme [5] is a combination of the Lax-Friedrichs scheme and the leapfrog scheme. First, the half-steps must be computed as below:

$$u_{j\pm 1/2}^{n+1/2} = \frac{1}{2}(u_j^n + u_{j\pm 1}^n) \mp \frac{\alpha}{2}(u_{j\pm 1}^n - u_j^n). \tag{13}$$

These are then used in the computation of a leapfrog half-step:

$$u_j^{n+1} = u_j^n - \alpha(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}). \tag{14}$$

Substitution then yields the numerical scheme as it was implemented in our program:

$$u_j^{n+1} = u_j^n - \frac{\alpha}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\alpha^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{15}$$

4.2.1 Boundary Conditions

Our first implementation of Lax-Wendroff applied to the wave equation implemented Dirichlet boundary conditions, with the solution at the endpoints being set to 0. This caused numerical instability in the solution which worsened as the number of time steps used in the discretization increased.

Our solution to this was to implement Sommerfeld boundary conditions (also called radiative boundary conditions). These boundary conditions allow the wave to freely move out of our system when it reaches the boundary by applying a numerical scheme for basic advection. The following numerical scheme was implemented to allow the waves to advect from the system:

$$u_{j+1}^{n+1} = u_j^n - u_j^{n+1}Q + u_{j+1}^n Q \tag{16}$$

for the outer (positive x) edge and

$$u_j^{n+1} = u_{j+1}^n - u_{j+1}^{n+1}Q + u_j^n Q \tag{17}$$

for the inner (negative x) edge, where

$$Q = \frac{(1-\alpha)}{(1+\alpha)}. \tag{18}$$

5 Application: Fukushima Daiichi Nuclear Disaster

The shallow water equation requires a choice of a particular seafloor shape $H = H(x, y)$. To test the shallow-water equation, the geography of eastern Japan, specifically Ōkuma, Fukushima will be used and a tsunami resulting from an earthquake with its epicenter at the same location as the Tōhoku earthquake will be simulated. If the tsnumai's wave front reaches the Fukushima Daiichi Nuclear Power Plant, it will be demonstrated whether or not the shallow-wave equation is suitable for simulating not only tsunamis, but areas that may be at risk of flooding from tsunamis.

6 Interpolating Sea Floor: Fukushima Daiichi Nuclear Disaster

A Bézier curve is a parametric curve used to model smooth curves. In order to develop the sea floor map (without having to define every single point), we decided to use a cubic Bézier Curve. A cubic Bézier curve takes in 4 control points and interpolates along the 4 points to develop a smooth, continuous curve. The control points are defined as two points at the ends, one at $\frac{1}{3}$ of the way, and another at $\frac{2}{3}$ of the way. Now, provided the control points, we have the following explicit form:

$$B(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3, \quad 0 \le t \le 1 \tag{19}$$

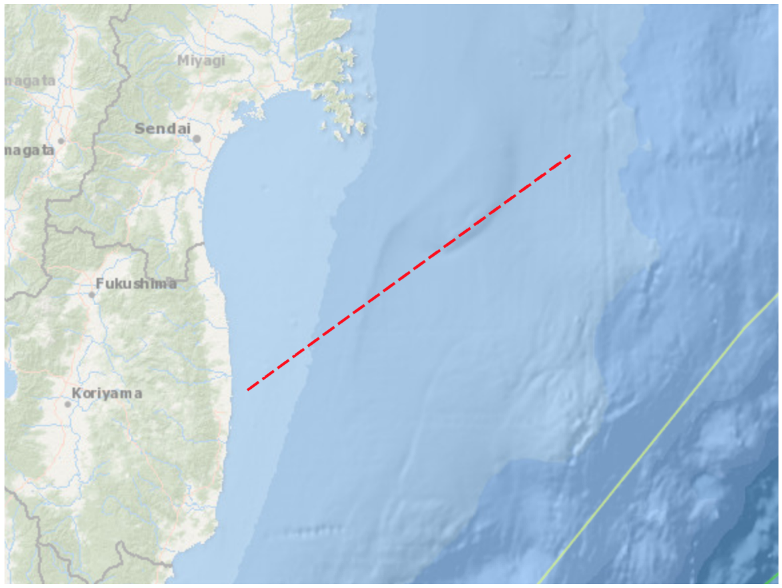
Where P is the control point. We can use this explicit form to find the derivative at any point t:

$$B'(t) = 3(1-t)^2(P_1 - P_0) + 6t(1-t)(P_2 - P_1) + 3t^2(P_3 - P_2), \quad 0 \le t \le 1 \tag{20}$$

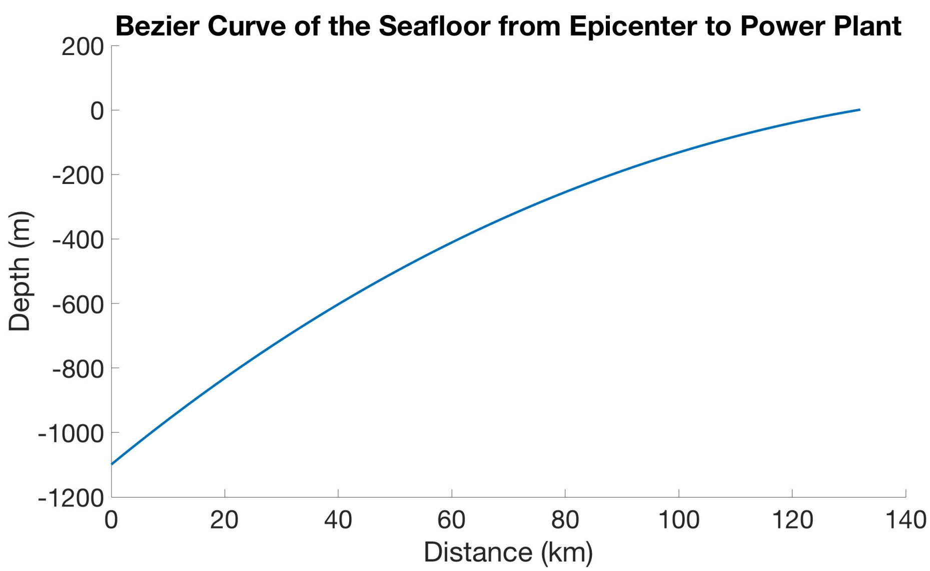
We can extend the idea of a Bézier curve to develop a Bézier surface, which is a 2D linearly interpolated surface that is made using 16 control points. The 16 control points create a 4×4 grid of points. In order to find a single point on the surface, say (u, v), we first create 4 new points by evaluating (19) at u using 4 sets of 4 control points. That is, we calculate $B(u)$ using control points 1 through 4, then 5 through 8, 9 through 12, and finally 13 through 16. We then use the 4 values of $B(u)$ as control points, and evaluate $B(v)$ to finally find the point on surface (u, v). This method is repeated on all points on the grid, in order to create the Bézier surface.

6.1 Implementation

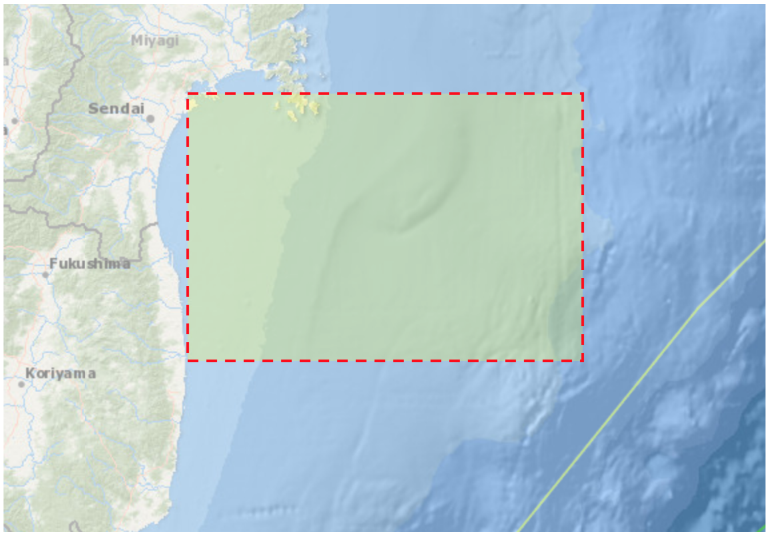
For our application with the Fukushima Daiichi Nuclear Power Plant, we derived the 4 control points for the sea floor using the National Oceanic and Atmospheric Administration's (NOAA) Bathymetric Data Viewer. The viewer allows users to gain elevation data at any coordinate. Thus, we drew a line between the epicenter of the earthquake to the nuclear power plant, and derived the 4 control points from there.



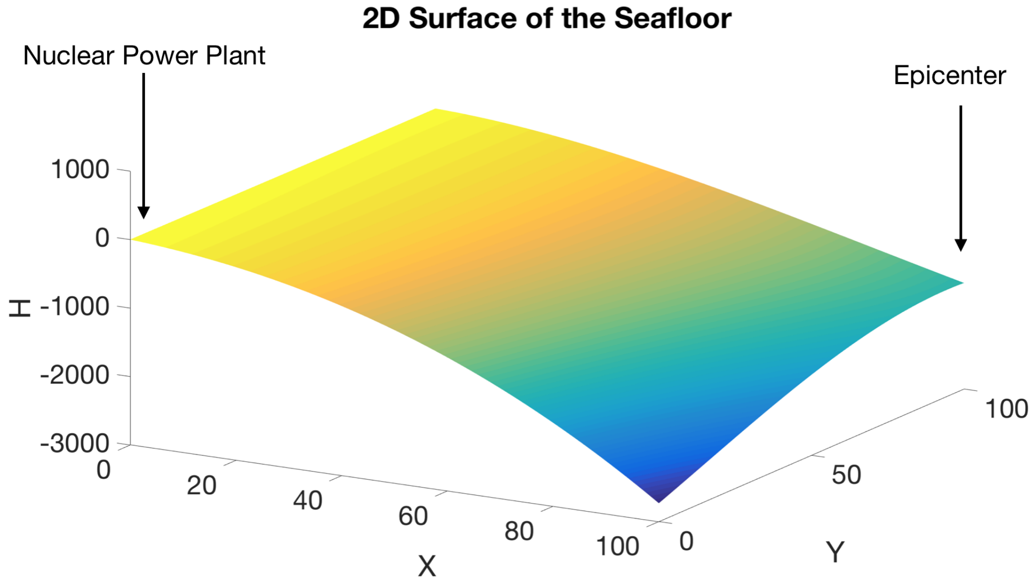
Using the control points, we can then apply the cubic Bézier curve to derive the following sea floor curve:



Similarly, we can also draw a rectangle with the diagonal being the line between the epicenter of the earthquake to the nuclear power plant. This rectangle is then used to define the 16 control points in order to derive the 2D interpolated sea floor surface.



Using the control points, we linearly interpolate all the points within the grid, and derive the following sea floor.



The key benefit of using the Bézier cubic interpolation is that we can create approximations of the sea floor very easily with only a few defined points. This gives us the liberty to apply our numerical method to any geographical location, and even test it with fictional surfaces.

Using Equation 20 for the derivative at any point t, we now generate three-dimensional surface plots describing the partial derivatives with respect to x and y of the Bezier surface. These derivatives for x and y are stored in two matrices, each of size 100x100, where the i,

j element of each represents the partial derivative at x location, i, and y location, j. The images below are useful visual representations of these derivatives.

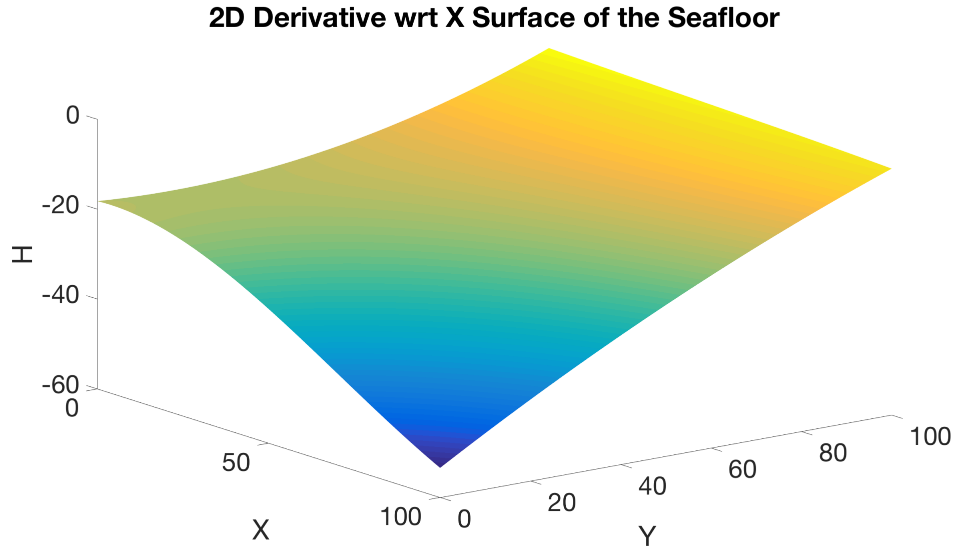


Figure 3: This is a surface plot of the derivative at each grid point with respect to x

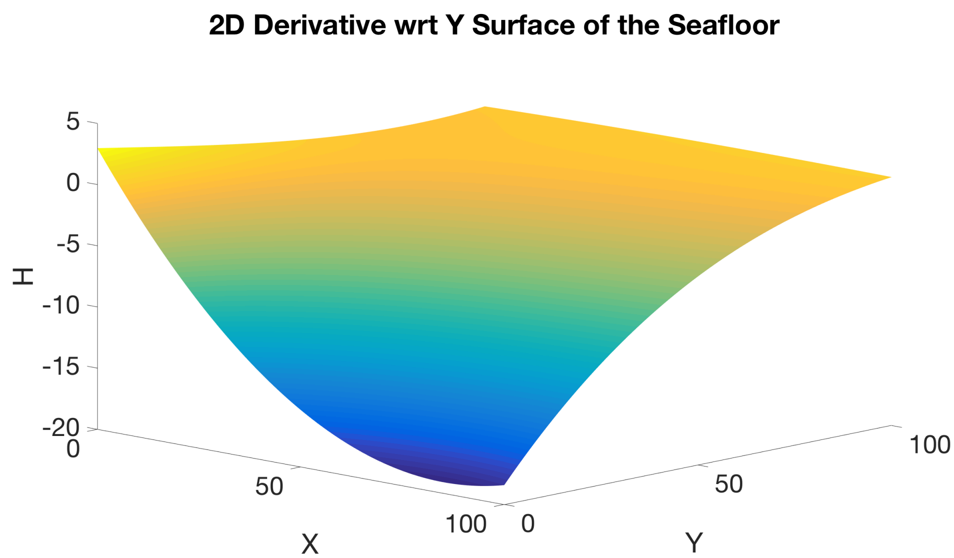


Figure 4: This is a surface plot of the derivative at each grid point with respect to y

These derivatives at each grid point are necessary for stepping forward in our model in order to better simulate the dynamics of the wave. This allows us to estimate the amplitude of the wave once reaching the nuclear power plant, which is assumed to be located a neglibible distance from the coast at which the elevation is set to the value 1 meter.

7 Results

References

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