VE 492 Homework9

Due: 23:59, July. 22^{th}

Question 1: Maximum Likelihood Estimation

We will begin with a short derivation. Consider a probability distribution with a domain that consists of $|X|$ different values. We get to observe N total samples from this distribution. We use n_i to represent the number of the N samples for which outcome i occurs. Our goal is to estimate the probabilities $\theta_i, i = 1,2...|X| - 1$ of each of the events. The probability of the last outcome, $|X|$, equals $1 - \sum_{i=1}^{|X|-1} \theta_i$.

In *maximum likelihood estimation*, we choose the θ_i that maximize the likelihood of the observed samples,

$$L(\text{samples}, \theta) \propto (1 - \theta_1 - \theta_2 - \ldots - \theta_{|X|-1})^{n_{|X|}} \prod_{i=1}^{|X|-1} \theta_i^{n_i}$$

For this derivation, it is easiest to work with the log of the likelihood. Maximizing log-likelihood also maximizes likelihood, since the quantities are related by a monotonic transformation. Taking logs we obtain

$$\theta^{\text{ML}} = \underset{\theta}{\text{argmax}} n_{|X|} \log(1 - \theta_1 - \theta_2 - \ldots - \theta_{|X|-1}) + \sum_{i=1}^{|X|-1} n_i \log \theta_i$$

Setting derivatives with respect to θ_i equal to zero, we obtain $|X| - 1$ equations in the $|X| - 1$ unknowns, $\theta_1, \theta_2, \ldots, \theta_{|X|-1}$:

$$\frac{-n_{|X|}}{1 - \theta_1^{\text{ML}} - \theta_2^{\text{ML}} - \ldots - \theta_{|X|-1}^{\text{ML}}} + \frac{n_i}{\theta_i^{\text{ML}}} = 0$$

Multiplying by $\theta_i(1 - \theta_1 - \theta_2 - \ldots - \theta_{|X|-1})$ makes the original $|X| - 1$ nonlinear equations into $|X| - 1$ linear equations:

$$-n_{|X|}\theta_i^{\text{ML}} + n_i\left(1 - \theta_1^{\text{ML}} - \theta_2^{\text{ML}} - \ldots - \theta_{|X|-1}^{\text{ML}}\right) = 0$$

That is, the maximum likelihood estimation of θ can be found by solving a linear system of $|X| - 1$ equations in $|X| - 1$ unknowns. Doing so shows that the maximum likelihood estimate corresponds to simply the count for each outcome divided by the total number of samples. I.e., we have that:

$$\theta_i^{\text{ML}} = \frac{n_i}{N}$$ part 1-2

Notice: Please write each sub-question in one row, that is, there will be 3 rows for this question. And please use irreducible fractions for your answer.

Sample Answer:

1,1/2,1/3,1/4
2/5 (instead of 4/10),1/3,4/7
3/8,3/7,3/5

Part 1.

Now, consider a sampling process with 3 possible outcomes: R, G, and B. We observe the following sample counts:

outcome	R	G	B
count	3	1	7

1) What is the total sample count N? 11
2) What are the maximum likelihood estimates for the probabilities of each outcome?

$\theta_R^{ML} =$ 3/11

$\theta_G^{ML} =$ 1/11

$\theta_B^{ML} =$ 7/11

Part 2.

Now, use *Laplace smoothing* with strength $k = 3$ to estimate the probabilities of each outcome.

$P_{LAP,k}(x) = \frac{c(x)+k}{N+k|X|}$

X: num of types

$\theta_R^{LAP,3} =$ $3+3 / 11+\underset{k}{3}\cdot\underset{RGB}{3} = 6/20 = 3/10$

$\theta_G^{LAP,3} =$ $1+3 / 11+3\cdot 3 = 4/20 = 1/5$

$\theta_B^{LAP,3} =$ $7+3 / 11+3\cdot 3 = 10/20 = 1/2$

Part 3.

Now, consider Laplace smoothing in the limit $k \to \infty$. Fill in the corresponding probability estimates.

$\theta_R^{LAP,\infty} =$ 1/3

$\theta_G^{LAP,\infty} =$ 1/3

$\theta_B^{LAP,\infty} =$ 1/3

R 3 — G B 1, 7

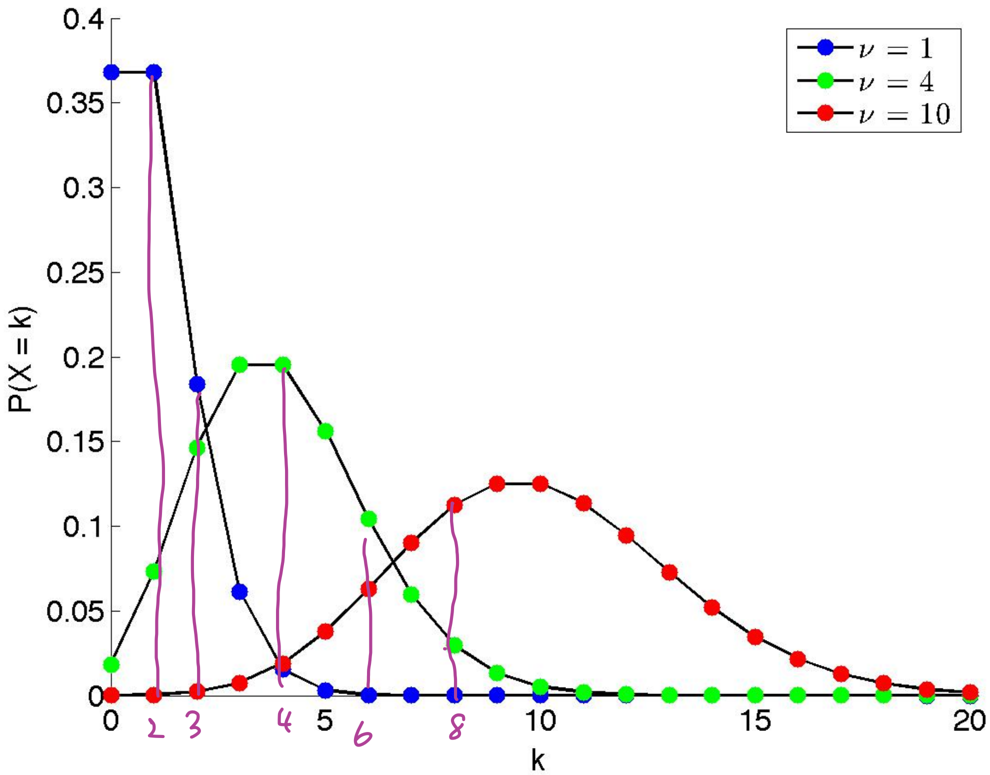
$$\lim_{k\to\infty} \frac{k+3}{11+k\cdot 3} = \lim_{k\to\infty} \frac{1+\frac{3}{k}}{\frac{11}{k}+3} = 1/3$$

Question 2: Poisson Parameter Evaluation

We will now consider maximum likelihood estimation in the context of a different probability distribution. Under the Poisson distribution, the probability of an event occurring $X = k$ times is:

$$P(X = k) = \frac{\nu^k e^{-\nu}}{k!}$$

Here ν is the parameter we wish to estimate. The distribution is plotted for several values of ν below.



On a sheet of scratch paper, work out the maximum likelihood estimate for ν, given observations of several k_i.

Hints: start by taking the product of the equation above over all the k_i, and then taking the log. Then, differentiate with respect to ν, set the result equal to 0, and solve for ν in terms of the k_i.

You observe the samples $k_1 = 6, k_2 = 3, k_3 = 8, k_4 = 4, k_5 = 2$. What is your maximum likelihood estimate of ν?

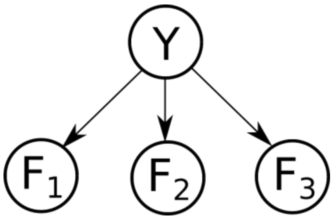
Sample Answer (rounded to 3 decimal places):
0.160

$\boxed{\frac{1}{n}\sum_{i=1}^{n} k_i}$

Poisson $= \frac{6+3+8+4+2}{5} = 4.6$

Question 3: Naive Bayes

In this question, we will train a Naive Bayes classifier to predict class labels Y as a function of input features F_i.



We are given the following 15 training points:

F_1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	0
F_2	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1
F_3	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
Y	A	A	A	A	A	A	A	A	A	A	B	B	B	B	C

Note: Please write your answer for each table in one row, that is, there will be 10 rows for this question. Besides, please use values rounded to 3 decimal places.

Sample Answer:
0.160,0.170,0.160
…
…
…
A
…
0.200,0.211,…

1) What is the maximum likelihood estimate of the prior $P(Y)$?

Y	$P(Y)$	
A	0.667	10/15
B	0.267	4/15
C	0.067	1/15

2) What are the maximum likelihood estimates of the conditional probability distributions? Fill in the tables below (the second and third are done for you).

F_1	Y	$P(F_1\|Y)$	
0	A		1/10
1	A		9/10
0	B		1/4
1	B		3/4
0	C		1
1	C		0

F_2	Y	$P(F_2\|Y)$
0	A	0.800
1	A	0.200
0	B	1.000
1	B	0.000
0	C	0.000
1	C	1.000

F_3	Y	$P(F_3\|Y)$
0	A	1.000
1	A	0.000
0	B	0.500
1	B	0.500
0	C	1.000
1	C	0.000

3) Now consider a new data point ($F_1 = 0,\ F_2 = 0,\ F_3 = 1$). Use your classifier to determine the joint probability of causes Y and this new data point, along with the posterior probability of Y given the new data: $P(Y)\prod_i P(F_i \mid Y)$

7

Y	$P(Y, F_1 = 0, F_2 = 0, F_3 = 1)$
A	0.000
B	0.033
C	0.000

10/15 × 1/10 × 8/10 × 0
4/15 × 1/4 × 1 × 0.5
1/15 × 1 × 0 × 0

8

Y	$P(Y \mid F_1 = 0, F_2 = 0, F_3 = 1)$
A	0
B	1
C	0

4) What label does your classifier give to the new data point? (Break ties alphabetically). Write capital letters only. B

5) Now use Laplace Smoothing with strength $k = 3$ to estimate the prior $P(Y)$ for the same data.

$P_{LAP,k}(x) = \frac{c(x)+k}{N+k|X|}$

X: num. of types (A, B, C)

Y	$P(Y)$
A	0.542
B	0.292
C	0.167

10+3 / 15+3·3
4+3 / 15+3·3
1+3 / 15+3·3

6) Use Laplace Smoothing with strength $k = 3$ to estimate the conditional probability distributions below (again, the second two are done for you).

(0,1)
3·2
k |X|

1+3 / 10+6
9+3 / 10+6
1+3 / 4+6
3+3 / 4+6
1+3 / 1+6
0+3 / 1+6

F_1	Y	$P(F_1 \mid Y)$
0	A	0.250
1	A	0.750
0	B	0.400
1	B	0.600
0	C	0.571
1	C	0.429

F_2	Y	$P(F_2 \mid Y)$
0	A	0.688
1	A	0.312
0	B	0.700
1	B	0.300
0	C	0.429
1	C	0.571

F_3	Y	$P(F_3 \mid Y)$
0	A	0.812
1	A	0.188
0	B	0.500
1	B	0.500
0	C	0.571
1	C	0.429

7) Now consider again the new data point ($F_1 = 0,\ F_2 = 0,\ F_3 = 1$). Use the Laplace-Smoothed version of your classifier to determine the joint probability of causes Y and this new data point, along with the posterior probability of Y given the new data:

Y	$P(Y, F_1 = 0, F_2 = 0, F_3 = 1)$
A	0.018
B	0.041
C	0.018

0.542 · 0.250 · 0.688 · 0.188
0.292 · 0.400 · 0.7 · 0.5
0.167 · 0.571 · 0.429 · 0.429

Y	$P(Y \mid F_1 = 0, F_2 = 0, F_3 = 1)$
A	0.234
B	0.532
C	0.234

$$\frac{0.0181}{0.018+0.041+0.018}$$

$$0.041 / (0.018+0.041+0.018)$$

8) What label does your (Laplace-Smoothed) classifier give to the new data point? (Break ties alphabetically). Write a single capital letter.

B

Question 4: Datasets

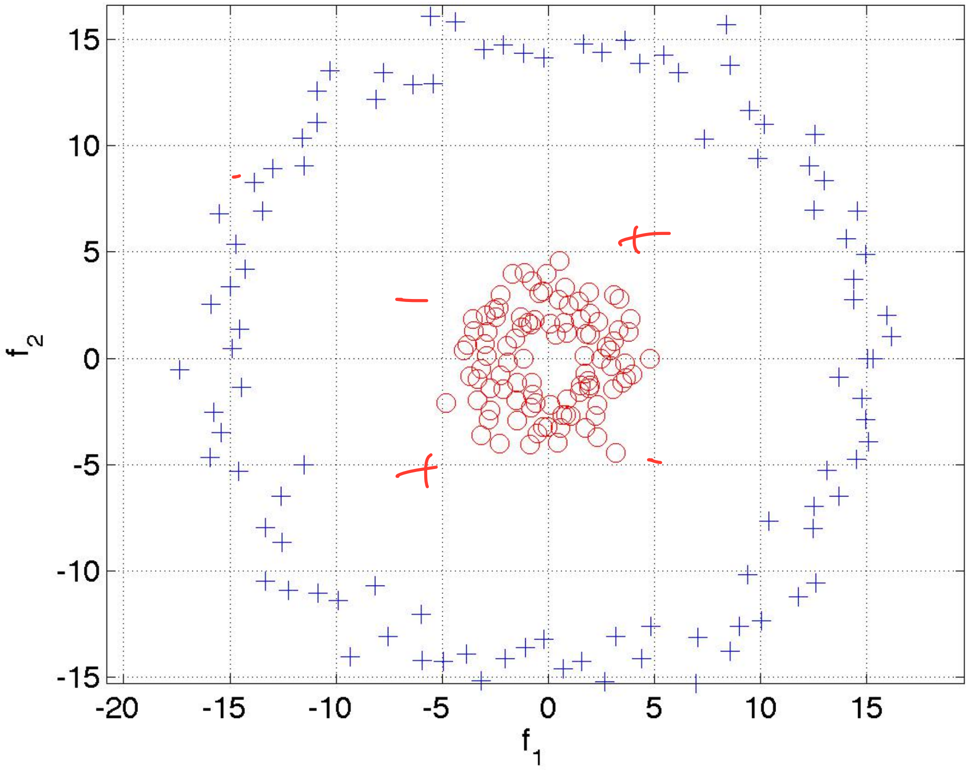
When training a classifier, it is common to split the available data into a training set, a hold-out set, and a test set, each of which has a different role.

Sample Answer:
A
A
A

1) Which data set is used to learn the conditional probabilities?
   A. Training Data
   B. Hold-Out Data
   C. Test Data
2) Which data set is used to tune the Laplace Smoothing hyperparameters?
   A. Training Data
   B. Hold-Out Data
   C. Test Data
3) Which data set is used for quantifying performance results?
   A. Training Data
   B. Hold-Out Data
   C. Test Data

Question 5: Linear Separability

Consider the data in the figure below.



The data is plotted as a function of two features, f_1 and f_2. As plotted, the data is not linearly separable. Which of the following candidate features f_3, when added, would cause the data to be linearly separable? Choose all possible answer(s).

A. $f_3 = |f_1| + |f_2|$

B. $f_3 = \sin(f_1)$

C. $f_3 = f_1^2 + f_2^2$

D. $f_3 = f_1^2$

E. $f_3 = f_1$

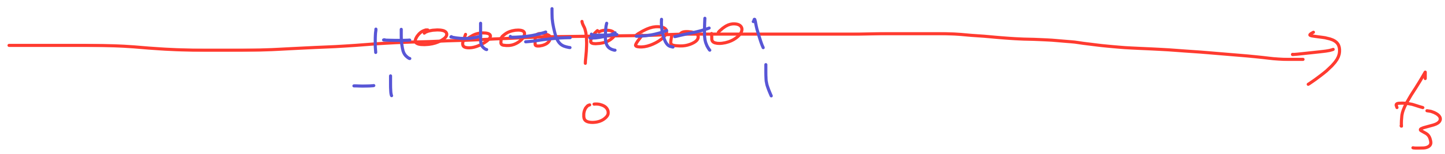
F. $f_3 = 1$

G. $f_3 = f_1 f_2$

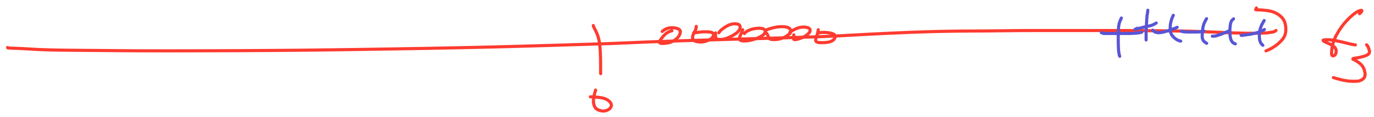
H. $f_3 = 1$ if $f_1 \in [-7,7]$ and $f_2 \in [-7,7]$, 0 $otherwise$

A ✓

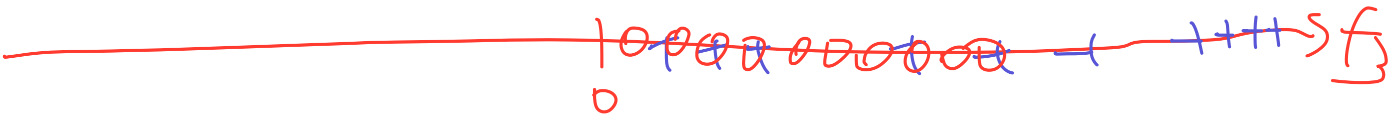
B ×



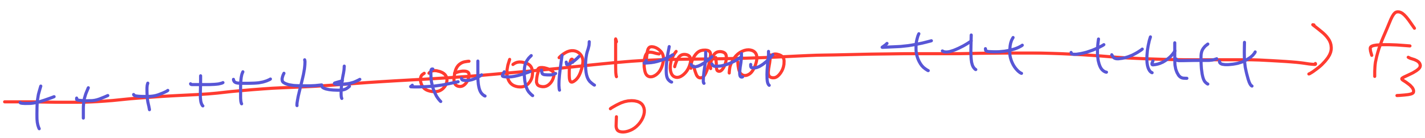
C ✓



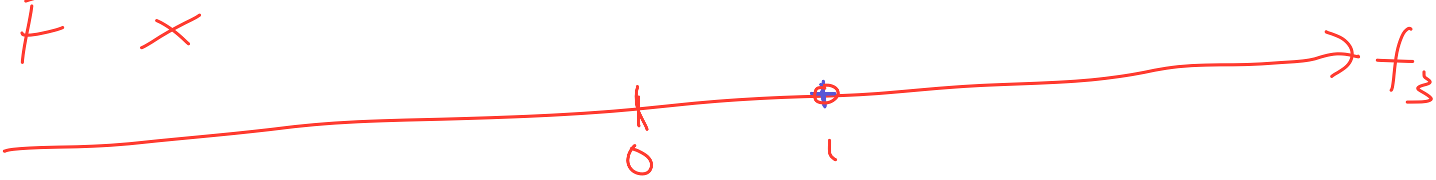
D ×



E ×



F ×



G ×

H ✓