Math 202A Notes

Jacob Krantz

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Contents

Chapter 1

Metric Spaces

Definition 1.0.1. Let X be a set, a metric on X is a function $d : X \times X \to \mathbb{R}$, such that

1. $d(x, x) = 0$, for all $x \in X$
2. if $d(x, y) = 0$, then $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Note that if we do not have that if $d(x, y) = 0$, then $x = y$, then we have a semimetric.

Definition 1.0.2. If $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we call the define the following norms:

1. $||v||_2 = (\sum |v_j|^2)^{\frac{1}{2}}$
2. $||v_1|| = \sum |r_j|$
3. $||v||_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$
4. $||v||_p = (\sum |v_j|^p)^{\frac{1}{p}}$

Definition 1.0.3. We can now define the following:

1. $d_2 := ||v - w||_2$

2. $d_1 := ||v - w||_1$

3. $d_\infty := ||v - w||_\infty$

4. $d_p := ||v - w||_p$

Example 1.0.1. Let (X, d) be a metric space, then let $Y \subset X$, the restriction of d to $Y \times Y \subset X \times X$ makes Y a metric space.

Example 1.0.2. $C([0,1]) = \mathbb{R}$-valued continuous functions on $[0,1]$.

Note 1.0.1. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. By a norm on V, we mean a function $||\cdot|| : V \to \mathbb{R}^+$ such that:

1. $||v|| = 0 \iff v = 0$

2. $||\alpha v|| = |\alpha| ||v||$

3. $||v + w|| \leq ||v|| + ||w||$

Example 1.0.3. From a norm on V, we get a metric on V by $d(v, w) = ||v - w||$. For $f \in C([0,1])$:

1. $||f||_2 = \left(\int_0^1 |f(t)|^2 dt \right)^{\frac{1}{2}}$

2. $||f||_1 = \int_0^1 |f(t)| dt$

3. $||f||_\infty = \sup\{|f(t)| : t \in [0,1]\}$

4. $||f||_p = \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}$

Definition 1.0.4. Let (X, d) be a metric space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. We say that this sequence converges to a point $x_* \in X$ if for all $\epsilon > 0$, there exists $N > 0$ such that for $n > N$, $d(x_n, x_*) < \epsilon$. [Note that this is the same as saying that $x_n \in$ oBall(x_*, ϵ), where oBall$(x_*, \epsilon) = \{y \in X \mid d(y, x_*) < \epsilon\}$.]

Definition 1.0.5. X is complete if every Cauchy sequence converges to some point of X.

Example 1.0.4. Some examples of complete metric spaces include $\mathbb{R}, \mathbb{R}^n, \mathbb{C}$.

Note 1.0.2. If S is a closed subset of $\mathbb{R}^n$, then S with the restricted metric is complete. Consider $C([0, 1]) : ||f||_\infty = \sup\{|f(t)| : t \in [0, 1]\}$. The uniform norm convergence for it is uniform convergence. If $\{f_n\}$ is Cauchy for $|| \cdot ||_\infty$, then for each $t_* \in [0, 1]$, then $\{f_n(t_*)\}$ is a Cauchy sequence, so it converges. Note that $f(t) = \lim(f_n(t))$, the uniform limit of continuous functions is continuous.

Definition 1.0.6. Let (X, d) be a metric space, and let S be a subset of X. We say that S is dense in X if every open ball in X contains a point of S.

Definition 1.0.7. Let (X, d) be a metric space, by a completion of X, we mean a metric space, $(\overline{X}, \bar{d})$, together with $j : X \to \overline{X}$ such that j is an isometry and j is dense in X.

Definition 1.0.8. An isometry is a function j such that $d(x, y) = d(j(x), j(y))$.

Example 1.0.5. Every metric space has a completion, and the completion is essentially unique. Let (X, d) be a metric space. Let $\text{CS}(X, d)$ be the set of all Cauchy sequences in (X, d). Try to define a distance on $\text{CS}(X, d)$: let $\{x_n\}, \{y_n\}$ be two Cauchy sequences. Consider $\{d(x_n, y_n)\}$, we claim it is Cauchy in $\mathbb{R}$. Set $\tilde{d}(\{x_n\}, \{y_n\}) = \lim\{d(x_n, y_n)\}$.

Note 1.0.3. Note that $d(x, y) \le d(x, z) + d(z, y)$ and $d(x, y) - d(x, z) \le d(z, y)$, so $|d(x, y) - d(x, z)| \le d(z, y)$ and $|d(x, z) - d(y, z)| \le d(x, y)$. Hence,

$$\begin{aligned}|d(x_n, y_n) - d(x_n, y_n)| &= |d(x_n, y_n - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \\ &\le |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \\ &\le d(y_n, y_m) + d(x_n, x_m) \to 0\end{aligned}$$

Now, let (X, d) be a semimetric space. We now define an equivalence relation on X, by if $d(x, y) = 0$, then $[x] = \{y : d(x, y) = 0\}$. Define ${}^{X}/_{\sim} := \{\text{equivalence classes}\}$. Define $\hat{d}$ on ${}^{X}/_{\sim}$ by $d([x], [y]) = d(x, y)$, well-defined. If $x' \in [x]$, $y' \in [y]$, then $d(x', y') \le d(x, x) + d(y, y) + d(x, y), d(x', y') = d(x, y)$, so $\hat{d}$ is a metric on ${}^{X}/_{\sim}$. Let $\tilde{d}$ on $\text{CS}(X, d)$ be the corresponding metric in the equivalence classes. The equivalence relation is $\{x_n\} \sim \{y_n\}$ if $\hat{d}(\{x_n\}, \{y_n\}) = 0$ or $\lim_{n\to\infty} d(x_n, y_n) = 0$. Embed (X, d) in ${}^{\text{CS}(X,d)}/_{\sim}$ by $x \mapsto$ Cauchy sequence, $x_n = x$, for all n, $\phi(x) = \{x_n = x\}, \tilde{d}(\phi(x), \phi(y)) = \lim d(x_n, y_n) = \lim d(x, y) = d(x, y)$, so ϕ is an isometry of X into $\text{CS}(X, d) \to {}^{\text{CS}(X,d)}/_{\sim}$. The image of X is dense in ${}^{\text{CS}(X,d)}/_{\sim}$. Let $\{x_n\}$ be any Cauchy sequence. Then, given any $\epsilon > 0$, there exists N such that for $m, n \ge N, d(x_m, x_n) < \epsilon$. Consider $\phi(x_N)$. Then, $d(\{x_n\}, \phi(x_n)) = \lim_{n\to\infty}\{d(x_n, x_N)\} < \epsilon$. To show that $({}^{\text{CS}(X,d)}/_{\sim}, d)$ is complete. For small m, let $\ldots \in \text{CS}(X, d)$, assume $\{S^m\}$ is a Cauchy sequence in $\text{CS}(X, d)$, for each k, find $x_k \in X$, such that $d(\phi(x_k), S^m) < \frac{1}{k}$, then $S = \{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, and $\tilde{d}(S^m, S)_{n\to\infty} \to 0$.

Definition 1.0.9. Let $(X, d_x), (Y, d_y)$ be metric spaces, $f : X \to Y$, and $x_0 \in X$, we say that f is continuous at x_0 if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x_0) < \delta$, then $d(f(x), f(x_0)) < \epsilon$, or equivalently, if $x \in \text{Ball}(x_0, \delta)$, then $f(x) \in \text{Ball}(f(x_0), \epsilon)$. For any open ball B about $f(x_0)$, there is an open ball C about$f(x_0)$ such that if $x \in B$, then $f(x) \in C$, or equivalently that $x \in f^{-1}(C)$, and $B \subseteq f^{-1}(C)$.

Definition 1.0.10. Let (X, d) be a metric space. If $A \subseteq X$ is an open subset (for d) if for each αA, there is an open ball about x contained in A.

Note 1.0.4. If f is continuous, i.e continuous at all points, let $\mathcal{O}$ be an open set in Y, let $x_0 \in f^{-1}(\mathcal{O})$, then $\mathcal{O}$ contains a ball about x_0 such that $x_0 \in C \subset f^{-1}(B)$, so $C \subseteq f^{-1}(\mathcal{O})$, so $f^{-1}(\mathcal{O})$ is open. Conversely, let f be any function from X to Y. If it is true that for any open set $\mathcal{O}$ in Y, $f^{-1}(\mathcal{O})$ is open in X, then f is continuous. Given any $\epsilon > 0$, let $\mathcal{O} = \text{Ball}(f(x_0), \epsilon)$, then $f^{-1}(\text{Ball}(f(x_0), \epsilon))$ is open. Hence, there is a ball $\text{Ball}(x_0, \delta)$ such that $\text{Ball}(x_0, \delta) \subseteq f^{-1}(\text{Ball}(f_0, \epsilon))$. The following are properties of the collection of open sets of a metric space:

1. An infinite union of open sets is open
2. A finite intersection of open sets is open. For $x_0 \in \mathcal{O}_1 \cap \mathcal{O}_2$, $\text{Ball}(x_0, r_1) \subseteq \mathcal{O}_1$, $\text{Ball}(x_0, r_2) \subseteq \mathcal{O}_2$. Let $r = \min\{r_1, r_2\}$, then $\text{Ball}(x_0, r) \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$.
3. X and $\varnothing$ are open.

Definition 1.0.11. Let X be a set. By a topology for X, we mean a collection $\mathcal{T}$ of subsets of X such that:

1. Arbitrary unions of elements of $\mathcal{T}$ are in $\mathcal{T}$.
2. Finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.
3. X and $\varnothing$ are elements of $\mathcal{T}$.

Definition 1.0.12. Let $\mathcal{T}$ be a topology of X. Then $A \subseteq X$ is closed if A' is open.

Note 1.0.5. Properties of closed sets:

1. Arbitrary intersections of closed sets are closed.
2. Finite unions of closed sets are closed.

3. X and $\varnothing$ are closed.

Definition 1.0.13. Let $A \subseteq X$. By the closure of A, we mean the smallest closed set that contains A, i.e. the intersection of all closed sets that contain A.

Definition 1.0.14. By the interior of A, we mean the biggest open set contained in A, i.e. the union of all open sets contained in A.

Definition 1.0.15. Let C be a closed set, and let $A \subseteq C$, we say that A is dense in C if $\overline{A} = C$.

Definition 1.0.16. Let X be a set, and let $\mathscr{S}$ be a collection of subsets of X, the smallest topology containing the intersection of topologies that contain $\mathscr{S}$ is said to be the topology generated by $\mathscr{S}$, and $\mathscr{S}$ is called a subbase for that topology. Note that if $\mathscr{C}$ is a collection of topologies for X, then $\bigcap\{\mathcal{T} \in \mathscr{C}\}$ is a topology for X.

Definition 1.0.17. Let X be a set, and let D be the collection of subsets of X. D is a topology for X, called the discrete topology for X. It is given by a metric:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

D is the biggest topology in X.

Definition 1.0.18. The smallest topology in X is $\{\varnothing, X\}$, called the indiscrete topology.

Note 1.0.6. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, such that:

$$\begin{array}{cc} \mathcal{T}_1 \subseteq & \mathcal{T}_2 \\ \text{smaller} & \text{larger} \\ \text{weaker} & \text{stronger.} \end{array}$$

Usually, we require that $\bigcup \mathscr{S} = X$. For $X = \mathbb{R}$, (a, b), $\mathscr{S} = \{(\infty, a), (b, +\infty)\}$.

Definition 1.0.19. A collection of subsets of X is a base for a topology is the set of all arbitrary unions of elements of $\mathscr{S}$ is a topology.

Example 1.0.6. $\mathscr{S} = \{(a, b) \subset \mathbb{R}\}$, $\mathbb{R}^2 = \{\text{open balls}\}$

Note 1.0.7. For $\mathscr{S}$ to be a base, it must have the property that if $A, B \in \mathscr{S}$, then $A \cap B$ must be a union of elements of $\mathscr{S}$.

Example 1.0.7. If $\mathscr{S}$ is any collection of subset of X, then the collection of all finite intersections of elements must be a topology.

Definition 1.0.20. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. f is continuous if for all open sets $\mathcal{O} \in \mathcal{T}_Y \implies f^{-1}(\mathcal{O}) \in \mathcal{T}_X$.

Note 1.0.8. Let Y be a set and $\mathscr{S} = \{A_\alpha\}$, let X be a set, and $f : X \to Y$ be a function. Then,

1. $f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$

2. $f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f(A_\alpha)$

3. If $A, b \in Y$, then $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$.

Example 1.0.8. Given $(X, \mathcal{T}_X)$ and $f : X \to Y$, let $\mathscr{S}$ be a subbase for $\mathcal{T}_Y$. Then f is continuous if $f^{-1}(A) \in \mathcal{T}_X$, for all $A \in \mathscr{S}$.

Example 1.0.9. Let X be a set and let $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let there be a quasifunction $f_\alpha : X_\alpha \to X$. Let $\mathcal{T}$ be the strongest topology such that all of the f_α's are continuous. Given α_0, f_α. If $A \subseteq X$, then if A is to be open, we must have that $\overline{f}_{\alpha_0}(A) \in \mathcal{T}_{\alpha_0}$. Now, let $\mathscr{S}_{\alpha_0} = \{A \subseteq : f_{\alpha_0}^{-1}(A) \in \mathcal{T}_{\alpha_0}\}$ is a topology for X; in fact, it is the strongest topology making f_{α_0} continuous. The strongest topology making all of the f_α continuous is the intersection of the $\mathscr{S}_\alpha$.

Example 1.0.10. Let $(X, \mathcal{T})$ be a topological space, let Y be a set. Then, $f : X \to Y$, $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\}$ is the strongest topology making f continuous. Usually, we want f to be onto Y.

Definition 1.0.21. We begin by defining an equivalence relation, $\sim$, on X by $x_1 \sim x_2$, if $f(x_1) = f(x_2)$. This gives a partition of X: the quotient of ${}^{X}/_{\sim}$, the quotient of X by $\sim$. This topology is called the quotient topology determined by f.

Definition 1.0.22. For $\sim$ on a set X, $B \subseteq X$ is saturated if when $x \in B$ and $x_1 \sim x$, for $x_1 \in B$.

Note 1.0.9. The open sets in the quotient topology in f on Y are in bijection with the saturated open sets of X.

Theorem 1.0.1. *Tietze Extension Theorem. Let $(X, \mathcal{T})$ be a normal topological space, and let $A \to \mathbb{R}$ be continuous. Then there is $\tilde{f} : X \to \mathbb{R}$, continuous that extends f, if $\tilde{f}|_A = f$. If $f : A \to [a, b], a, b \in \mathbb{R}$ then can arrange that $\tilde{f} : X \to [a, b]$.*

Proof. [Note that if $A \subseteq X$ is closed and if $B \subseteq A$ is closed in the relataive topology, then B is closed in X, $A \setminus B = A \cap O, O \in \mathcal{T}$, then $B = A \cap O'$, where A and O' are closed, as B is closed in X] Now, consider the first case of $f : A \to [0, 1]$. Let $C_0 = \{x \in A : f(x) \leq \frac{1}{3}\}, C_1 = \{x \in A : f(x) \geq \frac{2}{3}\}$, closed in A. Then, by Urysohn's Lemma, $\exists k : X \to [0, 1]$ with $k|_{C_0} = 0$, $k|_{C_1} = 1$. Let $g_1 = \frac{1}{3}k$, so $g_1 : X \to [0, \frac{1}{3}]$, $f - g_1|_A : A \to [0, \frac{2}{3}]$. Scale (?): If $h : A \to [o, r]$, then there exists g on X with $g : X \to [\frac{1}{3}r], h - g|_A A \to [0, \frac{2}{3}r]$. Apply this to $f - g_1|_A$, $r = \frac{2}{3}$. Thus there is $g_2 : X \to [0, \frac{1}{3}\frac{2}{3}], (f - g_1|A) - g_2|_A : X \to [0, \left(\frac{2}{3}\right)^2]$. Apply to $f - g_1|_A - g_2|_A$, $r = \left(\frac{2}{3}\right)^2$. So there is $g_3 : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^2], f - g_1|_A - g_2|_A - g_3|_A : X \to [0, \left(\frac{2}{3}\right)^3]$. Continue this for the nth case. Clearly we have that $g_n : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}]$, $f - \sum_{j=1}^{n} g_j|_A : X \to [0, \left(\frac{2}{3}\right)^n] \implies ||g_n||_\infty \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, define $\tilde{f} = \sum_{j=1}^{\infty} g_j$ cont, $||f - \sum^n g_j|A|| \leq \left(\frac{2}{3}\right)^n$. Hence, $\tilde{f}|_A = f$, $0 \leq g_n(x) \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, so $\sum_{j=1}^{\infty} g_j(x) \leq \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^{j-1} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{2}{3}\right)^j = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1$. If $f : A \to \mathbb{R}$, unbounded, then arctan $\mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2})$ is a homeomorphism. Let h be the arctan of $f : A \to (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [\frac{-\pi}{2}, \frac{\pi}{2}]$, as there is an equation $\tilde{h} : X \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ with $\tilde{h}|_A = h$. Let $B = \{\frac{-\pi}{2}, \frac{\pi}{2}\}$, a closed subset of $[\frac{\pi}{2}, \frac{\pi}{2}]$. Then take $B = \{\tilde{h}^{-1}\left(\frac{-\pi}{2}\right), \tilde{h}^{-1}\left(\frac{\pi}{2}\right)\} \subseteq X, A \subseteq X$... ∎

Definition 1.0.23. Let X be a set, $\mathcal{C}$ a collection of subsets of X. We say that $\mathcal{C}$ is a covering of X if

$$\bigcup\{A \in \mathcal{C}\} = X$$

. If $B \subseteq X, \mathcal{C}$ is a collection of subsets of X, we say that $\mathcal{C}$ covers $\mathcal{B}$ if $\mathcal{B} \subseteq \bigcup\{A \in \mathcal{C}\}$. If $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ is a subcover of $\mathcal{C}$ if $\mathcal{D}$ also is a c.

Definition 1.0.24. Let $(X, \mathcal{T})$ be a topological space. We say that it is compact if every open cover of X has a finite subcover.

Theorem 1.0.2. *If $(X, \mathcal{T})$ is compact and $A \subseteq X$, then the following are equivalent.*

1. *A is compact for the relative topology*
2. *If $\mathcal{C} \subseteq \mathcal{T}$ is a cover of A, then A has a finite subcover of $\mathcal{O}$.*

Proof. The open sets for the relative topology are of the form $A \cap \mathcal{O}$, $\mathcal{O} \in \mathcal{T}$. ■

Theorem 1.0.3. *If $(X, \mathcal{T})$ is compact and $A \subseteq X$ is closed then A is compact for the relative topology.*

Proof. Let $\mathcal{D} \subset \mathcal{T}$ be a collection of open sets that cover A. Since A is closed, A' is open, so $\mathcal{D} \cup ...$ is an open cover of X. ■

A compact subset of a topological space, even a compact one, need not be closed. For example, any set with at least 2 points within the discrete topology, every subset is compact.

Theorem 1.0.4. *Let $(X, \mathcal{T})$ be Hausdorff. Let $A \subseteq X$ be compact for the relative topology, then A is closed.*

Proof. Let $y \in X$, $y \notin A$. For each $x \in A$ find $\mathcal{U}_x, \mathcal{V}_x \in S$. Then the set of these $\mathcal{U}_x$ will cover A. So we have a finite subcover, $\mathcal{U}_{x_1}, \ldots \mathcal{U}_{x_n}$. Let $V = \mathcal{V}_{x_1} \cap \mathcal{V}_{x_2} \ldots \mathcal{V}_{x_n}$ be open, $y \in \mathcal{V}_1$, $V \cap A = \varnothing$. Thus A' is a union of open sets, so it is open. Thus, its compliment, A, is closed. ■

Theorem 1.0.5. *Let $(X, \mathcal{T})$ be compact and Hausdorff. For any closed subset A of X and any pf (?) $y \in X$, $y \notin A$, there are open sets u, v, disjoint, with $A \subseteq u$, $y \in V$.*

Definition 1.0.25. $(X, \mathcal{T})$ is regular for all $A \subseteq X$ closed and all $y \in X$, $y \notin A$.

Theorem 1.0.6. *Every compact Hausdorff space is normal.*

Proof. Let A, B be disjoint closed, (covered also (?)) subsets. By regularity, for each $y \in B$, there are disjoint open $\mathcal{U}_y,\ \mathcal{V}_y,\ A \subseteq \mathcal{U}_y,\ \mathcal{V}_y,\ y \in \mathcal{V}_y$. The $\{\mathcal{V}_y\}$ form an open cover of B, as by completion there is a finite subcover, $\{\mathcal{V}_{y_k}\}_{k \in I},\ I = \{1, \ldots, n\}$. ■

Theorem 1.0.7. *Tychonoff's Theorem*

Proof. Some stuff I missed. Let $(X_\lambda, \mathcal{T}_\lambda)$ compact top spaces. Let $X = \prod X_\lambda$ with the product topology. Want to show that X is compact. Let $\mathcal{C}$ be a collection of closed sets with FIP. Need to show that $\cap\{C \in \mathcal{C}\} \neq \varnothing$. By Zorn's Lemma, there is a collection $\mathcal{D}^*$ of elements of $X, \mathcal{C} \subseteq \mathcal{D}^*$, with $\mathcal{D}^*$ maximal among collection satisfying the FIP.

Lemma 1.0.8. *Let $\mathcal{D}$ be any collection of subsets of X maximal for FIP. Then the finite intersection of sets in $\mathcal{D}$ are in $\mathcal{D}$, and if $B \subset X$ and if $B \cap A \neq \varnothing$, for all $A \in \mathcal{D}$, then $B \in \mathcal{D}$.*

Proof. Let $\mathcal{D}'$ be the collection of all finite collection of elements of $\mathcal{D}$. Then $\mathcal{D}$ has FIP, and $\mathcal{D} \subseteq \mathcal{D}'$, so by maximality, $\mathcal{D} = \mathcal{D}'$. For the second statement, consider $\mathcal{D} \cup \{B\}$, then this has FIP, because $B \cap A_1 \cap \ldots \cap A_n = B \cap \left(\bigcap_{j=1}^n A_j \right)_{j \in \mathcal{D}} \neq \varnothing$. ∎

So $\mathcal{D} \cup \{B\}$ has FIP $\subseteq \mathcal{D}$. By maximality, $\mathcal{D} \cup \{B\} = \mathcal{D}, eB \in \mathcal{D}$, $\mathcal{C} \subseteq \mathcal{D}^*$. For each λ, $\{pi_\lambda(A) : A \in \mathcal{D}^*\}$ has FIP. Thus, $\{(\pi_\lambda(A)^- : A \in \mathcal{D}^*\} \subset X_\lambda$ has FIP, so since X_λ is compact, $\cap\{(\pi_\lambda(A))^- : A \in \mathcal{D}\} \neq \varnothing$. Choose $x_\lambda \in$ this set. Set $x_0 = \{x_\lambda\} \in X = \prod X_\lambda$. Want to show that $x_0 \in \cap\{C : C \in \mathcal{C}\}$, i.e., want $x_0 \in C$ for each $C \in \mathcal{C}$, suffices to show that $x_0 \notin C'$, which is open, for all $C \in \mathcal{C}$. So it suffices to show that for any $\mathcal{O}$ in base for product topology, if $x_0 \in \mathcal{O}$, then $\mathcal{O} \cap C$, $\mathcal{O} = \mathcal{U}_{\lambda_1} \times \mathcal{U}_{\lambda_2} \times \ldots \times \mathcal{U}_{\lambda_n} \times \prod_{\lambda \neq \lambda_1, \lambda j, \ldots \lambda_n}$, with $\mathcal{U}_{\lambda_j} \in J_{\lambda_j}$. By the definition of x_0, $x_{\lambda_j} \in \cap\{\pi_{\lambda_j}(A)^- : A \in \mathcal{D}^*\}$, for $j = 1, \ldots, n$. That is, for all $A \in \mathcal{D}^*$, $\mathcal{U}_j \cap \pi_{\lambda_j}(A) \neq \varnothing$. In other words, for all $A \in \mathcal{D}^*$, $A \cap \pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) \neq \varnothing$. Thus, $\pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) = \mathcal{D}^*$. Then, $\cap_{j=1}^n \pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) \in \mathcal{D}^*$, this intersection is just $\mathcal{O}$, so $\mathcal{O} \in \mathcal{D}^* \supseteq \mathcal{C}$, so $\mathcal{O} \cap C \neq \varnothing$ for all $C \in \mathcal{C}$. ∎

Note 1.0.10. Tychonoff's Theorem is equivalent to the axiom of choice. Let $\mathcal{C}$ be a collection of sets, $\mathcal{C} = \{X_\lambda\}_{\lambda \in \Lambda}$. Choose one element that is not in any X_λ, e.g ω = set of all subsets of $\cup X_\lambda$. Let $Y_\lambda = X_\lambda \cup \{\omega\}$, set $\mathcal{T}_\lambda = \{X_\lambda, \{\omega\}, Y_\lambda, \varnothing\}$. Then, let $Y = \prod_{\lambda \in \Lambda} Y_\lambda$, with the product topology. By Tychons, Y is compact. Consider $\{\pi_\lambda^{-1}(X_\lambda)\}$. Claim that this has FIP, where the inside of the set braces is closed. Given $\lambda_1, \ldots, \lambda_n$, $\pi_{\lambda_1}^{-1}(X_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(X_{\lambda_2} \cap \ldots \cap \pi_{\lambda_n}^{-1}(X_{\lambda_n})$. For $j = 1, \ldots, n$, choose $x_{\lambda_j} \in X_{\lambda_j}$. Define $x \in \prod Y_\lambda$ by $x_\lambda = x_{\lambda_j}$ if $\lambda = \lambda_j$,... got too long.

Chapter 2

Compactness in Metric Spaces

Note 2.0.1. Let (X, d) be a metric space, let $A \subseteq X$, and assume that $\bar{A}$ is compact for the relative topology. Then, for any $\epsilon > 0$, consider $\{\text{oBall}(x, \epsilon) : x \in A\} \supseteq \bar{A}$, with $\bar{A}$ is compact, so there is a finite subcover of $\bar{A}$, and so of A.

Definition 2.0.1. A subset A of a metric space (X, d) is said to be totally bounded if for any $\epsilon > 0$, it call be covered by a finite number of ϵ-balls.

Theorem 2.0.1. *Any subset of a compact subset of a metric space is totally bounded.*

Theorem 2.0.2. *If A is totally bounded subset of a metric space, then $\bar{A}$ is totally bounded.*

Proof. Let $\epsilon > 0$ be given, cover A by open $\text{Ball}(x_1, \frac{\epsilon}{2}), \ldots, \text{Ball}(x_n, \frac{\epsilon}{2})$. Then, $\text{Ball}(x_1, \epsilon), \ldots,$ $\text{Ball}(x_n, \epsilon)$ cover $\bar{A}$. ■

Theorem 2.0.3. *A metric that is not complete can be compact.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X (which is not complete) that does not have a limit. For each $x \in X$, it is not a limit of $\{x_n\}$, so there is an ϵ_x and an N_x such that for all $n > N_x$, there is $m > n$ so $x_m \notin \text{Ball}(x, 2\epsilon_x)$. By Cauchy, there is N so that if $m, n > N$, then $d(x_m, x_n) < \epsilon$, then for $m > N$, $m \geq N_\epsilon$, $x_m \in \text{Ball}(x, \epsilon)$. The oBall$(x, \epsilon_x)$ for an open cover of X, so if X were compact, there would be a finite subcover of X, Ball$(x_1, \epsilon_{x_1}), \ldots,$ Ball(x_n, ϵ_{x_n}), so $\{x_n\}$ asdksjasd aksd ja finite number of values, so by Cauchy, it will converge, which is a contradiction. ■

Theorem 2.0.4. *If X is complete, if $A \subset X$ is totally bounded, then $\bar{A}$ is compact.*

Proof. Proof of first theorem. Let $\mathcal{C}$ be an open cover, we want to find a finite subcover. Cover X by a finite number of balls of radius 1. If each B_j can be covered by a finite subcover of this collection, then get a finite subcover for X itself. At least one of the balls can be covered by a finite subcover, call it B'. ■

Theorem 2.0.5. *Let (X, d) be a complete metric space. Then, if (X, d) is totally bounded then it is compact.*

Proof. Let $\mathcal{C}$ be an open cover of X. We need to show it has a finite subcover. Suppose it does not. Let $B_1^1, \ldots, B_n^1$ be closed balls of radius 1 that cover X. Since there is no finite subcover of X, there is at least one j such that B_j^1 is not finitely covered by $\mathcal{C}$. Set $A_1 = B_j^1$. Cover A_1 by a finite number of closed balls of radius $\frac{1}{2}, B_1^2, \ldots, B_{n_2}^2$. Then, there is at least one j so that $A_1 \cap B_j^2$ is not finitely covered by $\mathcal{C}$. Let $A_2 = B_j \cap A_1 \neq \varnothing$, diameter of $A_2 \leq 1$. Cover A_2 by a finite number of closed balls of radius $\frac{1}{4}, B_1^3, \ldots, B_{n_3}^3$. At least one of the $A_2 \cap B_j^3$ cannot be finitely covered by $\mathcal{C}$, call that one A_3, etc. Diamter $A_3 \leq \frac{1}{2}$. Get a sequence $\{A_n\}$ of closed sets $A_n \supseteq A_{n+1}$, diameter $A_n \to 0$. For each n, choose $x_n \in A_n$. Then $\{x_n\}$ is a Cauchy sequence. By completeness, $\{x_n\}$ converges, say to x_*. Since $\mathcal{C}$ is a cover, there is $\mathcal{O} \in \mathcal{C}$ such that $x_* \in \mathcal{O}$. Thus, there is $\epsilon > 0$ such that Ball$(x_*, \epsilon) \leq \mathcal{O}$. Since $\{x_n\}$ converges to x_*, there is N such that $x_n \in$ Ball$(x_*, \frac{\epsilon}{2})$ for $n \geq N$, but there is N' such that if $n \geq N'$ then diam$(A_n) \leq \frac{\epsilon}{2}$, so $A_n \subseteq$ Ball$(x_*, \epsilon) \subseteq \mathcal{O} \in \mathcal{C}$, ie A_n is covered by a finite subcover. Contradiction. ■

Corollary 2.0.6. *Let (X, d) be a complete metric space, let $A \subseteq X$, with A totally bounded. Then $\bar{A}$ is compact.*

Corollary 2.0.7. $[a, b] \subseteq \mathbb{R}$, *the first is compact. Any closed bounded subset of $\mathbb{R}^n$ is compact.*

Example 2.0.1. Let X be a set, and let (M, d) be a metric space. Let $B_b(X, M)$ be the set of all bounded functions from X to M. Metric $d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$, let $\mathcal{T}$ be a topology for X, consider $C_b(X^{\mathcal{T}}, M) =$ continuous functions in $B_b(X, M)$. What are the compact subsets of C_b? What are the totally bounded subsets. Let J be a totally bounded subset of $C_b(X, M)$. Then, given $\epsilon > 0$, we can find $g_1, \ldots, g_n \in J$ such the Ball(g_j, ϵ), $j = 1, \ldots, n$ cover J. Given any $x \in X$, such taht $g_1, \ldots, g_n$ are continuous, there are open sets, $\mathcal{O}_1, \ldots, \mathcal{O}_n$, with $x \in \mathcal{O}_j$, for all j such that if $y \in \mathcal{O}_j$, then $d(g_j(x), g_j(y)) < \epsilon$, let $\mathcal{O} = \bigcap_{j=1}^n \mathcal{O}_j$, such that $x \in \mathcal{O}$. Then for any $y \in \mathcal{O}$, $d(g_j(x_, g_j(y)) < \epsilon$ for all j. Then for $f \in \mathcal{T}$, there is a j with $d_\infty(f, g_j) < \epsilon$, and so for $y \in \mathcal{O}$, $d(f(x), f(y)) \leq d(f(x), g_j(x)) + d(g_j(x), g_j(y)) + d(g_j(y), f(y)) < 3\epsilon$. Thus, given $x \in X$, for any $\epsilon > 0$, there is $\mathcal{O} \in J, x \in \mathcal{O}$ such that for $y \in \mathcal{O}$ has $d(f(x), f(y)) < \epsilon$, for all $f \in J$. The family f is equicontinuous at x. Since it is true for all x, we say that f is an equicontinuous set of functions. Also, for fixed x, given $f \in F$, there is g with $f \in$ Ball(g_j, ϵ), so that $d(f(x), g_j(x)) < \epsilon$, i.e., $\{f(x) : f \in F\} \subseteq M$ is covered by the balls Ball$(g_j(x), \epsilon)$, so it is totally bounded. Hence, F is pointwise totally bounded.

Theorem 2.0.8. *(Core of the Arzeli-Ascoli Theorem) Let $(X, \mathcal{T})$ be compact. Let $F \subseteq C(X, M)$. If F is equicontinuous and pointwise totally bounded, then F is totally bounded for d_∞.*

Proof. Let $\epsilon > 0$ be given. Then, by equicontinuity, for each $x \in X$, there is an open set $\mathcal{O}_x$, such that $x \in \mathcal{O}_x$ such that if $y \in \mathcal{O}_x$, then for all $f \in F$, we have $d(f(x), f(y)) < \epsilon$. The $\mathcal{O}_x$'s form an open cover of X, so there is a finite subcover $\mathcal{O}_{x_1}, \ldots, \mathcal{O}_{x_n}$. For each $j = 1, \ldots, n$, $\{f(x_j) : f \in F\}$ is totally bounded, so there is a finite subset, S_j such that the ϵ-balls about the points of S_j cover the aforementioned set. Let $S = \bigcup_j S_j$, a finite set in M. Let $\Psi = \{\psi : \{1, \ldots, n\} \to S\}$ a finite set. For each $\psi \in \Psi$, let $A_\psi = \{f \in F : f(x_j) \in \text{Ball}(\psi(j) \in S, \epsilon)\}$. The A_ψ's cover F. If $f, g \in A_\psi$, for any x, there is $y \in X$, there is j so that $y \in \mathcal{O}_{x_j}$. Then $d(f(x), g(x)) \leq d(f(y), f(x_j))(\leq \epsilon) + d(x_j < \epsilon, g_{x_j} \leq \epsilon)(\leq 2\epsilon) + d(g(x_j), g(y)) \leq \epsilon < 4\epsilon$, i.e. diameter $(A_\psi) < 4\epsilon$. ∎

Theorem 2.0.9. *(Arzela-Ascoli): Let $(X, \mathcal{T})$ be a complete metric space. Then, $F \subseteq C(X, M)$ is compact in d_∞ if it is closed and equicontinuous and pointwise totally bounded.*

Definition 2.0.2. Locally compact spaces. A topological space $(X, \mathcal{T})$ is locally compact if for each $x \in X$, there is a $\mathcal{O} \in \mathcal{T}, x \in \mathcal{O}, \bar{\mathcal{O}}$ is compact.

Chapter 3

Locally Compact Hausdorff Spaces

Note 3.0.1. LCH := "locally compact Hausdorff"

$(X, \mathcal{T})$ be a LCH space.

Lemma 3.0.1. *Let $C \subseteq X$ be compact. Then there is open $\mathcal{O}$ with $C \subseteq \mathcal{O}$, $\overline{\mathcal{O}}$ compact.*

Proof. For each $x \in C$, let $\mathcal{O}_x$ be open with $x \in \mathcal{O}_x$, $\overline{\mathcal{O}}$ compact. $\{\mathcal{O}\}_{x \in C}$ covers C, so there is a finite subcover $\mathcal{O}_{x_1}, \ldots \mathcal{O}_{x_n}$. Let $\mathcal{O} = \cup_{j=1}^{n} \mathcal{O}_{x_j}$, so $C \subseteq \mathcal{O}$, $\overline{\mathcal{O}} = \cup_{j=1}^{n} \overline{\mathcal{O}_{x_j}}$ is compact. ∎

Theorem 3.0.2. *Let $(X, \mathcal{T})$ be a LCH. Let $C = X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is open $\mathcal{U}$, $C \subseteq \mathcal{U}$, $\overline{\mathcal{U}}$ compact, $\overline{\mathcal{U}} \subseteq \mathcal{O}$.*

Proof. By the previous lemma, we can choose $\mathcal{O}_1$, $C \subseteq \mathcal{O}_1 \subseteq \overline{\mathcal{O}}_1$, the last of which is compact. Let $\mathcal{O}_2 = \mathcal{O} \cap \mathcal{O}_1$, see $C \subseteq \mathcal{O}_2 \subseteq \mathcal{O}$, where $\mathcal{O}_2$ is compact. So we can assume $\mathcal{O}$ has compact closure. $C \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}}$. Let $B = \overline{\mathcal{O}} \backslash \mathcal{O}$, closed $\subseteq \overline{\mathcal{O}}$. C, B are disjoint compact subsets of $\overline{\mathcal{O}}$. Because $\overline{\mathcal{O}}$ is compact, so normal, we can find disjoint relatively open $\mathcal{U}$, $\mathcal{V} \subseteq \overline{\mathcal{O}}$, with $C \subseteq \mathcal{U}$, $B \subset \mathcal{V}$. Then, $\mathcal{V}'$ is closed, $\mathcal{U} \subseteq \mathcal{V}'$. Thus, $\overline{\mathcal{U}} \subseteq V'$, so $\overline{\mathcal{U}} \cap B = \varnothing$. Thus, $\overline{U} \subseteq \mathcal{O}, \mathcal{U} \subseteq \mathcal{O}$. ∎

Theorem 3.0.3. *Let $(X, \mathcal{T})$ be LCH. Let $C \subseteq X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is a continuous $f : X \to [0, 1]$ with $f(x) = 1$, for $x \in C$ and $f(x) = 0$ for $x \notin \mathcal{O}$.*

Proof. Choose open $\mathcal{U}$ with $C \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}}$ (compact) $\subseteq \mathcal{O}$. Choose V with $C \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{O}, \overline{\mathcal{U}} - \mathcal{V}$ closed in $\mathcal{U}$, disjoint from C, so by Urysohn's Lemma, there exists $\tilde{f} : \overline{\mathcal{U}} \to [0,1]$, such that when $x \in C$, it evaluates to 1 and it evaluates to 0 for $x \in \overline{\mathcal{U}} - \mathcal{V}$. Let f be defined by $f(x) = \tilde{f}(x)$ if $x \in \overline{\mathcal{U}}$ and $f(x) = 0$ if $x \notin \overline{\mathcal{U}}$. We need f to be continuous. If $x \in \mathcal{U}$, then f is continuous at x, as $\tilde{f}$ is. If $x \notin \mathcal{U}$, then $x \notin \overline{\mathcal{V}}$, so $x \in X \backslash \overline{\mathcal{V}}$ open, on $X \backslash \overline{\mathcal{V}}$, $f(x) = 0$. ■

Definition 3.0.1. For $(X, \mathcal{T})$ LCH, let $C_c(X)$ be the set of continuous $\mathbb{R}-$valued functions on X "of compact support", i.e. there is a compact set outside of which $f \equiv 0$. $C_c(X)$ is an algebra for pointwise operations. $e, f, g \in C_c(X)$, then $f + g$, fg, $rf(r \in \mathbb{R}) \in C_c(X)$.

Note 3.0.2. $C_c(X) \subseteq C_b(X), || \cdot ||_\infty$, usually not complete if X is not compact. Its completion is the algebra of continuous functions that "vanish at infinity," $f \in C_\infty(X)$ if $\forall \epsilon > 0$, there is a compact set C_ϵ such that $|f(x)| \leq \epsilon$ for $x \notin C_\epsilon$. GL$(n, \mathbb{R})$ is locally compact.

Chapter 4

Measure Theory!!!

Note 4.0.1. Recall the first day of lecture: $C([0,1])$, for the L^1 and L^2 norms, we can have a Cauchy sequence. We want to figure out a way to accurately find the integral in these troublesome cases. We want a set and a family of subsets $\mathscr{F}$, and some function $\mu : \mathscr{F} \to \mathbb{R}^+$. We want additivity, i.e. if $E, F \in \mathscr{F}$, and if E and F are disjoint and $E \oplus F \in \mathscr{F}$, then $\mu(E \cup F) = \mu(E) + \mu(F)$. Also if $E, F \in \mathscr{F}$, $E \subseteq F$, $F = E \oplus (F \backslash E)$ (let $\oplus$ be the disjoint union), so $\mu(F) = \mu(E) + \mu(F \backslash E)$, i.e. $\mu(F \backslash E) = \mu(F) \backslash \mu(E)$.

Definition 4.0.1. Let X be a set and let R be a nonempty family of subsets of X. We say that R is a ring if R is closed under finite unions and differences of elements $E \backslash F$. This implies closed under finite intersection over $E \cap F = E \backslash (E \backslash F)$. If also $X \in R$, call $\mathscr{J}$ an algebra (or a field).

Definition 4.0.2. A finitely added measure or a ring R of sets is a finite $\mu : R \to \mathbb{R}^+$ such that if $E, F \in R$ and are disjoin, then $\mu(E \oplus F) = \mu(E) + \mu(F)$

Definition 4.0.3. A ring R is said to be a σ-ring of to so closed under taking countable unions of elements fo R, so we can take countable intersections.

Definition 4.0.4. A σ-algebra: $E = \cup_{n=}^{\infty} E_n$, then $\cap E_n = E \setminus \cup_{n=1}^{\infty} (E \setminus E_n)$

Definition 4.0.5. Let R be a σ-ring. By a measure on R we mean a function $\mu : R \to \mathbb{R}^+$, $\mathbb{R}^+ \cup \{+\infty\}$, $\mathbb{R}$, $\mathbb{R}^n$, Banach spaces, which is countable additive, i.e. if $\{E_n\}_n^{\infty}$ is a disjoint family of elements in R. Then,

$$\mu\left(\bigoplus_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Theorem 4.0.1. *Let $\mathscr{S}$ be a collection of rings (or algebras, or σ-algebras, or σ-rings, etc) of a given set X. Then the intersection of these rings is a ring (or ...).*

Definition 4.0.6. Given any collection of subsets of X, there is a smallest ring (...) that contains the collection, namely the intersection of all contains rings, etc.

Definition 4.0.7. Let $(X, \mathcal{T})$ be a topological space.

1. The σ-ring generated by $\mathcal{T}$ is called the σ-ring of Borel subsets of X.

Let $(X, \mathcal{T})$ be a LCH space, then the σ-ring generated by the compact subsets is called the σ-ring of Borel sets.

Note 4.0.2. $X = \mathbb{R}, \mathscr{S} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$