Math 202A Notes

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Contents

Chapter 1

Topology

1.1 Metric Spaces

Definition 1.1.1. Let X be a set, a metric on X is a function $d : X \times X \to \mathbb{R}$, such that

1. $d(x, x) = 0$, for all $x \in X$
2. if $d(x, y) = 0$, then $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Note that if we do not have that if $d(x, y) = 0$, then $x = y$, then we have a semimetric.

Definition 1.1.2. If $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we call the define the following norms:

1. $||v||_2 = (\sum |v_j|^2)^{\frac{1}{2}}$
2. $||v_1|| = \sum |r_j|$
3. $||v||_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$
4. $||v||_p = (\sum |v_j|^p)^{\frac{1}{p}}$

Definition 1.1.3. We can now define the following:

1. $d_2 := ||v - w||_2$
2. $d_1 := ||v - w||_1$
3. $d_\infty := ||v - w||_\infty$
4. $d_p := ||v - w||_p$

Example 1.1.1. Let (X, d) be a metric space, then let $Y \subset X$, the restriction of d to $Y \times Y \subset X \times X$ makes Y a metric space.

Example 1.1.2. $C([0, 1]) = \mathbb{R}$-valued continuous functions on $[0, 1]$.

Note 1.1.1. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. By a norm on V, we mean a function $||\cdot|| : V \to \mathbb{R}^+$ such that:

1. $||v|| = 0 \iff v = 0$
2. $||\alpha v|| = |\alpha|||v||$
3. $||v + w|| \leq ||v|| + ||w||$

Example 1.1.3. From a norm on V, we get a metric on V by $d(v, w) = ||v-w||$. For $f \in C([0, 1])$:

1. $||f||_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}$
2. $||f||_1 = \int_0^1 |f(t)| dt$
3. $||f||_\infty = \sup\{|f(t)| : t \in [0, 1]\}$
4. $||f||_p = \left(\int_0^1 |f(t)|^p dt\right)^{\frac{1}{p}}$

Definition 1.1.4. Let (X, d) be a metric space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. We say that this sequence converges to a point $x_* \in X$ if for all $\epsilon > 0$, there exists $N > 0$ such that for $n > N$, $d(x_n, x_*) < \epsilon$. [Note that this is the same as saying that $x_n \in \text{oBall}(x_*, \epsilon)$, where $\text{oBall}(x_*, \epsilon) = \{y \in X \mid d(y, x_*) < \epsilon\}$.]

Definition 1.1.5. X is complete if every Cauchy sequence converges to some point of X.

Example 1.1.4. Some examples of complete metric spaces include $\mathbb{R}, \mathbb{R}^n, \mathbb{C}$.

Note 1.1.2. If S is a closed subset of $\mathbb{R}^n$, then S with the restricted metric is complete. Consider $C([0,1]) : ||f||_\infty = \sup\{|f(t)| : t \in [0,1]\}$. The uniform norm convergence for it is uniform convergence. If $\{f_n\}$ is Cauchy for $||\cdot||_\infty$, then for each $t_* \in [0,1]$, then $\{f_n(t_*)\}$ is a Cauchy sequence, so it converges. Note that $f(t) = \lim(f_n(t))$, the uniform limit of continuous functions is continuous.

Definition 1.1.6. Let (X, d) be a metric space, and let S be a subset of X. We say that S is dense in X if every open ball in X contains a point of S.

Definition 1.1.7. Let (X, d) be a metric space, by a completion of X, we mean a metric space, $(\overline{X}, \bar{d})$, together with $j : X \to \overline{X}$ such that j is an isometry and j is dense in X.

Definition 1.1.8. An isometry is a function j such that $d(x, y) = d(j(x), j(y))$.

Example 1.1.5. Every metric space has a completion, and the completion is essentially unique. Let (X, d) be a metric space. Let $\text{CS}(X, d)$ be the set of all Cauchy sequences in (X, d). Try to define a distance on $\text{CS}(X, d)$: let $\{x_n\}, \{y_n\}$ be two Cauchy sequences. Consider $\{d(x_n, y_n)\}$, we claim it is Cauchy in $\mathbb{R}$. Set $\tilde{d}(\{x_n\}, \{y_n\}) = \lim\{d(x_n, y_n)\}$.

Note 1.1.3. Note that $d(x, y) \leq d(x, z) + d(z, y)$ and $d(x, y) - d(x, z) \leq d(z, y)$, so $|d(x, y) - d(x, z)| \leq d(z, y)$ and $|d(x, z) - d(y, z)| \leq d(x, y)$. Hence,

$$\begin{aligned}|d(x_n, y_n) - d(x_n, y_n)| &= |d(x_n, y_n - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \\ &\leq d(y_n, y_m) + d(x_n, x_m) \to 0\end{aligned}$$

Now, let (X, d) be a semimetric space. We now define an equivalence relation on X, by if $d(x, y) = 0$, then $[x] = \{y : d(x, y) = 0\}$. Define $X/\sim\ := \{\text{equivalence classes}\}$. Define $\hat{d}$ on $X/\sim$ by $d([x], [y]) = d(x, y)$, well-defined. If $x' \in [x]$, $y' \in [y]$, then $d(x', y') \leq d(x, x) + d(y, y) + d(x, y), d(x', y') = d(x, y)$, so $\hat{d}$ is a metric on $X/\sim$. Let $\tilde{d}$ on $\text{CS}(X, d)$ be the corresponding metric in the equivalence classes. The equivalence relation is $\{x_n\} \sim \{y_n\}$ if $\hat{d}(\{x_n\}, \{y_n\}) = 0$ or $\lim_{n\to\infty} d(x_n, y_n) = 0$. Embed (X, d) in $\text{CS}(X, d)/\sim$ by $x \mapsto$ Cauchy sequence, $x_n = x$, for all n, $\phi(x) = \{x_n = x\}, \tilde{d}(\phi(x), \phi(y)) = \lim d(x_n, y_n) = \lim d(x, y) = d(x, y)$, so ϕ is an isometry of X into $\text{CS}(X, d) \to \text{CS}(X, d)/\sim$. The image of X is dense in $\text{CS}(X, d)/\sim$. Let $\{x_n\}$ be any Cauchy sequence. Then, given any $\epsilon > 0$, there exists N such that for $m, n \geq N, d(x_m, x_n) < \epsilon$. Consider $\phi(x_N)$. Then, $d(\{x_n\}, \phi(x_n)) = \lim_{n\to\infty}\{d(x_n, x_N)\} < \epsilon$. To show that $(\text{CS}(X, d)/\sim, d)$ is complete. For small m, let $\ldots \in \text{CS}(X, d)$, assume $\{S^m\}$ is a Cauchy sequence in $\text{CS}(X, d)$, for each k, find $x_k \in X$, such that $d(\phi(x_k), S^m) < \frac{1}{k}$, then $S = \{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, and $\tilde{d}(S^m, S)_{n\to\infty} \to 0$.

Definition 1.1.9. Let $(X, d_x), (Y, d_y)$ be metric spaces, $f : X \to Y$, and $x_0 \in X$, we say that f is continuous at x_0 if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x_0) < \delta$, then $d(f(x), f(x_0)) < \epsilon$, or equivalently, if $x \in \text{Ball}(x_0, \delta)$, then $f(x) \in \text{Ball}(f(x_0), \epsilon)$. For any open ball B about $f(x_0)$, there is an open ball C about$f(x_0)$ such that if $x \in B$, then $f(x) \in C$, or equivalently that $x \in f^{-1}(C)$, and $B \subseteq f^{-1}(C)$.

Definition 1.1.10. Let (X, d) be a metric space. If $A \subseteq X$ is an open subset (for d) if for each αA, there is an open ball about x contained in A.

Note 1.1.4. If f is continuous, i.e continuous at all points, let $\mathcal{O}$ be an open set in Y, let $x_0 \in f^{-1}(\mathcal{O})$, then $\mathcal{O}$ contains a ball about x_0 such that $x_0 \in C \subset f^{-1}(B)$, so $C \subseteq f^{-1}(\mathcal{O})$, so $f^{-1}(\mathcal{O})$ is open. Conversely, let f be any function from X to Y. If it is true that for any open set $\mathcal{O}$ in Y, $f^{-1}(\mathcal{O})$ is open in X, then f is continuous. Given any $\epsilon > 0$, let $\mathcal{O} = \text{Ball}(f(x_0), \epsilon)$, then $f^{-1}(\text{Ball}(f(x_0), \epsilon))$ is open. Hence, there is a ball $\text{Ball}(x_0, \delta)$ such that $\text{Ball}(x_0, \delta) \subseteq f^{-1}(\text{Ball}(f_0, \epsilon))$. The following are properties of the collection of open sets of a metric space:

1. An infinite union of open sets is open
2. A finite intersection of open sets is open. For $x_0 \in \mathcal{O}_1 \cap \mathcal{O}_2$, $\text{Ball}(x_0, r_1) \subseteq \mathcal{O}_1$, $\text{Ball}(x_0, r_2) \subseteq \mathcal{O}_2$. Let $r = \min\{r_1, r_2\}$, then $\text{Ball}(x_0, r) \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$.
3. X and $\varnothing$ are open.

Definition 1.1.11. Let X be a set. By a topology for X, we mean a collection $\mathcal{T}$ of subsets of X such that:

1. Arbitrary unions of elements of $\mathcal{T}$ are in $\mathcal{T}$.
2. Finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.
3. X and $\varnothing$ are elements of $\mathcal{T}$.

Definition 1.1.12. Let $\mathcal{T}$ be a topology of X. Then $A \subseteq X$ is closed if A' is open.

Note 1.1.5. Properties of closed sets:

1. Arbitrary intersections of closed sets are closed.
2. Finite unions of closed sets are closed.

3. X and $\varnothing$ are closed.

Definition 1.1.13. Let $A \subseteq X$. By the closure of A, we mean the smallest closed set that contains A, i.e. the intersection of all closed sets that contain A.

Definition 1.1.14. By the interior of A, we mean the biggest open set contained in A, i.e. the union of all open sets contained in A.

Definition 1.1.15. Let C be a closed set, and let $A \subseteq C$, we say that A is dense in C if $\overline{A} = C$.

Definition 1.1.16. Let X be a set, and let $\mathscr{S}$ be a collection of subsets of X, the smallest topology containing the intersection of topologies that contain $\mathscr{S}$ is said to be the topology generated by $\mathscr{S}$, and $\mathscr{S}$ is called a subbase for that topology. Note that if $\mathscr{C}$ is a collection of topologies for X, then $\bigcap\{\mathcal{T} \in \mathscr{C}\}$ is a topology for X.

Definition 1.1.17. Let X be a set, and let D be the collection of subsets of X. D is a topology for X, called the discrete topology for X. It is given by a metric:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

D is the biggest topology in X.

Definition 1.1.18. The smallest topology in X is $\{\varnothing, X\}$, called the indiscrete topology.

Note 1.1.6. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, such that:

$$\begin{array}{cc} \mathcal{T}_1 \subseteq & \mathcal{T}_2 \\ \text{smaller} & \text{larger} \\ \text{weaker} & \text{stronger.} \end{array}$$

Usually, we require that $\bigcup \mathscr{S} = X$. For $X = \mathbb{R}$, (a, b), $\mathscr{S} = \{(\infty, a), (b, +\infty)\}$.

Definition 1.1.19. A collection of subsets of X is a base for a topology is the set of all arbitrary unions of elements of $\mathscr{S}$ is a topology.

Example 1.1.6. $\mathscr{S} = \{(a, b) \subset \mathbb{R}\}$, $\mathbb{R}^2 = \{\text{open balls}\}$

Note 1.1.7. For $\mathscr{S}$ to be a base, it must have the property that if $A, B \in \mathscr{S}$, then $A \cap B$ must be a union of elements of $\mathscr{S}$.

Example 1.1.7. If $\mathscr{S}$ is any collection of subset of X, then the collection of all finite intersections of elements must be a topology.

Definition 1.1.20. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. f is continuous if for all open sets $\mathcal{O} \in \mathcal{T}_Y \implies f^{-1}(\mathcal{O}) \in \mathcal{T}_X$.

Note 1.1.8. Let Y be a set and $\mathscr{S} = \{A_\alpha\}$, let X be a set, and $f : X \to Y$ be a function. Then,

1. $f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$

2. $f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha)$

3. If $A, B \subseteq Y$, then $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$.

Example 1.1.8. Given $(X, \mathcal{T}_X)$ and $f : X \to Y$, let $\mathscr{S}$ be a subbase for $\mathcal{T}_Y$. Then f is continuous if $f^{-1}(A) \in \mathcal{T}_X$, for all $A \in \mathscr{S}$.

Example 1.1.9. Let X be a set and let $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let there be a quasifunction $f_\alpha : X_\alpha \to X$. Let $\mathcal{T}$ be the strongest topology such that all of the f_α's are continuous. Given α_0, f_α. If $A \subseteq X$, then if A is to be open, we must have that $\bar{f}_{\alpha_0}(A) \in \mathcal{T}_{\alpha_0}$. Now, let $\mathscr{S}_{\alpha_0} = \{A \subseteq : f_{\alpha_0}^{-1}(A) \in \mathcal{T}_{\alpha_0}\}$ is a topology for X; in fact, it is the strongest topology making f_{α_0} continuous. The strongest topology making all of the f_α continuous is the intersection of the $\mathscr{S}_\alpha$.

Example 1.1.10. Let $(X, \mathcal{T})$ be a topological space, let Y be a set. Then, $f : X \to Y$, $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\}$ is the strongest topology making f continuous. Usually, we want f to be onto Y.

Definition 1.1.21. We begin by defining an equivalence relation, $\sim$, on X by $x_1 \sim x_2$, if $f(x_1) = f(x_2)$. This gives a partition of X: the quotient of ${}^{X}/_{\sim}$, the quotient of X by $\sim$. This topology is called the quotient topology determined by f.

Definition 1.1.22. For $\sim$ on a set X, $B \subseteq X$ is saturated if when $x \in B$ and $x_1 \sim x$, for $x_1 \in B$.

Note 1.1.9. The open sets in the quotient topology in f on Y are in bijection with the saturated open sets of X.

Note 1.1.10. We want the weakest topology to make all of the functions ot be continuous. For any B_α, any open set $\mathcal{O} \in \mathcal{T}_\alpha$ (where the topological space is $(Y_\alpha, \mathcal{T}_\alpha)$, we need $f_\alpha^{-1}(0) \subseteq X$. This weakest topology has a sub-base $\{f_\alpha^{-1}(0) : \mathcal{O} \in \mathcal{T}_\alpha\}$, which is called the conditional topology.

Example 1.1.11.

1. Given $(Y, \mathcal{T})$, let X be a subset of Y. $X \hookrightarrow^i Y$. The weakest topology making i continuous is $\{i^{-1}(\mathcal{O})\ \mathcal{O} \in \mathcal{T}\}$. $i^{-1}(0)$ can form the relative topology, $\{X \cap \mathcal{O} : \mathcal{O} \in \mathcal{T}_Y\}$.

2. Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be given. We can form the product topology, $X_1 \times X_2$, whose sub-base is $\mathcal{O} \times X_2$, $\mathcal{O} \in \mathcal{T}_1$, $X_1 \times \mathcal{U}, \mathcal{U} \in \mathcal{T}_2$, intersected: $\{\mathcal{O} \times \mathcal{U} : \mathcal{O} \in \mathcal{T}_1, \mathcal{U} \in \mathcal{T}_2\}$ is a sub-base. Furthermore, $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$. Then, form $\Pi_{\alpha \in A} X_\alpha$, functions f from A into $\cup X_\alpha$ such that $f(\alpha) \in X_\alpha$ used for all α. X_α is called the product topology, sub-base, π_α, for $\mathcal{O} \in \mathcal{T}_\alpha$, $X_1 \times \ldots \times \mathcal{O} \times \ldots$. We can only take finite intersections, so there can only be finitely many open sets.

3. $C([0,1]), ||\cdot||$. For each $h \in C([0,1])$, define linear functional, ϕ_n on $C([0,1])$ by

$$\phi_n(f) = \int_0^1 f(t)h(t)dt, C([0,1]) \to_{\phi_n} \mathbb{R}.$$

   We can then ask for the correspondingly weakest topology.

$$|\phi_n| \leq ||h||_\infty ||f||_1,$$

   where we chose h bounded.

Example 1.1.12. Special properties of topologies from metric spaces. If $x, y \in X$ and $x \neq y$, let $r = d(x,y) \neq 0$. Then, oBall$(x, \frac{r}{3})$ and oBall$(y, \frac{r}{3})$ are disjoint.

Definition 1.1.23. A topology $\mathcal{T}$ on X is Hausdorff is for any points $x, y, x \neq y$, there are open sets, $\mathcal{O}_x$ and $\mathcal{O}_y$, $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y$, and $\mathcal{O}_x \cap \mathcal{O}_y = \varnothing$.

Definition 1.1.24. The Separation Axioms:

1. T_2: Hausdorff
2. T_1: Given $x, y, x \neq y$, there exists $\mathcal{O}_x$ with $x \in \mathcal{O}_x$, $y \notin \mathcal{O}_x$ and there exists a similar $\mathcal{O}_y$.
3. T_0: Given $x, y, x \neq y$, there exists $\mathcal{O}$ such that only one of x or y is in $\mathcal{O}$.

Definition 1.1.25. A topology $\mathcal{T}$ is normal if for any two disjoint closed sets, A, B, there are disjoint open sets $\mathcal{O}_A$, $\mathcal{O}_B$, such that $A \subseteq \mathcal{O}_A, B \subseteq \mathcal{O}_B$.

Theorem 1.1.1. *Any topology that comes from a metric is normal.*

Proof. Let A, B be disjoint closed sets in (X, d). For each $x \in A$, B is closed so $x \notin B$. Can choose ϵ_x such that

$$\text{oBall}(x, \epsilon_x) \cap B = \varnothing.$$

Then, for each $y \in B$, we can choose ϵ_y such that oBall$(y, \epsilon_y) \cap A = \varnothing$.

$$\mathcal{O}_A = \bigcup_{x \in A} \text{oBall}\left(x, \frac{\epsilon_x}{3}\right), \mathcal{O}_B = \bigcup_{x \in B} \text{oBall}\left(y, \frac{\epsilon_y}{3}\right).$$

Note that $\mathcal{O}_A \cap \mathcal{O}_B = \varnothing$, as if $z \in \mathcal{O}_A \cap \mathcal{O}_B$, then there exists an $x \in A$, such that $z \in$ oBall$\left(x, \frac{\epsilon_x}{3}\right)$ and there exists $y \in B$, such that $z \in$ oBall$\left(y, \frac{\epsilon_y}{3}\right)$. Hence, $d(x, y) \leq \frac{\epsilon_x + \epsilon_y}{3}$. So, if $\epsilon = \max\{\epsilon_x, \epsilon_y\}$, this is bounded by $\frac{2\epsilon}{3}$. ■

Theorem 1.1.2. *(Urysohn's Lemma) Let $(X, \mathcal{T})$ be a normal topological space and if A, B are disjoint, closed sets in X, there exists a continuous map,*

$$f : X \to [0, 1] \subset \mathbb{R},$$

such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.

Proof. If $(X, \mathcal{T})$ is such that for every closed A, B which are disjoint, we have f, for $\mathcal{T}$ normal: If A, B are disjoint, $f : X \to [0,1]$, $f|_A = 0, f|_B = 1$, set $\mathcal{O}_A = \left\{x : f(x) < \dfrac{1}{3}\right\}, \mathcal{O}_B = \left\{x : f(x) > \dfrac{2}{3}\right\}$. Now, let $\mathcal{O}_A = \left\{x : f(x) > \dfrac{1}{3}\right\} \cap \mathcal{O}_B$.

Lemma 1.1.3. *If $(X, \mathcal{T})$ is normal, and if A is closed, $\mathcal{O}$ is open, $A \subseteq \mathcal{O}$, then there is an open set $\mathcal{U}$, such that $A \subseteq \mathcal{U} \subseteq \bar{\mathcal{U}}$.*

Proof. Note that $\mathcal{O}^C$ is closed, by definition, so, by normalily, there are open sets $\mathcal{U}, \mathcal{V}$, such that $A \subseteq \mathcal{U}$ and $\mathcal{O}^C \subseteq \mathcal{V}, \mathcal{V}^C \subseteq \mathcal{O}$. Then,

$$\mathcal{U} \subseteq \mathcal{V}^C \subseteq \mathcal{O}, \text{ so } A \subseteq \mathcal{U} \subseteq \bar{\mathcal{U}} \subseteq \mathcal{V}^C \subseteq \varnothing.$$

■

(Part) Given $(X, \mathcal{T})$ normal, A, B closed, disjoint, choose $\mathcal{O}_{\frac{1}{2}}$ such that $A \subseteq \mathcal{O}_{\frac{1}{2}} \subseteq \bar{\mathcal{O}}_{\frac{1}{2}} \subseteq B^C$. Then, choose $\mathcal{O}_{\frac{1}{4}}, \mathcal{O}_{\frac{3}{4}}$, such that

$$A \subseteq \mathcal{O}_{\frac{1}{4}} \subseteq \bar{\mathcal{O}}_{\frac{1}{4}} \subseteq \mathcal{O}_{\frac{1}{2}} \subseteq \bar{\mathcal{O}}_{\frac{1}{2}} \subseteq \mathcal{O}_{\frac{3}{4}} \subseteq \bar{\mathcal{O}}_{\frac{3}{4}} \subseteq B^C.$$

Then, choose $\mathcal{O}_{\frac{1}{8}}, \mathcal{O}_{\frac{3}{8}}, \mathcal{O}_{\frac{5}{8}}, \mathcal{O}_{\frac{7}{8}}$, such that ... Now, set $\mathcal{O}_1 = X$. Get a countable base subset, $\mathcal{O}_2$ of $[0,1]$, such that $0 \notin \mathcal{O}_2$, $1 \in \mathcal{O}_2$, and for each number $r \in \mathcal{O}_2$, we have an open set $\mathcal{O}_r$ such that if $r < s$, $\bar{\mathcal{O}}_r \subseteq \mathcal{O}_s$. Now, define the function $f(t)_{t \in [0,1]} := \inf\{r : r \in \mathcal{O}_r\}$. ■

Lemma 1.1.4. *Let $\mathbb{Q}$ be a countable dense subset of $[0,1]$, $0 \notin \mathbb{Q}$, $1 \in \mathbb{Q}$. $(X, \mathcal{T})$ is a normal topological space. Assume that for each $r \in \mathbb{Q}$, we have an open set $\mathcal{O}_r$, which satisfies if $r < s$, then $\mathcal{O}_r \subseteq \bar{\mathcal{O}}_r \subseteq \mathcal{O}s$ and $\mathcal{O}_1 = X$.*

Think of $\mathcal{O}_r$ as the set of x where $f(x) < r$, for $r \in BQ$. Set $f(x) = \inf\{r \in \mathbb{Q} : x \in \mathcal{O}_r\}$. We claim that f is continuous. Use the sub-base $(-\infty, a), (a, \infty)$. If $x \in f^{-1}((-\infty, a))$ iff $f(x) < a$, so there is $s \in \mathbb{Q}$ such that $s < a$, such that $x \in \mathcal{O}_s$. Then, for all $y \in \mathcal{O}_s$, $f(y) \leq s < a$, so $\mathcal{O}_s \subseteq f((-\infty, a))$. Thus, $f^{-1}((-\infty, a)) = \cup_{r<a} \mathcal{O}_r$ open. Then, $x \in f^{-1}((a, \infty))$ iff $f(x) > a$, so there is $s \in \mathbb{Q}$, $a < s < f(x)$ with $x \notin \mathcal{O}_s$, so there is a t such that $a < t < s < f(x)$ with $x \notin \bar{\mathcal{O}}_t \subset \mathcal{O}_s$, so $x \in \bar{\mathcal{O}}_t^{\,C}$ is open, so $f^{-1}((a, \infty)) = \cup_{t>a} \bar{\mathcal{O}}_t^{\,C}$ is open.

$(X, \mathcal{T})$ is normal, A, B be closed, disjoint sets. Choose a dense $\mathcal{O} \subset [0,1], 0 \notin \mathcal{O}, 1 \in \mathcal{O}$, such that $A \subseteq \mathcal{O}_r$, for all r. Then, $\mathcal{O}_1 \cap B = \varnothing$ because that $B \subseteq \mathcal{O}_1$. Then, note that:

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B. \end{cases}$$

Definition 1.1.26. Let X be a set, and let (M, d) be a complete metric space, and consider $f : X \to M$. We say that f is bounded if there is a $m_0 \in M, r \in \mathbb{R}^+$, such that $f(x) \in \text{Ball}(m_0, r)$, for all $x \in X$. For f, g bounded functions $X \to M$, $\{d(f(x), g(x))\}_{x \in X}$ is a bounded set in $\mathbb{R}$. Set $d_\infty(f, g) = \sup\{d(f(x), g(x)), x \in X\} \approx ||f - g||_\infty$. It is easy to show that d_∞ is a metric.

Let $B(X, (M, d))$ be the set of all bounded functions from X to M, with metric d_∞.

Proposition 1.1.1. *$B(X, (M, d))$ is complete for d_∞ (because (M, d) is complete).*

Proof. Let $\{f_n\}$ be a Cauchy sequence for d_∞. Then, for any $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence because $d(f_n(x), f_m(x)) \leq d_\infty(f_n, f_m)$. Call this limit $f(x)$. It is easy to show that f is bounded. To show that $\{f_n\}$ converges to f for d_∞, let $\epsilon > 0$ be given, and choose N_0, such that for $n, m \geq N_0$, we have $d_\infty(f_m, f_n) < \dfrac{\epsilon}{2}$. Thus, given any $x \in X$, there is $N_x > N_0$ such that for $n, m \geq N_x$, $d(f_n(x), f(x)) < \dfrac{\epsilon}{2}$. Then, for $n > N_0$, $d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \epsilon$, so $d(f_n, f) < \epsilon$. ∎

Proposition 1.1.2. *Let $(X, \mathcal{T})$ be a topological space, (M, d) be a complete metric space. Let $BC((X, \mathcal{T}), (M, d))$ be the set of bounded, continuous functions from X to M. Then, $BC((X, \mathcal{T}))$ is a closed subset of $(B(X, (M, d)), d_\infty)$ and is therefore complete.*

Proof. Let $\{f_n\}$ be a sequence in $CB(X, M)$ that converges for d_∞ to $f \in B(X, M)$, to show $f \in CB(X, M)$, to show continuous at any given $x \in X$, let $\epsilon > 0$ be given. Choose N such that for $n \geq N$, $d_\infty(f, f_n) < \dfrac{\epsilon}{3}$, such that f_n is continuous on X, there exists $\mathcal{O} \subset J$, such that $x \in \mathcal{O}$ and $d(f_n(y), f_n(x)) < \dfrac{\epsilon}{3}$. Then, for $y \in \mathcal{O}$, $d(f(y), f(x)) \leq d(f(y), f_n(y)) + d(f_n(y), f_n(x)) + d(f_n(x), f(x)) < \epsilon$. ∎

Theorem 1.1.5. *Tietze Extension Theorem. Let $(X, \mathcal{T})$ be a normal topological space, and let $A \to \mathbb{R}$ be continuous. Then there is $\tilde{f} : X \to \mathbb{R}$, continuous that extends f, if $\tilde{f}|_A = f$. If $f : A \to [a, b], a, b \in \mathbb{R}$ then can arrange that $\tilde{f} : X \to [a, b]$.*

Proof. [Note that if $A \subseteq X$ is closed and if $B \subseteq A$ is closed in the relataive topology, then B is closed in X, $A \setminus B = A \cap O, O \in \mathcal{T}$, then $B = A \cap O'$, where A and O' are closed, as B is closed in X] Now, consider the first case of $f : A \to [0,1]$. Let $C_0 = \{x \in A : f(x) \leq \frac{1}{3}\}, C_1 = \{x \in A : f(x) \geq \frac{2}{3}\}$, closed in A. Then, by Urysohn's Lemma, $\exists k : X \to [0,1]$ with $k|_{C_0} = 0$, $k|_{C_1} = 1$. Let $g_1 = \frac{1}{3}k$, so $g_1 : X \to [0, \frac{1}{3}]$, $f - g_1|_A : A \to [0, \frac{2}{3}]$. Scale (?): If $h : A \to [o, r]$, then there exists g on X with $g : X \to [\frac{1}{3}r], h - g|_A A \to [0, \frac{2}{3}r]$. Apply this to $f - g_1|_A$, $r = \frac{2}{3}$. Thus there is $g_2 : X \to [0, \frac{1}{3}\frac{2}{3}], (f - g_1|A) - g_2|_A : X \to [0, \left(\frac{2}{3}\right)^2]$. Apply to $f - g_1|_A - g_2|_A$, $r = \left(\frac{2}{3}\right)^2$. So there is $g_3 : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^2], f - g_1|_A - g_2|_A - g_3|_A : X \to [0, \left(\frac{2}{3}\right)^3]$. Continue this for the nth case. Clearly we have that $g_n : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}]$, $f - \sum_{j=1}^{n} g_j|_A : X \to [0, \left(\frac{2}{3}\right)^n] \implies ||g_n||_\infty \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, define $\tilde{f} = \sum_{j=1}^{\infty} g_j$ cont, $||f - \sum^n g_j|A|| \leq \left(\frac{2}{3}\right)^n$. Hence, $\tilde{f}|_A = f$, $0 \leq g_n(x) \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, so $\sum_{j=1}^{\infty} g_j(x) \leq \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^{j-1} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{2}{3}\right)^j = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1$. If $f : A \to \mathbb{R}$, unbounded, then arctan $\mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2})$ is a homeomorphism. Let h be the arctan of $f : A \to (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [\frac{-\pi}{2}, \frac{\pi}{2}]$, as there is an equation $\tilde{h} : X \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ with $\tilde{h}|_A = h$. Let $B = \{\frac{-\pi}{2}, \frac{\pi}{2}\}$, a closed subset of $[\frac{\pi}{2}, \frac{\pi}{2}]$. Then take $B = \{\tilde{h}^{-1}(\frac{-\pi}{2}), \tilde{h}^{-1}(\frac{\pi}{2})\} \subseteq X, A \subseteq X$... ■

Definition 1.1.27. Let X be a set, $\mathcal{C}$ a collection of subsets of X. We say that $\mathcal{C}$ is a covering of X if

$$\bigcup\{A \in \mathcal{C}\} = X$$

. If $B \subseteq X$, $\mathcal{C}$ is a collection of subsets of X, we say that $\mathcal{C}$ covers $\mathcal{B}$ if $\mathcal{B} \subseteq \bigcup\{A \in \mathcal{C}\}$. If $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ is a subcover of $\mathcal{C}$ if $\mathcal{D}$ also is a c.

Definition 1.1.28. Let $(X, \mathcal{T})$ be a topological space. We say that it is compact if every open cover of X has a finite subcover.

Theorem 1.1.6. *If $(X, \mathcal{T})$ is compact and $A \subseteq X$, then the following are equivalent.*

1. *A is compact for the relative topology*
2. *If $\mathcal{C} \subseteq \mathcal{T}$ is a cover of A, then A has a finite subcover of $\mathcal{O}$.*

Proof. The open sets for the relative topology are of the form $A \cap \mathcal{O}$, $\mathcal{O} \in \mathcal{T}$. ■

Theorem 1.1.7. *If $(X, \mathcal{T})$ is compact and $A \subseteq X$ is closed then A is compact for the relative topology.*

Proof. Let $\mathcal{D} \subset \mathcal{T}$ be a collection of open sets that cover A. Since A is closed, A' is open, so $\mathcal{D} \cup ...$ is an open cover of X. ■

A compact subset of a topological space, even a compact one, need not be closed. For example, any set with at least 2 points within the discrete topology, every subset is compact.

Theorem 1.1.8. *Let $(X, \mathcal{T})$ be Hausdorff. Let $A \subseteq X$ be compact for the relative topology, then A is closed.*

Proof. Let $y \in X$, $y \notin A$. For each $x \in A$ find $\mathcal{U}_x, \mathcal{V}_x \in S$. Then the set of these $\mathcal{U}_x$ will cover A. So we have a finite subcover, $\mathcal{U}_{x_1}, \ldots \mathcal{U}_{x_n}$. Let $V = \mathcal{V}_{x_1} \cap \mathcal{V}_{x_2} \ldots \mathcal{V}_{x_n}$ be open, $y \in \mathcal{V}_1$, $V \cap A = \varnothing$. Thus A' is a union of open sets, so it is open. Thus, its compliment, A, is closed. ■

Theorem 1.1.9. *Let $(X, \mathcal{T})$ be compact and Hausdorff. For any closed subset A of X and any pf (?) $y \in X$, $y \notin A$, there are open sets u, v, disjoint, with $A \subseteq u$, $y \in V$.*

Definition 1.1.29. $(X, \mathcal{T})$ is regular for all $A \subseteq X$ closed and all $y \in X$, $y \notin A$.

Theorem 1.1.10. *Every compact Hausdorff space is normal.*

Proof. Let A, B be disjoint closed, (covered also (?)) subsets. By regularity, for each $y \in B$, there are disjoint open $\mathcal{U}_y,\ \mathcal{V}_y,\ A \subseteq \mathcal{U}_y,\ \mathcal{V}_y,\ y \in \mathcal{V}_y$. The $\{\mathcal{V}_y\}$ form an open cover of B, as by completion there is a finite subcover, $\{\mathcal{V}_{y_k}\}_{k \in I},\ I = \{1, \ldots, n\}$. ■

Proposition 1.1.3. *Let $(X, \mathcal{T}_x), (Y, \mathcal{T}_y)$ be topological spaces, and let $f : X \to Y$ be continuous. Let $A \subseteq X$ be compact. Then, $f(A) = \{f(x) : x \in A\}$ is compact.*

Proof. Let $\mathcal{C}$ be a collection of open sets in Y that cover $f(A)$. Then, $\{f^{-1}(\mathcal{O}) : \mathcal{O} \in \mathcal{C}\}$ are a collection of open sets that cover A, so there must exist a finite subcover of A, $f^{-1}(\mathcal{O}_1), \ldots, f^{-1}(\mathcal{O}_n)$, so $\mathcal{O}_1, \ldots, \mathcal{O}_n$ cover $f(A)$. ■

Proposition 1.1.4. *Let $(X, \mathcal{T}_x)$ be a compact space, and let $(Y, \mathcal{T}_y)$ be a Hausdorff topological space. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.*

Proof. Let $A \subseteq X$ be closed in X. Then, A must be compact. By Proposition 1.1.3, $f(A)$ must be compact, so because Y if Hausdorff, $f(A)$ must also be closed. ■

We can rewrite compactness in a new way shortly.

Definition 1.1.30. Let $\mathcal{C}$ be a collection of subsets of a set X We say that $\mathcal{C}$ has the finite intersection property if given any $A_1, \ldots, A_n \in \mathcal{C}$, we have that:

$$\bigcap_{j=1}^{n} A_j \neq \varnothing.$$

Proposition 1.1.5. *$(X, \mathcal{T})$ is compact iff whenever $\mathcal{C}$ is a collection of closed subsets of X with the finite intersection property, then*

$$\bigcap (A \in \mathcal{C}) \neq \varnothing.$$

Lemma 1.1.11. *(Zorn's Lemma) If a poset has the property that every chain in P has an upper bound in P, then P has at least one maximal element.*

Theorem 1.1.12. *(Tychonoff's Theorem) Let Λ be an index set, and for each $\lambda \in \Lambda$, let $(X_\lambda, \mathcal{T}_\lambda)$ be a compact topological space. Let*

$$X = \prod_{\lambda \in \Lambda} X_\lambda,$$

with the product topology. Then X is compact.

Proof. Some stuff I missed. Let $(X_\lambda, \mathcal{T}_\lambda)$ compact top spaces. Let $X = \prod X_\lambda$ with the product topology. Want to show that X is compact. Let $\mathcal{C}$ be a collection of closed sets with FIP. Need to show that $\cap\{C \in \mathcal{C}\} \neq \varnothing$. By Zorn's Lemma, there is a collection $\mathcal{D}^*$ of elements of X, $\mathcal{C} \subseteq \mathcal{D}^*$, with $\mathcal{D}^*$ maximal among collection satisfying the FIP.

Lemma 1.1.13. *Let $\mathcal{D}$ be any collection of subsets of X maximal for FIP. Then the finite intersection of sets in $\mathcal{D}$ are in $\mathcal{D}$, and if $B \subset X$ and if $B \cap A \neq \varnothing$, for all $A \in \mathcal{D}$, then $B \in \mathcal{D}$.*

Proof. Let $\mathcal{D}'$ be the collection of all finite collection of elements of $\mathcal{D}$. Then $\mathcal{D}$ has FIP, and $\mathcal{D} \subseteq \mathcal{D}'$, so by maximality, $\mathcal{D} = \mathcal{D}'$. For the second statement, consider $\mathcal{D} \cup \{B\}$, then this has FIP, because $B \cap A_1 \cap \ldots \cap A_n = B \cap \left(\bigcap_{j=1}^n A_j\right)_{j \in \mathcal{D}} \neq \varnothing$. ■

So $\mathcal{D} \cup \{B\}$ has FIP $\subseteq \mathcal{D}$. By maximality, $\mathcal{D} \cup \{B\} = \mathcal{D}, eB \in \mathcal{D}$, $\mathcal{C} \subseteq \mathcal{D}^*$. For each λ, $\{pi_\lambda(A) : A \in \mathcal{D}^*\}$ has FIP. Thus, $\{(\pi_\lambda(A)^- : A \in \mathcal{D}^*\} \subset X_\lambda$ has FIP, so since X_λ is compact, $\cap\{(\pi_\lambda(A))^- : A \in \mathcal{D}\} \neq \varnothing$. Choose $x_\lambda \in$ this set. Set $x_0 = \{x_\lambda\} \in X = \prod X_\lambda$. Want to show that $x_0 \in \cap\{C : C \in \mathcal{C}\}$, i.e., want $x_0 \in C$ for each $C \in \mathcal{C}$, suffices to show that $x_0 \notin C'$, which is open, for all $C \in \mathcal{C}$. So it suffices to show that for any $\mathcal{O}$ in base for product topology, if $x_0 \in \mathcal{O}$, then $\mathcal{O} \cap C$, $\mathcal{O} = \mathcal{U}_{\lambda_1} \times \mathcal{U}_{\lambda_2} \times \ldots \times \mathcal{U}_{\lambda_n} \times \prod_{\lambda \neq \lambda_1, \lambda j, \ldots \lambda_n}$, with $\mathcal{U}_{\lambda_j} \in J_{\lambda_j}$. By the definition of x_0, $x_{\lambda_j} \in \cap\{\pi_{\lambda_j}(A)^- : A \in \mathcal{D}^*\}$, for $j = 1, \ldots, n$. That is, for all $A \in \mathcal{D}^*$, $\mathcal{U}_j \cap \pi_{\lambda_j}(A) \neq \varnothing$. In other words, for all $A \in \mathcal{D}^*$, $A \cap \pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) \neq \varnothing$. Thus, $\pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) = \mathcal{D}^*$. Then, $\cap_{j=1}^n \pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) \in \mathcal{D}^*$, this intersection is just $\mathcal{O}$, so $\mathcal{O} \in \mathcal{D}^* \supseteq \mathcal{C}$, so $\mathcal{O} \cap C \neq \varnothing$ for all $C \in \mathcal{C}$. ■

Note 1.1.11. Tychonoff's Theorem is equivalent to the axiom of choice. Let $\mathcal{C}$ be a collection of sets, $\mathcal{C} = \{X_\lambda\}_{\lambda \in \Lambda}$. Choose one element that is not in any X_λ, e.g $\omega =$ set of all subsets of $\cup X_\lambda$. Let $Y_\lambda = X_\lambda \cup \{\omega\}$, set $\mathcal{T}_\lambda = \{X_\lambda, \{\omega\}, Y_\lambda, \varnothing\}$. Then, let $Y = \prod_{\lambda \in \Lambda} Y_\lambda$, with the product topology. By Tychons, Y is compact. Consider $\{\pi_\lambda^{-1}(X_\lambda)\}$. Claim that this has FIP, where the inside of the set braces is closed. Given $\lambda_1, \ldots, \lambda_n$, $\pi_{\lambda_1}^{-1}(X_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(X_{\lambda_2} \cap \ldots \cap \pi_{\lambda_n}^{-1}(X_{\lambda_n})$. For $j = 1, \ldots, n$, choose $x_{\lambda_j} \in X_{\lambda_j}$. Define $x \in \prod Y_\lambda$ by $x_\lambda = x_{\lambda_j}$ if $\lambda = \lambda_j$,... got too long.

1.2 Compactness in Metric Spaces

Note 1.2.1. Let (X, d) be a metric space, let $A \subseteq X$, and assume that $\bar{A}$ is compact for the relative topology. Then, for any $\epsilon > 0$, consider $\{\text{oBall}(x, \epsilon) : x \in A\} \supseteq \bar{A}$, with $\bar{A}$ is compact, so there is a finite subcover of $\bar{A}$, and so of A.

Definition 1.2.1. A subset A of a metric space (X, d) is said to be totally bounded if for any $\epsilon > 0$, it call be covered by a finite number of ϵ-balls.

Theorem 1.2.1. *Any subset of a compact subset of a metric space is totally bounded.*

Theorem 1.2.2. *If A is totally bounded subset of a metric space, then $\bar{A}$ is totally bounded.*

Proof. Let $\epsilon > 0$ be given, cover A by open Ball$(x_1, \frac{\epsilon}{2}), \ldots,$ Ball$(x_n, \frac{\epsilon}{2})$. Then, Ball$(x_1, \epsilon), \ldots,$ Ball(x_n, ϵ) cover $\bar{A}$. ■

Theorem 1.2.3. *A metric that is not complete can be compact.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X (which is not complete) that does not have a limit. For each $x \in X$, it is not a limit of $\{x_n\}$, so there is an ϵ_x and an N_x such that for all $n > N_x$, there is $m > n$ so $x_m \notin$ Ball$(x, 2\epsilon_x)$. By Cauchy, there is N so that if $m, n > N$, then $d(x_m, x_n) < \epsilon$, then for $m > N$, $m \geq N_\epsilon$, $x_m \in$ Ball(x, ϵ). The oBall(x, ϵ_x) for an open cover of X, so if X were compact, there would be a finite subcover of X, Ball$(x_1, \epsilon_{x_1}), \ldots,$ Ball(x_n, ϵ_{x_n}), so $\{x_n\}$ asdksjasd aksd ja finite number of values, so by Cauchy, it will converge, which is a contradiction. ■

Theorem 1.2.4. *If X is complete, if $A \subset X$ is totally bounded, then $\bar{A}$ is compact.*

Proof. Proof of first theorem. Let $\mathcal{C}$ be an open cover, we want to find a finite subcover. Cover X by a finite number of balls of radius 1. If each B_j can be covered by a finite subcover of this collection, then get a finite subcover for X itself. At least one of the balls can be covered by a finite subcover, call it B'. ■

Theorem 1.2.5. *Let (X, d) be a complete metric space. Then, if (X, d) is totally bounded then it is compact.*

Proof. Let $\mathcal{C}$ be an open cover of X. We need to show it has a finite subcover. Suppose it does not. Let $B_1^1, \ldots, B_n^1$ be closed balls of radius 1 that cover X. Since there is no finite subcover of X, there is at least one j such that B_j^1 is not finitely covered by $\mathcal{C}$. Set $A_1 = B_j^1$. Cover A_1 by a finite number of closed balls of radius $\frac{1}{2}$, $B_1^2, \ldots, B_{n_2}^2$. Then, there is at least one j so that $A_1 \cap B_j^2$ is not finitely covered by $\mathcal{C}$. Let $A_2 = B_j \cap A_1 \neq \varnothing$, diameter of $A_2 \leq 1$. Cover A_2 by a finite number of closed balls of radius $\frac{1}{4}$, $B_1^3, \ldots, B_{n_3}^3$. At least one of the $A_2 \cap B_j^3$ cannot be finitely covered by $\mathcal{C}$, call that one A_3, etc. Diamter $A_3 \leq \frac{1}{2}$. Get a sequence $\{A_n\}$ of closed sets $A_n \supseteq A_{n+1}$, diameter $A_n \to 0$. For each n, choose $x_n \in A_n$. Then $\{x_n\}$ is a Cauchy sequence. By completeness, $\{x_n\}$ converges, say to x_*. Since $\mathcal{C}$ is a cover, there is $\mathcal{O} \in \mathcal{C}$ such that $x_* \in \mathcal{O}$. Thus, there is $\epsilon > 0$ such that $\text{Ball}(x_*, \epsilon) \leq \mathcal{O}$. Since $\{x_n\}$ converges to x_*, there is N such that $x_n \in \text{Ball}(x_*, \frac{\epsilon}{2})$ for $n \geq N$, but there is N' such that if $n \geq N'$ then $\text{diam}(A_n) \leq \frac{\epsilon}{2}$, so $A_n \subseteq \text{Ball}(x_*, \epsilon) \subseteq \mathcal{O} \in \mathcal{C}$, ie A_n is covered by a finite subcover. Contradiction. ■

Corollary 1.2.6. *Let (X, d) be a complete metric space, let $A \subseteq X$, with A totally bounded. Then $\bar{A}$ is compact.*

Corollary 1.2.7. $[a, b] \subseteq \mathbb{R}$*, the first is compact. Any closed bounded subset of $\mathbb{R}^n$ is compact.*

Example 1.2.1. Let X be a set, and let (M, d) be a metric space. Let $B_b(X, M)$ be the set of all bounded functions from X to M. Metric $d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$, let $\mathcal{T}$ be a topology for X, consider $C_b(X^{\mathcal{T}}, M) =$ continuous functions in $B_b(X, M)$. What are the compact subsets of C_b? What are the totally bounded subsets. Let J be a totally bounded subset of $C_b(X, M)$. Then, given $\epsilon > 0$, we can find $g_1, \ldots, g_n \in J$ such the $\text{Ball}(g_j, \epsilon)$, $j = 1, \ldots, n$ cover J. Given any $x \in X$, such taht $g_1, \ldots, g_n$ are continuous, there are open sets, $\mathcal{O}_1, \ldots, \mathcal{O}_n$, with $x \in \mathcal{O}_j$, for all j such that if $y \in \mathcal{O}_j$, then $d(g_j(x), g_j(y)) < \epsilon$, let $\mathcal{O} = \bigcap_{j=1}^n \mathcal{O}_j$, such that $x \in \mathcal{O}$. Then for any $y \in \mathcal{O}$, $d(g_j(x_, g_j(y)) < \epsilon$ for all j. Then for $f \in \mathcal{T}$, there is a j with $d_\infty(f, g_j) < \epsilon$, and so for $y \in \mathcal{O}$, $d(f(x), f(y)) \leq d(f(x), g_j(x)) + d(g_j(x), g_j(y)) + d(g_j(y), f(y)) < 3\epsilon$. Thus, given $x \in X$, for any $\epsilon > 0$, there is $\mathcal{O} \in J, x \in \mathcal{O}$ such that for $y \in \mathcal{O}$ has $d(f(x), f(y)) < \epsilon$, for all $f \in J$. The family f is equicontinuous at x. Since it is true for all x, we say that f is an equicontinuous set of functions. Also, for fixed x, given $f \in F$, there is g with $f \in \text{Ball}(g_j, \epsilon)$, so that $d(f(x), g_j(x)) < \epsilon$, i.e., $\{f(x) : f \in F\} \subseteq M$ is covered by the balls $\text{Ball}(g_j(x), \epsilon)$, so it is totally bounded. Hence, F is pointwise totally bounded.

Theorem 1.2.8. *(Core of the Arzela-Ascoli Theorem) Let $(X, \mathcal{T})$ be compact. Let $F \subseteq C(X, M)$. If F is equicontinuous and pointwise totally bounded, then F is totally bounded for d_∞.*

Proof. Let $\epsilon > 0$ be given. Then, by equicontinuity, for each $x \in X$, there is an open set $\mathcal{O}_x$, such that $x \in \mathcal{O}_x$ such that if $y \in \mathcal{O}_x$, then for all $f \in F$, we have $d(f(x), f(y)) < \epsilon$. The $\mathcal{O}_x$'s form an open cover of X, so there is a finite subcover $\mathcal{O}_{x_1}, \ldots, \mathcal{O}_{x_n}$. For each $j = 1, \ldots, n$, $\{f(x_j) : f \in F\}$ is totally bounded, so there is a finite subset, S_j such that the ϵ-balls about the points of S_j cover the aforementioned set. Let $S = \bigcup_j S_j$, a finite set in M. Let $\Psi = \{\psi : \{1, \ldots, n\} \to S\}$ a finite set. For each $\psi \in \Psi$, let $A_\psi = \{f \in F : f(x_j) \in \text{Ball}(\psi(j) \in S, \epsilon)\}$. The A_ψ's cover F. If $f, g \in A_\psi$, for any x, there is $y \in X$, there is j so that $y \in \mathcal{O}_{x_j}$. Then $d(f(x), g(x)) \le d(f(y), f(x_j))(\le \epsilon) + d(x_j < \epsilon, g_{x_j} \le \epsilon)(\le 2\epsilon) + d(g(x_j), g(y)) \le \epsilon < 4\epsilon$, i.e. diameter $(A_\psi) < 4\epsilon$. ■

Theorem 1.2.9. *(Arzela-Ascoli): Let $(X, \mathcal{T})$ be a complete metric space. Then, $F \subseteq C(X, M)$ is compact in d_∞ if it is closed and equicontinuous and pointwise totally bounded.*

Definition 1.2.2. Locally compact spaces. A topological space $(X, \mathcal{T})$ is locally compact if for each $x \in X$, there is a $\mathcal{O} \in \mathcal{T}, x \in \mathcal{O}, \bar{\mathcal{O}}$ is compact.

1.3 Locally Compact Hausdorff Spaces

Note 1.3.1. LCH := "locally compact Hausdorff"

$(X, \mathcal{T})$ be a LCH space.

Lemma 1.3.1. *Let $C \subseteq X$ be compact. Then there is open $\mathcal{O}$ with $C \subseteq \mathcal{O}$, $\overline{\mathcal{O}}$ compact.*

Proof. For each $x \in C$, let $\mathcal{O}_x$ be open with $x \in \mathcal{O}_x$, $\overline{\mathcal{O}}$ compact. $\{\mathcal{O}\}_{x \in C}$ covers C, so there is a finite subcover $\mathcal{O}_{x_1}, \ldots \mathcal{O}_{x_n}$. Let $\mathcal{O} = \cup_{j=1}^n \mathcal{O}_{x_j}$, so $C \subseteq \mathcal{O}$, $\overline{\mathcal{O}} = \cup_{j=1}^n \overline{\mathcal{O}_{x_j}}$ is compact. ■

Theorem 1.3.2. *Let $(X, \mathcal{T})$ be a LCH. Let $C = X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is open $\mathcal{U}$, $C \subseteq \mathcal{U}$, $\overline{\mathcal{U}}$ compact, $\overline{\mathcal{U}} \subseteq \mathcal{O}$.*

Proof. By the previous lemma, we can choose $\mathcal{O}_1$, $C \subseteq \mathcal{O}_1 \subseteq \overline{\mathcal{O}}_1$, the last of which is compact. Let $\mathcal{O}_2 = \mathcal{O} \cap \mathcal{O}_1$, see $C \subseteq \mathcal{O}_2 \subseteq \mathcal{O}$, where $\mathcal{O}_2$ is compact. So we can assume $\mathcal{O}$ has compact closure. $C \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}}$. Let $B = \overline{\mathcal{O}} \backslash \mathcal{O}$, closed $\subseteq \overline{\mathcal{O}}$. C, B are disjoint compact subsets of $\overline{\mathcal{O}}$. Because $\overline{\mathcal{O}}$ is compact, so normal, we can find disjoint relatively open $\mathcal{U}$, $\mathcal{V} \subseteq \overline{\mathcal{O}}$, with $C \subseteq \mathcal{U}$, $B \subset \mathcal{V}$. Then, $\mathcal{V}'$ is closed, $\mathcal{U} \subseteq \mathcal{V}'$. Thus, $\overline{\mathcal{U}} \subseteq V'$, so $\overline{\mathcal{U}} \cap B = \varnothing$. Thus, $\overline{U} \subseteq \mathcal{O}$, $\mathcal{U} \subseteq \mathcal{O}$. ∎

Theorem 1.3.3. *Let $(X, \mathcal{T})$ be LCH. Let $C \subseteq X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is a continuous $f : X \to [0, 1]$ with $f(x) = 1$, for $x \in C$ and $f(x) = 0$ for $x \notin \mathcal{O}$.*

Proof. Choose open $\mathcal{U}$ with $C \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}}$ (compact) $\subseteq \mathcal{O}$. Choose V with $C \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{O}$, $\overline{\mathcal{U}} - \mathcal{V}$ closed in $\mathcal{U}$, disjoint from C, so by Urysohn's Lemma, there exists $\tilde{f} : \overline{\mathcal{U}} \to [0, 1]$, such that when $x \in C$, it evaluates to 1 and it evaluates to 0 for $x \in \overline{\mathcal{U}} - \mathcal{V}$. Let f be defined by $f(x) = \tilde{f}(x)$ if $x \in \overline{\mathcal{U}}$ and $f(x) = 0$ if $x \notin \overline{\mathcal{U}}$. We need f to be continuous. If $x \in \mathcal{U}$, then f is continuous at x, as $\tilde{f}$ is. If $x \notin \mathcal{U}$, then $x \notin \overline{\mathcal{V}}$, so $x \in X \backslash \overline{\mathcal{V}}$ open, on $X \backslash \overline{\mathcal{V}}$, $f(x) = 0$. ∎

Definition 1.3.1. For $(X, \mathcal{T})$ LCH, let $C_c(X)$ be the set of continuous $\mathbb{R}-$valued functions on X "of compact support", i.e. there is a compact set outside of which $f \equiv 0$. $C_c(X)$ is an algebra for pointwise operations. $e, f, g \in C_c(X)$, then $f + g$, fg, $rf (r \in \mathbb{R}) \in C_c(X)$.

Note 1.3.2. $C_c(X) \subseteq C_b(X), || \cdot ||_\infty$, usually not complete if X is not compact. Its completion is the algebra of continuous functions that "vanish at infinity," $f \in C_\infty(X)$ if $\forall \epsilon > 0$, there is a compact set C_ϵ such that $|f(x)| \leq \epsilon$ for $x \notin C_\epsilon$. GL$(n, \mathbb{R})$ is locally compact.

Chapter 2

Measure Theory

2.1 Introduction to Measure Theory

Note 2.1.1. Recall the first day of lecture: $C([0,1])$, for the L^1 and L^2 norms, we can have a Cauchy sequence. We want to figure out a way to accurately find the integral in these troublesome cases. We want a set and a family of subsets $\mathscr{F}$, and some function $\mu : \mathscr{F} \to \mathbb{R}^+$. We want additivity, i.e. if $E, F \in \mathscr{F}$, and if E and F are disjoint and $E \oplus F \in \mathscr{F}$, then $\mu(E \cup F) = \mu(E) + \mu(F)$. Also if $E, F \in \mathscr{F}$, $E \subseteq F$, $F = E \oplus (F \backslash E)$ (let $\oplus$ be the disjoint union), so $\mu(F) = \mu(E) + \mu(F \backslash E)$, i.e. $\mu(F \backslash E) = \mu(F) \backslash \mu(E)$.

Definition 2.1.1. Let X be a set and let R be a nonempty family of subsets of X. We say that R is a ring if R is closed under finite unions and differences of elements $E \backslash F$. This implies closed under finite intersection over $E \cap F = E \backslash (E \backslash F)$. If also $X \in R$, call $\mathscr{J}$ an algebra (or a field).

Definition 2.1.2. A finitely added measure or a ring R of sets is a finite $\mu : R \to \mathbb{R}^+$ such that if $E, F \in R$ and are disjoint, then $\mu(E \oplus F) = \mu(E) + \mu(F)$

Definition 2.1.3. A ring R is said to be a σ-ring of to so closed under taking countable unions of elements fo R, so we can take countable intersections.

Definition 2.1.4. A σ-algebra: $E = \cup_{n=}^{\infty} E_n$, then $\cap E_n = E \setminus \cup_{n=1}^{\infty} (E \setminus E_n)$

Definition 2.1.5. Let R be a σ-ring. By a measure on R we mean a function $\mu : R \to \mathbb{R}^+$, $\mathbb{R}^+ \cup \{+\infty\}$, $\mathbb{R}$, $\mathbb{R}^n$, Banach spaces, which is countable additive, i.e. if $\{E_n\}_n^\infty$ is a disjoint family of elements in R. Then,

$$\mu\left(\bigoplus_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Theorem 2.1.1. *Let $\mathscr{S}$ be a collection of rings (or algebras, or σ-algebras, or σ-rings, etc) of a given set X. Then the intersection of these rings is a ring (or ...).*

Definition 2.1.6. Given any collection of subsets of X, there is a smallest ring (...) that contains the collection, namely the intersection of all contains rings, etc.

Definition 2.1.7. Let $(X, \mathcal{T})$ be a topological space.

1. The σ-ring generated by $\mathcal{T}$ is called the σ-ring of Borel subsets of X.

Let $(X, \mathcal{T})$ be a LCH space, then the σ-ring generated by the compact subsets is called the σ-ring of Borel sets.

Note 2.1.2. $X = \mathbb{R}, \mathscr{S} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$

Note 2.1.3. Let $P = \{[a, b) \subseteq \mathbb{R} : a < b\}$.

Definition 2.1.8. Let X be a set, P a collection of subsets. We say that P is a pre-ring if

1. For $E, F \in P$, we have that $E \cap F \in P$
2. For $E, F \in P$, there are $G_1, \ldots, G_n \in P$, such that $E \backslash F = \bigoplus^n G_j$.

Note 2.1.4. Let α be a non-decreasing left-continuous function $\alpha : \mathbb{R} \to \mathbb{R}$, if $s < t$, then $\alpha(s) \leq \alpha(t)$. Now, given α, define $\mu_\alpha([a, b)) = \alpha(b) - \alpha(a) \geq 0$.

Theorem 2.1.2. *μ_α on P is countably additive.*

Proof. Need: if $[a_0, b_0) = \bigoplus_{n=}^\infty [a_n, b_n)$, then $\mu_\alpha([a_0, b_0)) = \sum_{n=0}^\infty \mu_\alpha([a_n, b_n))$. Need to show $\geq$: Suffices to show that for each n, $\mu_\alpha([a_0, b_0)) \geq \sum^n \mu_\alpha([a_j, b_j))$. we know that the $[a_j, b_j)$ are disjoint. We can renumber these intervals so that $a_1 < a_2 < \ldots < a_n$. Since disjoint, $b_j \leq a_{j+1}$ for $j = 1, \ldots, n$, $\alpha(b_1) - \alpha(a_1) + \alpha(b_2) - \alpha(a_2) + \ldots + \alpha(b_n) - \alpha(a_n) = -\alpha(a_1) + (\alpha(b_1) - \alpha(a_2))(\leq 0) + \ldots + (\alpha(b_{n-1}) - \alpha(a_n))(\leq 0) + \alpha(b_n) \leq \alpha(b_n) - \alpha(a_1) \leq \alpha(b_0) - \alpha(a_0) = \mu_\alpha([a_0, b_0))$. We now need $\mu_\alpha([a_0, b_0)) \leq \sum_{j=1}^\infty \mu_\alpha([a_j, n_j))$. Let $\epsilon > 0$ be given. Choose ϵ_j's, $\epsilon_j > 0$, $\sum^\infty \epsilon_j \leq \frac{\epsilon}{2}$, where $\epsilon_j = \frac{\epsilon}{2^{j+1}}$. Choose $b_0' < b_0$, such that (since α is left continuous), $\alpha(b_0') + \frac{\epsilon}{2} \geq \alpha(b_0)$, for each j, choose $a_j' < a_j$ such that $\alpha(a_j') + \epsilon_j \geq \alpha(a_j)$, $\alpha(a_j') < \alpha(a_j)$. Then, $[a_0, b_0'] \subseteq \cup_{j=1}^\infty (a_j', b_j)$, so there is a finite subcover. Remember finite subcover $\mathcal{C}$ as follows. Let (a_1', b_1) be the interval in $\mathcal{C}$, with smallest a_1. Assume $b_1 \leq b_0'$. Let (a_2', b_2) the interval in $\mathcal{C}$ that contains b_1 and has smallest a_2', so $a_2' < b_2$. Continue $\ldots$ $(a_j', b_j), a_{j+1} < b_j$. As soon as $b_j > b_0'$, STOP. $\mu_\alpha([a_0, b_0]) = \alpha(b_0) - \alpha(a_0) \leq \alpha(b_0') + \frac{\epsilon}{2} - \alpha(a_0) \leq \alpha(t_0') + \frac{\epsilon}{2} - \alpha(a_0') \leq \alpha(t_n) - \alpha(a_0') + \frac{\epsilon}{2}$, $b_n > b_0'$, $a_{j+1}' \leq b_j, \alpha(a_{j+1}') \leq \alpha(b_j), \alpha(b_j) - \alpha(a_j') \geq 0$. ■

Insert stuff in picture above.

Definition 2.1.9. A premeasure is afunction μ defined on a semiring P, $\mu : P \to \mathbb{R}^+$, and is countably additive. Each μ_α is a pre-measure.

Theorem 2.1.3. *$\mu : P \to \mathbb{R}^+$ just finitely added. Then, if $E \in P$ containes $\bigoplus_{j=1}^n F_j$. Then, $\mu(E) \geq \sum \mu(F_j)$.*

Proof. $E = \oplus H_n \oplus E_n \oplus F_j$, $\mu(E) = \sum \mu(H_n)(\geq 0) + \sum \mu(E \cap F_j)(= F_j)$ ∎

Definition 2.1.10. Let $\mathcal{C}$ be a collection of sets

$[a_0, b_0'] \subset \cup_{j=1}^n (a_j', b_j)$ overlapping, $b_j > a_{j+1}'$, $a_1' < a_0$, $b_n > b_0'$. Then $\alpha(b_0') - \alpha(a_0) \leq \sum \alpha(b_j) - \alpha(a_j)$.

Proof.

$$\begin{aligned}\sum \alpha(b_j) - \alpha(a_j) &= \alpha(b_n) - (\alpha(a_n') - \alpha(b_{n-1}))(< 0) - \ldots - (\alpha(a_2') - \alpha(b_1))(< 0) - \alpha(a_1') \\ &\geq \alpha(b_n) - \alpha(a_1') \\ &\geq \alpha(b_0') - \alpha(a_0).\end{aligned}$$

∎

We saw that if $E \supseteq \oplus_{j=1}^n F_j$, for μ on every P, then $\mu(E) \geq \sum \mu(F_j)$.

Definition 2.1.11. Let $\mathscr{F}$ be a family of subsets of X. let $\mu : \mathscr{F} \to \mathbb{R} \cup \{+\infty\}$, we say that μ is countably additive if whenever we have that $E \subseteq \cup_{j=1}^\infty F_j$, then $\mu(E) \leq \sum \mu(F_j)$.

Definition 2.1.12. μ on $\mathscr{F}$ is monotone if $E \supseteq F$ implies that $\mu(E) \supseteq \mu(F)$.

Theorem 2.1.4. *Let P be a semiring, $\mu : P \to \mathbb{R}$, countably additive $E = \oplus_{j=1}^\infty F_j$. Then μ is countably subadditive, $E \subseteq \cup F_j$ want $\mu(E) \leq \sum \mu(F_j)$.*

Proof. Then, $E \subseteq \cup F_j \cap E$, and by μ monotone, $\mu(F_j \cap E) \leq \mu(F_j)$, so it suffices to show that for $E = \cup^\infty F_j$, then disjointage: set H_j(not really in P) $= F_j \setminus \cup_{k<j} F_k$. $H_1 = F_1$. Then, $E = \bigoplus H_j$. Note that $H_j = \oplus_{k=1}^{n_k} G_{jk}$, with $G_{jk} \in P$. Thus, $E = \oplus G_{jk} \in P$. Next, by the countable additivity of μ, we must have that:

$$\begin{aligned}\mu(E) = \sum_{j,k} \mu(G_{jk}) &= \sum_j \sum_{k=1}^{n_j} \mu(G_{jk}) \\ &\leq \sum_j \mu(F_j).\end{aligned}$$

Note that $\oplus_k G_{jk} \subseteq F_j$ and $\sum_k \mu(G_{jk}) \leq \mu(F_j)$. ■

Let $\mathscr{F}$ be a family of subsets of a set X, and let μ be any function from $\mathscr{F} \to \mathbb{R}^+ \cup \{+\infty\}$. For any $A \subseteq X$, set $\mu^*(A) = \inf\{\sum \mu(F_j) : F_j \in \mathscr{F}, A \subseteq \cup_{j=1}^\infty F_j\}$. Let $\mathscr{H}(\mathscr{F}) = \{A \subseteq X : \exists \{F_j\}_{j=1}^\infty \subseteq \mathscr{F},$ with $A \subseteq \cup_{j=1}^\infty F_j\}$. It is clear that $\mathscr{H}(\mathscr{F})$ is a σ-ring, this is hereditary (i.e. if $A \in \mathscr{H}(\mathscr{F})$ and $B \subseteq A$, then $B \in \mathscr{H}(\mathscr{F})$). Finally, note that the $F_j's$ cover A. Set $\mu^*(\varnothing) = 0$.

Example 2.1.1. Let $X = \mathbb{R}$, then let $\mathscr{F}$ be a collection of all finite subsets of $\mathbb{R}$, $\mathscr{H}(\mathscr{F}) =$ countable subsets of $\mathbb{R}$.

Example 2.1.2. Properties:

1. Monotone.
2. μ^* is countably sub-additive.

Proof. (2): Let A, $\{B_j\}_{j=1}^\infty$ be in $\mathscr{H}(\mathscr{F})$, $A \subseteq \cup B_j$. Want $\mu^*(A) \leq \sum \mu^*(B_j$. Let $\epsilon > 0$ be given, choose $\{\epsilon_j > 0\}$ with $\sum_{j=1}^\infty \epsilon_j < \epsilon$, for each j, choose $\{F_k^j\}_{k=1}^\infty$ with $B_j \subseteq \cup_k F_k^j$ but $\sum_k \mu(F_k^j) \leq \mu^*(B_j) + \epsilon_j$. Then, $A \subseteq \cup_{j,k} F_k^j$, so

$$\begin{aligned}\mu^*(A) \leq \sum_{j,k} \mu(F_k^j) &= \sum_j \sum_k \mu(F_k^j) \\ &\leq \sum_j (\mu(B_j)...\end{aligned}$$

■

Definition 2.1.13. Let $\mathscr{H}$ be a hereditary σ-ring of subsets of X. By an outer measure on $\mathscr{H}$, we mean a finite $\mathcal{V} : \mathscr{H} \to \mathbb{R}^+ \cup \{+\infty\}$ that is monotone and countably subadditive, $\mathcal{V}(\varnothing) = 0$.

Let P be a semiring, and let μ be a premeasure on P, i.e. μ is countably additive. Let μ^* be the corresponding outer measure on $\mathscr{H}(P)$.

Theorem 2.1.5. *For any $E \in P$, $\mu^*(E) = \mu(E)$, i.e. μ^* is an exterior of μ to all of $\mathscr{H}(P)$.*

Proof. $\mu^*(E) = \inf\{\sum \mu(F_j) : E \subseteq \cup F_j\}$, so $\mu(E) \leq \mu^*(E)$, but μ is countably additive, so $\mu(E) \leq \sum \mu(F_j)$. For E_n, $\mu(E) = \mu^*(E)$. ■

Let $\mathcal{V}$ be an outer measure on $\mathscr{H}$. Let $E \in \mathscr{H}$. We say that E splits all sets in $\mathscr{H}$ if for any $A \in \mathscr{H}$, $\mathcal{V}(A) = \mathcal{V}(A \cap E) + \mathcal{V}(A \backslash E)$ (Note that $A = A \cap E \oplus A \backslash E$. By subadditive, we have $\leq$, so we have that $\mathcal{V}(A) \geq$. Let $\mathscr{S}(\mathcal{V}) = \{E \in \mathscr{H} : E$ splits all sets in $\mathscr{H}\}$, with $\varnothing \in \mathscr{S}$.

Theorem 2.1.6. *$\mathscr{S}(\mathcal{V})$ is a σ-ring, and $\mathcal{V}|_{\mathscr{J}}$ is coubntably additive and therefore a measure.*

Proof. Let $E, F \in \mathscr{S}(\mathcal{V})$. We want $E \cup F \in \mathscr{S}(\mathcal{V})$. Let $A \in \mathscr{H}$, we want that $\mathcal{V}(A) \geq \mathcal{V}(A \cap (E \cup F)) + \mathcal{V}(A \backslash (E \cup F)) = \mathcal{V}(A \cap E \oplus (A \backslash E) \cap F) \leq \mathcal{V}(A \cap E) + \mathcal{V}((A \backslash E) \cap F) + mathcalV((A \backslash E) \backslash F) = \mathcal{V}(A \cap E) + \mathcal{V}(A \backslash E) = \mathcal{V}(A)$, because $F \in \mathscr{S}(\mathcal{V})$, $E \in \mathscr{S}(\mathcal{V})$.

Now, we want to show that if $E, F \in \mathscr{S}(\mathcal{V})$ the $E \backslash F \in \mathscr{S}(\mathcal{V})$. Let $A \in \mathscr{H}$. We want $\mathcal{V}(A) =^? \mathcal{V}(A \cap (E \backslash F)) + \mathcal{V}(A \backslash (E \backslash F)) = \mathcal{V}((A \cap E) \backslash F) + \mathcal{V}((A \backslash E) \cup (A \cup F))(\mathcal{V}((A \backslash E) \oplus (A \cap F \cap E)) \leq \mathcal{V}((A \cap E) \backslash F) + \mathcal{V}(A \backslash E) + \mathcal{V}(A \cap F \cap E) = \mathbb{V}(A \cap E) + \mathcal{V}(A \backslash E) = \mathcal{V}(A)$. ■

$\mathscr{H}$ is hereditary σ-ring of subsets of X, ν is an outer measure defined on $\mathscr{H}$, $M(\nu) = \{E \in \mathscr{H} : \nu(A \cap E) + \nu(A \backslash E) = \nu(A), \forall A \in \mathscr{H}\}$. We saw that $M(\nu)$, the ν-measurable sets is a ring. We now claim that if $E, F \in M(\nu)$, $E \cap F = \varnothing$, then for all $A \in \mathscr{H}$, $\nu(A \cap E \oplus A \cap F) = \nu(A \cap E) + \nu(A \cap F)$.

Proof. E splits $A \cap (E \oplus F)$, or equivalently $\nu((A \cap (E \oplus F)) \cap E) + \nu((A \cap (E \oplus F)) \backslash E) = \nu((A \cap E) \oplus (A \cap F))$. ■

Theorem 2.1.7. *$M(v)$ is a σ-ring, and ν is countably additive on $M(\nu)$.*

Proof. Let $\{E_j\}_j^\infty \subseteq M(\nu)$. Let $G = \cup_{j=}^\infty E_j$. We want to show that $G \in M(\nu)$. Given A, we need to show that G splits A. Can disjointize the E_j's, so $G = \oplus_{j=}^\infty F_j$, $F_j \in M(\nu)$. Hence,

$$\begin{aligned}
\nu(A) &= \nu(A \cap \oplus_{j=1}^n f_j) + \nu(A \backslash \oplus_{j=1}^n F_j \\
&= \sum^n \nu(A \cap F_j +'' \\
&\geq \sum^n \nu(A \cap F_j) + \nu(A \backslash G) \text{ by nu monotone} \\
\nu(A) &\geq \sum^\infty \nu(A \cap F_j) + \nu(A \backslash G) \geq (\text{countably additive})\nu(A \cap G) + \nu(A \backslash G) \geq \nu(A).
\end{aligned}$$

Hence, $M(\nu)$ is a σ-ring. ■

Note 2.1.5. For a set X, define

$$\begin{aligned}
\nu(A) &= 1, A \neq \varnothing \\
\nu(\varnothing) &= 0.
\end{aligned}$$

Theorem 2.1.8. *Let $(\mathscr{P}, \mu)$ be a premeasure. Let μ^* be the corresponding outer measure on $\mathscr{H}(\mathscr{P})$. Then, $\mathscr{P} \subseteq M(\mu^*)$. Define*

$$\mu^*(A) = \inf \left\{ \sum \mu(E_j) : E_j \in \mathscr{P}, A \subseteq \cup E_j \right\}.$$

Proof. Let $E, F \in \mathscr{P}$, $E \backslash F = \oplus^n G_j, G_j \in \mathscr{P}$. Hence, $\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \backslash F)$, so $\mu(E)$=$\mu(E \cap F) + \mu^*(E \backslash F)$. THen, let $E \in \mathscr{P}$, then let $A \in \mathscr{H}(\mathscr{P})$, we need $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$. Now, let $\epsilon > 0$ be given, and choose $\{F_j\}_{j=1}^n \subset \mathscr{P}, A \subseteq \cup^n F_j, \mu^*(A) + \epsilon \geq \sum^n \mu(F_j)$. Then, $\epsilon + \mu(A) \geq \sum^n \mu(F_j) = \sum^n \mu(F_j \cap E) + \sum^n \mu^*(F_j \backslash E) = \sum \mu(\cup F_j \cap E) \geq \mu^*(A \cap E)(\text{monotone}) + \mu^*(A \backslash E)(\text{countably additive}) \geq \mu^*(A)$. Since ϵ is arbitrary, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$. Hence, $E \in M(\mu^*)$. Thus, $\mathscr{P} \subseteq M(\mu^*)$. ■

$\mathscr{H}, \nu M(\nu)$. If $A \in M(\nu)$ abd if $\nu(A) = 0$, then $A = \varnothing$, then for any $B \subseteq A$, $B \in M(\nu)$ (with $\nu(B) = 0$), "complete," given any $D \in \mathscr{H}$, $\nu(D) \supseteq \nu(D \cap B) + \nu(D \backslash B)$, by monotone.

Note 2.1.6. If $(\mathscr{P}, \mu)$ is a premeasure then μ^* on $M(\mu^*)$ is a complete measure. Can restrict μ^* to the $\mathscr{S}(\mu) = \sigma$-ring generated by $\mathscr{P}$, $\mathscr{S}(\mu) \subseteq M(\mu^*)$, but μ on $\mathscr{S}(\mu)$ need not be complete. For α a left-cont non-decreasing function, μ_α^* on $M(\mu_\alpha)$ is called a Lebesgure-Stieltjes measure, which is complete its restriction to $\mathscr{S}(\mathscr{P}$ is called a Borel-Stieltjes measure. Maybe not be complete. $\mathscr{S}(\mathscr{P})$ are the Borel sets in $\mathbb{R}$. But different α's maybe have different $M(\mu^*)$. When using just one measure on $\mathbb{R}$, we usually use $M(\mu_\alpha^*$. When using many of the μ_α's, use $\mathscr{S}(\mathscr{P})$, because they are all defined on $\mathscr{S}(\mathscr{P})$, if considering α's with $\lim_{t\to+\infty}(\alpha(t) - \lim_{t\to-\infty}\alpha(t)) = 1$. Then, the μ_α have $\mu_\alpha(\mathbb{R}) = 1$. The μ_α are the (Borel) probability measures on $\mathbb{R}$. Next, note that in the case of $\alpha(t) = t$, gives Lebesgue measuer on $\mathbb{R}$. It is the translation invariant.

$[a, b), [a + c, b + c), b - a = (b + c) - (a + c)$.

Definition 2.1.14. A measure μ or σ-rings is said to be σ-finite if for all $E \in \mathscr{S}$, there are $\{F_j\} \subset \mathscr{S}$ with $\mu(F_j) < \infty$ and $E \subseteq \cup F_j$.

Theorem 2.1.9. *For $\mu, \mathscr{S}, \mu^*$, $\mu^*(A) = \inf\{\sum^\infty \mu(E_j) : A \subseteq \cup^\infty E_j, E_j \in \mathscr{S}\}$, we can disjointize $A \subseteq \oplus F_j \sum \mu(F_j) \leq \sum \mu(E_j), \sum \mu(F_j) = \mu(\oplus F_j), \mu^* = \ldots$*

Theorem 2.1.10. *Let $(\mu, \mathscr{S}, \mu)$ be a measure space. Let $M(\mu^*)$ be the μ^*-measureable sets the $\mathscr{S} \subseteq M(\mu^*$. We can then consider $\mathscr{S} \subseteq \mathscr{S}_1 \subseteq M(\mu^*)$. Then, the restriction of μ^* to $\mathscr{S}_1$ is the largest extension of μ to $\mathscr{S}_1$.*

Proof. Let ν be another extension of μ to $\mathscr{S}$. Then, for $A \in \mathscr{S}_1$. ■

Midterm is on next Thursday :(
$(\mathcal{P}, \mu), \mathscr{H}(\mathcal{P}), \mu^*, M(\mu^*) \supseteq \mathscr{S}(\mathcal{P})$ is a σ-ring. For any $A \in \mathscr{H}(\mathcal{P}), \mu^*(A) = \inf\{\mu^*(E) : E \in M(\mu^*), A \subseteq E\}$. Then, for each n, choose $E_n \supseteq A$ such that $\mu^*(E_n) \leq \mu^*(A) + 1/n$. Then, set $E = \bigcap E_n \supseteq A, \mu^*(E) = \mu^*(A)$.

Theorem 2.1.11. *Assume that $(\mathcal{P}, \mu)$ is σ-finite. For all $A \in \mathscr{H}(\mathcal{P})$ there are $\{E_n\} \subseteq \mathcal{P}, \mu(E_n) < \infty$ and $A \subseteq \bigcup E_n$. THen, for any σ-ring $\mathscr{S}$, $\mathscr{S}(\mathcal{P}) \subseteq \mathscr{S} \subseteq M(\mu^*)$, μ on $\mathscr{S}$ on $\mathscr{S}(\mathcal{P})$, and any extension, μ', of μ, then $\mu'(F) = \mu^*(F)$, for any $F \in \mathscr{S}$ (so extension μ' is unique).*

Proof. Part 1: Assume that $F \in \mathscr{S}, F \subseteq E \in \mathscr{S}(\mathcal{P}), \mu(E) < \infty . E = E \cap F \oplus E \backslash F$. $\mu'(E) = \mu(E) = \mu^*(E \cap F) + \mu^*(E \backslash F) \geq \mu'(E \cap F) + \mu'(E \backslash F) = \mu'(E)$. But $\infty > \mu^*(E \cap F) \geq \mu'(E \cap F), \infty > \mu^*(E \backslash F) \geq \mu(E \backslash)$. Thus, $\mu^*(E \cap F) = \mu'(E \cap F)$, $\mu^*(F) = \mu'(F)$.

For general $F \in \mathscr{S}$, assume μ is σ-finite, then there exists $\{E_j\} : F \subseteq \bigcup E_j$, $\mu(E_j) < \infty$, can disjointize, so assume that $F \subseteq \oplus E_j$. Then, $\mu'(F) = \sum \mu'(F \cap E_j) = \sum \mu^*(F \cap E_j) = \mu^*(\oplus F \cap E_j) = \mu^*(F)$. ∎

2.2 Continuity Properties of Measures

Theorem 2.2.1. *Let $(X, \mathscr{S}, \mu)$ be a measure space. Let $\{E_j\} \subset \mathscr{S}$, increasing, i.e. $E_{j+1} \supseteq E_j$. Let $E = \cup^\infty E_j$. Then, $\mu(E) = \lim \mu(E_j)$.*

Proof. $E = E_1 \oplus (E_2 \backslash E_1) \oplus (E_3 \backslash E_2) \cdots (E_{j+1} \backslash E)$. Hence, it must be the case that

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_{j+1} \backslash E_j) + \mu(E_1).$$

Then, $\mu(E_n) = \mu(E_1) + \mu(E_2 \backslash E_1) + \ldots + \mu(E_n \backslash E_{n-1})$ partial sum. Thus, $\mu(E_n) \to \mu(E)$. ∎

Theorem 2.2.2. *$\{E_j\}$, $E_{j+1} \subseteq E_j$, $E = \bigcap E_j$. $\mu(E_j) \to \mu(E)$, and if $(\mu(E_1) < \infty$, then $\mu(E_j) \to \mu(E)$.*

Proof. See online notes (hopefully?). ∎

Example 2.2.1. A counterexample, $\mathbb{R}, M$ Lebesgue: $E_j = [j, \infty)$. $\mu(E_j) = \infty, \bigcap E_j = \varnothing \to 0$.

$\mathbb{R}$, Lebesgue measure, μ_α, $\alpha([a, b)) = b - a$. Translation movement.

$$\mathbb{R}/\mathbb{Z} \to T$$

$$t \mapsto e^{2\pi i t},$$

fundamaental domain $[0, 1)$, transfer Lebesgue measure restricted to $[0, 1)$ onto S^1. Then, we get a rotation invariant measure on T, with $\mu(T) = 1$. In the group T, let G be the subgroup of elements of finite order, $\{e^{2\pi i \frac{m}{n}}, m, n \in \mathbb{Z}\}$. G is a countable subgroup (Dense in T). Consider $T/G = \{\text{cosets}\}$, which is uncountable. Let $A \subset T$ consist of a closure of one point for each coset of G, each element of T is in one coset. Thus, $T = \oplus_{r\in G} rA$. Given $z \in T$, there is $a \in A$, in the same coset as z, i.e., $z = ra$. By translation of invariance, $\mu(rA) = \mu(A)$ for all $r \in G$, but G is countable,

$$1 = \mu(T) = \sum_{r\in G} \mu(rA) = \sum_{r\in G} \mu(A).$$

Hence, A is not measurable.

Note 2.2.1. Berkeley professor Solovey showed that you cannot show that there are non-measurable sets without something like the axiom of choice.

2.3 Introduction to Integration

$(X, \mathcal{S})$, $\mathcal{S}$ is a ring of subsets of X. Let B be a vector space. Given $E \in \mathcal{S}$,

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

If $b \in B$,

$$b\chi_E(x) = \begin{cases} b & x \in E \\ 0 & x \notin E. \end{cases}$$

Definition 2.3.1. By a simple B-valued function on X, we mean $f : X \to B$ that has finite range, and for any $b \in \text{range}(f)$, $b \neq 0$, $f^{-1}(b) \in \mathcal{S}$. Thus,

$$f = \sum b_j \chi_{E_j},$$

where the b_j are not equal to 0 (or $f \equiv 0$), E_j's are disjoint and in $\mathcal{S}$. If

$$f = \sum^n b_j \chi_{E_j},$$

with the E_j's disjoint, but the b_j's not distinct and b_j maybe 0.

Lemma 2.3.1. *Let*

$$f = \sum^n b_j \chi_{E_j},$$

$E_j \in \mathcal{S}$ disjoint, b_j disjoint, $\neq 0$. Let $F \in \mathcal{S}$, $c \in B$, set $g = c\chi_F$. Then, $f + g$ is a SMF.

Proof. Let $E_{n+1} := F \backslash \oplus E_j$. Then

$$f = \sum_{j=1}^{n+1} b_j E_j,$$

where $b_{n+1} = 0$, $F = \oplus(F \cap E_j)$, $E_j = (E_j \cap F) \oplus (E_j \backslash F)$. Note that $F \subseteq \oplus_{j=1}^{n+1}$. Then,

$$f = \sum^{n+1} b_j \chi_{E_j \cap F} + \sum^{n+1} b_j \chi_{E_j \backslash F},$$

$$g = \sum^{n+1} c \chi_{F \cap E_j}.$$

So

$$f + g = \sum^{n+1} (b_j + c) \chi_{E_j \cap F} + \sum^{n+1} b_j \chi_{E_j \backslash F},$$

where $E_j \cap F, E_j \backslash F \in \mathcal{S}$. ■

Lemma 2.3.2. *If f, g are SMF's, then so is $f + g$.*

Proof. Let

$$f = \sum b_j \chi_{E_j},$$

and

$$g = \sum c_k \chi_{F_k},$$

then $f + c_1 \chi_{F_1}$. ■

Let μ be a finitely additive measure on $\mathcal{S}$. By a simple, μ-integrable function, we mean a SMF

$$f = \sum b_j \chi_{E_j},$$

with disjoint E_j and distinct, nonzero b_j, such that $\mu(E_j) < \infty$ for all j. Then,

$$\int b \chi_E d\mu = b\mu(E), \quad \mu(E) < \infty.$$

Definition 2.3.2. We define the integral as:

$$\int f d\mu = \sum b_j \mu(E_j').$$

Lemma 2.3.3. *If*

$$f = \sum^{n} b_j \chi_{E_j}$$

is SIF, if $F \in \mathcal{S}$, $\mu(E) < \infty$ and $c \in B$, then $f + g$ is a SIF and

$$\int (f+g) d\mu = \int f d\mu + \int g d\mu.$$

Proof. Let $E_{n+1} = F \backslash \oplus E_j$, then $f + g$ (refer to above), so $f + g$ is SIF. Then,

$$\begin{aligned}
\int (f+g) d\mu &= \sum (b_j + c)\mu(E_j \cap F) + \sum (b_j \mu(E_j \backslash F) \\
&= \sum b_j \mu(E_j \cap F) + \sum b_j \mu(E_j \backslash F) + \sum c\mu(E_j \cap F) = \int f d\mu + \int g d\mu \\
&= \sum b_j \mu(E_j).
\end{aligned}$$

■

Lemma 2.3.4. *If f is SMF, if $\alpha \in \mathbb{R}, \mathbb{C}$, then αf,*

$$f = \sum b_j \chi_{E_j} \quad \alpha f = \sum (\alpha b_j) \chi_{E_j},$$

SMF$(X, \mathcal{S}, B)$ forms a vector space under pointwise operations, SIF$(X, \mathcal{S}, \mu, B)$.

Note 2.3.1. SIF$(X, \mathcal{S}, \mu, B)$, and

$$f \mapsto \int f d\mu$$

is a linear operator.

If $f \in$ SIF$(X, \mathcal{S}, \mu, \mathbb{R})$ and if $f \geq 0$, then

$$\int f d\mu \geq 0, f = \sum b_j \chi_{E_j}, b_j \in \mathbb{R}, b_j \geq 0,$$

we have that

$$\int f d\mu = \sum b_j \mu(E_j \geq 0,$$

for $f, g \in$ SIF$(X, \mathcal{S}, \mu, \mathbb{R})$, we say that $f \geq g$ if $f(x) \geq g(x)$ for any x, or equivalently, $f - g \geq 0$. If $f \geq g$, then

$$\int f d\mu \geq \int g d\mu.$$

Let B have a norm $||\cdot||, ||\cdot||_B$. For f any B-valued function, define

$$x \mapsto ||f(x)||$$

is $\mathbb{R}^+$-valued, if f is a SMF,

$$f = \sum b_j \chi_{E_j},$$

then $||f(x)|| = \sum ||b_j|| \chi_{E_j}$, so $x \mapsto ||f(x)||$ is SMF. If f is SMF, then $x \mapsto ||f(x)||$ is SMF.

Definition 2.3.3. $||\cdot||_1$ on SIF$(X, \mathcal{S}, \mu, B)$ by

$$||f||_1 = \int ||f(x)|| d\mu(x).$$

Note 2.3.2. Some properties of this include:

1. $||\alpha f||_1 = \int ||\alpha f(x)|| d\mu(x) = |\alpha| \cdot ||f||_1$.
2. $||f + g||_1 \leq ||f||_1 + ||g||_1$. Then,

$$\int ||f(x) + g(x)|| d\mu(x) \leq \int (||f(x)|| + ||g(x)||) d\mu(x) = ||f||_1 + ||g||_1,$$

   so $||\cdot||_1$ is a norm on SIF.

If f is SIF and

$$||f|| = \int f d\mu = 0,$$

then

$$||f|| = \sum |b_j| \chi_E(x), 0 = ||f||_1 = \sum |b_j| \mu(E_j) \implies \mu(E_j) = 0, \forall j.$$

Let $N(X, \mathcal{S}, \mu) = \{E \in \mathcal{S} : \mu(E) = 0\}$, where N stands for null sets, ring. Let $\mathcal{N} = \{\text{SIF} f : ||f||_1 = 0\}$, then $\mathcal{N}$ is a vector space of SIF, SIF$/\mathcal{N}$ is a vector space, and $||\cdot||_1$ drops to give a norm on SIF$/\mathcal{N}$. (SIF$/\mathcal{N}, ||\cdot||_1$). We need to find the completion. Let $\{b_j\}$ be a Cauchy sequence in B. Then, $f_j = b_j \chi_E$, $\{f_j\}$ is a Cauchy sequence for $||\cdot||_1$. We need B to be complete, so we need a Banach space. Let $\{E_j\}$ be a disjoint collection of $\subseteq \mathcal{S}$, $\mu(E_j) \leq \frac{1}{2^j}$. Choose $b \in B$, $||b|| = 1$. Let

$$f_n = \sum_{j=1}^{n} b\chi_{E_j} = b\chi_{\bigoplus_{j=1}^{n} E_j},$$

where $\{f_j\}$ is a Cauchy sequence for $|| \cdot ||_1$. Should converge to

$$\sum_{j=1}^{\infty} b\chi_{E_j} = b\chi_{\bigoplus_{j=1}^{\infty} E_j},$$

and note that

$$\mu\left(\bigoplus_{j=1}^{n} E_j\right) = \sum^{n} \mu(E_j) = \sum^{n} \frac{1}{2^j}.$$

Definition 2.3.4. $(X, \mathcal{S})$, S is a σ-ring. A B-valued function on X is said to be $\mathcal{S}$-measurable if there is a sequence $\{f_n\}$ if SMF that converges pointwise to f, for (for $|| \cdot ||_B$).

Midterm scores: median 23, average 21, high 30 (multiple people), low 2.

Definition 2.3.5. Let $(X, \mathcal{S})$ be a measurable space, i.e., $\mathcal{S}$ is a σ-ring of subsets of X. Let B be a Banach space. Hence, the MSF. A function

$$f : X \to B$$

is $\mathcal{S}$-measurable if there is a sequence $\{f_n\}$ of MSF's that converges to f pointwise. $M(X, \mathcal{S}, B)$.

Example 2.3.1. We can now define some properties, as follows:

1. If $f, g \in M(X, \mathcal{S}, B)$, then $f + g \in M(X, \mathcal{S}, B)$, the set of measurable functions, and define

   $$(f + g)(x) := f(x) + g(x).$$

   If $\{f_n\} \subset$ MSF, $f_n \to f$, if $\{g_n\} \subset$ MSF, $g_n \to g$, then $f_n + g_n \subset$ MSF, $f_n + g_n \to f + g$. Note that if $z \in \mathbb{R}$ or $\mathbb{C}$, and if $f \in M(X, \mathcal{S}, B)$, then $zf \in M(X, \mathcal{S}, B)$.

2. If $f \in M(X, \mathcal{S}, B)$ and if $h \in M(X, \mathcal{S}, \mathbb{R}$ or $\mathbb{C})$, then $hf \in M(X, \mathcal{S}, B)$.

3. If $f \in M(X, \mathcal{S}, B)$, then $x \mapsto ||f(x)||$ is in $M(X, \mathcal{S}, \mathbb{R})$.

4. If $f \in M(X, \mathcal{S}, \mathbb{R}$ or $\mathbb{C})$, then $x \mapsto |f(x)|$ is in $M(X, \mathcal{S}, \mathbb{R})$.

5. If $f, g \in M(X, \mathcal{S}, \mathbb{R})$, then $f \vee g$ is in $M(X, \mathcal{S}, \mathbb{R})$, where

$$f \vee g = \frac{f + g + |f - g|}{2},$$

and $f \wedge g$...

Note 2.3.3. If $f \in M(X, \mathcal{S}, B)$, and if $\{f_n\} \subset$ MSF , $f_n \to f$, then $\bigcup_{n=1}^{\infty}$ range(f_n) (call this M) which is countable. Then, range$(f) \subseteq \overline{M}$, which is separable, i.e. has a countable dense set.

Example 2.3.2. A property includes: if $f \in M(X, \mathcal{S}, B)$, then range$(f)$ is separable. Let $\{f_n\}$ be a sequence of functions that have the property that range(f_n) is separable, for each n, let $f_n \to f$. Then, f has this property.

Proof: Let D_n be a countable dense subset of range(f_n), and let $D = \cup_{n=1}^{\infty} D_n$ is countable (by an argument similar to showing there is a bijection from the naturals to the rationals), and range$(f) \subseteq \overline{D}$.

Proposition 2.3.1. *Let $\{f_n\}$ be a sequence of B-valued functions on X, and suppose that each f_n has the property that for any open $\mathcal{U} \subseteq B$, $f_n^{-1}(\mathcal{U} \backslash \{0\}) \in \mathcal{S}$, then if $f_n \to f$, then f also has this property.*

Proof. Let $\mathcal{U} \subseteq B$ be open. Then, $x \in f^{-1}(\mathcal{U} \backslash \{0\})$ iff $f(x) \in \mathcal{U} \backslash \{0\}$. [For any n, let $\mathcal{U}_n = \{\nu \in \mathcal{U} :\ \text{dist}(nu, \mathcal{U}') > \dfrac{1}{n}\}$]. Which is true if and only if there exists n such that $f(x) \in \mathcal{U}_n$ and there is K such that for $k' \geq k$, $f_k(x) \in \mathcal{U}_n$ (i.e. $x \in f_{k'}^{-1}(\mathcal{U}_n \backslash \{0\})$, which is true if and only if, there exists n and there exists k such that

$$x \in \bigcap f_k^{-1}(\mathcal{U}_n \backslash \{0\}).$$

However, this is true if and only if

$$x \in \underbrace{\bigcup_{n=1}^{\infty} \bigcup_{k}^{\infty} \underbrace{\bigcap_{k' \geq k} \underbrace{f_{k'}^{-1}(\mathcal{U}_n \backslash \{0\})}_{\in \mathcal{S}}}_{\in \mathcal{S}}}_{f^{-1}(\mathcal{U}\backslash\{0\}=}.$$

Thus, $f^{-1}(\mathcal{U}\backslash\{0\}) \in \mathcal{S}$. ■

Corollary 2.3.5. *If $f \in M(X, \mathcal{S}, B)$, then for any open $\mathcal{U} \subseteq B$, $f^{-1}(\mathcal{U}\backslash\{0\}) \in \mathcal{S}$.*

Corollary 2.3.6. *If $f : X \to B$ is the pointwise limit of $\{f_n\} \subset M(X, \mathcal{S}, B)$, then for f as above ($\in \mathcal{S}$).*

Theorem 2.3.7. *Let $(X, \mathcal{S})$, B be given. If $f : X \to B$ satisfies:*

1. *range(f) is separable*
2. *$f^{-1}(\mathcal{U}\backslash\{0\}) \in \mathcal{S}$, for any open $\mathcal{U} \subseteq B$.*

Then, $f \in M(X, \mathcal{S}, B)$.

Proof. Let $\{b_i\}$ be a sequence in range(f) that is dense. For i, j, let $C_{ji} = \{x \in X : f(x) \in \text{oBall}(b_i, \frac{1}{j})\backslash\{0\}\} \in \mathcal{S}$. We now want to disjointize carefully. First, order the pairs lexicographically, i.e. in "dictionary order." Say that $(j, i) < (l, k)$ if $j < l$, or when $j = l$, if $i < k$,

$$E_{ji}^n = C_{ji} \backslash \bigcup \{C_{lk} : (j, i) < (l, k) \leq (n, n)\} \in \mathcal{S},$$

for $(j, i) \leq (n, n)$. Now, let

$$f_n = \sum_{j \leq n, i \leq n} b_i \chi_{E_{ji}^n},$$

f_n is a SMF. [[Note that MSF $\cong$ SMF.]] We now claim that $f_n \to f$ pointwise. To see this, let $x \in X$ be given, and let $\epsilon > 0$ be given. Choose j_0 so that $\frac{1}{j_0} < \epsilon$. Then, choose i_0 so that there is $i \leq i_0$ with $||f(x) - b_{i_0}|| < \epsilon$. Then, let $n = \max\{j_0, i_0\}$, then find the biggest $(j_1, i_1) \leq (n, n)$ such that $||f(x) - b_i|| < \frac{1}{j}$. Then, $x \in E^n_{j_1,i_1}$, and will not be in any other $E^n_{l,k}$, $(l, k) \leq (n, n)$, so

$$||f_n(x) - b_{i_1}|| \leq \frac{1}{j_1} < \epsilon,$$

so $||f(x) - f_n(x)|| < \epsilon$. ∎

Corollary 2.3.8. *$M(X, \mathcal{S}, B)$ is "closed" under taking pointwise limits of sequence in it.*

If we have $(X, \mathcal{S}, \mu)$, a measurable space, let $\mathcal{N}(\mu)$ be a σ-ring and the set of null sets for μ, i.e. $E \in \mathcal{N}(\mu) \iff E \in \mathcal{S}, \mu(E) = 0$. A property $P(x)$ that depends on $x \in X$ is said to be satisfied almost everywhere (a.e.), if the set of x's where it fails is contained in a null set, "almost surely." Let f be a B-valued function defined a.e. We can say that a sequence $\{f_n\}$ on X converges to f a.e. f is μ- measurable if it is the limit a.e.of SMF.

Theorem 2.3.9. *(Egoroff) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of B-valued measurable functions. Let $E \in \mathcal{S}, \mu(E) < \infty$. Assume that $\{f_n\}$ converges, on E, to a function f. Then, for every $\epsilon > 0$, we have that there must be a $F \subseteq E, F \in \mathcal{S}$, with $\mu(E \backslash F) < \epsilon$, such that, on F, the seqeunce converges uniformly to F.*

Proof. ∎

Example 2.3.3. Example about characteristic functions. Maybe it will be on the midterm.

Definition 2.3.6. Let $\{f_n\}$ be a sequence of B-valued functions of X. Let $E \in \mathcal{S}$. We say that $\{f_n\}$ converges "almost uniformly" on E to a function f is for all $\epsilon > 0$, there is $F \subseteq E$, with $\mu(E \setminus F) < \epsilon$ and $f_n \to f$ on F uniformly.

Definition 2.3.7. Uniformly Cauchy if for all $\epsilon > 0$, there is N such that if $m, n \geq N$, then $||f_m(x) - f_n(x)|| < \epsilon$, for all $x \in F$, $||f_m - f_n||_{\infty,E}$.

Definition 2.3.8. Let $\{f_n\}$ be a sequence of functions on X. We say that this sequence is "almost uniformly Cauchy" on E if for all $\epsilon > 0$, there is $F \in \mathcal{S}, F \subset E$ with $\mu(E \setminus F) < \epsilon$ and $\{f_n\}$ is uniformly Cauchy on F.

Proposition 2.3.2. *If $\{f_n\}$ converges almost uniformly on E to f, then $\{f_n\}$ converges to f, a.e.*

Proof. For each n, let $E_n \subset E$, $\mu(E \setminus E_n) < \frac{1}{n}$, and $\{f_n\}$ converges uniformly on F_n. Then, let $F = \bigcup_n E_n$, $\mu(E \setminus F) \leq \mu(E \setminus E_n)$ for all n, so $\mu(E \setminus F) = 0$. Then, if $x \in F$, then there is n with $x \in E_n$, so $f_n(x) \to f$. ∎

Proposition 2.3.3. *(B-complete) If $\{f_n\}$ is almost uniformly Cauchy on $E \in \mathcal{S}$, then there is a function f define a.e. on E, so defined on $F \subseteq E$, $\mu(E \setminus F) = 0$, such taht $\{f_n\}$ converges almost uniformly on F.*

Proof. For each m, choose $E_m \subseteq E$, such that

$$\mu(E \setminus E_m) < \frac{1}{m}$$

and $\{f_n\}$ on E_m is uniformly Cauchy, so pointwise Cauchy, so define f on E_n by $f(x) = \lim f_n(x)$. Thus, f is well-defined on

$$F = -\bigcup E_m,$$

we must have that $\mu(E \setminus F) = 0$. Then, $f_n \to f$ uniformly on E_m, given $\epsilon > 0$, choose m such that $\mu(E \setminus E_m) < \epsilon$, so $\{f_n\}$ converges almost uniformly to f on E. ∎

Example 2.3.4 ([[[[[[[Very important example).]]]]]]]] On $[0,1]$, Lebesgue measure, there is a norm-Cauchy sequence of SMF that does not converge pointwise for any point in $[0,1]$. Note that $\{f(x)\}_{x\in[0,1]}$ does not converge. For f a SMF, $f = \sum b_j \chi_{E_j}$, $\mu(E_j) < \infty$. Next, note that

$$\int f d\mu = \sum b_j \mu(E_j), ||f||_1 = \int ||f(x)|| d\mu.$$

Next, note that

$$\chi_{[0,\frac{1}{2}]}, \chi_{[\frac{1}{2},1]}$$

Keep going for diving the interval by $\frac{1}{3}$, etc. Then, $||f_n||_1 \to 0$, $||f_n - 0||_1 \to 0$.

$$\mu\left(\{x : ||f(x)(\to 0) - f_n(x)|| < \epsilon\}\right) \ldots$$

Proposition 2.3.4. *If $\{f_n\} \to f$, almost uniformly, and $\{g_n\}$ a.u., for B-valued function, then $f_n + g_n \to f + g$, a.u., $rf_n \to rf$ a.u.*

Proof. No proof given in class :((. ■

Definition 2.3.9. Let $\{f_n\}$ a sequence, with $f \in M(X, \mathcal{S}, \mu, B)$, we say that $\{f_n\}$ converges to f "in measure" if, for all $\epsilon > 0$,

$$\mu\left(\{x : ||f(x) - f_n(x)|| > \epsilon\}\right) \xrightarrow{n\to\infty} 0.$$

Definition 2.3.10. $\{f_n\}$ is Cauchy in measure if

$$\forall \epsilon > 0, \mu\left(\{x : ||f_m(x) - f_n(x)|| > \epsilon\}\right) \xrightarrow{m,n\to\infty} 0.$$

Example 2.3.5. If $\{f_n\} \to \{f\}$ in measure, and $\{g_n\} \to g$, in measure, then $\{f_n + g_n\} \to f + g$ in measure. Let $\epsilon > 0$ be given. Then, choose N, such that for $n \geq N$, $\{x : ||f(x) - f_n(x)|| > \frac{\epsilon}{2} \cup \{x :\}$

Example 2.3.6. If $\{f_n\}]tof$ in measure, then $rf_n \to rf$ in measure.

Example 2.3.7. The following is a vector space:

$$(X, \mathcal{S}, \mu),\ \text{ISF}(X, \mathcal{S}, \mu).$$

Next, note that

$$\int f d\mu,$$

$$||f||_1 = \int ||f(x)||_B d\mu,$$

$$||\int f d\mu|| \leq ||f||_1.$$

2.4 Convergence in Measure

For $\{f_n\}, f \in M(X, \mathcal{S}, \mu)$. Given $\epsilon > 0$, consider

$$\mu(\{x : ||f(x) - f_n(x)|| > \epsilon\}) \xrightarrow{n} 0.$$

Cauchy in measure:

$$\mu\left(\{x : ||f_m(x) - f_n(x)||_B > \epsilon\}\right) \xrightarrow{m,n\to\infty} 0.$$

$$E_{m,n} = \{x : ||f_m(x) - f_n(x)|| > \epsilon\},$$

$$\chi_{E_{m,n}} \leq \frac{||f_m(x) - f_n(x)||}{\epsilon}.$$

If f_n's are ISF,

$$\int \chi_{E_{mn}} d\mu \leq \int \frac{||f_m(x) - f_n(x)||}{\epsilon} d\mu(x),$$

$$\mu(E_{mn}) \leq ||f_m - f_n||_1.$$

So, if

$$||f_m - f_n||_1 \xrightarrow{m,n\to\infty} 0,$$

then

$$\mu(E_{mn}^{\epsilon}) \to 0.$$

Proposition 2.4.1. *If $\{f_n\}$ is a sequence of ISF that is Cauchy for $||\cdot||_1$, then it is Cauchy in measure.*

Theorem 2.4.1. *(Riesz-Weyl) Let $\{f_n\} \subset M(X, \mathcal{S}, \mu, B)$, that is Cauchy in measure. Then, there is a subsequence that is almost uniformly Cauchy.*

Let (X, d) be a metric space, and let $\{x_n\}$ be a Cauchy sequence in X. Produce a "rapidly Cauchy subsequence," $\{x_{n_j}\}$. Let some $\delta > 0$ be given. Then, choose n_1 such that if $m, n \geq n_1$, then $d(x_m, x_n) < \dfrac{\delta}{2}$. x_{n_1}. Choose $n_2 > n_1$, such that $\ldots$ $d(x_m, x_n) < \dfrac{\delta}{2^2}$, $d(x_{n_1}, x_{n_2}) < \dfrac{\delta}{2}$. Then, choose $n_3 > n_2 \ldots$, $d(x_m, x_n) < \dfrac{\delta}{2^3}$, $d(x_{n_2}, x_{n_3})\dfrac{\delta}{2^2}$, continue on in this pattern to achieve:

$$\sum_{j=1}^{\infty} d(x_{n_{j+1}} - x_{n_j}) < \delta,$$

which is the characteristic of the rapidly Cauchy subsequence.

Proof. (Proof of Riesz-Weyl) Let $n_1 = 1$, then choose n_2 such that for $m, n \geq n_2$,

$$\mu\left(\left\{x : ||f_m(x) - f_n(x)|| > \frac{1}{2}\right\}\right) < \frac{1}{2}.$$

Now, choose $n_3 > n_2$ such that for $m, n \geq n_3$,

$$\mu\left(\left\{x : ||f_m(x) - f_n(x)|| > \frac{1}{2^2}\right\}\right) < \frac{1}{2^2}.$$

Again, continue on in this method to see that, for $n_{j+1} > n_j$, such that $m, n > n_{j+1}$

$$\mu\left(\left\{x : ||f_m(x) - f_n(x)|| > \frac{1}{2^j}\right\}\right) < \frac{1}{2^j}.$$

We now claim that $\{f_{n_j}\}$ is almost uniformly Cauchy. [Side note(?): Given $f \in M$, let $C_f = \{x : f(x) \neq 0\} \in \mathcal{S}$, we call this C_f the carrier of f. $\{f_n\}$, let $E = \bigcup C_{f_n} \in \mathcal{S}$, then for $x \notin E$, $f_n(x) = 0$, for all x.] Let $\epsilon > 0$ be given. Let

$$E_j = \left\{x : ||f_{n_{j+1}}(x) - f_{n_j}(x)|| > \frac{1}{2^j}\right\}.$$

Choose j_0 large enough that

$$\sum_{j=j_0}^{\infty} 2^{-j} < \epsilon.$$

Let

$$F = E \setminus \bigcup_{j=j_0}^{\infty} E_j.$$

Then, as

$$\mu(E_j) < \frac{1}{2^j}, \mu(E \backslash F) < \epsilon.$$

We next claim that $\{f_n\}$ is uniformly Cauchy on F, let $\delta > 0$ be given. Suppose $j > k$. Then,

$$\begin{aligned} ||f_{n_1}(x) - f_{n_k}(x)|| &= \left|\left|f_{n_j}(x) - f_{n_{j-1}}(x) + f_{n_{j-1}}(x) - f_{n_{j-2}}(x) + \ldots\right|\right| \\ &\leq \left|\left|f_{n_j}(x) - f_{n_{j-1}}(x)\right|\right| + \left|\left|f_{n_{j-1}}(x) - f_{n_{j-2}}(x)\right|\right| + \ldots + \left|\left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|\right|. \end{aligned}$$

For n_j's $> j_0$, considering the final inequality above, write to align with the above:

$$\frac{1}{2^j} + \frac{1}{2^{j-1}} + \ldots + 2^k.$$

For $x \in F$, choose $k \geq j_0$, such that

$$\sum_{\ell=k}^{\infty} 2^{-\ell} < \delta,$$

then $j, k \geq K, < \delta$. ■

Corollary 2.4.2. *If $\{f_n\}$ is a sequence that is Cauchy in measure, then there is a subsequence that converges a.u. to a function f.*

Proposition 2.4.2. *If $\{f_n\} \in M$ converge to f a.u. then, E, f_n converges to f in measure.*

Proof. Given $\epsilon > 0$,

$$\mu(\{x : ||f(x) - f_n(x)|| > \epsilon\}) \leq \mu(E \setminus F) < \delta, \text{ for } n > N.$$

Next, choose $F \subset E, \mu(E \setminus F) < \delta$, such that $f_n \to f$, uniformly on F, choose N such that, for $n \geq N$,

$$||f(x) - f_n(x)|| < \epsilon,$$

for $x \in F$. ■

Proposition 2.4.3. *If $\{f_n\}$ converges in measure to f, and if $\{f_n\}$ also converges in measure to g a.e. then $f = g$, a.e.*

Proof. $||f(x) - g(x)|| \le ||f(x) - f_n(x)|| + ||f_n(x) - g(x)||$,

$$\{x : ||f(x) - g(x)|| > \epsilon\} \subseteq \{x : ||f(x) - f_n(x)|| \ge \frac{\epsilon}{2}\} \cup \{x : ||f_n(x) - g(x)|| \ge \frac{\epsilon}{2}\}.$$

Then basically take μ of the above and add the union. Then, this goes to zero, as $n \to \infty$. Note that this holds, as

$$\{x : ||f(x) - g(x)|| \ne 0\} \subseteq \bigcup_{n=1}^{\infty} \{x : ||f(x) - g(x)|| > \frac{1}{n}\},$$

so

$$\mu(\{x : ||f(x) - g(x)|| > \epsilon\}) = 0, \ \forall \epsilon,$$

let ϵ never seen through $\frac{1}{n}$. ■

Proposition 2.4.4. *Let $\{f_n\}$ be a sequence of functions that are Cauchy in measure, and if a subsequence $\{f_{n_j}\}$ converges to a function f in measure, then $\{f_n\}$ converges to f in measure.*

Proof.

$$\{x : ||f(x) - f_n(x)|| > \epsilon\} \subseteq \left\{x : ||f(x) - f_{n_j}(x) > \frac{\epsilon}{2}\right\} + \left\{x : ||f_{n_j}(x) - f_n(x)|| > \frac{\epsilon}{2}\right\}.$$

Now, let δ be given. Choose N such that, for $m, n > N$, μ(right summand) $< \frac{\delta}{2}$, and μ(left summand) $< \frac{\delta}{2}$, for $n_j > N$. ■

Next lecture.

$(X, \mathcal{S}, \mu, B)$, MSF, ISF. Then, let $\{f_n\}$ be a sequence of ISF, Cauchy for $|| \cdot ||_1$ ("mean Cauchy"). Then, $\{f_n\}$ is Cauchy in measure, then there is a subsequence that is a.u. Cauchy, so it converges a.u. to a function f. Then, $\{f_n\}$ converges to f is measure. Also, f is a.e. unique.

Proposition 2.4.5. *Let $\{f_n\}, \{g_n\}$ be mean-Cauchy sequence of ISF taht are equivalent, i.e. $||f_n - g_n||_1 \xrightarrow{n} 0$. If $\{f_n\} \to f$ in measure, then $\{g_n\}$ converges to f in measure.*

Proof. Consider the sequence, $f_1, g_1, f_2, g_2, \ldots$. This is a mean Cauchy sequence, and the subsequence $\{f_n\}$ converges to f in measure, so it is Cauchy in measure. So this sequence of f_i, g_i converges to f in measure, so $\{g_n\}$ converges to f in measure, for each equivalence class of mean Cauchy sequences, there is a function f to which they all converge in measure, f a.e. unique. ■

Proposition 2.4.6. *Let $\{f_n\}$ and $\{g_n\}$ be mean Cauchy sequences of ISF and assume that tehy both converge to f in measure. Then, $\{f_n\}$ and $\{g_n\}$ are equivalent.*

Proof. So there are subsequences, $\{f_{n_k}\}$, $\{g_{m_k}\}$ that converge to f a.u. Let $h_k = f_{n_k} - g_{m_k}$, $\{h_n\}$ for a mean Cauchy sequence, and $h_k \to 0$ a.u. We need that $||h_k|| \to 0$.

Lemma 2.4.3. *If $\{h_k\}$ is a mean Cauchy sequence of ISF, such that $h_k]to0$ a.u. then $||h_k||_1 \to 0$.*

Proof. Let $\epsilon > 0$ be given. Choose N such that for $m, n \geq N$, we have that $||h_m - h_n|| < \epsilon$. Let $E = C_{h_N} = \{x : h_N(x) \neq 0\}$,. then

$$\begin{aligned} m \geq N, \epsilon > ||h_N - h_m||_1 \\ &= \int ||h_N(x) - h_m(x)|| d\mu(x) \\ &\geq \int_{E'} ||h_N(x) - h_m(x)|| d\mu(x) \\ &= \int_{E'} ||h_n(x)|| d\mu(x), \end{aligned}$$

$\mu(E) < \infty$, $\{h_n\}$ converges a.u. so it also converges a.u. on E, so can find $F \subset E$, $\mu(E \setminus F) < \dfrac{\epsilon}{a}$ such that $\{h_n\}$ converges uniformly to 0 on F. Choose $n \geq N$ such that

$$||h_n(x)|| \leq \frac{\epsilon}{\mu(F)},$$

for $x \in F$, then

$$\int_F ||h_n(x)|| d\mu < \epsilon,$$

for $n > N$, (???)

$$\int_{E \setminus F} ||h_N(x)|| \leq \mu(E \setminus F) ||h_N(x)||_\infty < \epsilon.$$

For $n > N_1$, $||h_n||_1 < 4\epsilon$. ∎

∎

Proposition 2.4.7. *Let $f \in M(X, \mathcal{S}, \mu, B)$. The following are equivalent:*

1. *There is a mean Cauchy sequence of ISF that converges to f in measure.*

2. *"" f a.u.*

3. *"" f a.e.*

Proof. $(1) \implies (2)$ is the Riesz - Weyl Theorem. $(2) \implies (3)$ is an earlier proposition. $(3) \implies (1)$ $\{f_n\}$ mean Cauchy, then there exists a subsequence converging to some g a.e. but $f_n \to f$ a.e. so $g = f$ a.e. ∎

Definition 2.4.1. Let $f \in M(X, \mathcal{S}, \mu, B)$. Then f is μ-integrable if there is a mean Cauchy sequence of ISF that converges to f

1. in measure, or

2. a.u.

3. a.e.

There is a bijection between equivalence classes of mean Cauchy sequences of ISF and equivalence classes of integrable functions where the second case of equivalence classes is for almost everywhere equivalence.

Proposition 2.4.8. *Let $\{f_n\}$ be a mean Cauchy sequence of ISF, then*

$$\{\int f_n d\mu\}$$

is a Cauchy sequence in B (so converges to an element of B).

Proof.

$$\begin{aligned}\left|\left|\int_E f_n d\mu - \int_E f_m d\mu\right|\right| &= \left|\left|\int_E (f_n - f_m) d\mu\right|\right| \\ &\leq \int_E ||f_n(x) - f_m(x)||\, d\mu(x) \\ &\leq ||f_n - f_m|| \xrightarrow{m,n} 0.\end{aligned}$$

∎

$\mathscr{L}^1(X, \mathcal{S}, \mu, B)$ be the set of μ-integrable functions.

Definition 2.4.2. Let $f \in \mathscr{L}^1$, the

$$\int f d\mu$$

is defined to be the

$$\lim \int f_n d\mu,$$

for any Cauchy sequence on ISF that converge to f in measure, a.u., a.e.

Example 2.4.1. Some properties include:

1. If $f, g \in \mathscr{L}^1$, then $f + g \in \mathscr{L}^1$.
2. $$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$
3. If $r \in \mathbb{R}$ (or in $\mathbb{C}$), then $rf \in \mathscr{L}^1$ and
$$\int_E r f d\mu = r \int_E f d\mu.$$
4. If $f \in \mathscr{L}^1(X, \mathcal{S}, \mu, B)$, then $(x \mapsto ||f(x)||) \in \mathscr{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$.
5. If $f \in \mathscr{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ and if $f \geq 0$, then
$$\int f d\mu \geq 0.$$
Thus, if $f, g \in \mathscr{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$, with $f \geq g$, then
$$\int f d\mu \geq \int g d\mu.$$
6. If $f \in \mathscr{L}^1(X, \mathcal{S}, \mu, B)$, then
$$|| \int f d\mu|| \leq \int ||f(x)|| d\mu.$$

7. Set
$$||f||_1 = \int ||f(x)||d\mu(x),$$
then $||f+g||_1 \leq ||f||_1 + ||g||_1$. Also, we have that $||rf||_1 = |r| \cdot ||f||_1$. Hence, $||\cdot||_1$ is a seminorm on $\mathscr{L}^1$.

8. $||f||_1 = 0 \iff f(x) = 0$, a.e.

   Proof. If $f(x) = 0$, a.e. then $||f(x)|| = 0$ a.e. so
   $$\int ||f(x)||d\mu(x) = 0.$$
   If $||f||_1 = 0$, then the constant sequence 0 converges to f a.e.
   $$\int f = \lim \int 0 = 0.$$
   ∎

Example 2.4.2. Let $\mathcal{N}$ be a vector subspace of $\mathscr{L}^1$. Set $L^1(X, \mathcal{S}, \mu, B) = \mathscr{L}^1(X, \mathcal{S}, \mu, B)/\mathcal{N}$. Then, $||\cdot||_1$ is a norm on $L^1(X, \mathcal{S}, \mu, B)$. Then, $L^1(\ )$ is complete for this norm.

Definition 2.4.3. If $\{f_n\}$ is a sequence in $\mathscr{L}^1, L^1$ that converges to $f \in \mathscr{L}^1$, for $||\cdot||_1$, i.e. $||f - f_n||_1 \to 0$, we say that $\{f_n\}$ converges to f in mean. "Mean Cauchy Sequences in $\mathscr{L}^1, L^1$."

If $\{f_n\}$ is a mean Cauchy sequence in $\mathscr{L}^1$, [If $\{f_n\}$ is a mean Cauchy sequence of ISF, $\{f_n\} \to f$, then $||f - f_n||_1 \to 0$.] for each n, choose ISF with
$$||f_n - g_n|| < \frac{1}{2^n},$$
then $\{g_n\}$ is a mean Cauchy sequence of ISF $\to f$. Then, $||f_n - f|| \to 0, \leq ||f_n - g_n||_1 + ||g_n - f||_1$.

Note 2.4.1. Thus, $L^1(X, \mathcal{S}, \mu, B)$ is a Banach space.

Definition 2.4.4. Carrier$(f) = C_f = \{x \in X : f(x) \neq 0_B$.

Note 2.4.2. If $f \in \mathscr{L}^1(X, \mathcal{S}, \mu, B)$:

1. C_f is σ-finite.

   Proof. Let $\{f_n\}$ be a sequence of ISF with $f_n \to f$, a.e. then $\mu(C_{f_n}) < \infty$ and

   $$C_f \subseteq \bigcup_{n=1}^{\infty} C_{f_n}.$$

   ■

2. Let $f \in \mathscr{L}^1$, $\mathbb{R}$-valued, let $E \in \mathcal{S}$, with $\chi_E \leq f$, then $\mu(E) < \infty$. Let $\{F_n\} \uparrow C_f$, with $\mu(F_n) < \infty$, let $E_n = E \cap F_n, \mu(E_n) < \infty, \chi_{E_n} \leq f, \chi_n \in \mathscr{L}^1$. Then,

   $$\mu(E_n) \leftarrow \int \chi_{E_n} d\mu \leq \int f d\mu,$$

   so for all n,

   $$\mu(E_n) \leq \int f d\mu < \infty,$$

   $E_n \uparrow E$, so

   $$\mu(E) \leq \int f d\mu.$$

Definition 2.4.5. For $\{f_n\} \subset \mathscr{L}^1$, we say that $\{f_n\}$ is Cauchy in mean if

$$||f_m - f_n||_1 \xrightarrow{m,n} 0,$$

and we say that $\{f_n\}$ converges in mean to f if

$$||f - f_n||_1 \to 0.$$

Proposition 2.4.9. *If $\{f_n\} \subseteq \mathscr{L}^1$ is mean Cauchy then $\{f_n\}$ is Cauchy in measure.*

Proof. For any given $\epsilon > 0$, for m, n, set $E_{mn} = \{x : ||f_m(x) - f_n(x)|| > \epsilon\}$ then

$$\chi_{E_{mn}} \leq \left(x \mapsto \frac{||f_m(x) - f_n(x)||}{\epsilon}\right),$$

thus $\chi_{E_{mn}}$ is in $\mathscr{L}^1$, so $\mu(E_{mn}) \leq \frac{1}{\epsilon}||f_m - f_n||_1 \xrightarrow{m,n} 0$. Similarly, if $\{f_n\}$ converges to f in mean, then $\{f_n\} \to f$ in measure. ■

Definition 2.4.6. Let $f \in \mathscr{L}^1$. Then, the "indefinite integral" of f, denoted by μ_f, defined by

$$\mu_f(E) = \underbrace{\int_E f(x) d\mu(x)}_{\int \chi_E f d\mu} \in B.$$

Proposition 2.4.10. *μ_f is a (B-valued) measure (finite).*

Example 2.4.3. If $E, F \in \mathcal{S}$ and $E \cap F = \varnothing$, then

$$\int_{E \oplus F} f d\mu = \int_E f d\mu + \int_F f d\mu,$$

$f_n \to f$, $\{f_n\}$ are ISF.

Proof. (Proposition 2.4.10) μ_f is finitely additive. Let

$$E = \bigoplus^{\infty} E_n$$

, we want

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n),$$

choose ISF g with $||f-g||_1 < \frac{\epsilon}{3}$. μ_g is countably additive:

$$g = \sum^{n} b_j \chi_{F_j}$$
$$\mu_g(E) = \sum b_j \mu(E \cap F_j).$$

So find N such that for $n \geq N$,

$$||\mu_g(E) - \sum_{n=1}^{k} \mu_g(E_n)|| < \frac{\epsilon}{3}.$$

Then, for $n \geq N$,

$$\begin{aligned}
||\mu_f(E) = \sum_{k=1}^{n} \mu_f(E_k)||_B &\leq ||\mu_f(E) - \mu_g(E)||_1 + ||\mu_g(E) - \sum^{n} \mu_g(E_k)|| + ||\sum^{n} \mu_g(E_k) - \sum^{n} \mu_f(E_k)|| \\
&\leq ||\int_E (f-g)d\mu|| + \frac{\epsilon}{3} + ||\int_{\bigoplus^n E_k} (f-g)d\mu \\
&\leq ||f-g||_1 + \frac{\epsilon}{3} + ||f-g||_1 \leq \epsilon.
\end{aligned}$$

Note that

$$||\int f d\mu|| \leq \int ||f(x)||d\mu(x) = ||f||_1.$$

■

Definition 2.4.7.

$$\int_X f d\mu$$

$A \subseteq X$ is locally $\mathcal{S}$-measurable if $A \cap E \in \mathcal{S}$ whenever $E \in \mathcal{S}$, then X is locally $\mathcal{S}$-measurable, as is $X \setminus \underbrace{E}_{\in \mathcal{S}}$ is locally $\mathcal{S}$-measurable. If A is locally $\mathcal{S}$-measurable,

$$\int_A f d\mu = \int_{A \cap C_f} f d\mu \quad \mu_f.$$

Proposition 2.4.11. *Let $f \in \mathscr{L}^1$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then $||\mu_f(E)|| < \epsilon$.*

Proof. Given $\epsilon > 0$, choose ISF g with $||f - g||_1 < \frac{\epsilon}{2}$, for any $E \in \mathcal{S}$,

$$||\mu_f(E) \leq \underbrace{||\mu_f(E) - \mu_g(E)||}_{\leq ||f-g||_1 \leq \frac{\epsilon}{2}} + ||\mu_g(E)||,$$

so $||\mu_f(E)|| < \epsilon$, $||\mu_g(E)|| = ||\int_E g(x)d\mu|| \leq \int_E ||g(x)||d\mu \leq \mu(E)||g||_\infty \leq \frac{\epsilon}{2}$, so we choose $\delta = \frac{\epsilon}{2 + 2||g||_\infty}$. Use this δ and we are done. ∎

Proposition 2.4.12. *Let $f \in \mathscr{L}^1$, then for every $\epsilon > 0$, there is $E \in \mathcal{S}$, $\mu(E) < \infty$ with*

$$\left|\left|\int_{X \setminus E} f d\mu\right|\right| < \epsilon.$$

Proof. Choose g to be an ISF, $||f - g||_1 < \epsilon$. Let $E = C_g$. Then,

$$\left|\left|\int_{X \setminus E} f d\mu\right|\right| = \left|\left|\int_{X \setminus E} (f - g) d\mu\right|\right| \leq ||f - g||_1 < \epsilon.$$

∎

Theorem 2.4.4. *(Lebesgue Dominated Convergence Theorem) Let $\{f_n\} \subset \mathscr{L}^1$, with $f_n \to f$ a.e. dominated by g. Assume there is $g \in \mathscr{L}^1(\ldots, \mathbb{R})$ such taht $||f_n(x)|| \leq g(x)$ for all x, all n. Then, $\{f_n\}$ is a mean Cauchy sequence. (Thus, $f \in \mathscr{L}^1$ and*

$$\int f d\mu = \lim \int f_n d\mu.)$$

Proof. Let $\epsilon > 0$ be given. Choose E with

$$\int_{X \setminus E} g < \frac{\epsilon}{6}.$$

Then,

$$\left|\left|\int_{X \setminus E} (f_m(x) - f_n(x)) d\mu(x)\right|\right| \leq \int_{X \setminus E} ||f_m(x)))d\mu + \int_{X \setminus E} ||f_n(x)||d\mu \leq 2\int_{X \setminus E} g(x)d\mu < \frac{\epsilon}{3}.$$

By Egoroff's Theorem, given any $\delta > 0$, there is $F \subset E$ with $\mu(E \setminus F) < \delta$, such that $f_n \to f$ uniformly. Then,

$$\left|\left|\int_{E\setminus F} ||f_m(x) - f_n(x)||d\mu\right|\right| \leq 2\int_{E\setminus F} gd\mu = 2\mu_g(E \setminus F).$$

Can choose δ so that if $\mu(G) < \delta$, then $\mu_g(G) < \frac{\epsilon}{6}$. We then choose $\delta > 0$ so that (the last sentence regarding δ). We then get that $2\mu_g(E \setminus F) < \frac{\epsilon}{3}$. We then choose N such that if $m, n \geq N$, $||f_m(x) - f_n(x)|| < \frac{1}{\mu(F)} \cdot \frac{\epsilon}{3}$. Note that

$$\left|\left|\int_F f_m(x) - f_n(x)\right|\right| \leq \int_F ||f_m(x) - f_n(x)||d\mu,$$

we need

$$||f_n(x) - f_n(x)||_{x\in F} < \frac{\epsilon}{3} \cdot \frac{1}{\mu(F)}.$$

... ■

Theorem 2.4.5. *(Monotone Convergence Theorem) Only for $\mathbb{R}$-valued functions. Let $\{f_n\} \in \mathscr{L}^1(\ldots, \mathbb{R})$, $m < n \implies f_m(x) \leq f_n(x) \forall x, m$. If there is a $c \in \mathbb{R}$ such that*

$$\int f_m d\mu \leq c,$$

for all m, then $\{f_n\}$ is a mean Cauchy sequence, that converges a.e. to some function $f \in \mathscr{L}^1$ and

$$\int f d\mu = \lim \int f_n d\mu.$$

Proof. If $m < n$, then

$$\int f_m d\mu < \int f_n d\mu \leq c,$$

then $\left\{\int f_n d\mu\right\}$ is an increasing sequence in $\mathbb{R}$ bounded above by c, thus $\left\{\int f_n d\mu\right\}$ is a Cauchy sequence. But, for $m < n$,

$$\int (f_n(x) - f_m(x))d\mu = \int |f_n(x) - f_m(x)|d\mu = ||f_n - f_m||_1,$$

so $\{f_n\}$ is mean Cauchy. ■

If $f \in M(X, \mathcal{S}, \mu, \mathbb{R})$, $f \geq 0$, then if f is not integrable, then set

$$\int f d\mu = +\infty.$$

Proposition 2.4.13. *Let $f \in M(X, \mathcal{S}, \mu, B)$, and suppose there is $g \in \mathscr{L}^1(\ldots, \mathbb{R})$, such that $||f(x)||_B \leq g(x)$ a.e. Then, $f \in \mathscr{L}^1(X, \mathcal{S}, \mu, B)$.*

Proof. Suppose f is measurable, then there exists a sequence, $\{f_n\}$ of MSF such that $f_n \to f$ a.e. Then, for each n, set

$$g_n(x) = \begin{cases} f_n(x) & ||f_n(x)|| \leq 2g(x) \\ 0 & \ldots > \end{cases}$$

Let $E_n = \{x : 2g(x) - \|f_n(x)\| \geq 0\}, \chi_{E_n} \leq g$, so $g_n = \chi_{E_n} f_n \in$ ISF. Then, $\|g_n\| \leq 2g$. Thus, $g_n \to f$ a.e. so by LDCT, $f \in \mathscr{L}^1$. ■

Definition 2.4.8. Given $(X, \mathcal{S}, \mu, B)$, for $p > 0$, set $\mathscr{L}^p = \{f$-measurable, B-valued functions : $x \mapsto \|f(x)\|^p \in \mathscr{L}^1\}$. If $f, g \in \mathscr{L}^p, \|f(x) + g(x)\|^p \leq (\|f(x)\| + \|g(x)\|)^p \leq (2\max(\|f(x)\|, \|g(x)\|)^p \leq 2^p \max(\|f(x)\|, \|g(x))^p \leq 2^p(\max\{\|f(x)\|^p, \|g(x)\|^p\}) \leq 2^p(\|f(x)\|^p + \|g(x)\|^p) \in \mathscr{L}^1$. So, $\|f + g\|^p \in \mathscr{L}^1$.

Proposition 2.4.14. *$\mathscr{L}^p$ is a vector space with pointwise operations.*

Definition 2.4.9. Set

$$\|f\|_p = \left(\int \|f(x)\|^p d\mu(x)\right)^{\frac{1}{p}}.$$

If $\|f\|_p = 0$, then $f \equiv 0$ a.e.

Theorem 2.4.6. *If $1 < p < \infty$, then $\|f\|_p$ is a (semi) norm.*

Note 2.4.3. For $1 < p < \infty$, define q by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.4.7. *For any* $r, s \in \mathbb{R}^+$,

$$rs \leq \frac{r^p}{p} + \frac{s^q}{q}.$$

Proof. Fix r and set

$$\varphi(s) = \frac{r^p}{p} + \frac{s^q}{q} - rs.$$

We want to show that $\varphi(s) \geq 0$, for all s. Then,

$$\varphi(s) \xrightarrow{s \to \infty} +\infty$$

and

$$\varphi(s) \xrightarrow{s \to 0} \frac{r^p}{p}.$$

Then, note that $\varphi'(s) = s^{q-1} - r$, but the critical part is where $s^{\frac{q}{p}} = s^{q-1} = r$, $s = r^{\frac{p}{q}}$. Then,

$$\varphi(r \cdot \frac{p}{q} = \frac{r^p}{p} + \frac{(r^{\frac{p}{q}})^q}{q} - r^{1+\frac{p}{q}} = p ... = 0.$$

■

Proposition 2.4.15. *(Holder's Inequality) If* $f \in \mathscr{L}^p$ *and* $g \in \mathscr{L}^q$, *then* $x \mapsto \|f(x)\| \cdot \|g(x)\| \in \mathscr{L}^1$, *and*

$$\int \|f(x)\| \cdot \|g(x)\| d\mu(x) \leq \|f\|_p \cdot \|g\|_q.$$

Proof. Let

$$r = \frac{\|f(x)\|}{\|f\|_p}, s = \frac{\|g(x)\|}{\|g\|_q}.$$

Then,

$$\frac{\|f(x)\| \cdot \|g(x)\|}{\|f\|_p \cdot \|g\|_q} \leq \frac{\|f(x)\|^p}{p\|f\|_p^p} + \frac{\|g(x)\|^q}{q\|g\|_q^q},$$

so $x \mapsto \|f(x)\| \cdot \|g(x)\| \in \mathscr{L}^1$, as each of the previous are. Note that $g_n \to f$ a.e. Then,

$$\frac{\int \|f(x)\| \cdot \|g(x)\|}{\|f\|_p \|g\|_q} \leq \frac{\int \|f(x)\|^p d\mu}{p \cdot \|f\|_p^p} + \frac{\int \|g(x)\|^q d\mu}{q\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

■

Proposition 2.4.16. *(Minkowski's Inequality) Note that here we have that $1 < p < \infty$. Let $f, g \in \mathscr{L}^p$. Then, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. $\|f(x) - g(x)\|^p = \underbrace{\|f(x) + g(x)\|}_{\in \mathscr{L}^p} \cdot \underbrace{\|f(x) + g(x)\|^{p-1}}_{\in \mathscr{L}^q} = \dfrac{p}{q}$. But this is less than or equal to $(\|f(x)\| + \|g(x)\|) = \underbrace{\|f(x)\|}_{\in \mathscr{L}^p} \cdot \underbrace{\|f(x) + q(x)\|^{\frac{p}{q}}}_{\in \mathscr{L}^q} + \|g(x)\| \cdot \|f(x) + g(x)\|^{\frac{p}{q}}$. But then,

$$\begin{aligned}\int \|f(x) + g(x)\|^p d\mu &\leq \int \|f(x)\| \cdot \|f(x) + g(x)\|^{\frac{p}{q}} d\mu + \int \|g(x)\| \cdot \|f(x) + g(x)\|^{\frac{p}{q}} \\ &\leq \|f\|_p \|x \mapsto \|f(x) + g(x)\|^{\frac{p}{q}}\|_q + \|g\|_p \cdot \|f(x) + g(x)\|^{\frac{p}{q}}\|_q \\ &= (\|f\|_p + \|g\|_p)(\int \|f(x) + g(x)\|^p)^{\frac{1}{q}} = (\|f+g\|_p)^{\frac{p}{q}} \\ &= (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{\frac{p}{q}} \ldots\end{aligned}$$

■

Note 2.4.4. So $\mathscr{L}^p$ is a normed vector space. Is L^p complete?

Definition 2.4.10. Convergence in p-mean if $\|f - f_n\| \to 0$, p-mean Cauchy, $\|f_m - f_n\|_p \to 0$.

Proposition 2.4.17. *If $\{f_n\}$ is a p-mean Cauchy sequence, then it is Cauchy in measure.*

Proof. Given $\epsilon > 0$, $E_{mn} = \{x : \|f_n(x) - f\| > \epsilon\} = \{x : \|f_m(x) - f_n(x)\|^p > \epsilon^p\}$, so

$$\chi_{E_{mn}}(x) \leq \frac{\|f_m(x) - f_n(x)\|^p}{\epsilon^p},$$

so

$$\mu(\chi_{E_{mn}}) \leq \frac{\|f_n - f_m\|_p^p}{\epsilon^p} \to 0.$$

So by the Riesz-Weyl theorem, there is a subsequence that converges a.u. to a function f, which is measurable. We want that for $f \in L^p$ and $\|f - f_n\|_p \to 0$. Continue here next lecture. ∎

Note 2.4.5. Let $\{f_{n\}}$ be a sequence of $\mathbb{R}^+$-valued measurable functions, or a sequence in $\mathbb{R}^+$ itself. Then, for $n > m$, let $h_{nm} = f_m \wedge f_{m+1} \wedge f_{m+2} \wedge \ldots \wedge f_n$, where $\wedge$ is a minimum, where $h_{mn} \downarrow$ as $n \to \infty$. Let $g_m =$

Lemma 2.4.8. *(Fatou's Lemma) For $\{f_n\} \subset L^1(X, \mathcal{S}, \mu)$, for all $n, f_n \geq 0$. Then,*

$$\int \underline{\lim} f_n d\mu \leq \underline{\lim} \int f_n d\mu.$$

Proof. For $m > n, h_n \leq h_m \leq f_m \implies \int h_n d\mu \leq \int f_m d\mu$ for all $m > n$. Then, it follows that $\int h_n d\mu \leq \underline{\lim}_m \int f_m d\mu$,

$$\int h_n d\mu \uparrow \int \underline{\lim} f_n d\mu \implies \int \underline{\lim} f_n d\mu \leq \underline{\lim} \int f_n d\mu.$$

∎

Theorem 2.4.9. *$L^p(X, \mathcal{S}, \mu, B)$ is complete.*

Proof. Let $\{f_n\}$ be a p-mean Cauchy sequence. Then $\{f_n\}$ is Cauchy in measure, so it has a subsequence that converges pointwise to some f(may not be in l^p). Let $\{f_n\}$ be the subsequence. Then,

$$\|f - f_n\|_p^p = \int \|f(x) - f_n(x)\|^p d\mu(x).$$

Then, $\|f(x) - f_n(x)\|^p = \lim_{m\to\infty} \|f_m(x) - f_n(x)\|^p$. Then,

$$\int \lim_{n\to\infty} \|f_n(x) - f_m(x)\|^p \leq \underline{\lim} \int \|f_m(x) - f_n(x)\|^p d\mu = \underline{\lim} \|f_m(x) - f_n(x)\|_p^p \to 0.$$

Then,

$$\int \|f(x) - f_n(x)\|^p d\mu(x) \to 0 \implies f \in L^p$$

and $\|f - f_n\|_p \to 0$. ■

If $\mathcal{H}$ is a Hilbert space. then for $f, g \in L^2(X, \mathcal{S}, \mu, \mathcal{H})$ is also a Hilbert space! We can define the inner product as

$$\langle f, g\rangle = \int \langle f(x), g(x)\rangle d\mu.$$

Then, it follows that $\underbrace{|\langle f(x), g(x)\rangle|}_{L^1} \leq \underbrace{\|f(x)\| \cdot \|g(x)\|}_{L^1}$ by Minkowski. Now, given a measure space $(X, \mathcal{S}, \mu)$, and let $R \subset \mathcal{S}$ be a ring, and assume that R generates $\mathcal{S}$. Consider ISF$(X, R, \mu|_R)$ is dense in each $L^p(X, \mathcal{S}, \mu, B)$ for $1 \leq p < \infty$. Consider now the outer measure given by $\mu|_R$. Namely,

$$\mu_R^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(E_n) : A \subseteq \bigcup E_n, E_n \in R \right\}.$$

Then $\mathcal{S}$ consists of measurable sets for μ_R^*. Then, given $E \in \mathcal{S}$ with $\mu(E) < \infty, \forall \epsilon > 0$, there is $\{F_n\} \subset R$ with $E \subset \bigcup F_n$ and

$$\sum^{\infty} \mu(F_n) \leq \mu(E) + \epsilon.$$

Then, disjointize the F_n, $E = \bigoplus^{\infty} F_n \cap E$, so $\mu(E) = \sum \mu(E \cap F_n)$. We can then find n such that $G = \bigoplus_{m=1}^{n} F_m$ such that $\mu(E) - \epsilon \leq \mu(G) \leq \mu(E) + \epsilon$. Then, $\mu(E) - \epsilon \leq \mu(G \cap E)$, so $\mu(E \setminus G) + \mu(G \setminus E) < 2\epsilon$. We then also have that

$$\|\chi_E - \chi_G\| = \chi_{E\Delta G},$$

so $\|\chi_E - \chi_G\|^p = \chi_{E\Delta G} \implies \|\chi_E - \chi_G\|_p^p = \mu(E\Delta G) < 2\delta$. Then, for $b \in B$, $\|b\chi_E - b\chi_G\|_p = \|b\| \cdot \|\chi_E - \chi_G\|_p \leq \|b\|(2\epsilon) \implies$ for any ISF, for (X, R, μ, B).

Theorem 2.4.10. *For R a ring that generates $\mathcal{S}$, ISF(X, R, μ, B) is dense in $L^p(X, \mathcal{S}, \mu, B)$ for $1 \leq p < \infty$.*

For $X = \mathbb{R}$, α non-decreasing and left continuous, μ_α, consider $\mathcal{P} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, $f_n \to \chi_{[a,b)}$ pointwise a.u. Thus, each $\chi_{[a,b)}$ can be approximated in the p-norm by continuous functions if compact support $C_c(\mathbb{R})$, so the same is true for χ_E with $E \in \mathbb{R}$. Then, $C_c(\mathbb{R})$ are dense in $L^p(X, \mathcal{S}, \mu, B)$ for all $1 \leq p < \infty$.

Chapter 3

Product Measure

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be measure spaces and let B be a Banach space. Then, note that $(Y, \mathcal{T}, \nu, B)$ is a Banach space itself, so we can consider the following:

$$L^1(X, \mathcal{S}, \mu, L^1(Y, \mathcal{T}, \nu, B)).$$

Then, given $E \in \mathcal{S}, F \in \mathcal{T}, b \in B$, we can set

$$f(x, y) = \chi_E(x)\chi_F(y)b.$$

Note that we also need that $\mu(E) < \infty$ and $\nu(F) < \infty$. Then, fix $f(x, -) \in L^1(Y, \mathcal{T}, \nu, B)$. Then,

$$\int f d\nu = b\chi_E(x)\mu(F).$$

Hence, it follows that

$$\int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = b\mu(E)\nu(F).$$

You can also do this in the opposite way by taking the other L^1 space to be the Banach space of the other. We now want to find a measure on $X \times Y$, for $E \in \mathcal{S}, F \in \mathcal{T}$, we want $\chi_{E\times F}$ to be integrable. That is, we want

$$\int \chi_{E\times F} = \mu(E)\nu(F).$$

Definition 3.0.1. Let $S \otimes T$ be the σ-ring generated by $E \times F$, for $E \in \mathcal{S}, F \in \mathcal{T}$. In fact, $\mathcal{P} = \{E \times F : E \in \mathcal{S}, F \in \mathcal{T}\}$ is a semi-ring that generates $\mathcal{S} \otimes \mathcal{T}$. Then, we can finally define $(\mu \otimes \nu)(E \times F) = \mu(E)\nu(F)$ on $\mathcal{P}$.

Theorem 3.0.1. *$\mu \otimes \nu$ is countably additive on $\mathcal{P}$.*

Proof. Let $E \times F = \bigoplus_{n=1}^{\infty} E_n \times F_n$. Then, $\chi_{E\times F}$, $\chi_{\bigoplus^m E_n \times F_n} \uparrow \chi_{E\times F}$ pointwise. Then, fix x_0. Then,

$$\chi_{\bigoplus_{n=1}^m}(x_0, y) \uparrow \chi_{E\times F}(x_0, y) = \chi_E(x_0)\chi_F(y).$$

Thus,

$$\int \chi_{E\times F}(x_0, y) d\mu(y) = \lim \int \chi_{\bigoplus^m E_n \times F_n}(x_0, y) d\mu(y) = \lim_{m\to\infty} \sum^m \chi_{E_n}(x_0)\nu(F_n).$$

Hence, by the Monotone Convergence Theorem,

$$\int \chi_E(x)\nu(F) d\mu(x) = \lim \sum \int \chi_{E_n}(x)\nu(F_n) d\mu(x).$$

It then follows that

$$\mu(E)\nu(F) = (\mu \otimes \nu)(E \times F) = \lim \sum^{\infty} \mu(E_n)\nu(F_n) = \sum (\mu \otimes \nu)(E_n \times F_n).$$

■

Then, let $(\mu \otimes \nu)^*$ be the outer measure on $X \times Y$. Then, $(X \otimes Y)^*$ restricts to a measure on $(\mu \otimes \nu)^*$-measurable sets. Thus, $\mathcal{S} \otimes \mathcal{T}$ is contained in the measurable sets and $\mathcal{P}$ is contained in the $(\mu \otimes \nu)^*$-measurable sets. Finally, it follows that $(\mu \otimes \nu)^*|_{\mathcal{S}\otimes\mathcal{T}}$ is a measure, $\mu \otimes \nu$, called the product measure on $\mathcal{S} \otimes \mathcal{T}$. Consider $(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$, we can construct $(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu), L^1(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu, B)$, given $f \in L^1(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu, B)$,

$$\int f d(\mu \otimes \nu) = \int \left(\int f(x, y) d\nu(y) \right) d\mu(x) = \int \left(\int f(x, y) d\mu(x) \right) d\nu(x),$$

which is Fubini's Theorem. Prove for $f = \chi_G, G \in \mathcal{S} \otimes \mathcal{T}$.

Proposition 3.0.1. *If μ and ν are σ-finite, then so is $\mu \otimes \nu$.*

Proof. Let $R = \{G \in \mathcal{S} \otimes \mathcal{T} : G \subseteq \bigcup^{\infty} G_n, (\mu \otimes \nu)(G_n) < \infty\}$. R is a σ-ring, and it contains all rectangles $E \times F$, for $E \in \mathcal{S}, F \in \mathcal{T}$, because $E \subseteq \bigcup^{\infty} E_n, \mu(E_n) < \infty, F \subseteq \bigcup^{\infty} F_n, \nu(F_n) < \infty$. Then, $E \times F \subseteq \bigcup_{m,n}^{\infty} E_m \times F_n, (\mu \otimes \nu)(E_m \times F_n) = \mu(E_m) \cup (F_n) < \infty$. For $x \in X$, let $G_x = \{y : (x, y) \in G\}$. For $y \in Y$, let $G^y = \{x : (x, y) \in G\}$. If is a function on $X \times Y$, set $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$. ■

Proposition 3.0.2. *Assume that μ and ν are σ-finite, (so $\mu \otimes \nu$ is a σ-finite). Let $G \in \mathcal{S} \otimes \mathcal{T}$. Then,*

1. *For each $x \in X$, $G_x \in \mathcal{T}$, and for each $y \in Y$, $G^y \in \mathcal{S}$.*
2. *$x \mapsto \nu(G_x)$ is $\mathcal{S}$-measurable, $y \mapsto \mu(G^y)$ is $\mathcal{T}$-measurable. In particular, $\{x : \mu(G_x) = +\infty\} \in \mathcal{S}$.*
3. *Finally,*
$$(\mu \otimes \nu)(G) = \int \nu(G_x)\, d\mu(x) = \int \mu(G^y)\, d\nu(y).$$

Proof. Let $\mathcal{S} = \{G \in \mathcal{S} \otimes \mathcal{T} : 1, 2, 3 \text{ alone hold}\}$. We want to show that $\mathcal{S} = \mathcal{S} \otimes \mathcal{T}$. [WTF happened here?! If $G = E \times F : E \in \mathcal{S}, F \in \mathcal{T}$, then $G_x = \{y : (x, y) \in E \times F\}$ and $\chi_{G_x}(y) = \chi_E(x)\chi_F(g)$. Also, $\underbrace{x \mapsto \nu(G_x)}_{\mathcal{S}\text{-measurable}} = \chi_E(x)\nu(F)$.] Suppose that we have that $G_n \in \mathcal{S}, G_n \uparrow G$. We want that $G \in \mathcal{S}$.

1. $(G_n)_x \uparrow G_x$, so $G_x \in \mathcal{T}, G_n^y \uparrow G$, so $G^y \in \mathcal{S}$.
2. $\nu((G_n)_x) \uparrow \nu(G_x)$, so $(X \mapsto \nu((G_n)_x)) \uparrow (x \mapsto \nu(G_x))$.
3. Note that
$$(\mu \otimes \nu)(G_n) = \int \nu((G_n)_x)\, d\mu(x) \uparrow_{\text{MCT}} \int \nu(G_x)\, d\mu(x),$$
since $G_n \uparrow G, (\mu \otimes \nu)(G_n) \uparrow (\mu \otimes \nu)(G)$.

Next step, let $G \subseteq E \times F, \mu(E) < \infty, \nu(F) < \infty$, and suppose that $\{G_n\} \subseteq \mathcal{S}, G_n \downarrow G$. Then, we claim that $G \in \mathcal{S}$.

1. $(G_n)_x \in \mathcal{T}, (G_n)_x \downarrow G_x$, so $G_x \in \mathcal{T}$. By the same argument, we also have that $G^y \in \mathcal{T}$.
2. Again, by the same argument, we have that $(x \mapsto \nu((G_n)_x) \downarrow \nu(G_x)$, so $x \mapsto \nu(G_x)$ is $\mathcal{S}$-measurable.
3. Because everything is in $E \times F$, of finite measure, then we have that
$$(\mu \otimes \nu)(G_n) = \int \nu((G_n)_x)\, d\mu(x) \downarrow \int \nu(G_x)\, d\mu(x) \downarrow (\mu \otimes \nu)(G).$$

Look only in $E \times F$, so in effect, assume that $\mu(\underbrace{X}_{E}) < \infty, \nu(\underbrace{Y}_{F}) < \infty$. Then, $\mathcal{S}$ is closed under increasing unions and decreasing intersections. We now claim that $\mathcal{S}$ is a σ-ring. ■

Definition 3.0.2. Let M be a collection of subsets of a set X. If M is closed under countable increasing unions and countable decreasing intersection, it is called a "monotone class" of sets.

Note 3.0.1. Any collection C of subsets of X is contained in a smallest monotone class, namely the intersection of all monotone classes containing C. Call it the monotone class generated by C.

Lemma 3.0.2. *Let R be a ring of sets in X. Let $M(R)$ be the monotone class generated by R. Then $M(R) = \mathcal{S}(R) \leftarrow$ the σ-ring generated by R.*

Proof. $\mathcal{S}(R)$ is a monotone class, so $M(R) \subseteq \mathcal{S}(R)$. First show that $M(R)$ is a ring. Let $E \in M(R)$, and see $L(E) = \{F \in M(R) : E \setminus F, F \setminus E, E \cap F \in M(R)\}$. Then, $L(E)$ is a monotone class. Because, if $\{F_n\} \subseteq L(E), F_n \uparrow F \in M(R)$. Then, $\underbrace{E \setminus F_n}_{\in M(R)} \downarrow \underbrace{E \setminus F}_{\in M(R)}, F_n \backslash E \uparrow F \backslash E, E \cap F_n \downarrow E \cap F$. Thus, $F \in L(E)$. Similarly, if $\{F_n\} \subset L(E)$, and $F_n \downarrow F$, so $F \in L(E)$.

Note 3.0.2. If $F \in L(E)$, then $E \in L(F)$.

Let $A \in R$, let $E \in L(A)$. Then, $A \in L(E)$, so $R \subset L(E)$, so $L(E) = M(R)$.
$L(A)$
$R \subseteq M(R)$. Then for any $B \in R, B \in L(A)$, so $R \subseteq L(A) \subseteq M(R)$, so $L(A) = M(R)$. Hence, by all of the above, we see that $M(R)$ is a ring. Finally, if $\{E_n\} \subseteq M(R)$ and let

$$E = \bigcup E_n,$$

then

$$\underbrace{\bigcup^{k} E_n}_{\in M(R)} \uparrow E,$$

so $E \in M(R)$, so $M(R)$ is a σ-ring, so $= \mathcal{S}(R)$. ∎

$(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu), \mu \otimes \nu, \mathcal{S} = \{$good subsets of $\mathcal{S} \otimes \mathcal{T}\}$. If $G \in \mathcal{S} \otimes \mathcal{T}, G \subseteq E \times F, \mu(E) < \infty, \nu(F) < \infty \implies G \in \mathcal{S}$. For the general case, note that $G \in \mathcal{S} \otimes \mathcal{T}, G \subseteq E \times F$, with $E, F \sigma$-finite.

$$E = \bigcup^{\infty} E_n, F = \bigcup^{\infty} F_n, \mu(E_n) < \infty, \mu(F_n) < \infty.$$

Let

$$E^m = \bigcup^m E_n, F^m = \bigcup^m F^m, E^m \times F^m \uparrow E \times F.$$

$G_m = G \cap E^m \times F^m$, so $G_m \in \mathcal{S}, G_m \uparrow G$.

Lemma 3.0.3. *If* $\{G_m\} \in \mathcal{S}, G_m \uparrow G$, *then* $G \in \mathcal{S}$.

Proof. For $x \in X, (G_m)_x \uparrow G_x, G^y_m \uparrow G^y$. $x \to \nu((G_m)_x) \uparrow \nu(G_x), \mu(G^y_m) \uparrow \mu(G^y)$, so $x \to \nu(G_x)$ is $\mathcal{S}$-measurable, $y \to \mu(G^y)$ is $\mathcal{T}$-measurable. Next, by the Monotone Convergence Theorem,

$$\int \nu(G_x)\, d\mu(x) = \lim \int \nu((G_m)_x)\, d\mu(x) = \lim(\mu \otimes \nu)(G_m) = (\mu \otimes \nu)(G).$$

$$\int \left(\int \chi_G(x,y)\, d\mu(x) \right) d\nu(g) = \int \mu(G^y)\, d\nu(x) = (\mu \otimes \nu)(G).$$

Thus, $G \in \mathcal{S}$. ∎

Let B be a Banach space, and let $G \in \mathcal{S} \otimes \mathcal{T}$. Assume that $(\mu \otimes \nu)(G) < \infty$. Let $f = b\chi_G$, i.e. $f(x,y) = b\chi_G(x,y)$. We then have that $f_x = b\chi_{G_x}$ is $\mathcal{T}$-measurable. Then f^g is $\mathcal{S}$-measurable. Then,

$$\int f_x(y)\, d\nu(x) = b\nu(G_x),$$

f_x is ν-integrable for a.e. x, undefined in a null set, i.e. a set when $\nu(G_x) = \infty$. Then,

$$x \mapsto \int f_x(y)\, d\nu$$

is $\mathcal{S}$-measurable, and integrable and

$$\int \left(\int f_x(y)\, d\nu(y) \right)\, d\mu(x) = b(\mu \otimes \nu)(G) = \int f\, d(\mu \otimes \nu).$$

f on $X \times Y$, $f_x = f(x,y) = f^y(x)$. Same for

$$\int \left(\int f^y(x)\, d\mu(x) \right)\, d\nu(y) = \int f\, d(\mu \otimes \nu).$$

If f is $\mu \otimes \nu$ ISF, B-valued, then $x \mapsto f^y(x)$ is $\mathcal{S}$-measurable, $y \mapsto f_x(y)$ is $\mathcal{T}$-measurable, and f^y is μ-integrable a.e. f^y is μ-integrable a.e.

$$y \mapsto \int f^y(x)\, d\mu(x)$$

is $\mathcal{T}$-measurable, ν-integrable a.e. ν.

$$x \mapsto \int f_x(y)\, d\nu(y)$$

is $\mathcal{S}$-measurable, μ-measurable, μ.

$$\int \left(\int f_x \, d\nu \right) \, d\mu = \int f d(\mu \otimes \nu) = \int \left(\int f^y \, d\mu \right) d\nu.$$

Proposition 3.0.3. *Let f be $\mathcal{S} \otimes \mathcal{T}$-measurable, $\mathbb{R}$-valued, $f \geq 0$. Then, there exists $\{f_n\}$ of $\mathcal{S} \otimes \mathcal{T}$-measurable simple functions (MSF), $f_n \geq 0, f_n \uparrow f$ pointwise.*

Proof. Then $(f_n)_x \uparrow f_x, f_n^y \uparrow f^y$, so f_x is $\mathcal{T}$-measurable, f^y is $\mathcal{S}$-measurable. Then, by the Monotone Convergence Theorem,

$$\int (f_n)_x \, d\nu \uparrow \int f_x \, d\nu, \int f_n^y \, d\mu \uparrow \int f^y \, d\mu,$$

so

$$x \mapsto \int f_x \, d\nu$$

is $\mathcal{S}$-measurable,

$$y \mapsto \int f^y \, d\mu$$

is $\mathcal{T}$-measurable. Then, again by the Monotone Convergence Theorem,

$$\int \left(\int f_x \, d\nu \right) \, d\mu(x) = \lim \int \left(\int (f_n)_x \, d\nu \right) \, d\mu = \lim \int f_n \, d(\mu \otimes \nu) = \int f \, d(\mu \otimes \nu),$$

where the last equality again follows from the Monotone Convergence Theorem. ■

If f is $\mu \otimes \nu$ integrable, so

$$\int f \, d(\mu \otimes \nu) < \infty.$$

We then have that

$$x \mapsto \int f_x(y) \, d\nu(y)$$

is finite a.e. and

$$\int \left(\int f_x(y) \, d\nu(y) \right) \, d\mu(x) = \int f \, d(mu \otimes \nu)$$

and

$$\int \left(\int f^y(x) \, d\mu(x) \right) \, d\nu(y) = \int f \, d(\mu \otimes \nu).$$

Theorem 3.0.4. *(Tonelli's Theorem)*

Proof. ... picture 1 ■

Theorem 3.0.5. *(Fubini's Theorem) If $f \in \mathscr{L}^1(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu, B)$, $\mu, \nu\sigma$-finite. Then, $x \mapsto f^y(x)$ is integrable a.e. μ, $y \mapsto f_x(y)$ is ν-integrable a.e. and*

$$y \to \int f^y(x)\, d\mu(x)$$

is ν-integrable,

$$x \to \int f_x(y)\, d\nu(y)$$

is μ-integrable a.e. and

$$\int f(x,y)\, d\mu(x,y) = \int \left(\int f(x,y)\, d\mu(x) \right)\, d\mu(y) = \int \left(\int f(x,y)\, d\nu(y) \right)\, d\mu(x).$$

Proof. Let $\{f_n\}$ be the sequence of ISF converging to f, $\|f_n(x,y)\| \le 2g(x,y)$, g-integrable e.g. $g(x,y) = \|f(x,y)\|$. $f_n^y \to f^y$, dominated by $2g^y$, so f^y is integrable whenever $2g^y$ is integrable, so off of a null set, so

$$\int f_n^y(x)\, d\mu \xrightarrow{\text{LDCT}} \int f^y(x)\, d\mu(x), \int (f_n)_x\, d\nu(y) \to \int f_x(y)\, d\nu(y).$$

... picture 2 ■

Note 3.0.3. Probability. $(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu)$. If $f \in \mathscr{L}^1(X, \mathcal{S}, \mu), g \in \mathscr{L}^1(Y, \mathcal{T}, \nu)$. $\mu(X) = 1, \nu(Y) = 1$. If we have a bunch of $(X_n, \mathcal{S}_n, \mu_n), n = 1, 2, \ldots, N$. Then, $X = X_1 \times \ldots \times X_N, \mathcal{S} = \mathcal{S}1 \otimes \mathcal{S}_2 \ldots \mathcal{S}_n, \mu = \mu_1 \otimes \mu_2 \otimes \ldots \mu_N$, then $E_1 \times E_2 \times \ldots E_n, E_j \in \mathcal{S}_j$.

f is B-valued, $\mathcal{S} \otimes \mathcal{T}$-measurable. For f to be integrable for $\mu \otimes \nu$, it suffices to show that $(x,y) = \|f(x,y)\|$ is integrable. The actual statement: if $x \to g^y(x) = \|f(x,y)\|$ is integrable a.e. x and if y

Note 3.0.4. The final exam is on Friday, December 20th from 8 - 11 AM in 160 Kroeber. Professor Rieffel will have the same office hours next week.

Note 3.0.5. For any $F \in \mathcal{S}(\mathcal{P})$ and any ϵ, there us $E \in \mathcal{P}$, with

$$\mu((E \setminus F)\nu(F \setminus E)) < \epsilon.$$

$\implies$

$\implies$ Every ISF can be approx ...Thus, $C_C(\mathbb{R})$ are dense in $L^p(\mathbb{R})$, for $1 \leq p < \infty$.

Chapter 4

Integral Operators

Let K be a $m \times n$ matrix. Then K determines a linear operator, T_K, from $\mathbb{R}^n \to \mathbb{R}^m$.

$$(T_K \xi)_{j, \xi \in \mathbb{R}} = \sum_{k=1}^{n} K_{jk} \xi_k.$$

Now, given $(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$, given $K \in M(X \times Y, \mathcal{S} \otimes \mathcal{T})$ can try to form an operator T_K, defined from $M(Y)$ to $M(X)$ by

$$(T_K)\xi)(x) = \int K(x, y)\xi(y) \; d\nu(y).$$

Given $p, 1 \leq p < \infty$. Suppose that $K \in L^p(X \times Y, \ldots)$ i.e. $(x, y) \mapsto |K(x, y)|^p \in L^1(X \times Y, \mu \otimes \nu)$. Then, Fubini tells us that for a.e. μ x,

$$y \to |K(x, y)|^p \in L^1(Y).$$

Thus,

$$y \mapsto |K(x, y)| \in L^p,$$

so

$$(y \mapsto K(x, y)) \in L^p.$$

Then, for $\xi \in L^q(Y)$,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$y \mapsto K(x, y)\xi(y) \in L^1(Y).$$

Hence, we have that

$$\int K(x, y)\xi(y) \; d\nu(y)$$

is well-defined. By Hölder:

$$\left|\int K(x,y)\xi(y)\ dy\right| \leq \|K(x,\cdot)\|_p\|\xi\|_q.$$

Then, we see that:

$$\begin{aligned}\left|\int K(x,y)\xi(y)\ d\nu\right|^p &\leq \|K(x,\cdot)\|_p^p\|\xi\|_q^p \\ &= \int \underbrace{|K(x,y)|^p}_{\in L^1}\ dy\|\xi\|_q^p.\end{aligned}$$

Then, by Fubini's Theorem, we see that

$$\left(x \to \int |K(x,y)|^p\ dy\right) \in L^1(X),$$

so

$$x \to \int K(x,y)\ d\nu(y) \in L^p.$$

Hence, we see that

$$\begin{aligned}\left|\int K(x,y)\xi(y)\ d\nu(y)\right| &\leq \left(\int |K(x,y)||\xi(y)\ dy\right)^p \\ &\leq \int |K(x,y)|^p\|\xi\|_q^p.\end{aligned}$$

We want to show that $|T_K\xi(x)|^p \in L^1$. Then, we see that:

$$\begin{aligned}\int |T_K\xi(x)|^p\ d\mu(x) &= \int \underbrace{\left|\int K(x,y)\xi(y)\ d\nu(y)\right|^p}\ d\mu(x) \\ &\leq \int\int |K(x,y)|^p\ d\nu(y)\|\xi\|_q^p\ d\mu(x) \\ &= \left(\int |K(x,y)|^p\ d(\mu\otimes\nu)\right)\|\xi\|_q^p \\ &= \|K\|_p^p\|\xi\|_q^p.\end{aligned}$$

Hence, we see that

$$\|T_K\xi\|_p \leq \|K\|_p\|\xi\|_q.$$

If V and W are vector spaces and if $T : V \to W$ is a linear operator, we say that T is bounded if there is a constant $r > 0$ such that for all $v \in V$, $\|Tv\|_W \leq r\|v\|_V$. The smallest constant r is called the norm of T, $\|T\|$,

$$\begin{aligned}\|T\| &= \sup\{\|T(v)\| : \|v\| \leq 1\} \\ &= \sup\left\{\frac{\|Tv\|}{\|v\|} : \text{ for all } v \neq 0\right\}.\end{aligned}$$

If T is bounded, then for $v_1, v_2 \in V$, we can see that:

$$\begin{aligned}\|Tv_1 - Tv_2\|_W &= \|T(v_1 - v_2)\| \\ &\leq \|T\| \cdot \|v_1 - v_2\|_V,\end{aligned}$$

so T is Lipschitz from V to W. Hence, we see that T is uniformly continuous. To show that a linear operator is continuous, it suffices to show continuity at 0 by Homework 3. If it is continuous at 0, given Ball$(0_W, 1)$, there is Ball$(0_V, r)$, such that if $v \in$ Ball$(0_V, r)$, then $Tv \in$ Ball$(0_W, 1)$. Equivalently, if $\|v\| < r$, then $\|Tv\| \leq 1$, or if $\|v\| \leq 1$, then $\|T(v)\| \leq \frac{1}{r}$.

Example 4.0.1. (Unbouned Operator) $L^1([0,1]) \supset C^\infty([0,1]), Tf = f' = \dfrac{df}{dt}$.

Example 4.0.2. Most important, L^2, $K \in L^2(X \times Y)$,

$$T_K : L^2(Y) \to L^2(X).$$

If

$$X = Y,$$

then we have that T_K is a Hilbert-Schmidt operator.

$L^p(\mathbb{R}, \mu)$. For $t \in \mathbb{R}$, define $\mathcal{U}_t$ on $L^p(\mathbb{R})$ by

$$\begin{aligned}(T_t\xi)(s) &= \xi(s-t), \\ \|T_t\xi\|_p^p &= \int |f(s-t)| \, ds \\ &= \int |f(s)|^p \, ds \\ &= \|\xi\|_p^p.\end{aligned}$$

Thus, we see that $\|T_t\xi\|_p = \|\xi\|_p$, i.e. T_t is an isometry. If $T_tT_r = T_{t+r}$, so $t \to T_t$ is a group homomorphism from $\mathbb{R}$ to the group of isometries of L^p. For given ξ, we see that $t \to T_t\xi$ is continuous.

Proof. Check for $\xi \in C_C(\mathbb{R})$, $t_n \to t_0$, then $T_{t_n}\xi \to T_{t_0}\xi$ in norm $\|\cdot\|_\infty$. Now, $C_C(\mathbb{R})$ is dense in L^p. Given a group G, a homomorphism α of $G \to \mathrm{Aut}(V)$, where V is a vector space, we say that α is a representation of G. ■

Note 4.0.1. Final Exam, December 20th from 8-11 in 160 Kroeber Hall.

Example 4.0.3. Let $(X, \mathcal{S}, \mu)$ be a measure space and $1 \leq p < \infty$, $L^p(X, \mathcal{S}, \mu)$, where

$$(\mathbb{R}, \mathcal{S}, \mu)$$

with μ being the Lebesgue measure, for $t \in \mathbb{R}$ we have the linear operator T_t, with

$$(T_t\xi)(s) = \xi(s-t),$$

$t \to T_t$ is strong operator continuous, i.e. for $\xi \in L^p(\mathbb{R}), t \mapsto T_t\xi \in L^p$ is continuous.

Note 4.0.2. Let G be the identity group with the counting measure, $\ell^p(G)$. For $x \in G$,

$$(T_x\xi)(y) = \xi(x^{-1}y).$$

Example 4.0.4. Let B be a Banach space, and let $t \mapsto T_t \in \mathcal{B}(B)$, $\|T_t b\| = \|b\|$, where T_t is an isometry, and assume that $t \mapsto T_t$ is strong operator continuous. Then, let $f \in L^1(\mathbb{R})$. For $b \in B$, define

$$T_f b = \int_{\mathbb{R}} f(t) T_t b \, dt.$$

Then, we see that:

$$\begin{aligned} \|f(t)T_t b\| &= |f(t)| \cdot \|T_t b\| \\ &= |f(t)| \cdot \|b\| \in L^1(\mathbb{R}, \ldots, B). \end{aligned}$$

Hence, it must be the case that $T_t b$ is well-defined. Next, we note that:

$$\begin{aligned}\|T_f b\| &= \left\| \int f(t) T_t b \; dt \right\| \\ &\leq \int |f(t)| \cdot \frac{\|T_t b\|}{\|b\|} \; dt \\ &= \|f\|_1 \cdot \|b\|.\end{aligned}$$

So, it must be the case that $T_f \in \mathcal{B}(B), \|T_f\| \leq \|f\|_1$. Then, let $f, g \in L^1(\mathbb{R}), T_f(T_g b)$. Then,

$$\begin{aligned}T_f(T_g b) &= \int f(t) T_t \left(\int g(s) T_s b \; ds \right) \; dt \\ &= \int f(t) \left(\int g(s) T_{t+s} b \; ds \right) \; dt \\ &= \int f(t) \left(\int g(s-t) T_s b \; ds \right) \; dt.\end{aligned}$$

Further,

$$(t, s) \mapsto \|f(t) g(s-t) T_s b\|.$$

But then,

$$\begin{aligned}\int \|f(t) g(s-t) T_s b\| \; ds &= \int |f(t)| \cdot |g(s-t)| \; ds \cdot \|b\| \\ &= \int |f(t)| \cdot |g(s)| \; ds \cdot \|b\| \\ &= |f(t)| \cdot \|g\|_1 \|b\|.\end{aligned}$$

Hence,

$$\begin{aligned}\int \left(\int \|f(t) g(s-t) T_s b\| \; ds \right) \; dt &= \int |f(t)| \cdot \|g\|_1 \|b\| \; dt \\ &= \|f\|_1 \cdot \|g\|_1 \cdot \|b\|.\end{aligned}$$

Next, by Tonelli's Theorem. we see that

$$\int \|f(t) g(s-t) T_s b\| \; ds$$

is in $L^2(\mathbb{R})$. Furthermore,

$$T_f T_g b = \int \left(\int f(t) g(s-t) \; dt \right) T_s b \; ds.$$

As a result of applying Fubini's Theorem, we see that:

$$s \mapsto \int f(t) g(s-t) \; dt$$

exists a.e. and gives a function in $L^1(\mathbb{R})$.

Definition 4.0.1. Define:

$$(f * g)(s) = \int f(t)g(s-t)\,dt,$$

for $f * g \in L^1(\mathbb{R})$. This is the convolution product.

Note 4.0.3. Note that $T_f T_g = T_{f*g}$, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. This means that $L^1(G)$ into a Banach algebra. (Must check associativity, $(f*g)*h = f*(g*h)$.) We note that

$$\begin{aligned} T_{(f*g)*h} &= T_{f*g}T_h \\ &= (T_f T_g)T_h \\ &= T_f(T_g T_h) \\ &= T_f T_{g*h} \\ &= T_{f*(g*h)}. \end{aligned}$$

Example 4.0.5. Let $f \in L^1(\mathbb{R}), \xi \in L^p(\mathbb{R})$, then

$$\left(\int f(t)T_t\xi\,dt\right)(s) = \int f(t)\xi(s-t)\,dt \in L^p(\mathbb{R}).$$

We notice that $L^p(\mathbb{R})$ is a module over the ring $L^1(\mathbb{R})$.

Example 4.0.6. Let B be a Banach space. Let $B' = \{\varphi : B \to \mathbb{R}(\text{or } \mathbb{C}),\ \text{linear, continuous}\} = \mathcal{B}(B, \mathbb{R}(\mathbb{C}))$, $\|\varphi\|$, B' is a normal vector space, complete for this norm, so it is a Banach space (called the dual space).

Example 4.0.7. Let $B = L^p(X, \ldots, \mathbb{R}(\mathbb{C})), 1 < p < \infty$. Let q be

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $g \in L^q$. For $f \in L^p$, $fg \in L^1$. Hölder's Inequality, and

$$\left|\int fg\right| \leq \|f\|_p \cdot \|g\|_q.$$

Hence, if we define φ_g by

$$\varphi_g(f) = \int fg.$$

Then, we would have that:

$$|\varphi_g(f)| \leq \|f\|_p \cdot \|g\|_q,$$

so

$$\|\varphi_g\| \leq \|g\|_q.$$

Proposition 4.0.1. *Claim:* $\|\varphi_g\| = \|g\|_q$. *Let*

$$f(x) = \frac{\bar{g(x)}}{|g(x)|} \cdot |g|^{\frac{q}{p}} \in L^p.$$

Then, it must be the case that:

$$\varphi_g(f) = \int g(x) \cdot \frac{\bar{g(x)}}{|g(x)|} \cdot |g(x)|^{\frac{q}{p}} = \int |g(x)| \cdot |g(x)|^{\frac{q}{p}} = \int |g(x)|^{\frac{q}{p}+1} = \int |g(x)|^q = \|g\|_q^q.$$

We also have that:

$$\|f\|_p^p = \int \frac{|\bar{g(x)}|}{|g(x)|} \cdot |g(x)|^q \|g\|_q^q, \|f\|_p = \|g\|_q^{\frac{q}{p}}.$$

Then, we have that:

$$\|\varphi_g\| \leq \|g\|_q,$$

so

$$\|g\|_q^q = |\varphi_g(f)|^q \leq \|\varphi_g\|^q \cdot \|f\|_p^q = \|g\|_q \cdot \|g\|_q^{\frac{q}{p}} = \|g\|_q^{\frac{q}{p}+1} = \|g\|_q^q.$$

Note 4.0.4. $L^1(X, \mu)'$, $L^\infty(X, \mu, B)$ = the vector space of essentially bounded functions. Then, $f \in L^\infty$ if there exists some r such that

$$\mu(\{x : \|f(x)\| > r\}) = 0.$$

We then set

$$\|f\|_\infty = \inf\{r : \mu(\{x : \|f(x) > r\}) = 0\}.$$

If $g \in L^\infty$. Then, define φ_g on $L^1(X, \mathbb{R})$ by

$$\varphi_g(f) = \underbrace{\int f(x)g(x)\ d\mu(x)}_{\underbrace{|f(x)g(x)|}_{L^1} \leq r|f(x)|}.$$

Then, we note that:

$$(L^1(X))' = L^\infty(X)$$

and

$$\|\varphi_g\| \leq \|g\|_\infty.$$

We can then show that:

$$\|\varphi_g\| = \|g\|_\infty.$$

Note that:

$$(L^\infty(X))' \supseteq L^1(X).$$

Finally, note that the double dual is just the initial space.

Note 4.0.5. End of 202A :(