Math 202A Notes

Jacob Krantz

October 22, 2019

Contents

Chapter 1

Topology

1.1 Metric Spaces

Definition 1.1.1. Let X be a set, a metric on X is a function $d : X \times X \to \mathbb{R}$, such that

1. $d(x, x) = 0$, for all $x \in X$
2. if $d(x, y) = 0$, then $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Note that if we do not have that if $d(x, y) = 0$, then $x = y$, then we have a semimetric.

Definition 1.1.2. If $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we call the define the following norms:

1. $||v||_2 = (\sum |v_j|^2)^{\frac{1}{2}}$
2. $||v_1|| = \sum |r_j|$
3. $||v||_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$
4. $||v||_p = (\sum |v_j|^p)^{\frac{1}{p}}$

Definition 1.1.3. We can now define the following:

1. $d_2 := ||v - w||_2$
2. $d_1 := ||v - w||_1$
3. $d_\infty := ||v - w||_\infty$
4. $d_p := ||v - w||_p$

Example 1.1.1. Let (X, d) be a metric space, then let $Y \subset X$, the restriction of d to $Y \times Y \subset X \times X$ makes Y a metric space.

Example 1.1.2. $C([0, 1]) = \mathbb{R}$-valued continuous functions on $[0, 1]$.

Note 1.1.1. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. By a norm on V, we mean a function $||\cdot|| : V \to \mathbb{R}^+$ such that:

1. $||v|| = 0 \iff v = 0$
2. $||\alpha v|| = |\alpha|||v||$
3. $||v + w|| \leq ||v|| + ||w||$

Example 1.1.3. From a norm on V, we get a metric on V by $d(v, w) = ||v - w||$. For $f \in C([0, 1])$:

1. $||f||_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}$
2. $||f||_1 = \int_0^1 |f(t)| dt$
3. $||f||_\infty = \sup\{|f(t)| : t \in [0, 1]\}$
4. $||f||_p = \left(\int_0^1 |f(t)|^p dt\right)^{\frac{1}{p}}$

Definition 1.1.4. Let (X, d) be a metric space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. We say that this sequence converges to a point $x_* \in X$ if for all $\epsilon > 0$, there exists $N > 0$ such that for $n > N$, $d(x_n, x_*) < \epsilon$. [Note that this is the same as saying that $x_n \in \text{oBall}(x_*, \epsilon)$, where $\text{oBall}(x_*, \epsilon) = \{y \in X \mid d(y, x_*) < \epsilon\}$.]

Definition 1.1.5. X is complete if every Cauchy sequence converges to some point of X.

Example 1.1.4. Some examples of complete metric spaces include $\mathbb{R}, \mathbb{R}^n, \mathbb{C}$.

Note 1.1.2. If S is a closed subset of $\mathbb{R}^n$, then S with the restricted metric is complete. Consider $C([0,1]) : ||f||_\infty = \sup\{|f(t)| : t \in [0,1]\}$. The uniform norm convergence for it is uniform convergence. If $\{f_n\}$ is Cauchy for $||\cdot||_\infty$, then for each $t_* \in [0,1]$, then $\{f_n(t_*)\}$ is a Cauchy sequence, so it converges. Note that $f(t) = \lim(f_n(t))$, the uniform limit of continuous functions is continuous.

Definition 1.1.6. Let (X, d) be a metric space, and let S be a subset of X. We say that S is dense in X if every open ball in X contains a point of S.

Definition 1.1.7. Let (X, d) be a metric space, by a completion of X, we mean a metric space, $(\overline{X}, \bar{d})$, together with $j : X \to \overline{X}$ such that j is an isometry and j is dense in X.

Definition 1.1.8. An isometry is a function j such that $d(x, y) = d(j(x), j(y))$.

Example 1.1.5. Every metric space has a completion, and the completion is essentially unique. Let (X, d) be a metric space. Let CS(X, d) be the set of all Cauchy sequences in (X, d). Try to define a distance on CS(X, d): let $\{x_n\}, \{y_n\}$ be two Cauchy sequences. Consider $\{d(x_n, y_n)\}$, we claim it is Cauchy in $\mathbb{R}$. Set $\tilde{d}(\{x_n\}, \{y_n\}) = \lim\{d(x_n, y_n)\}$.

Note 1.1.3. Note that $d(x, y) \leq d(x, z) + d(z, y)$ and $d(x, y) - d(x, z) \leq d(z, y)$, so $|d(x, y) - d(x, z)| \leq d(z, y)$ and $|d(x, z) - d(y, z)| \leq d(x, y)$. Hence,

$$\begin{aligned} |d(x_n, y_n) - d(x_n, y_n)| &= |d(x_n, y_n - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \\ &\leq d(y_n, y_m) + d(x_n, x_m) \to 0 \end{aligned}$$

Now, let (X, d) be a semimetric space. We now define an equivalence relation on X, by if $d(x, y) = 0$, then $[x] = \{y : d(x, y) = 0\}$. Define $X/\sim$:= $\{$equivalence classes$\}$. Define $\hat{d}$ on $X/\sim$ by $d([x], [y]) = d(x, y)$, well-defined. If $x' \in [x]$, $y' \in [y]$, then $d(x', y') \leq d(x, x) + d(y, y) + d(x, y), d(x', y') = d(x, y)$, so $\hat{d}$ is a metric on $X/\sim$. Let $\tilde{d}$ on CS(X, d) be the corresponding metric in the equivalence classes. The equivalence relation is $\{x_n\} \sim \{y_n\}$ if $\hat{d}(\{x_n\}, \{y_n\}) = 0$ or $\lim_{n\to\infty} d(x_n, y_n) = 0$. Embed (X, d) in CS$(X, d)/\sim$ by $x \mapsto$ Cauchy sequence, $x_n = x$, for all n, $\phi(x) = \{x_n = x\}, \tilde{d}(\phi(x), \phi(y)) = \lim d(x_n, y_n) = \lim d(x, y) = d(x, y)$, so ϕ is an isometry of X into CS$(X, d) \to$ CS$(X, d)/\sim$. The image of X is dense in CS$(X, d)/\sim$. Let $\{x_n\}$ be any Cauchy sequence. Then, given any $\epsilon > 0$, there exists N such that for $m, n \geq N, d(x_m, x_n) < \epsilon$. Consider $\phi(x_N)$. Then, $d(\{x_n\}, \phi(x_n)) = \lim_{n\to\infty}\{d(x_n, x_N)\} < \epsilon$. To show that (CS$(X, d)/\sim, d)$ is complete. For small m, let $\ldots \in$ CS(X, d), assume $\{S^m\}$ is a Cauchy sequence in CS(X, d), for each k, find $x_k \in X$, such that $d(\phi(x_k), S^m) < \frac{1}{k}$, then $S = \{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, and $\tilde{d}(S^m, S)_{n\to\infty} \to 0$.

Definition 1.1.9. Let $(X, d_x), (Y, d_y)$ be metric spaces, $f : X \to Y$, and $x_0 \in X$, we say that f is continuous at x_0 if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x_0) < \delta$, then $d(f(x), f(x_0)) < \epsilon$, or equivalently, if $x \in$ Ball(x_0, δ), then $f(x) \in$ Ball$(f(x_0), \epsilon)$. For any open ball B about $f(x_0)$, there is an open ball C about$f(x_0)$ such that if $x \in B$, then $f(x) \in C$, or equivalently that $x \in f^{-1}(C)$, and $B \subseteq f^{-1}(C)$.

Definition 1.1.10. Let (X, d) be a metric space. If $A \subseteq X$ is an open subset (for d) if for each αA, there is an open ball about x contained in A.

Note 1.1.4. If f is continuous, i.e continuous at all points, let $\mathcal{O}$ be an open set in Y, let $x_0 \in f^{-1}(\mathcal{O})$, then $\mathcal{O}$ contains a ball about x_0 such that $x_0 \in C \subset f^{-1}(B)$, so $C \subseteq f^{-1}(\mathcal{O})$, so $f^{-1}(\mathcal{O})$ is open. Conversely, let f be any function from X to Y. If it is true that for any open set $\mathcal{O}$ in Y, $f^{-1}(\mathcal{O})$ is open in X, then f is continuous. Given any $\epsilon > 0$, let $\mathcal{O} = \text{Ball}(f(x_0), \epsilon)$, then $f^{-1}(\text{Ball}(f(x_0), \epsilon))$ is open. Hence, there is a ball $\text{Ball}(x_0, \delta)$ such that $\text{Ball}(x_0, \delta) \subseteq f^{-1}(\text{Ball}(f_0, \epsilon))$. The following are properties of the collection of open sets of a metric space:

1. An infinite union of open sets is open
2. A finite intersection of open sets is open. For $x_0 \in \mathcal{O}_1 \cap \mathcal{O}_2$, $\text{Ball}(x_0, r_1) \subseteq \mathcal{O}_1$, $\text{Ball}(x_0, r_2) \subseteq \mathcal{O}_2$. Let $r = \min\{r_1, r_2\}$, then $\text{Ball}(x_0, r) \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$.
3. X and $\varnothing$ are open.

Definition 1.1.11. Let X be a set. By a topology for X, we mean a collection $\mathcal{T}$ of subsets of X such that:

1. Arbitrary unions of elements of $\mathcal{T}$ are in $\mathcal{T}$.
2. Finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.
3. X and $\varnothing$ are elements of $\mathcal{T}$.

Definition 1.1.12. Let $\mathcal{T}$ be a topology of X. Then $A \subseteq X$ is closed if A' is open.

Note 1.1.5. Properties of closed sets:

1. Arbitrary intersections of closed sets are closed.
2. Finite unions of closed sets are closed.

3. X and $\varnothing$ are closed.

Definition 1.1.13. Let $A \subseteq X$. By the closure of A, we mean the smallest closed set that contains A, i.e. the intersection of all closed sets that contain A.

Definition 1.1.14. By the interior of A, we mean the biggest open set contained in A, i.e. the union of all open sets contained in A.

Definition 1.1.15. Let C be a closed set, and let $A \subseteq C$, we say that A is dense in C if $\overline{A} = C$.

Definition 1.1.16. Let X be a set, and let $\mathscr{S}$ be a collection of subsets of X, the smallest topology containing the intersection of topologies that contain $\mathscr{S}$ is said to be the topology generated by $\mathscr{S}$, and $\mathscr{S}$ is called a subbase for that topology. Note that if $\mathscr{C}$ is a collection of topologies for X, then $\bigcap\{\mathcal{T} \in \mathscr{C}\}$ is a topology for X.

Definition 1.1.17. Let X be a set, and let D be the collection of subsets of X. D is a topology for X, called the discrete topology for X. It is given by a metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

D is the biggest topology in X.

Definition 1.1.18. The smallest topology in X is $\{\varnothing, X\}$, called the indiscrete topology.

Note 1.1.6. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, such that:

$$\begin{array}{cc} \mathcal{T}_1 \subseteq & \mathcal{T}_2 \\ \text{smaller} & \text{larger} \\ \text{weaker} & \text{stronger.} \end{array}$$

Usually, we require that $\bigcup \mathscr{S} = X$. For $X = \mathbb{R}$, (a, b), $\mathscr{S} = \{(\infty, a), (b, +\infty)\}$.

Definition 1.1.19. A collection of subsets of X is a base for a topology is the set of all arbitrary unions of elements of $\mathscr{S}$ is a topology.

Example 1.1.6. $\mathscr{S} = \{(a, b) \subset \mathbb{R}\}$, $\mathbb{R}^2 = \{\text{open balls}\}$

Note 1.1.7. For $\mathscr{S}$ to be a base, it must have the property that if $A, B \in \mathscr{S}$, then $A \cap B$ must be a union of elements of $\mathscr{S}$.

Example 1.1.7. If $\mathscr{S}$ is any collection of subset of X, then the collection of all finite intersections of elements must be a topology.

Definition 1.1.20. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. f is continuous if for all open sets $\mathcal{O} \in \mathcal{T}_Y \implies f^{-1}(\mathcal{O}) \in \mathcal{T}_X$.

Note 1.1.8. Let Y be a set and $\mathscr{S} = \{A_\alpha\}$, let X be a set, and $f : X \to Y$ be a function. Then,

1. $f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$

2. $f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f(A_\alpha)$

3. If $A, b \in Y$, then $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$.

Example 1.1.8. Given $(X, \mathcal{T}_X)$ and $f : X \to Y$, let $\mathscr{S}$ be a subbase for $\mathcal{T}_Y$. Then f is continuous if $f^{-1}(A) \in \mathcal{T}_X$, for all $A \in \mathscr{S}$.

Example 1.1.9. Let X be a set and let $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let there be a quasifunction $f_\alpha : X_\alpha \to X$. Let $\mathcal{T}$ be the strongest topology such that all of the f_α's are continuous. Given α_0, f_α. If $A \subseteq X$, then if A is to be open, we must have that $\overline{f}_{\alpha_0}(A) \in \mathcal{T}_{\alpha_0}$. Now, let $\mathscr{S}_{\alpha_0} = \{A \subseteq : f_{\alpha_0}^{-1}(A) \in \mathcal{T}_{\alpha_0}\}$ is a topology for X; in fact, it is the strongest topology making f_{α_0} continuous. The strongest topology making all of the f_α continuous is the intersection of the $\mathscr{S}_\alpha$.

Example 1.1.10. Let $(X, \mathcal{T})$ be a topological space, let Y be a set. Then, $f : X \to Y$, $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\}$ is the strongest topology making f continuous. Usually, we want f to be onto Y.

Definition 1.1.21. We begin by defining an equivalence relation, $\sim$, on X by $x_1 \sim x_2$, if $f(x_1) = f(x_2)$. This gives a partition of X: the quotient of ${}^{X}\!/_{\sim}$, the quotient of X by $\sim$. This topology is called the quotient topology determined by f.

Definition 1.1.22. For $\sim$ on a set X, $B \subseteq X$ is saturated if when $x \in B$ and $x_1 \sim x$, for $x_1 \in B$.

Note 1.1.9. The open sets in the quotient topology in f on Y are in bijection with the saturated open sets of X.

Note 1.1.10. We want the weakest topology to make all of the functions ot be continuous. For any B_α, any open set $\mathcal{O} \in \mathcal{T}_\alpha$ (where the topological space is $(Y_\alpha, \mathcal{T}_\alpha)$, we need $f_\alpha^{-1}(0) \subseteq X$. This weakest topology has a sub-base $\{f_\alpha^{-1}(0) : \mathcal{O} \in \mathcal{T}_\alpha\}$, which is called the conditional topology.

Example 1.1.11.

1. Given $(Y, \mathcal{T})$, let X be a subset of Y. $X \hookrightarrow^i Y$. The weakest topology making i continuous is $\{i^{-1}(\mathcal{O})\ \mathcal{O} \in \mathcal{T}\}$. $i^{-1}(0)$ can form the relative topology, $\{X \cap \mathcal{O} : \mathcal{O} \in \mathcal{T}_Y\}$.

2. Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be given. We can form the product topology, $X_1 \times X_2$, whose sub-base is $\mathcal{O} \times X_2$, $\mathcal{O} \in \mathcal{T}_1$, $X_1 \times \mathcal{U}, \mathcal{U} \in \mathcal{T}_2$, intersected: $\{\mathcal{O} \times \mathcal{U} : \mathcal{O} \in \mathcal{T}_1, \mathcal{U} \in \mathcal{T}_2\}$ is a sub-base. Furthermore, $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$. Then, form $\Pi_{\alpha \in A} X_\alpha$, functions f from A into $\cup X_\alpha$ such that $f(\alpha) \in X_\alpha$ used for all α. X_α is called the product topology, sub-base, π_α, for $\mathcal{O} \in \mathcal{T}_\alpha$, $X_1 \times \ldots \times \mathcal{O} \times \ldots$. We can only take finite intersections, so there can only be finitely many open sets.

3. $C([0,1]), ||\cdot||$. For each $h \in C([0,1])$, define linear functional, ϕ_n on $C([0,1])$ by

$$\phi_n(f) = \int_0^1 f(t)h(t)dt, C([0,1]) \to_{\phi_n} \mathbb{R}.$$

We can then ask for the correspondingly weakest topology.

$$|\phi_n| \leq ||h||_\infty ||f||_1,$$

where we chose h bounded.

Example 1.1.12. Special properties of topologies from metric spaces. If $x, y \in X$ and $x \neq y$, let $r = d(x,y) \neq 0$. Then, oBall$(x, \frac{r}{3})$ and oBall$(y, \frac{r}{3})$ are disjoint.

Definition 1.1.23. A topology $\mathcal{T}$ on X is Hausdorff is for any points $x, y, x \neq y$, there are open sets, $\mathcal{O}_x$ and $\mathcal{O}_y$, $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y$, and $\mathcal{O}_x \cap \mathcal{O}_y = \varnothing$.

Definition 1.1.24. The Separation Axioms:

1. T_2: Hausdorff
2. T_1: Given $x, y, x \neq y$, there exists $\mathcal{O}_x$ with $x \in \mathcal{O}_x$, $y \notin \mathcal{O}_x$ and there exists a similar $\mathcal{O}_y$.
3. T_0: Given $x, y, x \neq y$, there exists $\mathcal{O}$ such that only one of x or y is in $\mathcal{O}$.

Definition 1.1.25. A topology $\mathcal{T}$ is normal if for any two disjoint closed sets, A, B, there are disjoint open sets $\mathcal{O}_A$, $\mathcal{O}_B$, such that $A \subseteq \mathcal{O}_A, B \subseteq \mathcal{O}_B$.

Theorem 1.1.1. *Any topology that comes from a metric is normal.*

Proof. Let A, B be disjoint closed sets in (X, d). For each $x \in A$, B is closed so $x \notin B$. Can choose ϵ_x such that

$$\text{oBall}(x, \epsilon_x) \cap B = \varnothing.$$

Then, for each $y \in B$, we can choose ϵ_y such that oBall$(y, \epsilon_y) \cap A = \varnothing$.

$$\mathcal{O}_A = \bigcup_{x \in A} \text{oBall}\left(x, \frac{\epsilon_x}{3}\right), \mathcal{O}_B = \bigcup_{x \in B} \text{oBall}\left(y, \frac{\epsilon_y}{3}\right).$$

Note that $\mathcal{O}_A \cap \mathcal{O}_B = \varnothing$, as if $z \in \mathcal{O}_A \cap \mathcal{O}_B$, then there exists an $x \in A$, such that $z \in$ oBall $\left(x, \frac{\epsilon_x}{3}\right)$ and there exists $y \in B$, such that $z \in$ oBall $\left(y, \frac{\epsilon_y}{3}\right)$. Hence, $d(x, y) \leq \frac{\epsilon_x + \epsilon_y}{3}$. So, if $\epsilon = \max\{\epsilon_x, \epsilon_y\}$, this is bounded by $\frac{2\epsilon}{3}$. ∎

Theorem 1.1.2. *(Urysohn's Lemma) Let $(X, \mathcal{T})$ be a normal topological space and if A, B are disjoint, closed sets in X, there exists a continuous map,*

$$f : X \to [0, 1] \subset \mathbb{R},$$

such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.

Proof. If $(X, \mathcal{T})$ is such that for every closed A, B which are disjoint, we have f, for $\mathcal{T}$ normal: If A, B are disjoint, $f : X \to [0,1]$, $f|_A = 0, f|_B = 1$, set $\mathcal{O}_A = \left\{x : f(x) < \frac{1}{3}\right\}, \mathcal{O}_B = \left\{x : f(x) > \frac{2}{3}\right\}$. Now, let $\mathcal{O}_A = \left\{x : f(x) > \frac{1}{3}\right\} \cap \mathcal{O}_B$.

Lemma 1.1.3. *If $(X, \mathcal{T})$ is normal, and if A is closed, $\mathcal{O}$ is open, $A \subseteq \mathcal{O}$, then there is an open set $\mathcal{U}$, such that $A \subseteq \mathcal{U} \subseteq \bar{\mathcal{U}}$.*

Proof. Note that $\mathcal{O}^C$ is closed, by definition, so, by normalily, there are open sets $\mathcal{U}, \mathcal{V}$, such that $A \subseteq \mathcal{U}$ and $\mathcal{O}^C \subseteq \mathcal{V}, \mathcal{V}^C \subseteq \mathcal{O}$. Then,

$$\mathcal{U} \subseteq \mathcal{V}^C \subseteq \mathcal{O}, \text{so } A \subseteq \mathcal{U} \subseteq \bar{\mathcal{U}} \subseteq \mathcal{V}^C \subseteq \varnothing.$$

■

(Part) Given $(X, \mathcal{T})$ normal, A, B closed, disjoint, choose $\mathcal{O}_{\frac{1}{2}}$ such that $A \subseteq \mathcal{O}_{\frac{1}{2}} \subseteq \bar{\mathcal{O}}_{\frac{1}{2}} \subseteq B^C$. Then, choose $\mathcal{O}_{\frac{1}{4}}, \mathcal{O}_{\frac{3}{4}}$, such that

$$A \subseteq \mathcal{O}_{\frac{1}{4}} \subseteq \bar{\mathcal{O}}_{\frac{1}{4}} \subseteq \mathcal{O}_{\frac{1}{2}} \subseteq \bar{\mathcal{O}}_{\frac{1}{2}} \subseteq \mathcal{O}_{\frac{3}{4}} \subseteq \bar{\mathcal{O}}_{\frac{3}{4}} \subseteq B^C.$$

Then, choose $\mathcal{O}_{\frac{1}{8}}, \mathcal{O}_{\frac{3}{8}}, \mathcal{O}_{\frac{5}{8}}, \mathcal{O}_{\frac{7}{8}}$, such that ... Now, set $\mathcal{O}_1 = X$. Get a countable base subset, $\mathcal{O}_2$ of $[0,1]$, such that $0 \notin \mathcal{O}_2$, $1 \in \mathcal{O}_2$, and for each number $r \in \mathcal{O}_2$, we have an open set $\mathcal{O}_r$ such that if $r < s$, $\bar{\mathcal{O}}_r \subseteq \mathcal{O}_s$. Now, define the function $f(t)_{t \in [0,1]} := \inf\{r : r \in \mathcal{O}_r\}$. ■

Lemma 1.1.4. *Let $\mathbb{Q}$ be a countable dense subset of $[0,1]$, $0 \notin \mathbb{Q}$, $1 \in \mathbb{Q}$. $(X, \mathcal{T})$ is a normal topological space. Assume that for each $r \in \mathbb{Q}$, we have an open set $\mathcal{O}_r$, which satisfies if $r < s$, then $\mathcal{O}_r \subseteq \bar{\mathcal{O}}_r \subseteq \mathcal{O}s$ and $\mathcal{O}_1 = X$.*

Think of $\mathcal{O}_r$ as the set of x where $f(x) < r$, for $r \in BQ$. Set $f(x) = \inf\{r \in \mathbb{Q} : x \in \mathcal{O}_r\}$. We claim that f is continuous. Use the sub-base $(-\infty, a), (a, \infty)$. If $x \in f^{-1}((-\infty, a))$ iff $f(x) < a$, so there is $s \in \mathbb{Q}$ such that $s < a$, such that $x \in \mathcal{O}_s$. Then, for all $y \in \mathcal{O}_s$, $f(y) \leq s < a$, so $\mathcal{O}_s \subseteq f((-\infty, a))$. Thus, $f^{-1}((-\infty, a)) = \cup_{r<a} \mathcal{O}_r$ open. Then, $x \in f^{-1}((a, \infty))$ iff $f(x) > a$, so there is $s \in \mathbb{Q}$, $a < s < f(x)$ with $x \notin \mathcal{O}_s$, so there is a t such that $a < t < s < f(x)$ with $x \notin \bar{\mathcal{O}}_t \subset \mathcal{O}_s$, so $x \in \bar{\mathcal{O}}_t^{\,C}$ is open, so $f^{-1}((a, \infty)) = \cup_{t>a} \bar{\mathcal{O}}_t^{\,C}$ is open.

$(X, \mathcal{T})$ is normal, A, B be closed, disjoint sets. Choose a dense $\mathcal{O} \subset [0,1], 0 \notin \mathcal{O}, 1 \in \mathcal{O}$, such that $A \subseteq \mathcal{O}_r$, for all r. Then, $\mathcal{O}_1 \cap B = \varnothing$ because that $B \subseteq \mathcal{O}_1$. Then, note that:

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B. \end{cases}$$

Definition 1.1.26. Let X be a set, and let (M, d) be a complete metric space, and consider $f : X \to M$. We say that f is bounded if there is a $m_0 \in M, r \in \mathbb{R}^+$, such that $f(x) \in \text{Ball}(m_0, r)$, for all $x \in X$. For f, g bounded functions $X \to M$, $\{d(f(x), g(x))\}_{x \in X}$ is a bounded set in $\mathbb{R}$. Set $d_\infty(f, g) = \sup\{d(f(x), g(x)), x \in X\} \approx ||f - g||_\infty$. It is easy to show that d_∞ is a metric.

Let $B(X, (M, d))$ be the set of all bounded functions from X to M, with metric d_∞.

Proposition 1.1.1. *$B(X, (M, d))$ is complete for d_∞ (because (M, d) is complete).*

Proof. Let $\{f_n\}$ be a Cauchy sequence for d_∞. Then, for any $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence because $d(f_n(x), f_m(x)) \leq d_\infty(f_n, f_m)$. Call this limit $f(x)$. It is easy to show that f is bounded. To show that $\{f_n\}$ converges to f for d_∞, let $\epsilon > 0$ be given, and choose N_0, such that for $n, m \geq N_0$, we have $d_\infty(f_m, f_n) < \dfrac{\epsilon}{2}$. Thus, given any $x \in X$, there is $N_x > N_0$ such that for $n, m \geq N_x$, $d(f_n(x), f(x)) < \dfrac{\epsilon}{2}$. Then, for $n > N_0$, $d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \epsilon$, so $d(f_n, f) < \epsilon$. ■

Proposition 1.1.2. *Let $(X, \mathcal{T})$ be a topological space, (M, d) be a complete metric space. Let $BC((X, \mathcal{T}), (M, d))$ be the set of bounded, continuous functions from X to M. Then, $BC((X, \mathcal{T}))$ is a closed subset of $(B(X, (M, d)), d_\infty)$ and is therefore complete.*

Proof. Let $\{f_n\}$ be a sequence in $CB(X, M)$ that converges for d_∞ to $f \in B(X, M)$, to show $f \in CB(X, M)$, to show continuous at any given $x \in X$, let $\epsilon > 0$ be given. Choose N such that for $ngeqN$, $d_{infty}(f, f_n) < \dfrac{\epsilon}{3}$, such that f_n is continuous on X, there exists $\mathcal{O} \subset J$, such that $x \in \mathcal{O}$ and $d(f_n(y), f_n(x)) < \dfrac{\epsilon}{3}$. Then, for $y \in \mathcal{O}$, $d(f(y), f(x)) \leq d(f(y), f_n(y)) + d(f_n(y), f_n(x)) + d(f_n(x), f(x)) < \epsilon$. ■

Theorem 1.1.5. *Tietze Extension Theorem. Let $(X, \mathcal{T})$ be a normal topological space, and let $A \to \mathbb{R}$ be continuous. Then there is $\tilde{f} : X \to \mathbb{R}$, continuous that extends f, if $\tilde{f}|_A = f$. If $f : A \to [a, b], a, b \in \mathbb{R}$ then can arrange that $\tilde{f} : X \to [a, b]$.*

Proof. [Note that if $A \subseteq X$ is closed and if $B \subseteq A$ is closed in the relataive topology, then B is closed in X, $A \setminus B = A \cap O, O \in \mathcal{T}$, then $B = A \cap O'$, where A and O' are closed, as B is closed in X] Now, consider the first case of $f : A \to [0,1]$. Let $C_0 = \{x \in A : f(x) \leq \frac{1}{3}\}, C_1 = \{x \in A : f(x) \geq \frac{2}{3}\}$, closed in A. Then, by Urysohn's Lemma, $\exists k : X \to [0,1]$ with $k|_{C_0} = 0$, $k|_{C_1} = 1$. Let $g_1 = \frac{1}{3}k$, so $g_1 : X \to [0, \frac{1}{3}]$, $f - g_1|_A : A \to [0, \frac{2}{3}]$. Scale (?): If $h : A \to [o, r]$, then there exists g on X with $g : X \to [\frac{1}{3}r], h - g|_A A \to [0, \frac{2}{3}r]$. Apply this to $f - g_1|_A$, $r = \frac{2}{3}$. Thus there is $g_2 : X \to [0, \frac{1}{3}\frac{2}{3}], (f - g_1|A) - g_2|_A : X \to [0, \left(\frac{2}{3}\right)^2]$. Apply to $f - g_1|_A - g_2|_A, r = \left(\frac{2}{3}\right)^2$. So there is $g_3 : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^2], f - g_1|_A - g_2|_A - g_3|_A : X \to [0, \left(\frac{2}{3}\right)^3]$. Continue this for the nth case. Clearly we have that $g_n : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}]$, $f - \sum_{j=1}^{n} g_j|_A : X \to [0, \left(\frac{2}{3}\right)^n] \implies ||g_n||_\infty \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, define $\tilde{f} = \sum_{j=1}^{\infty} g_j$ cont, $||f - \sum^n g_j|A|| \leq \left(\frac{2}{3}\right)^n$. Hence, $\tilde{f}|_A = f$, $0 \leq g_n(x) \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, so $\sum_{j=1}^{\infty} g_j(x) \leq \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^{j-1} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{2}{3}\right)^j = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1$. If $f : A \to \mathbb{R}$, unbounded, then arctan $\mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2})$ is a homeomorphism. Let h be the arctan of $f : A \to (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [\frac{-\pi}{2}, \frac{\pi}{2}]$, as there is an equation $\tilde{h} : X \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ with $\tilde{h}|_A = h$. Let $B = \{\frac{-\pi}{2}, \frac{\pi}{2}\}$, a closed subset of $[\frac{\pi}{2}, \frac{\pi}{2}]$. Then take $B = \{\tilde{h}^{-1}(\frac{-\pi}{2}), \tilde{h}^{-1}(\frac{\pi}{2})\} \subseteq X, A \subseteq X$... ∎

Definition 1.1.27. Let X be a set, $\mathcal{C}$ a collection of subsets of X. We say that $\mathcal{C}$ is a covering of X if

$$\bigcup\{A \in \mathcal{C}\} = X$$

. If $B \subseteq X$, $\mathcal{C}$ is a collection of subsets of X, we say that $\mathcal{C}$ covers $\mathcal{B}$ if $\mathcal{B} \subseteq \bigcup\{A \in \mathcal{C}\}$. If $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ is a subcover of $\mathcal{C}$ if $\mathcal{D}$ also is a c.

Definition 1.1.28. Let $(X, \mathcal{T})$ be a topological space. We say that it is compact if every open cover of X has a finite subcover.

Theorem 1.1.6. *If $(X, \mathcal{T})$ is compact and $A \subseteq X$, then the following are equivalent.*

1. *A is compact for the relative topology*
2. *If $\mathcal{C} \subseteq \mathcal{T}$ is a cover of A, then A has a finite subcover of $\mathcal{O}$.*

Proof. The open sets for the relative topology are of the form $A \cap \mathcal{O}$, $\mathcal{O} \in \mathcal{T}$. ∎

Theorem 1.1.7. *If $(X, \mathcal{T})$ is compact and $A \subseteq X$ is closed then A is compact for the relative topology.*

Proof. Let $\mathcal{D} \subset \mathcal{T}$ be a collection of open sets that cover A. Since A is closed, A' is open, so $\mathcal{D} \cup ...$ is an open cover of X. ■

A compact subset of a topological space, even a compact one, need not be closed. For example, any set with at least 2 points within the discrete topology, every subset is compact.

Theorem 1.1.8. *Let $(X, \mathcal{T})$ be Hausdorff. Let $A \subseteq X$ be compact for the relative topology, then A is closed.*

Proof. Let $y \in X$, $y \notin A$. For each $x \in A$ find $\mathcal{U}_x, \mathcal{V}_x \in S$. Then the set of these $\mathcal{U}_x$ will cover A. So we have a finite subcover, $\mathcal{U}_{x_1}, \ldots \mathcal{U}_{x_n}$. Let $V = \mathcal{V}_{x_1} \cap \mathcal{V}_{x_2} \ldots \mathcal{V}_{x_n}$ be open, $y \in \mathcal{V}_1$, $V \cap A = \varnothing$. Thus A' is a union of open sets, so it is open. Thus, its compliment, A, is closed. ■

Theorem 1.1.9. *Let $(X, \mathcal{T})$ be compact and Hausdorff. For any closed subset A of X and any pf (?) $y \in X$, $y \notin A$, there are open sets u, v, disjoint, with $A \subseteq u$, $y \in V$.*

Definition 1.1.29. $(X, \mathcal{T})$ is regular for all $A \subseteq X$ closed and all $y \in X$, $y \notin A$.

Theorem 1.1.10. *Every compact Hausdorff space is normal.*

Proof. Let A, B be disjoint closed, (covered also (?)) subsets. By regularity, for each $y \in B$, there are disjoint open $\mathcal{U}_y,\ \mathcal{V}_y,\ A \subseteq \mathcal{U}_y,\ \mathcal{V}_y,\ y \in \mathcal{V}_y$. The $\{\mathcal{V}_y\}$ form an open cover of B, as by completion there is a finite subcover, $\{\mathcal{V}_{y_k}\}_{k \in I},\ I = \{1, \ldots, n\}$. ■

Proposition 1.1.3. *Let $(X, \mathcal{T}_x), (Y, \mathcal{T}_y)$ be topological spaces, and let $f : X \to Y$ be continuous. Let $A \subseteq X$ be compact. Then, $f(A) = \{f(x) : x \in A\}$ is compact.*

Proof. Let $\mathcal{C}$ be a collection of open sets in Y that cover $f(A)$. Then, $\{f^{-1}(\mathcal{O}) : \mathcal{O} \in \mathcal{C}\}$ are a collection of open sets that cover A, so there must exist a finite subcover of A, $f^{-1}(\mathcal{O}_1), \ldots, f^{-1}(\mathcal{O}_n)$, so $\mathcal{O}_1, \ldots, \mathcal{O}_n$ cover $f(A)$. ■

Proposition 1.1.4. *Let $(X, \mathcal{T}_x)$ be a compact space, and let $(Y, \mathcal{T}_y)$ be a Hausdorff topological space. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.*

Proof. Let $A \subseteq X$ be closed in X. Then, A must be compact. By Proposition 1.1.3, $f(A)$ must be compact, so because Y if Hausdorff, $f(A)$ must also be closed. ■

We can rewrite compactness in a new way shortly.

Definition 1.1.30. Let $\mathcal{C}$ be a collection of subsets of a set X We say that $\mathcal{C}$ has the finite intersection property if given any $A_1, \ldots, A_n \in \mathcal{C}$, we have that:

$$\bigcap_{j=1}^{n} A_j \neq \varnothing.$$

Proposition 1.1.5. *$(X, \mathcal{T})$ is compact iff whenever $\mathcal{C}$ is a collection of closed subsets of X with the finite intersection property, then*

$$\bigcap (A \in \mathcal{C}) \neq \varnothing.$$

Lemma 1.1.11. *(Zorn's Lemma) If a poset has the property that every chain in P has an upper bound in P, then P has at least one maximal element.*

Theorem 1.1.12. *(Tychonoff's Theorem) Let Λ be an index set, and for each $\lambda \in \Lambda$, let $(X_\lambda, \mathcal{T}_\lambda)$ be a compact topological space. Let*

$$X = \prod_{\lambda \in \Lambda} X_\lambda,$$

with the product topology. Then X is compact.

Proof. Some stuff I missed. Let $(X_\lambda, \mathcal{T}_\lambda)$ compact top spaces. Let $X = \prod X_\lambda$ with the product topology. Want to show that X is compact. Let $\mathcal{C}$ be a collection of closed sets with FIP. Need to show that $\cap\{C \in \mathcal{C}\} \neq \varnothing$. By Zorn's Lemma, there is a collection $\mathcal{D}^*$ of elements of X, $\mathcal{C} \subseteq \mathcal{D}^*$, with $\mathcal{D}^*$ maximal among collection satisfying the FIP.

Lemma 1.1.13. *Let $\mathcal{D}$ be any collection of subsets of X maximal for FIP. Then the finite intersection of sets in $\mathcal{D}$ are in $\mathcal{D}$, and if $B \subset X$ and if $B \cap A \neq \varnothing$, for all $A \in \mathcal{D}$, then $B \in \mathcal{D}$.*

Proof. Let $\mathcal{D}'$ be the collection of all finite collection of elements of $\mathcal{D}$. Then $\mathcal{D}$ has FIP, and $\mathcal{D} \subseteq \mathcal{D}'$, so by maximality, $\mathcal{D} = \mathcal{D}'$. For the second statement, consider $\mathcal{D} \cup \{B\}$, then this has FIP, because $B \cap A_1 \cap \ldots \cap A_n = B \cap \left(\bigcap_{j=1}^n A_j\right)_{j\in\mathcal{D}} \neq \varnothing$. ■

So $\mathcal{D} \cup \{B\}$ has FIP $\subseteq \mathcal{D}$. By maximality, $\mathcal{D} \cup \{B\} = \mathcal{D}, eB \in \mathcal{D}$, $\mathcal{C} \subseteq \mathcal{D}^*$. For each λ, $\{pi_\lambda(A) : A \in \mathcal{D}^*\}$ has FIP. Thus, $\{(\pi_\lambda(A)^- : A \in \mathcal{D}^*\} \subset X_\lambda$ has FIP, so since X_λ is compact, $\cap\{(\pi_\lambda(A))^- : A \in \mathcal{D}\} \neq \varnothing$. Choose $x_\lambda \in$ this set. Set $x_0 = \{x_\lambda\} \in X = \prod X_\lambda$. Want to show that $x_0 \in \cap\{C : C \in \mathcal{C}\}$, i.e., want $x_0 \in C$ for each $C \in \mathcal{C}$, suffices to show that $x_0 \notin C'$, which is open, for all $C \in \mathcal{C}$. So it suffices to show that for any $\mathcal{O}$ in base for product topology, if $x_0 \in \mathcal{O}$, then $\mathcal{O} \cap C$, $\mathcal{O} = \mathcal{U}_{\lambda_1} \times \mathcal{U}_{\lambda_2} \times \ldots \times \mathcal{U}_{\lambda_n} \times \prod_{\lambda\neq\lambda_1,\lambda j,\ldots\lambda_n}$, with $\mathcal{U}_{\lambda_j} \in J_{\lambda_j}$. By the definition of x_0, $x_{\lambda_j} \in \cap\{\pi_{\lambda_j}(A)^- : A \in \mathcal{D}^*\}$, for $j = 1, \ldots, n$. That is, for all $A \in \mathcal{D}^*$, $\mathcal{U}_j \cap \pi_{\lambda_j}(A) \neq \varnothing$. In other words, for all $A \in \mathcal{D}^*$, $A \cap \pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) \neq \varnothing$. Thus, $\pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) = \mathcal{D}^*$. Then, $\cap_{j=1}^n \pi_{\lambda_j}^{-1}(\mathcal{U}_{\lambda_j}) \in \mathcal{D}^*$, this intersection is just $\mathcal{O}$, so $\mathcal{O} \in \mathcal{D}^* \supseteq \mathcal{C}$, so $\mathcal{O} \cap C \neq \varnothing$ for all $C \in \mathcal{C}$. ■

Note 1.1.11. Tychonoff's Theorem is equivalent to the axiom of choice. Let $\mathcal{C}$ be a collection of sets, $\mathcal{C} = \{X_\lambda\}_{\lambda\in\Lambda}$. Choose one element that is not in any X_λ, e.g $\omega =$ set of all subsets of $\cup X_\lambda$. Let $Y_\lambda = X_\lambda \cup \{\omega\}$, set $\mathcal{T}_\lambda = \{X_\lambda, \{\omega\}, Y_\lambda, \varnothing\}$. Then, let $Y = \prod_{\lambda\in\Lambda} Y_\lambda$, with the product topology. By Tychons, Y is compact. Consider $\{\pi_\lambda^{-1}(X_\lambda)\}$. Claim that this has FIP, where the inside of the set braces is closed. Given $\lambda_1, \ldots, \lambda_n$, $\pi_{\lambda_1}^{-1}(X_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(X_{\lambda_2} \cap \ldots \cap \pi_{\lambda_n}^{-1}(X_{\lambda_n})$. For $j = 1, \ldots, n$, choose $x_{\lambda_j} \in X_{\lambda_j}$. Define $x \in \prod Y_\lambda$ by $x_\lambda = x_{\lambda_j}$ if $\lambda = \lambda_j$,... got too long.

1.2 Compactness in Metric Spaces

Note 1.2.1. Let (X, d) be a metric space, let $A \subseteq X$, and assume that $\bar{A}$ is compact for the relative topology. Then, for any $\epsilon > 0$, consider $\{\text{oBall}(x, \epsilon) : x \in A\} \supseteq \bar{A}$, with $\bar{A}$ is compact, so there is a finite subcover of $\bar{A}$, and so of A.

Definition 1.2.1. A subset A of a metric space (X, d) is said to be totally bounded if for any $\epsilon > 0$, it call be covered by a finite number of ϵ-balls.

Theorem 1.2.1. *Any subset of a compact subset of a metric space is totally bounded.*

Theorem 1.2.2. *If A is totally bounded subset of a metric space, then $\bar{A}$ is totally bounded.*

Proof. Let $\epsilon > 0$ be given, cover A by open $\text{Ball}(x_1, \frac{\epsilon}{2}), \ldots, \text{Ball}(x_n, \frac{\epsilon}{2})$. Then, $\text{Ball}(x_1, \epsilon), \ldots,$ $\text{Ball}(x_n, \epsilon)$ cover $\bar{A}$. ∎

Theorem 1.2.3. *A metric that is not complete can be compact.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X (which is not complete) that does not have a limit. For each $x \in X$, it is not a limit of $\{x_n\}$, so there is an ϵ_x and an N_x such that for all $n > N_x$, there is $m > n$ so $x_m \notin \text{Ball}(x, 2\epsilon_x)$. By Cauchy, there is N so that if $m, n > N$, then $d(x_m, x_n) < \epsilon$, then for $m > N$, $m \geq N_\epsilon$, $x_m \in \text{Ball}(x, \epsilon)$. The $\text{oBall}(x, \epsilon_x)$ for an open cover of X, so if X were compact, there would be a finite subcover of X, $\text{Ball}(x_1, \epsilon_{x_1}), \ldots, \text{Ball}(x_n, \epsilon_{x_n})$, so $\{x_n\}$ asdksjasd aksd ja finite number of values, so by Cauchy, it will converge, which is a contradiction. ∎

Theorem 1.2.4. *If X is complete, if $A \subset X$ is totally bounded, then $\bar{A}$ is compact.*

Proof. Proof of first theorem. Let $\mathcal{C}$ be an open cover, we want to find a finite subcover. Cover X by a finite number of balls of radius 1. If each B_j can be covered by a finite subcover of this collection, then get a finite subcover for X itself. At least one of the balls can be covered by a finite subcover, call it B'. ∎

Theorem 1.2.5. *Let (X, d) be a complete metric space. Then, if (X, d) is totally bounded then it is compact.*

Proof. Let $\mathcal{C}$ be an open cover of X. We need to show it has a finite subcover. Suppose it does not. Let $B_1^1, \ldots, B_n^1$ be closed balls of radius 1 that cover X. Since there is no finite subcover of X, there is at least one j such that B_j^1 is not finitely covered by $\mathcal{C}$. Set $A_1 = B_j^1$. Cover A_1 by a finite number of closed balls of radius $\frac{1}{2}$, $B_1^2, \ldots, B_{n_2}^2$. Then, there is at least one j so that $A_1 \cap B_j^2$ is not finitely covered by $\mathcal{C}$. Let $A_2 = B_j \cap A_1 \neq \varnothing$, diameter of $A_2 \leq 1$. Cover A_2 by a finite number of closed balls of radius $\frac{1}{4}$, $B_1^3, \ldots, B_{n_3}^3$. At least one of the $A_2 \cap B_j^3$ cannot be finitely covered by $\mathcal{C}$, call that one A_3, etc. Diamter $A_3 \leq \frac{1}{2}$. Get a sequence $\{A_n\}$ of closed sets $A_n \supseteq A_{n+1}$, diameter $A_n \to 0$. For each n, choose $x_n \in A_n$. Then $\{x_n\}$ is a Cauchy sequence. By completeness, $\{x_n\}$ converges, say to x_*. Since $\mathcal{C}$ is a cover, there is $\mathcal{O} \in \mathcal{C}$ such that $x_* \in \mathcal{O}$. Thus, there is $\epsilon > 0$ such that Ball$(x_*, \epsilon) \leq \mathcal{O}$. Since $\{x_n\}$ converges to x_*, there is N such that $x_n \in$ Ball$(x_*, \frac{\epsilon}{2})$ for $n \geq N$, but there is N' such that if $n \geq N'$ then diam$(A_n) \leq \frac{\epsilon}{2}$, so $A_n \subseteq$ Ball$(x_*, \epsilon) \subseteq \mathcal{O} \in \mathcal{C}$, ie A_n is covered by a finite subcover. Contradiction. ■

Corollary 1.2.6. *Let (X, d) be a complete metric space, let $A \subseteq X$, with A totally bounded. Then $\bar{A}$ is compact.*

Corollary 1.2.7. $[a, b] \subseteq \mathbb{R}$*, the first is compact. Any closed bounded subset of* $\mathbb{R}^n$ *is compact.*

Example 1.2.1. Let X be a set, and let (M, d) be a metric space. Let $B_b(X, M)$ be the set of all bounded functions from X to M. Metric $d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$, let $\mathcal{T}$ be a topology for X, consider $C_b(X^{\mathcal{T}}, M) =$ continuous functions in $B_b(X, M)$. What are the compact subsets of C_b? What are the totally bounded subsets. Let J be a totally bounded subset of $C_b(X, M)$. Then, given $\epsilon > 0$, we can find $g_1, \ldots, g_n \in J$ such the Ball(g_j, ϵ), $j = 1, \ldots, n$ cover J. Given any $x \in X$, such taht $g_1, \ldots, g_n$ are continuous, there are open sets, $\mathcal{O}_1, \ldots, \mathcal{O}_n$, with $x \in \mathcal{O}_j$, for all j such that if $y \in \mathcal{O}_j$, then $d(g_j(x), g_j(y)) < \epsilon$, let $\mathcal{O} = \bigcap_{j=1}^n \mathcal{O}_j$, such that $x \in \mathcal{O}$. Then for any $y \in \mathcal{O}$, $d(g_j(x, g_j(y)) < \epsilon$ for all j. Then for $f \in \mathcal{T}$, there is a j with $d_\infty(f, g_j) < \epsilon$, and so for $y \in \mathcal{O}$, $d(f(x), f(y)) \leq d(f(x), g_j(x)) + d(g_j(x), g_j(y)) + d(g_j(y), f(y)) < 3\epsilon$. Thus, given $x \in X$, for any $\epsilon > 0$, there is $\mathcal{O} \in J, x \in \mathcal{O}$ such that for $y \in \mathcal{O}$ has $d(f(x), f(y)) < \epsilon$, for all $f \in J$. The family f is equicontinuous at x. Since it is true for all x, we say that f is an equicontinuous set of functions. Also, for fixed x, given $f \in F$, there is g with $f \in$ Ball(g_j, ϵ), so that $d(f(x), g_j(x)) < \epsilon$, i.e., $\{f(x) : f \in F\} \subseteq M$ is covered by the balls Ball$(g_j(x), \epsilon)$, so it is totally bounded. Hence, F is pointwise totally bounded.

Theorem 1.2.8. *(Core of the Arzeli-Ascoli Theorem) Let $(X, \mathcal{T})$ be compact. Let $F \subseteq C(X, M)$. If F is equicontinuous and pointwise totally bounded, then F is totally bounded for d_∞.*

Proof. Let $\epsilon > 0$ be given. Then, by equicontinuity, for each $x \in X$, there is an open set $\mathcal{O}_x$, such that $x \in \mathcal{O}_x$ such that if $y \in \mathcal{O}_x$, then for all $f \in F$, we have $d(f(x), f(y)) < \epsilon$. The $\mathcal{O}_x$'s form an open cover of X, so there is a finite subcover $\mathcal{O}_{x_1}, \dots, \mathcal{O}_{x_n}$. For each $j = 1, \dots, n$, $\{f(x_j) : f \in F\}$ is totally bounded, so there is a finite subset, S_j such that the ϵ-balls about the points of S_j cover the aforementioned set. Let $S = \bigcup_j S_j$, a finite set in M. Let $\Psi = \{\psi : \{1, \dots, n\} \to S\}$ a finite set. For each $\psi \in \Psi$, let $A_\psi = \{f \in F : f(x_j) \in \text{Ball}(\psi(j) \in S, \epsilon)\}$. The A_ψ's cover F. If $f, g \in A_\psi$, for any x, there is $y \in X$, there is j so that $y \in \mathcal{O}_{x_j}$. Then $d(f(x), g(x)) \leq d(f(y), f(x_j))(\leq \epsilon) + d(x_j < \epsilon, g_{x_j} \leq \epsilon)(\leq 2\epsilon) + d(g(x_j), g(y)) \leq \epsilon < 4\epsilon$, i.e. diameter $(A_\psi) < 4\epsilon$. ■

Theorem 1.2.9. *(Arzela-Ascoli): Let $(X, \mathcal{T})$ be a complete metric space. Then, $F \subseteq C(X, M)$ is compact in d_∞ if it is closed and equicontinuous and pointwise totally bounded.*

Definition 1.2.2. Locally compact spaces. A topological space $(X, \mathcal{T})$ is locally compact if for each $x \in X$, there is a $\mathcal{O} \in \mathcal{T}, x \in \mathcal{O}, \bar{\mathcal{O}}$ is compact.

1.3 Locally Compact Hausdorff Spaces

Note 1.3.1. LCH := "locally compact Hausdorff"

$(X, \mathcal{T})$ be a LCH space.

Lemma 1.3.1. *Let $C \subseteq X$ be compact. Then there is open $\mathcal{O}$ with $C \subseteq \mathcal{O}$, $\overline{\mathcal{O}}$ compact.*

Proof. For each $x \in C$, let $\mathcal{O}_x$ be open with $x \in \mathcal{O}_x$, $\overline{\mathcal{O}}$ compact. $\{\mathcal{O}\}_{x \in C}$ covers C, so there is a finite subcover $\mathcal{O}_{x_1}, \dots \mathcal{O}_{x_n}$. Let $\mathcal{O} = \cup_{j=1}^n \mathcal{O}_{x_j}$, so $C \subseteq \mathcal{O}$, $\overline{\mathcal{O}} = \cup_{j=1}^n \overline{\mathcal{O}_{x_j}}$ is compact. ■

Theorem 1.3.2. *Let $(X, \mathcal{T})$ be a LCH. Let $C = X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is open $\mathcal{U}$, $C \subseteq \mathcal{U}$, $\overline{\mathcal{U}}$ compact, $\overline{\mathcal{U}} \subseteq \mathcal{O}$.*

Proof. By the previous lemma, we can choose $\mathcal{O}_1$, $C \subseteq \mathcal{O}_1 \subseteq \overline{\mathcal{O}}_1$, the last of which is compact. Let $\mathcal{O}_2 = \mathcal{O} \cap \mathcal{O}_1$, see $C \subseteq \mathcal{O}_2 \subseteq \mathcal{O}$, where $\mathcal{O}_2$ is compact. So we can assume $\mathcal{O}$ has compact closure. $C \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}}$. Let $B = \overline{\mathcal{O}} \backslash \mathcal{O}$, closed $\subseteq \overline{\mathcal{O}}$. C, B are disjoint compact subsets of $\overline{\mathcal{O}}$. Because $\overline{\mathcal{O}}$ is compact, so normal, we can find disjoint relatively open $\mathcal{U}$, $\mathcal{V} \subseteq \overline{\mathcal{O}}$, with $C \subseteq \mathcal{U}$, $B \subset \mathcal{V}$. Then, $\mathcal{V}'$ is closed, $\mathcal{U} \subseteq \mathcal{V}'$. Thus, $\overline{\mathcal{U}} \subseteq V'$, so $\overline{\mathcal{U}} \cap B = \varnothing$. Thus, $\overline{U} \subseteq \mathcal{O}$, $\mathcal{U} \subseteq \mathcal{O}$. ■

Theorem 1.3.3. *Let $(X, \mathcal{T})$ be LCH. Let $C \subseteq X$ be compact, $\mathcal{O}$ open, $C \subseteq \mathcal{O}$. Then there is a continuous $f : X \to [0, 1]$ with $f(x) = 1$, for $x \in C$ and $f(x) = 0$ for $x \notin \mathcal{O}$.*

Proof. Choose open $\mathcal{U}$ with $C \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}}$ (compact) $\subseteq \mathcal{O}$. Choose V with $C \subseteq \mathcal{V} \subseteq \bar{\mathcal{V}} \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{O}$, $\overline{\mathcal{U}} - \mathcal{V}$ closed in $\mathcal{U}$, disjoint from C, so by Urysohn's Lemma, there exists $\tilde{f} : \overline{\mathcal{U}} \to [0, 1]$, such that when $x \in C$, it evaluates to 1 and it evaluates to 0 for $x \in \overline{\mathcal{U}} - \mathcal{V}$. Let f be defined by $f(x) = \tilde{f}(x)$ if $x \in \overline{\mathcal{U}}$ and $f(x) = 0$ if $x \notin \overline{\mathcal{U}}$. We need f to be continuous. If $x \in \mathcal{U}$, then f is continuous at x, as $\tilde{f}$ is. If $x \notin \mathcal{U}$, then $x \notin \bar{\mathcal{V}}$, so $x \in X \backslash \bar{\mathcal{V}}$ open, on $X \backslash \bar{\mathcal{V}}$, $f(x) = 0$. ■

Definition 1.3.1. For $(X, \mathcal{T})$ LCH, let $C_c(X)$ be the set of continuous $\mathbb{R}-$valued functions on X "of compact support", i.e. there is a compact set outside of which $f \equiv 0$. $C_c(X)$ is an algebra for pointwise operations. $e, f, g \in C_c(X)$, then $f + g$, fg, $rf (r \in \mathbb{R}) \in C_c(X)$.

Note 1.3.2. $C_c(X) \subseteq C_b(X), || \cdot ||_\infty$, usually not complete if X is not compact. Its completion is the algebra of continuous functions that "vanish at infinity," $f \in C_\infty(X)$ if $\forall \epsilon > 0$, there is a compact set C_ϵ such that $|f(x)| \leq \epsilon$ for $x \notin C_\epsilon$. GL$(n, \mathbb{R})$ is locally compact.

Chapter 2

Measure Theory

2.1 Introduction to Measure Theory

Note 2.1.1. Recall the first day of lecture: $C([0,1])$, for the L^1 and L^2 norms, we can have a Cauchy sequence. We want to figure out a way to accurately find the integral in these troublesome cases. We want a set and a family of subsets $\mathscr{F}$, and some function $\mu : \mathscr{F} \to \mathbb{R}^+$. We want additivity, i.e. if $E, F \in \mathscr{F}$, and if E and F are disjoint and $E \oplus F \in \mathscr{F}$, then $\mu(E \cup F) = \mu(E) + \mu(F)$. Also if $E, F \in \mathscr{F}$, $E \subseteq F$, $F = E \oplus (F \backslash E)$ (let $\oplus$ be the disjoint union), so $\mu(F) = \mu(E) + \mu(F \backslash E)$, i.e. $\mu(F \backslash E) = \mu(F) \backslash \mu(E)$.

Definition 2.1.1. Let X be a set and let R be a nonempty family of subsets of X. We say that R is a ring if R is closed under finite unions and differences of elements $E \backslash F$. This implies closed under finite intersection over $E \cap F = E \backslash (E \backslash F)$. If also $X \in R$, call $\mathscr{J}$ an algebra (or a field).

Definition 2.1.2. A finitely added measure or a ring R of sets is a finite $\mu : R \to \mathbb{R}^+$ such that if $E, F \in R$ and are disjoint, then $\mu(E \oplus F) = \mu(E) + \mu(F)$

Definition 2.1.3. A ring R is said to be a σ-ring of to so closed under taking countable unions of elements fo R, so we can take countable intersections.

Definition 2.1.4. A σ-algebra: $E = \cup_{n=}^{\infty} E_n$, then $\cap E_n = E \setminus \cup_{n=1}^{\infty} (E \setminus E_n)$

Definition 2.1.5. Let R be a σ-ring. By a measure on R we mean a function $\mu : R \to \mathbb{R}^+$, $\mathbb{R}^+ \cup \{+\infty\}$, $\mathbb{R}$, $\mathbb{R}^n$, Banach spaces, which is countable additive, i.e. if $\{E_n\}_n^\infty$ is a disjoint family of elements in R. Then,

$$\mu\left(\bigoplus_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Theorem 2.1.1. *Let $\mathscr{S}$ be a collection of rings (or algebras, or σ-algebras, or σ-rings, etc) of a given set X. Then the intersection of these rings is a ring (or ...).*

Definition 2.1.6. Given any collection of subsets of X, there is a smallest ring (...) that contains the collection, namely the intersection of all contains rings, etc.

Definition 2.1.7. Let $(X, \mathcal{T})$ be a topological space.

1. The σ-ring generated by $\mathcal{T}$ is called the σ-ring of Borel subsets of X.

Let $(X, \mathcal{T})$ be a LCH space, then the σ-ring generated by the compact subsets is called the σ-ring of Borel sets.

Note 2.1.2. $X = \mathbb{R}, \mathscr{S} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$

Note 2.1.3. Let $P = \{[a, b) \subseteq \mathbb{R} : a < b\}$.

Definition 2.1.8. Let X be a set, P a collection of subsets. We say that P is a pre-ring if

1. For $E, F \in P$, we have that $E \cap F \in P$
2. For $E, F \in P$, there are $G_1, \ldots, G_n \in P$, such that $E \backslash F = \bigoplus^n G_j$.

Note 2.1.4. Let α be a non-decreasing left-continuous function $\alpha : \mathbb{R} \to \mathbb{R}$, if $s < t$, then $\alpha(s) \leq \alpha(t)$. Now, given α, define $\mu_\alpha([a,b)) = \alpha(b) - \alpha(a) \geq 0$.

Theorem 2.1.2. *μ_α on P is countably additive.*

Proof. Need: if $[a_0, b_0) = \bigoplus_{n=}^{\infty}[a_n, b_n)$, then $\mu_\alpha([a_0, b_0)) = \sum_{n=0}^{\infty} \mu_\alpha([a_n, b_n))$. Need to show $\geq$: Suffices to show that for each n, $\mu_\alpha([a_0, b_0)) \geq \sum^n \mu_\alpha([a_j, b_j))$. we know that the $[a_j, b_j)$ are disjoint. We can renumber these intervals so that $a_1 < a_2 < \ldots < a_n$. Since disjoint, $b_j \leq a_{j+1}$ for $j = 1, \ldots, n$, $\alpha(b_1) - \alpha(a_1) + \alpha(b_2) - \alpha(a_2) + \ldots + \alpha(b_n) - \alpha(a_n) = -\alpha(a_1) + (\alpha(b_1) - \alpha(a_2))(\leq 0) + \ldots + (\alpha(b_{n-1}) - \alpha(a_n))(\leq 0) + \alpha(b_n) \leq \alpha(b_n) - \alpha(a_1) \leq \alpha(b_0) - \alpha(a_0) = \mu_\alpha([a_0, b_0))$. We now need $\mu_\alpha([a_0, b_0)) \leq \sum_{j=1}^{\infty} \mu_\alpha([a_j, n_j))$. Let $\epsilon > 0$ be given. Choose ϵ_j's, $\epsilon_j > 0$, $\sum^\infty \epsilon_j \leq \frac{\epsilon}{2}$, where $\epsilon_j = \frac{\epsilon}{2^{j+1}}$. Choose $b_0' < b_0$, such that (since α is left continuous), $\alpha(b_0') + \frac{\epsilon}{2} \geq \alpha(b_0)$, for each j, choose $a_j' < a_j$ such that $\alpha(a_j') + \epsilon_j \geq \alpha(a_j)$, $\alpha(a_j') < \alpha(a_j)$. Then, $[a_0, b_0'] \subseteq \cup_{j=1}^{\infty}(a_j', b_j)$, so there is a finite subcover. Remember finite subcover $\mathcal{C}$ as follows. Let (a_1', b_1) be the interval in $\mathcal{C}$, with smallest a_1. Assume $b_1 \leq b_0'$. Let (a_2', b_2) the interval in $\mathcal{C}$ that contains b_1 and has smallest a_2', so $a_2' < b_2$. Continue $\ldots$ $(a_j', b_j), a_{j+1} < b_j$. As soon as $b_j > b_0'$, STOP. $\mu_\alpha([a_0, b_0]) = \alpha(b_0) - \alpha(a_0) \leq \alpha(b_0') + \frac{\epsilon}{2} - \alpha(a_0) \leq \alpha(t_0') + \frac{\epsilon}{2} - \alpha(a_0') \leq \alpha(t_n) - \alpha(a_0') + \frac{\epsilon}{2}$, $b_n > b_0'$, $a_{j+1}' \leq b_j, \alpha(a_{j+1}') \leq \alpha(b_j), \alpha(b_j) - \alpha(a_j') \geq 0$. ■

Insert stuff in picture above.

Definition 2.1.9. A premeasure is afunction μ defined on a semiring P, $\mu : P \to \mathbb{R}^+$, and is countably additive. Each μ_α is a pre-measure.

Theorem 2.1.3. *$\mu : P \to \mathbb{R}^+$ just finitely added. Then, if $E \in P$ containes $\bigoplus_{j=1}^n F_j$. Then, $\mu(E) \geq \sum \mu(F_j)$.*

Proof. $E = \oplus H_n \oplus E_n \oplus F_j$, $\mu(E) = \sum \mu(H_n)(\geq 0) + \sum \mu(E \cap F_j)(= F_j)$ ∎

Definition 2.1.10. Let $\mathcal{C}$ be a collection of sets

$[a_0, b_0'] \subset \cup_{j=1}^n (a_j', b_j)$ overlapping, $b_j > a_{j+1}'$, $a_1' < a_0$, $b_n > b_0'$. Then $\alpha(b_0') - \alpha(a_0) \leq \sum \alpha(b_j) - \alpha(a_j)$.

Proof.

$$\begin{aligned}\sum \alpha(b_j) - \alpha(a_j) &= \alpha(b_n) - (\alpha(a_n') - \alpha(b_{n-1}))(< 0) - \ldots - (\alpha(a_2') - \alpha(b_1))(< 0) - \alpha(a_1') \\ &\geq \alpha(b_n) - \alpha(a_1') \\ &\geq \alpha(b_0') - \alpha(a_0).\end{aligned}$$

∎

We saw that if $E \supseteq \oplus_{j=1}^n F_j$, for μ on every P, then $\mu(E) \geq \sum \mu(F_j)$.

Definition 2.1.11. Let $\mathscr{F}$ be a family of subsets of X. let $\mu : \mathscr{F} \to \mathbb{R} \cup \{+\infty\}$, we say that μ is countably additive if whenever we have that $E \subseteq \cup_{j=1}^\infty F_j$, then $\mu(E) \leq \sum \mu(F_j)$.

Definition 2.1.12. μ on $\mathscr{F}$ is monotone if $E \supseteq F$ implies that $\mu(E) \supseteq \mu(F)$.

Theorem 2.1.4. *Let P be a semiring, $\mu : P \to \mathbb{R}$, countably additive $E = \oplus_{j=1}^\infty F_j$. Then μ is countably subadditive, $E \subseteq \cup F_j$ want $\mu(E) \leq \sum \mu(F_j)$.*

Proof. Then, $E \subseteq \cup F_j \cap E$, and by μ monotone, $\mu(F_j \cap E) \leq \mu(F_j)$, so it suffices to show that for $E = \cup^{\infty} F_j$, then disjointage: set H_j(not really in P) $= F_j \backslash \cup_{k<j} F_k$. $H_1 = F_1$. Then, $E = \bigoplus H_j$. Note that $H_j = \oplus_{k=1}^{n_k} G_{jk}$, with $G_{jk} \in P$. Thus, $E = \oplus G_{jk} \in P$. Next, by the countable additivity of μ, we must have that:

$$\begin{aligned} \mu(E) = \sum_{j,k} \mu(G_{jk}) &= \sum_j \sum_{k=1}^{n_j} \mu(G_{jk}) \\ &\leq \sum_j \mu(F_j). \end{aligned}$$

Note that $\oplus_k G_{jk} \subseteq F_j$ and $\sum_k \mu(G_{jk}) \leq \mu(F_j)$. ∎

Let $\mathscr{F}$ be a family of subsets of a set X, and let μ be any function from $\mathscr{F} \to \mathbb{R}^+ \cup \{+\infty\}$. For any $A \subseteq X$, set $\mu^*(A) = \inf\{\sum \mu(F_j) : F_j \in \mathscr{F}, A \subseteq \cup_{j=1}^{\infty} F_j\}$. Let $\mathscr{H}(\mathscr{F}) = \{A \subseteq X : \exists \{F_j\}_{j=1}^{\infty} \subseteq \mathscr{F}, \text{ with } A \subseteq \cup_{j=1}^{\infty} F_j\}$. It is clear that $\mathscr{H}(\mathscr{F})$ is a σ-ring, this is hereditary (i.e. if $A \in \mathscr{H}(\mathscr{F})$ and $B \subseteq A$, then $B \in \mathscr{H}(\mathscr{F})$). Finally, note that the $F_j's$ cover A. Set $\mu^*(\varnothing) = 0$.

Example 2.1.1. Let $X = \mathbb{R}$, then let $\mathscr{F}$ be a collection of all finite subsets of $\mathbb{R}$, $\mathscr{H}(\mathscr{F}) =$ countable subsets of $\mathbb{R}$.

Example 2.1.2. Properties:

1. Monotone.
2. μ^* is countably sub-additive.

Proof. (2): Let A, $\{B_j\}_{j=1}^{\infty}$ be in $\mathscr{H}(\mathscr{F})$, $A \subseteq \cup B_j$. Want $\mu^*(A) \leq \sum \mu^*(B_j$. Let $\epsilon > 0$ be given, choose $\{\epsilon_j > 0\}$ with $\sum_{j=1}^{\infty} \epsilon_j < \epsilon$, for each j, choose $\{F_k^j\}_{k=1}^{\infty}$ with $B_j \subseteq \cup_k F_k^j$ but $\sum_k \mu(F_k^j) \leq \mu^*(B_j) + \epsilon_j$. Then, $A \subseteq \cup_{j,k} F_k^j$, so

$$\begin{aligned} \mu^*(A) \leq \sum_{j,k} \mu(F_k^j) &= \sum_j \sum_k \mu(F_k^j) \\ &\leq \sum_j (\mu(B_j)... \end{aligned}$$

∎

Definition 2.1.13. Let $\mathscr{H}$ be a hereditary σ-ring of subsets of X. By an outer measure on $\mathscr{H}$, we mean a finite $\mathcal{V} : \mathscr{H} \to \mathbb{R}^+ \cup \{+\infty\}$ that is monotone and countably subadditive, $\mathcal{V}(\varnothing) = 0$.

Let P be a semiring, and let μ be a premeasure on P, i.e. μ is countably additive. Let μ^* be the corresponding outer measure on $\mathscr{H}(P)$.

Theorem 2.1.5. *For any $E \in P$, $\mu^*(E) = \mu(E)$, i.e. μ^* is an exterior of μ to all of $\mathscr{H}(P)$.*

Proof. $\mu^*(E) = \inf\{\sum \mu(F_j) : E \subseteq \cup F_j\}$, so $\mu(E) \leq \mu^*(E)$, but μ is countably additive, so $\mu(E) \leq \sum \mu(F_j)$. For E_n, $\mu(E) = \mu^*(E)$. ■

Let $\mathcal{V}$ be an outer measure on $\mathscr{H}$. Let $E \in \mathscr{H}$. We say that E splits all sets in $\mathscr{H}$ if for any $A \in \mathscr{H}$, $\mathcal{V}(A) = \mathcal{V}(A \cap E) + \mathcal{V}(A \backslash E)$ (Note that $A = A \cap E \oplus A \backslash E$. By subadditive, we have $\leq$, so we have that $\mathcal{V}(A) \geq$. Let $\mathscr{S}(\mathcal{V}) = \{E \in \mathscr{H} : E$ splits all sets in $\mathscr{H}\}$, with $\varnothing \in \mathscr{S}$.

Theorem 2.1.6. *$\mathscr{S}(\mathcal{V})$ is a σ-ring, and $\mathcal{V}|_{\mathscr{J}}$ is coubntably additive and therefore a measure.*

Proof. Let $E, F \in \mathscr{S}(\mathcal{V})$. We want $E \cup F \in \mathscr{S}(\mathcal{V})$. Let $A \in \mathscr{H}$, we want that $\mathcal{V}(A) \geq \mathcal{V}(A \cap (E \cup F)) + \mathcal{V}(A \backslash (E \cup F)) = \mathcal{V}(A \cap E \oplus (A \backslash E) \cap F) \leq \mathcal{V}(A \cap E) + \mathcal{V}((A \backslash E) \cap F) + mathcalV((A \backslash E) \backslash F) = \mathcal{V}(A \cap E) + \mathcal{V}(A \backslash E) = \mathcal{V}(A)$, because $F \in \mathscr{S}(\mathcal{V})$, $E \in \mathscr{S}(\mathcal{V})$.

Now, we want to show that if $E, F \in \mathscr{S}(\mathcal{V})$ the $E \backslash F \in \mathscr{S}(\mathcal{V})$. Let $A \in \mathscr{H}$. We want $\mathcal{V}(A) =^? \mathcal{V}(A \cap (E \backslash F)) + \mathcal{V}(A \backslash (E \backslash F)) = \mathcal{V}((A \cap E) \backslash F) + \mathcal{V}((A \backslash E) \cup (A \cup F))(\mathcal{V}((A \backslash E) \oplus (A \cap F \cap E)) \leq \mathcal{V}((A \cap E) \backslash F) + \mathcal{V}(A \backslash E) + \mathcal{V}(A \cap F \cap E) = \mathbb{V}(A \cap E) + \mathcal{V}(A \backslash E) = \mathcal{V}(A)$. ■

$\mathscr{H}$ is hereditary σ-ring of subsets of X, ν is an outer measure defined on $\mathscr{H}$, $M(\nu) = \{E \in \mathscr{H} : \nu(A \cap E) + \nu(A \backslash E) = \nu(A), \forall A \in \mathscr{H}\}$. We saw that $M(\nu)$, the ν-measurable sets is a ring. We now claim that if $E, F \in M(\nu)$, $E \cap F = \varnothing$, then for all $A \in \mathscr{H}$, $\nu(A \cap E \oplus A \cap F) = \nu(A \cap E) + \nu(A \cap F)$.

Proof. E splits $A \cap (E \oplus F)$, or equivalently $\nu((A \cap (E \oplus F)) \cap E) + \nu((A \cap (E \oplus F)) \backslash E) = \nu((A \cap E) \oplus (A \cap F))$. ■

Theorem 2.1.7. *$M(v)$ is a σ-ring, and ν is countably additive on $M(\nu)$.*

Proof. Let $\{E_j\}_j^\infty \subseteq M(\nu)$. Let $G = \cup_{j=}^\infty E_j$. We want to show that $G \in M(\nu)$. Given A, we need to show that G splits A. Can disjointize the E_j's, so $G = \oplus_{j=}^\infty F_j$, $F_j \in M(\nu)$. Hence,

$$\begin{aligned}\nu(A) &= \nu(A \cap \oplus_{j=1}^n f_j) + \nu(A \backslash \oplus_{j=1}^n F_j \\ &= \sum^n \nu(A \cap F_j +'' \\ &\geq \sum^n \nu(A \cap F_j) + \nu(A \backslash G) \text{ by nu monotone} \\ \nu(A) &\geq \sum^\infty \nu(A \cap F_j) + \nu(A \backslash G) \geq (\text{countably additive})\nu(A \cap G) + \nu(A \backslash G) \geq \nu(A).\end{aligned}$$

Hence, $M(\nu)$ is a σ-ring. ■

Note 2.1.5. For a set X, define

$$\begin{aligned}&\nu(A) = 1, A \neq \varnothing \\ &\nu(\varnothing) = 0.\end{aligned}$$

Theorem 2.1.8. *Let $(\mathscr{P}, \mu)$ be a premeasure. Let μ^* be the corresponding outer measure on $\mathscr{H}(\mathscr{P})$. Then, $\mathscr{P} \subseteq M(\mu^*)$. Define*

$$\mu^*(A) = \inf\left\{\sum \mu(E_j) : E_j \in \mathscr{P}, A \subseteq \cup E_j\right\}.$$

Proof. Let $E, F \in \mathscr{P}$, $E \backslash F = \oplus^n G_j, G_j \in \mathscr{P}$. Hence, $\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \backslash F)$, so $\mu(E)_=\mu(E \cap F) + \mu^*(E \backslash F)$. THen, let $E \in \mathscr{P}$, then let $A \in \mathscr{H}(\mathscr{P})$, we need $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$. Now, let $\epsilon > 0$ be given, and choose $\{F_j\}_{j=1}^n \subset \mathscr{P}, A \subseteq \cup^n F_j, \mu^*(A) + \epsilon \geq \sum^n \mu(F_j)$. Then, $\epsilon + \mu(A) \geq \sum^n \mu(F_j) = \sum^n \mu(F_j \cap E) + \sum^n \mu^*(F_j \backslash E) = \sum \mu(\cup F_j \cap E) \geq \mu^*(A \cap E)(\text{monotone}) + \mu^*(A \backslash E)(\text{countably additive}) \geq \mu^*(A)$. Since ϵ is arbitrary, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$. Hence, $E \in M(\mu^*)$. Thus, $\mathscr{P} \subseteq M(\mu^*)$. ■

$\mathscr{H}, \nu M(\nu)$. If $A \in M(\nu)$ abd if $\nu(A) = 0$, then $A = \varnothing$, then for any $B \subseteq A$, $B \in M(\nu)$ (with $\nu(B) = 0$), "complete," given any $D \in \mathscr{H}$, $\nu(D) \supseteq \nu(D \cap B) + \nu(D \backslash B)$, by monotone.

Note 2.1.6. If $(\mathscr{P}, \mu)$ is a premeasure then μ^* on $M(\mu^*)$ is a complete measure. Can restrict μ^* to the $\mathscr{S}(\mu) = \sigma$-ring generated by $\mathscr{P}$, $\mathscr{S}(\mu) \subseteq M(\mu^*)$, but μ on $\mathscr{S}(\mu)$ need not be complete. For α a left-cont non-decreasing function, μ_α^* on $M(\mu_\alpha)$ is called a Lebesgure-Stieltjes measure, which is complete its restriction to $\mathscr{S}(\mathscr{P}$ is called a Borel-Stieltjes measure. Maybe not be complete. $\mathscr{S}(\mathscr{P})$ are the Borel sets in $\mathbb{R}$. But different α's maybe have different $M(\mu^*)$. When using just one measure on $\mathbb{R}$, we usually use $M(\mu_\alpha^*$. When using many of the μ_α's, use $\mathscr{S}(\mathscr{P})$, because they are all defined on $\mathscr{S}(\mathscr{P})$, if considering α's with $\lim_{t\to+\infty}(\alpha(t) - \lim_{t\to-\infty}\alpha(t)) = 1$. Then, the μ_α have $\mu_\alpha(\mathbb{R}) = 1$. The μ_α are the (Borel) probability measures on $\mathbb{R}$. Next, note that in the case of $\alpha(t) = t$, gives Lebesgue measuer on $\mathbb{R}$. It is the translation invariant.

$[a, b), [a + c, b + c), b - a = (b + c) - (a + c)$.

Definition 2.1.14. A measure μ or σ-rings is said to be σ-finite if for all $E \in \mathscr{S}$, there are $\{F_j\} \subset \mathscr{S}$ with $\mu(F_j) < \infty$ and $E \subseteq \cup F_j$.

Theorem 2.1.9. *For $\mu, \mathscr{S}, \mu^*$, $\mu^*(A) = \inf\{\sum^\infty \mu(E_j) : A \subseteq \cup^\infty E_j, E_j \in \mathscr{S}\}$, we can disjointize $A \subseteq \oplus F_j \sum \mu(F_j) \leq \sum \mu(E_j), \sum \mu(F_j) = \mu(\oplus F_j), \mu^* = \dots$*

Theorem 2.1.10. *Let $(\mu, \mathscr{S}, \mu)$ be a measure space. Let $M(\mu^*)$ be the μ^*-measureable sets the $\mathscr{S} \subseteq M(\mu^*$. We can then consider $\mathscr{S} \subseteq \mathscr{S}_1 \subseteq M(\mu^*)$. Then, the restriction of μ^* to $\mathscr{S}_1$ is the largest extension of μ to $\mathscr{S}_1$.*

Proof. Let ν be another extension of μ to $\mathscr{S}$. Then, for $A \in \mathscr{S}_1$. ■

Midterm is on next Thursday :(
$(\mathcal{P}, \mu), \mathscr{H}(\mathcal{P}), \mu^*, M(\mu^*) \supseteq \mathscr{S}(\mathcal{P})$ is a σ-ring. For any $A \in \mathscr{H}(\mathcal{P}), \mu^*(A) = \inf\{\mu^*(E) : E \in M(\mu^*), A \subseteq E\}$. Then, for each n, choose $E_n \supseteq A$ such that $\mu^*(E_n) \leq \mu^*(A) + 1/n$. Then, set $E = \bigcap E_n \supseteq A, \mu^*(E) = \mu^*(A)$.

Theorem 2.1.11. *Assume that $(\mathcal{P}, \mu)$ is σ-finite. For all $A \in \mathscr{H}(\mathcal{P})$ there are $\{E_n\} \subseteq \mathcal{P}, \mu(E_n) < \infty$ and $A \subseteq \bigcup E_n$. THen, for any σ-ring $\mathscr{S}$, $\mathscr{S}(\mathcal{P}) \subseteq \mathscr{S} \subseteq M(\mu^*)$, μ on $\mathscr{S}$ on $\mathscr{S}(\mathcal{P})$, and any extension, μ', of μ, then $\mu'(F) = \mu^*(F)$, for any $F \in \mathscr{S}$ (so extension μ' is unique).*

Proof. Part 1: Assume that $F \in \mathscr{S}, F \subseteq E \in \mathscr{S}(\mathcal{P}), \mu(E) < \infty. E = E \cap F \oplus E \backslash F$. $\mu'(E) = \mu(E) = \mu^*(E \cap F) + \mu^*(E \backslash F) \geq \mu'(E \cap F) + \mu'(E \backslash F) = \mu'(E)$. But $\infty > \mu^*(E \cap F) \geq \mu'(E \cap F), \infty > \mu^*(E \backslash F) \geq \mu(E \backslash)$. Thus, $\mu^*(E \cap F) = \mu'(E \cap F)$, $\mu^*(F) = \mu'(F)$.

For general $F \in \mathscr{S}$, assume μ is σ-finite, then there exists $\{E_j\} : F \subseteq \bigcup E_j$, $\mu(E_j) < \infty$, can disjointize, so assume that $F \subseteq \oplus E_j$. Then, $\mu'(F) = \sum \mu'(F \cap E_j) = \sum \mu^*(F \cap E_j) = \mu^*(\oplus F \cap E_j) = \mu^*(F)$. ∎

2.2 Continuity Properties of Measures

Theorem 2.2.1. *Let $(X, \mathscr{S}, \mu)$ be a measure space. Let $\{E_j\} \subset \mathscr{S}$, increasing, i.e. $E_{j+1} \supseteq E_j$. Let $E = \cup^{\infty} E_j$. Then, $\mu(E) = \lim \mu(E_j)$.*

Proof. $E = E_1 \oplus (E_2 \backslash E_1) \oplus (E_3 \backslash E_2) \cdots (E_{j+1} \backslash E)$. Hence, it must be the case that

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_{j+1} \backslash E_j) + \mu(E_1).$$

Then, $\mu(E_n) = \mu(E_1) + \mu(E_2 \backslash E_1) + \ldots + \mu(E_n \backslash E_{n-1})$ partial sum. Thus, $\mu(E_n) \to \mu(E)$. ∎

Theorem 2.2.2. $\{E_j\}$, $E_{j+1} \subseteq E_j$, $E = \bigcap E_j$. *$\mu(E_j) \to \mu(E)$, and if $(\mu(E_1) < \infty$, then $\mu(E_j) \to \mu(E)$.*

Proof. See online notes (hopefully?). ∎

Example 2.2.1. A counterexample, $\mathbb{R}, M$ Lebesgue: $E_j = [j, \infty)$. $\mu(E_j) = \infty, \bigcap E_j = \varnothing \to 0$.

$\mathbb{R}$, Lebesgue measure, μ_α, $\alpha([a, b)) = b - a$. Translation movement.

$$\mathbb{R}/\mathbb{Z} \to T$$

$$t \mapsto e^{2\pi i t},$$

fundamaental domain $[0, 1)$, transfer Lebesgue measure restricted to $[0, 1)$ onto S^1. Then, we get a rotation invariant measure on T, with $\mu(T) = 1$. In the group T, let G be the subgroup of elements of finite order, $\{e^{2\pi i \frac{m}{n}}, m, n \in \mathbb{Z}\}$. G is a countable subgroup (Dense in T). Consider $T/G = \{\text{cosets}\}$, which is uncountable. Let $A \subset T$ consist of a closure of one point for each coset of G, each element of T is in one coset. Thus, $T = \oplus_{r \in G} rA$. Given $z \in T$, there is $a \in A$, in the same coset as z, i.e., $z = ra$. By translation of invariance, $\mu(rA) = \mu(A)$ for all $r \in G$, but G is countable,

$$1 = \mu(T) = \sum_{r \in G} \mu(rA) = \sum_{r \in G} \mu(A).$$

Hence, A is not measurable.

Note 2.2.1. Berkeley professor Solovey showed that you cannot show that there are non-measurable sets without something like the axiom of choice.

2.3 Introduction to Integration

$(X, \mathcal{S})$, $\mathcal{S}$ is a ring of subsets of X. Let B be a vector space. Given $E \in \mathcal{S}$,

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

If $b \in B$,

$$b\chi_E(x) = \begin{cases} b & x \in E \\ 0 & x \notin E. \end{cases}$$

Definition 2.3.1. By a simple B-valued function on X, we mean $f : X \to B$ that has finite range, and for any $b \in \text{range}(f)$, $b \neq 0$, $f^{-1}(b) \in \mathcal{S}$. Thus,

$$f = \sum b_j \chi_{E_j},$$

where the b_j are not equal to 0 (or $f \equiv 0$), E_j's are disjoint and in $\mathcal{S}$. If

$$f = \sum^{n} b_j \chi_{E_j},$$

with the E_j's disjoint, but the b_j's not distinct and b_j maybe 0.

Lemma 2.3.1. *Let*

$$f = \sum^{n} b_j \chi_{E_j},$$

$E_j \in \mathcal{S}$ disjoint, b_j disjoint, $\neq 0$. Let $F \in \mathcal{S}$, $c \in B$, set $g = c\chi_F$. Then, $f + g$ is a SMF.

Proof. Let $E_{n+1} := F \backslash \oplus E_j$. Then

$$f = \sum_{j=1}^{n+1} b_j E_j,$$

where $b_{n+1} = 0$, $F = \oplus(F \cap E_j)$, $E_j = (E_j \cap F) \oplus (E_j \backslash F)$. Note that $F \subseteq \oplus_{j=1}^{n+1}$. Then,

$$f = \sum^{n+1} b_j \chi_{E_j \cap F} + \sum^{n+1} b_j \chi_{E_j \backslash F},$$

$$g = \sum^{n+1} c\chi_{F \cap E_j}.$$

So

$$f + g = \sum^{n+1} (b_j + c)\chi_{E_j \cap F} + \sum^{n+1} b_j \chi_{E_j \backslash F},$$

where $E_j \cap F, E_j \backslash F \in \mathcal{S}$. ■

Lemma 2.3.2. *If f, g are SMF's, then so is $f + g$.*

Proof. Let

$$f = \sum b_j \chi_{E_j},$$

and

$$g = \sum c_k \chi_{F_k},$$

then $f + c_1 \chi_{F_1}$. ■

Let μ be a finitely additive measure on $\mathcal{S}$. By a simple, μ-integrable function, we mean a SMF

$$f = \sum b_j \chi_{E_j},$$

with disjoint E_j and distinct, nonzero b_j, such that $\mu(E_j) < \infty$ for all j. Then,

$$\int b\chi_E d\mu = b\mu(E), \ \ \mu(E) < \infty.$$

Definition 2.3.2. We define the integral as:

$$\int f d\mu = \sum b_j \mu(E_j').$$

Lemma 2.3.3. *If*

$$f = \sum^{n} b_j \chi_{E_j}$$

is SIF, if $F \in \mathcal{S}$, $\mu(E) < \infty$ and $c \in B$, then $f + g$ is a SIF and

$$\int (f+g)d\mu) = \int f d\mu + \int g d\mu.$$

Proof. Let $E_{n+1} = F \backslash \oplus E_j$, then $f + g$ (refer to above), so $f + g$ is SIF. Then,

$$\begin{aligned}\int (f+g)d\mu &= \sum (b_j + c)\mu(E_j \cap F) + \sum (b_j \mu(E_j \backslash F) \\ &= \sum b_j \mu(E_j \cap F) + \sum b_j \mu(E_j \backslash F) + \sum c\mu(E_j \cap F) = \int f d\mu + \int g d\mu \\ &= \sum b_j \mu(E_j).\end{aligned}$$

■

Lemma 2.3.4. *If f is SMF, if $\alpha \in \mathbb{R}, \mathbb{C}$, then αf,*

$$f = \sum b_j \chi_{E_j} \quad \alpha f = \sum (\alpha b_j)\chi_{E_j},$$

SMF$(X, \mathcal{S}, B)$ forms a vector space under pointwise operations, SIF$(X, \mathcal{S}, \mu, B)$.

Note 2.3.1. SIF$(X, \mathcal{S}, \mu, B)$, and

$$f \mapsto \int f d\mu$$

is a linear operator.

If $f \in$ SIF$(X, \mathcal{S}, \mu, \mathbb{R})$ and if $f \geq 0$, then

$$\int f d\mu \geq 0, f = \sum b_j \chi_{E_j}, b_j \in \mathbb{R}, b_j \geq 0,$$

we have that

$$\int f d\mu = \sum b_j \mu(E_j \geq 0,$$

for $f, g \in$ SIF$(X, \mathcal{S}, \mu, \mathbb{R})$, we say that $f \geq g$ if $f(x) \geq g(x)$ for any x, or equivalently, $f - g \geq 0$. If $f \geq g$, then

$$\int f d\mu \geq \int g d\mu.$$

Let B have a norm $||\cdot||, ||\cdot||_B$. For f any B-valued function, define

$$x \mapsto ||f(x)||$$

is $\mathbb{R}^+$-valued, if f is a SMF,

$$f = \sum b_j \chi_{E_j},$$

then $||f(x)|| = \sum ||b_j|| \chi_{E_j}$, so $x \mapsto ||f(x)||$ is SMF. If f is SMF, then $x \mapsto ||f(x)||$ is SMF.

Definition 2.3.3. $||\cdot||_1$ on SIF$(X, \mathcal{S}, \mu, B)$ by

$$||f||_1 = \int ||f(x)|| d\mu(x).$$

Note 2.3.2. Some properties of this include:

1. $||\alpha f||_1 = \int ||\alpha f(x)|| d\mu(x) = |\alpha| \cdot ||f||_1$.
2. $||f + g||_1 \leq ||f||_1 + ||g||_1$. Then,

$$\int ||f(x) + g(x)|| d\mu(x) \leq \int (||f(x)|| + ||g(x)||) d\mu(x) = ||f||_1 + ||g||_1,$$

so $||\cdot||_1$ is a norm on SIF.

If f is SIF and

$$||f|| = \int f d\mu = 0,$$

then

$$||f|| = \sum |b_j| \chi_E(x), 0 = ||f||_1 = \sum |b_j| \mu(E_j) \implies \mu(E_j) = 0, \forall j.$$

Let $N(X, \mathcal{S}, \mu) = \{E \in \mathcal{S} : \mu(E) = 0\}$, where N stands for null sets, ring. Let $\mathcal{N} = \{$SIF$f : ||f||_1 = 0\}$, then $\mathcal{N}$ is a vector space of SIF, SIF$/\mathcal{N}$ is a vector space, and $||\cdot||_1$ drops to give a norm on SIF$/\mathcal{N}$. (SIF$/\mathcal{N}, ||\cdot||_1)$. We need to find the completion. Let $\{b_j\}$ be a Cauchy sequence in B. Then, $f_j = b_j \chi_E$, $\{f_j\}$ is a Cauchy sequence for $||\cdot||_1$. We need B to be complete, so we need a Banach space. Let $\{E_j\}$ be a disjoint collection of $\subseteq \mathcal{S}$, $\mu(E_j) \leq \frac{1}{2^j}$. Choose $b \in B$, $||b|| = 1$. Let

$$f_n = \sum_{j=1}^{n} b\chi_{E_j} = b\chi_{\oplus_{j=1}^{n} E_j},$$

where $\{f_j\}$ is a Cauchy sequence for $||\cdot||_1$. Should converge to

$$\sum_{j=1}^{\infty} b\chi_{E_j} = b\chi_{\infty E_j},$$

and note that

$$\mu\left(\bigoplus_{j=1}^{n} E_j\right) = \sum^{n} \mu(E_j) = \sum^{n} \frac{1}{2^j}.$$

Definition 2.3.4. $(X, \mathcal{S})$, S is a σ-ring. A B-valued function on X is said to be $\mathcal{S}$-measurable if there is a sequence $\{f_n\}$ if SMF that converges pointwise to f, for (for $||\cdot||_B$).