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Least squares hedge for exotic options

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Abstract

Hedging is a core part of financial risk management. When financial institutions issue derivative contracts, they typically hedge their exposures by forming dynamic replicating portfolios consisting of the underlying asset and a risk-free bond. However, dynamic replication requires continuous rebalancing to maintain equivalence with the derivative's payoff, which can be operationally demanding and model-dependent. A static replicating portfolio, in contrast, would eliminate the need for continuous adjustments.

This thesis introduces the least squares hedge as a method for constructing static hedges for various exotic options. The approach builds on the static option replication framework of Derman, Ergener, and Kani, where the boundary conditions of an exotic option are replicated using available vanilla options. In the least squares hedge, the replicating portfolio is determined by solving a linear least squares problem that minimizes the deviation between the portfolio's and the derivative's boundary values.

A case study compares the least squares hedge with conventional dynamic hedging methods and demonstrates its effectiveness in reducing risk. The results show that the static hedge not only eliminates a substantial portion of the risk, but also achieves this at a lower cost than the fair value implied by dynamic pricing models. Hence, the least squares hedge provides a flexible and efficient framework for risk management and may reveal arbitrage opportunities under certain market conditions.

Keywords Hedging, exotic options, financial engineering, quantitative finance, arbitrage

Contents

1 Introduction

Financial markets encompass a vast array of products of varying complexity and specialization. As financial products become more specialized, their customer base tends to narrow, leading to low market liquidity. To help the markets clear, liquidity is provided by financial specialists known as market makers, who quote both buy and sell prices, called *bids* (highest buy quote) and *asks* (lowest sell quote), respectively. The spread between the bid and ask prices works in favor of the market maker. If they can simultaneously sell at the ask price and buy at the bid price, they should be able to profit the spread in a risk-free fashion. However, in illiquid markets this is rarely the case.

Grossman and Miller [1] model the price behavior in markets when buyers and sellers do not arrive simultaneously. They show that while market makers are willing to act as counterparties to facilitate trades, they require a premium for carrying the taken risk until market imbalances disappear. This premium is seen as a temporary price adjustment, which reverses once the market maker is able to offset their position through an opposing trade. The premium, in the form of the bid–ask spread, is the market makers compensation for carrying the risk in the interim.

Grossman and Miller characterize liquid markets as those in which liquidity demands occur frequently or the cost of time is low, meaning that either the duration of risk exposure is short or the cost of hedging that risk is minimal. While the frequency of liquidity demands is inherent to the specific market or product and beyond the market maker's control, the cost of time can be reduced through advancements in financial engineering.

The seminal work by Fischer Black and Myron Scholes on their namesake option pricing model gave market makers in equity derivatives desks the ability to hedge their issued options against the movements of the underlying asset [2]. In this *delta hedging*, the option writer offsets the option's price sensitivity by taking a position in the underlying asset that responds in equal, but opposite magnitude to the changes in the asset's value. Black and Scholes also observed that as the position is riskless, it should provide a return equal to that of a risk-free asset. Thus, a market maker who is able to fully hedge their portfolio should be able to cover their funding costs with the risk-free return while profiting from the bid-ask spread.

Due to the popularity of the Black-Scholes model, similar pricing models have also been developed for other financial instruments. As an example, Vasicek [3] as well as Cox, Ingersoll and Ross [4] have introduced models for the term structure of interest rates. Garman and Kohlhagen [5] among others (see, e.g. [6]), have in turn introduced models for pricing FX options. Thus, there is a plenty of academic literature for hedging various financial instruments based on their sensitivity to the underlying.

However, as the aphorism from statistics goes: "all models are wrong, but some are useful". Indeed, while groundbreaking at the time, the Black-Scholes model has some known shortfalls [7]. Many alternative models have been proposed to remedy them. For example, Heston introduced a model with stochastic volatility term, which better reflects the nature of volatility observed in the markets [8].

Still, even if a perfect model were found, sensitivity-based hedging has some fundamental disadvantages. Sensitivities only hold for infinitesimal changes in the value of the variable, and thus as the asset price process evolves, the position needs to be continuously rehedged. In addition, the choice of the sensitivity or sensitivities to be hedged can affect the effectiveness of protection. For example, once the delta risk is eliminated, the main remaining source of risk is volatility [9] (for a more thorough study on profit-and-loss attribution of a delta-hedged position, see [10] and references there in).

For a trader concerned about volatility risk, Neuberger proposed a contract that would be an alternative to dynamic vega-hedging [9] (for a more complete discussion on trading volatility, see e.g. [11] and references there in). This *log-contract*[1], is a futures-style contract, but unlike traditional futures or options, its settlement price is equal to the logarithm of the price of the underlying. Such specialized contracts fall under the umbrella of over the counter *exotic options*. And while the buyer of the e.g. log-contract can benefit from the more complex instrument, the issuer still faces the need to hedge the taken risk. However, given the increased complexity, a pricing equation may not be readily available. Thus, indirect methods need to be applied.

To price arbitrary *state-contingent claims*, Breeden and Litzenberger proposed using a replicating portfolio of call options [13]. Then, the sensitivities of the portfolio's call options can be used to hedge the taken risk. Although this ends up recreating the issues associated with dynamic hedging of option issuances, it also proposes an interesting prospect of using vanilla options to hedge exotic options. The added benefit of using vanilla options in the hedging is that the hedge could be made static and model-invariant.

Exotic options can be divided into two broad categories. These are *path-dependent* and *path-independent* exotic options. Path-dependent exotic options can change their characteristics at critical price levels before the settlement of the contract. Such exotic options include e.g. barrier options. Path-independent exotic options in turn derive their value purely from the value of the underlying at the settlement date. The aforementioned log-contract is an example of a path-independent exotic option. Due to the inherent differences in the contract types, different hedging approaches need to also be used.

Indeed, Carr, Ellis and Gupta have developed a method for statically hedging some path-dependent exotics with vanilla options by utilizing an extension (and restriction) of the put-call parity, they have called *put-call symmetry* [14]. Similarly, Derman

[1] Another related product would be the volatility swap, which payoff also contains a log-contract [12]

Ergener and Kani have shown how to create static replicating portfolios for various boundary conditions under the Black-Scholes assumptions [15]. They coined their method the *static option replication*, in contrast to the dynamic option replication of Black and Scholes.

However, both of the aforementioned static hedging methods fail to provide a good way to form the static hedges from a finite set of options for arbitrary state-contingent claims. This thesis attempts to remedy this by introducing the *least squares hedge*. This least squares hedge works for path-independent exotic options without making any assumptions about the stock price process or the option pricing equation. For path-dependent exotic options, it can be considered as an extension of the static option replication method. The basic principle behind it is that under the no-arbitrage assumption all portfolios providing the same cashflow should be valued the same. Thus, it is enough to find the portfolio that has the closest state-contingent claim to that of the hedged exotic option. This is an optimization problem, which, as will be shown, can be simplified to a least squares problem. In addition to hedging exotic options, the method is also suitable for valuing them when paired with a chosen option pricing formula.

This thesis is structured as follows. In Chapter 2 the Black-Scholes model is introduced from the point of view of hedging issued vanilla options. Chapter 3 introduces the Breeden-Litzenberger approach to pricing exotic options. Together, these two chapters work as an introduction to the sensitivity based dynamic approach for hedging. In Chapter 4 the least squares hedging method is presented, along with the static options replication method by Derman, Ergener and Kani. The dynamic and static approaches are compared in Chapter 5. Finally, Chapter 6 concludes the thesis with some observations.

2 Dynamic hedging of European options

An option is a financial contract, in which the buyer of the contract is given the right to purchase (call option) or sell (put option) an asset at a predefined price (strike price) at a predefined time in the future (maturity date). This thesis mainly considers call options on the common stock of some company. Furthermore, "ideal conditions" in the market as outlined by Black and Scholes [2] are assumed to hold. Under ideal conditions, the value of the call option depends only on the time t and the value of the underlying S_t, along with known option- and market-specific constants. Thus, an option writer, who has sold a call option, should be able to hedge their position by going long on an appropriate amount Δ of the underlying stock.

To see this, let C be a function giving the value of a call option, K be the strike price, T be the maturity date, σ be the volatility, and S_t the value of the underlying at time t. The hedged position would then be

$$\Pi_\Delta(S_t) = -C(K, T, \sigma; t, S_t) + \Delta S_t,$$

where the semicolon is used to separate the constants and the variables. Assume the value of the underlying changes by some infinitesimal amount dS_t. The change in the value of the position would then be

$$\begin{aligned}\Pi_\Delta(S_t + dS_t) - \Pi_\Delta(S_t) &= C(K, T, \sigma; t, S_t + dS_t) - \Delta(S_t + dS_t) \\ &\quad - (C(K, T, \sigma; t, S_t) - \Delta S_t) \\ &= C(K, T, \sigma; t, S_t + dS_t) - C(K, T, \sigma; t, S_t) - \Delta dS_t.\end{aligned}$$

For a sufficiently small dS_t the change in position value would be $\Pi(S_t + dS_t) - \Pi(S_t) \approx 0$. Then

$$C(K, T, \sigma; t, S_t + dS_t) - C(K, T, \sigma; t, S_t) - \Delta dS_t \approx 0$$

implying

$$\Delta \approx \frac{C(K, T, \sigma; t, S_t + dS_t) - C(K, T, \sigma; t, S_t)}{dS_t},$$

which in the limit gives

$$\Delta = \lim_{dS_t \to 0} \frac{C(K, T, \sigma; t, S_t + dS_t) - C(K, T, \sigma; t, S_t)}{dS_t} = \frac{\partial C}{\partial S}. \tag{2.1}$$

With the result from Equation (2.1) the hedged position becomes

$$\Pi_\Delta = -C(K\,,\ T\,,\ \sigma;\ t\,,\ S_t) + \frac{\partial C}{\partial S} S_t. \tag{2.2}$$

However, Equation (2.2) is only useful if the function C is known and differentiable. What that function is depends on the chosen pricing model. In this thesis, the function C always refers to the Black-Scholes option pricing formula. For completeness, it is derived in Chapter 2.1.

2.1 Black-Scholes option pricing formula

Assume that the stock price S_t follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{2.3}$$

where μ and σ are constants known as the drift and the volatility respectively, and B_t is the Brownian motion. Let $C(K\,,\ T\,,\ \sigma;\ t\,,\ S_t)$ be some as of yet unknown function. By Itô's lemma [16], the function C follows the process

$$dC = \left(\frac{\partial C}{\partial S}\mu S_t + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial C}{\partial S}\sigma S_t dB_t. \tag{2.4}$$

The hedged position is given by Equation (2.2). This hedged position follows the process

$$d\Pi_\Delta = -dC + \frac{\partial C}{\partial S} dS_t. \tag{2.5}$$

Substituting the processes from Equations (2.3) and (2.4) to Equation (2.5) gives

$$d\Pi_\Delta = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\right) dt. \tag{2.6}$$

Note that the Brownian motion is not present in Equation (2.6). Thus, the non-deterministic part of the stock price process has been eliminated and the position should be risk-free. Since the position is riskless it's rate of return should be the same as that of a risk-free asset. Denote the risk-free interest rate by r. Then

$$d\Pi = r\Pi dt.$$

Substituting this to Equation (2.6) gives

$$\left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\right) dt = r\left(-C + \frac{\partial C}{\partial S} S_t\right) dt,$$

which can be reorganized as

$$\frac{\partial C}{\partial t} = rC - rS_t\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S^2}. \tag{2.7}$$

The partial differential equation in Equation (2.7) is known as the Black-Scholes equation. Solving it for C gives the so called Black-Scholes formula that can be used to price options. However, to make the solution unique, boundary conditions need to be introduced. These are

$$\begin{aligned} C(K, T, \sigma; t, 0) &= 0 & \forall\ t \\ \lim_{S_t \to \infty} C(K, T, \sigma; t, S_t) &= \lim_{S_t \to \infty} S_t - K\exp(-r(T-t)) & \forall\ t \\ C(K, T, \sigma; T, S_T) &= \max\{0, S_T - K\} & \forall\ S_T. \end{aligned} \tag{2.8}$$

The first boundary condition in Equation (2.8) can be interpreted as if the company goes bankrupt, the stock of the company becomes permanently worthless, i.e. $S_t = 0$. In such a case, the option will also become worthless. The second boundary condition holds when the option goes deep in-the-money. Then the option can be assumed to also expire in-the-money and the option value should be the discounted payoff. The final boundary condition is the state-contingent claim for the call option.

With a change of variables

$$\begin{aligned} \tau &= T - t \\ u &= C\exp(r\tau) \\ x &= \ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau, \end{aligned}$$

the Equation (2.7) becomes a diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2},$$

which can be solved with the boundary conditions from Equation (2.8) to obtain the Black-Scholes formula as seen in Equation (2.9). A visualization of the call option price surface is in Figure 1.

$$\begin{aligned} C(K, T, \sigma; t, S_t) &= S_t\mathcal{N}(d_+) - K\exp(-r(T-t))\mathcal{N}(d_-) \\ d_\pm &= \frac{\ln(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ \mathcal{N}(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(-\frac{1}{2}y^2\right)\, dy. \end{aligned} \tag{2.9}$$

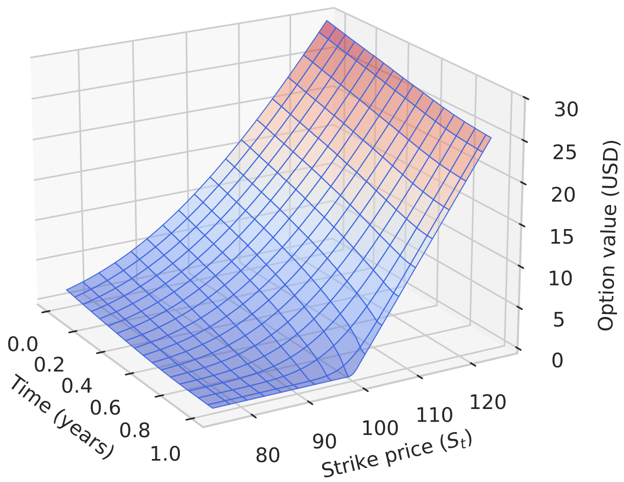

Figure 1: An example option price surface calculated using the Black-Scholes option pricing formula from Equation (2.9). In this example the risk-free rate r is 0.02 and the volatility σ is 0.25.

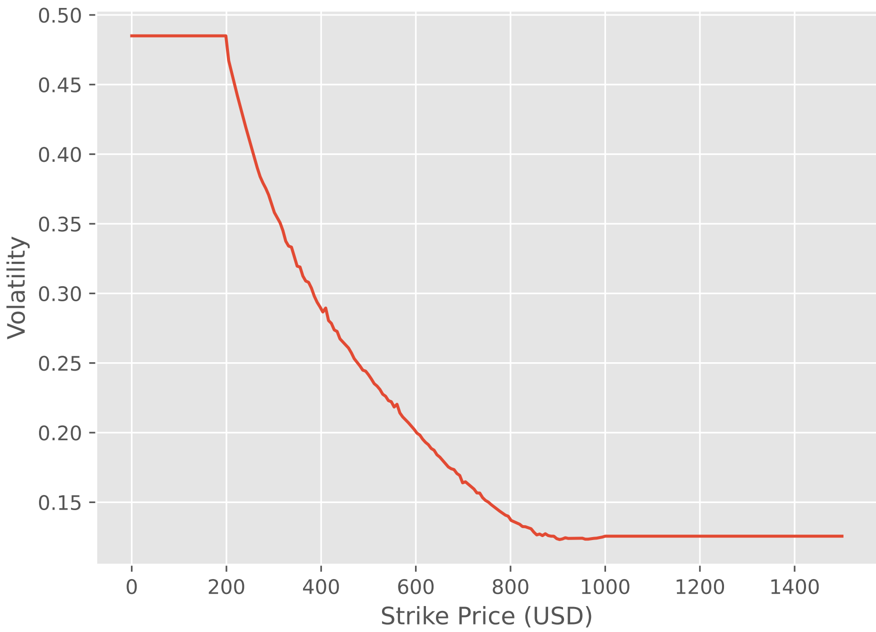

Figure 2: Volatility curve for $SPY options maturing on 2027-12-17. The option prices were quoted after market close on 2025-08-15.

By applying the put-call parity [17] with discount factor $\exp(-r(T-t))$, the value of a put option under the Black-Scholes model can be given as

$$P(K\,,\; T\,,\; \sigma;\; t\,,\; S_t) = K\exp(-r(T-t))\mathcal{N}(-d_-) - S_t\mathcal{N}(-d_+), \tag{2.10}$$

where $\mathcal{N}$ and $d_\pm$ are as in Equation (2.9).

2.2 Market constants and implied volatility

The Black-Scholes formula in Equation (2.9) is dependent on two market-specific constants - the risk-free rate r and the volatility σ. In order to properly estimate the sensitivities, a good choice of constants is required.A market maker employed by a bank typically funds their positions through loans obtained from the bank's treasury, which may, for instance, be linked to an interbank offered rate. Consequently, the interbank offered rate serves as the effective risk-free rate. For traders without access to such funding, the government bond yields from major nations (e.g., United States and Germany) are the least risky available investment. For both cases, it can be assumed that the risk-free rate is known with certainty.

The volatility is less clear as a concept. Black and Scholes define it as the standard deviation of the stock returns [2] in their derivation of the Black-Scholes formula. However, the historical standard deviation can be problematic, as it assigns equal weight to all past observations. In practice, more recent stock price developments typically carry more relevant information. Two widely used models that incorporate this idea are the Exponentially Weighted Moving Average model (EWMA) [18] and the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) [19, 20] model.

All of the aforementioned volatility models still suffer from a fundamental flaw - their estimates are backwards looking. The sensitivities given by the option pricing model would be more useful if they were based on the assumed future development of the stock price, not the past development. Such an estimate can be calculated by solving the volatility $\sigma_I(T\ ,\ K;\ t\ ,\ S_t)$ for which the observed option price C^* and the one given by the option pricing model C are equal. This estimate of volatility is known as the option *implied volatility*.

The implied volatility can be found via numerical root finding methods such as the bisection method. However, unlike the volatility estimates that are based on stock price data and are invariant of the option parameters, the implied volatility varies between available options. Not only does the implied volatility vary between options of different maturities, but also with options of different strike prices for the same maturity. This behavior is known as the *volatility smile* (or volatility skew) and is characterized by volatilities that decrease exponentially as a function of the strike price. An example of volatility smile is visualised in Figure 2.

The drawback with using the implied volatilities is that they are only observable for options with strikes that are actively traded in the market. For other strike prices, the corresponding volatilities must be obtained through interpolation or extrapolation. A commonly used interpolation method (see, e.g., [21]) is the cubic spline method [22].

The cubic spline interpolation can be done directly in the strike price-volatility space or, as Bliss and Panigirtzoglou propose, in the delta-volatility space [21, 23]. They show that the interpolation in delta-volatility space can provide slightly more stable results, but with the added complexity of needing to convert the delta to strike price in order to calculate the needed sensitivities. However, to keep the interpretation of the implied volatility simple in the coming Chapters, this thesis uses interpolation in the strike price-volatility space.

As there exists a finite number of options in the markets, interpolation can only cover a finite region of the strike price-volatility space. Beyond the smallest and greatest available strike prices the implied volatility needs to be extrapolated. Carr and Wu propose using flat extrapolation [24] where beyond the available strikes the last available implied volatility is used. This has the benefit of being independent of the interpolation method or possible parametric model such as SVI [25]. Additionally, flat extrapolation is numerically safe as the implied volatilities will not grow beyond observed levels, guaranteeing that the tails have a limited impact on the integrals in Chapter 3.

2.3 Delta-vega hedging

Knowing the risk-free rate and the volatility, the pricing formula in Equation (2.9), can be utilized to determine the position in Equation (2.2). The amount of the underlying asset needed in the hedged position would be

$$\Delta = \frac{\partial C}{\partial S} = \mathcal{N}(d_+). \tag{2.11}$$

The sensitivity of the option price to the value of the underlying is colloquially known as the *delta*, and a position where the delta is zero is delta-hedged. Other sensitivities are also given their own Greek letter and together they are known as “the Greeks”. A non-exhaustive list of the Greeks is given in Equation (2.12)

$$\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} \\
\Gamma &= \frac{\partial^2 C}{\partial S^2} \\
\nu &= \frac{\partial C}{\partial \sigma} \\
\Theta &= \frac{\partial C}{\partial t} \\
\rho &= \frac{\partial C}{\partial r}.
\end{aligned} \tag{2.12}$$

If the ideal conditions as outlined by Black and Scholes hold, then the delta-hedge should be enough to remove all risk. However, this is not the case, as both the risk-free interest rate (usually in the context of the term structure) [3, 4] and the volatility [8] can be modeled as a stochastic process. Of these, the stochastic volatility, as taken into account in the Heston model, is the more common choice for hedging.

From the Black-Scholes formula in Equation (2.9), the sensitivity to volatility can be stated as

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{S_t \exp(-d_+^2/2)\sqrt{T-t}}{\sqrt{2\pi}}.$$

Assume that a market maker has sold a call option $C_1(K, T_1, \sigma; t, S_t)$. To vega neutralize this position they would need to go long on some amount η of an equivalent call option $C_2(K, T_2, \sigma; t, S_t)$, where $T_2 > T_1$. This hedged position can be written as

$$\Pi_\nu(\sigma) = \eta C_2(K, T_2, \sigma; t, S_t) - C_1(K, T_1, \sigma; t, S_t).$$

If the volatility σ were to change by some small amount $d\sigma$, the value of the hedged position would change by

$$\begin{aligned}\Pi_\nu(\sigma + d\sigma) - \Pi_\nu(\sigma) = \eta C_2(K\,,\ T_2\,,\ \sigma + d\sigma;\ t\,,\ S_t) \\ - C_1(K\,,\ T_1\,,\ \sigma + d\sigma;\ t\,,\ S_t) \\ - (\eta C_2(K\,,\ T_2\,,\ \sigma;\ t\,,\ S_t) - C_1(K\,,\ T_1\,,\ \sigma;\ t\,,\ S_t)).\end{aligned} \tag{2.13}$$

Setting $\Pi_\nu(\sigma + d\sigma) - \Pi_\nu(\sigma) \approx 0$ in Equation (2.13) gives

$$\eta \approx \frac{C_1(K\,,\ T_1\,,\ \sigma + d\sigma;\ t\,,\ S_t) - C_1(K\,,\ T_1\,,\ \sigma;\ t\,,\ S_t)}{C_2(K\,,\ T_2\,,\ \sigma + d\sigma;\ t\,,\ S_t) - C_2(K\,,\ T_2\,,\ \sigma;\ t\,,\ S_t)}.$$

Finally, in the limit the value of η becomes

$$\begin{aligned}\eta &= \lim_{d\sigma \to 0} \frac{C_1(K\,,\ T_1\,,\ \sigma + d\sigma;\ t\,,\ S_t) - C_1(K\,,\ T_1\,,\ \sigma;\ t\,,\ S_t)}{C_2(K\,,\ T_2\,,\ \sigma + d\sigma;\ t\,,\ S_t) - C_2(K\,,\ T_2\,,\ \sigma;\ t\,,\ S_t)} \\ &= \frac{\nu_1}{\nu_2}.\end{aligned}$$

Let Π_ν be the vega-neutral position. In order to both vega and delta protect the sold option, the position Π_ν needs to be delta-hedged, i.e. some amount α of the underlying needs to be purchased. The amount α should be the position's sensitivity to the change in the value of the underlying

$$\alpha = \frac{\partial \Pi_\nu}{\partial S} = \eta \frac{\partial C_2}{\partial S} + \frac{\partial C_1}{\partial S} = \frac{\nu_1}{\nu_2}\Delta_2 - \Delta_1. \tag{2.14}$$

From Equation (2.14) the delta-vega hedged position can be stated as

$$\Pi_{\Delta,\nu} = \eta C_2 + \alpha S_t - C_1. \tag{2.15}$$

In comparison to delta hedging, delta-vega hedging is already significantly more complicated. Additionally, from the point of view of an option writer, dynamically rehedging the vega is very impractical, as they would need to constantly issue new option contracts in appropriate amounts to maintain the protection. If this were possible, the option writer should either enter into a long contract on the same option they have sold, or go long on an equivalent put option and a single share of the underlying. These approaches would match the state-contingent claim of the sold option and thus statically hedge the obligations at settlement. In practice, this may not be possible and a market maker must carry some risk due to imperfect hedging. This becomes even more apparent in the case of exotic options as will be shown in Chapter 3.

3 Dynamic hedging of state-contingent claims

To hedge state-contingent claims on exotic options, a trader would either require a pricing model for each contract type or a way to generalize e.g. the Black-Scholes model from vanilla options to more complicated products. Indeed, both are possible, but this thesis will focus on the latter. Breeden and Litzenberger were the first to observe that arbitrary state-contingent claims can be replicated using just call options, given that a continuity of strike prices prevails. Their work builds on the concept of state prices introduced by Arrow and Debreu in their general equilibrium model [26]. The state prices are introduced in Chapter 3.1 and the Breeden-Litzenberger model in Chapter 3.2. For a more formal derivation of the completeness of the replicating portfolio, see [27] and [28].

3.1 State prices

Arrow and Debreu consider the so called Arrow-Debreu securities (elementary claims in Breeden and Litzenberger's work), which are securities that pay an unit amount of some numeraire if a given future state is realized and nothing otherwise. Let $\mathcal{S} = \{s_T^{(1)}, s_T^{(2)}, ...s_T^{(n)}\}$ be the set of n possible future states at period T and S_T be a a random variable giving the future state at period T. An Arrow-Debreu security on state $s_T^{(i)} \in \mathcal{S}$ would then have the payoff

$$v^{(i)}(T\,,\ S_T) = \begin{cases} 1 & \text{if } S_T = s_T^{(i)} \\ 0 & \text{otherwise.} \end{cases}$$

The present value of an Arrow-Debreu security is known as the state's *state price*. Arrow and Debreu propose that the state price is equal to the discounted probability of the state occurring. To see why such would be the case consider a simple example.

Assume the future states are tied to the results of some sport's game, which can end in one of two ways - either the home team wins and the away team loses or the away team wins and the home team loses. Denote these states by $s_T^{(H)}$ and $s_T^{(A)}$ for the home team winning and the away team winning, respectively. The random variable S_T associates a probability for both states occuring as $\mathbb{P}[S_T = s_T^{(H)}] = p_H$ and $\mathbb{P}[S_T = s_T^{(A)}] = p_A$ such that the events $S_T = s_T^{(H)}$ and $S_T = s_T^{(A)}$ are mutually exclusive and collectively exhaustive.

Clearly, if an investor were to buy an Arrow-Debreu security on both states $s_T^{(H)}$ and $s_T^{(A)}$, they would be guaranteed a unit payoff in the given numeraire. The position would be equivalent to a riskless asset and it's present value equal to the unit amount discounted to present day. The state prices at period t can be written as

$$v^{(H)}(t\,,\ S_T) + v^{(A)}(t\,,\ S_T) = e^{-r(T-t)} = e^{-r(T-t)}(w_H + w_A),$$

where the weights $w_H > 0$ and $w_A > 0$ sum up to one and are associated with the states $s_T^{(H)}$ and $s_T^{(A)}$, respectively. Arrow and Debreu propose that the value of the Arrow-Debreu security is directly proportional to the probability of the state occuring. Thus, must hold that $w_H = p_H$ and $w_A = p_A$. The state prices are then

$$\begin{aligned} v^{(H)}(t,\ S_T) &= e^{-r(T-t)} p_H \\ v^{(A)}(t,\ S_T) &= e^{-r(T-t)} p_A. \end{aligned}$$

More generally, with a set $\mathcal{S}$ of n states, this result can be stated as

$$v^{(i)}(t,\ S_T) = e^{-r(T-t)} \mathbb{P}[S_T = s_T^{(i)}], \tag{3.1}$$

where $s_T^{(i)} \in \mathcal{S}$

Assume there exists some state-contingent claim E on the states in set $\mathcal{S}$ that an investor would like to price. With the state prices, as defined in Equation (3.1), the present value of the state-contingent claim on period t becomes

$$E(T;\ t,\ S_t) = \sum_{i=1}^{n} E(T;\ T,\ s_T^{(i)}) v^{(i)}(t,\ S_t). \tag{3.2}$$

3.2 Breeden-Litzenberger

As the consideration is expanded to arbitrary state-contingent claims, the set of states $\mathcal{S}$ needs to be redefined. Let the set of possible future states be $\mathcal{S} = \{s \in \mathbb{R} \mid s \geq 0\}$, i.e. the non-negative real line. These states represent the possible future values that the stock price process S_T can take at period T.

Assume that there exist call options with strike prices for each of the states in the set $\mathcal{S}$ (meaning there are call options infinitely densely on the non-negative real line). Under these assumptions, Breeden and Litzenberger propose using three call options with strikes $K - dS_t$, K and $K + dS_t$, for some small and positive value dS_t, to form a portfolio

$$\begin{aligned} e(K,\ T;\ t,\ S_t) = \frac{1}{dS_t}&([C(K - dS_t,\ T,\ \sigma;\ t,\ S_t) - C(K,\ T,\ \sigma;\ t,\ S_t)] \\ &- [C(K,\ T,\ \sigma;\ t,\ S_t) - C(K + dS_t,\ T,\ \sigma;\ t,\ S_t)]). \end{aligned} \tag{3.3}$$

Consider the portfolio in Equation (3.3). By going long $1/dS_t$ units of the call option with strike $K - dS_t$, the portfolio provides unit payoff if $S_T = K$ and no payoff if $S_T \leq K - dS_t$. By shorting $2/dS_t$ units of the call option with strike K and going long $1/dS_t$ units of the call option with strike $K + dS_t$, the portfolio provides no payoff if $S_T \geq K + dS_t$. Thus, the payoff deviates from the definition of an Arrow-Debreu

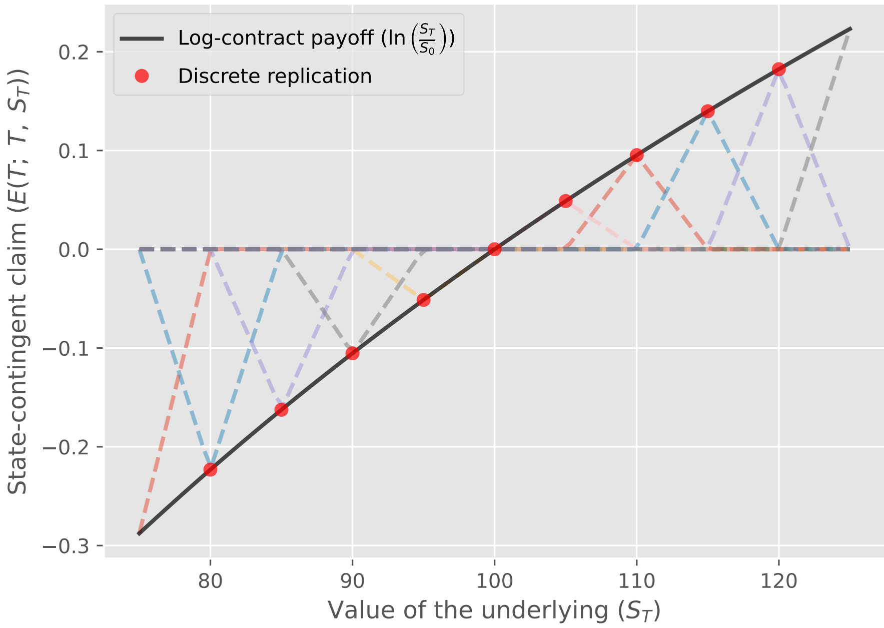

Figure 3: State-contingent claim of a log-contract and the replication of it in discrete points using the Breeden-Litzenberger method. The points represent the discrete replication using an elementary claim (Arrow-Debreu security) to match the wanted payoff. The dashed lines represent the option payoffs.

security only on the intervals $(K - dS_t\,,\ K)$ and $(K\,,\ K + dS_t)$. So in the limit $dS_t \to 0$ the portfolio in Equation (3.3) approaches the Arrow-Debreu security.

Breeden and Litzenberger observed that in the limit the portfolio from Equation (3.3) divided by the step size dS_t becomes

$$\lim_{dS_t \to 0} \frac{e(K\,,\ T;\ t\,,\ S_t)}{dS_t} = \frac{\partial^2 C}{\partial S^2}. \tag{3.4}$$

Let $E(T;\ t\,,\ S_t)$ be a function giving the value of an exotic option maturing at time T, on asset S_t at time t. The exotic option has a known state-contingent payoff $E(T;\ T\,,\ S_T)$ and potentially some boundary conditions or cashflows for $t < T$. By applying the expression from Equation (3.4), the function E can be defined as

$$E(T;\ t\,,\ S_t) = \int_t^T \int_0^\infty E(\tau;\ \tau\,,\ K) \frac{\partial^2 C(K\,,\ \tau\,,\ \sigma;\ t\,,\ S_t)}{\partial S^2}\, dK d\tau. \tag{3.5}$$

A discrete example of a replicating portfolio of a single state-contingent claim is in

Figure 3. Note that so far no assumptions have been made about the function C. However, to hedge the exotic options, the function C needs to be known and twice differentiable. As in the original paper by Breeden and Litzenberger [13], in this thesis the Black-Scholes formula from Equation (2.9) will be used. Then the second partial derivative (or Γ as in Equation (2.12)) is

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\mathcal{N}'(d_+)}{S_t \sigma \sqrt{T-t}}, \tag{3.6}$$

where $\mathcal{N}'(z) = \exp(-z^2/2)/\sqrt{2\pi}$ is the standard normal probability density function. Substituting Γ from Equation (3.6) into Equation (3.5) gives the pricing formula

$$E(T;\ t,\ S_t) = \int_t^T \int_0^\infty E(\tau;\ \tau,\ K) \frac{\mathcal{N}'(d_+)}{S_t \sigma \sqrt{\tau - t}}\, dK d\tau, \tag{3.7}$$

from which the sensitivities for a given state-contingent claim E can be calculated. In this thesis, the delta Δ and the vega ν sensitivities of the exotic option are of specific interest. Under the Breeden-Litzenberger model, these are

$$\begin{aligned}
\Delta^{(BL)} &= \frac{\partial E}{\partial S_t} \\
&= \int_t^T \int_0^\infty -E(\tau;\ \tau,\ K) \frac{d_+ \mathcal{N}'(d_+) + \sigma\sqrt{\tau - t}\mathcal{N}'(d_+)}{(S_t \sigma \sqrt{\tau - t})^2}\, dK d\tau \\
\nu^{(BL)} &= \frac{\partial E}{\partial \sigma} \\
&= \int_t^T \int_0^\infty -E(\tau;\ \tau,\ K) \frac{\sigma d_+ \mathcal{N}'(d_+)(d_+/\sigma - \sqrt{\tau - t}) - \mathcal{N}'(d_+)}{S_t \sigma^2 \sqrt{\tau - t}}\, dK d\tau.
\end{aligned} \tag{3.8}$$

The delta and vega hedging of an issued exotic option follows the same procedure as for vanilla options described in Chapter 2.3, with the only difference being that the sensitivities are determined by Equation (3.8). As such the procedure is not repeated here. However, it is worth noting that as this is the case, the drawbacks from dynamically hedging vanilla options also extend to exotic options.

3.3 Implied volatilities and option implied probability density

Chapter 2.2 discusses the implied volatilities of options calculated from their market prices. For the price E from Equation (3.7) to reflect the market expectations, the volatility term in the integrand should vary with the maturity date and the strike price. This leads to an expression

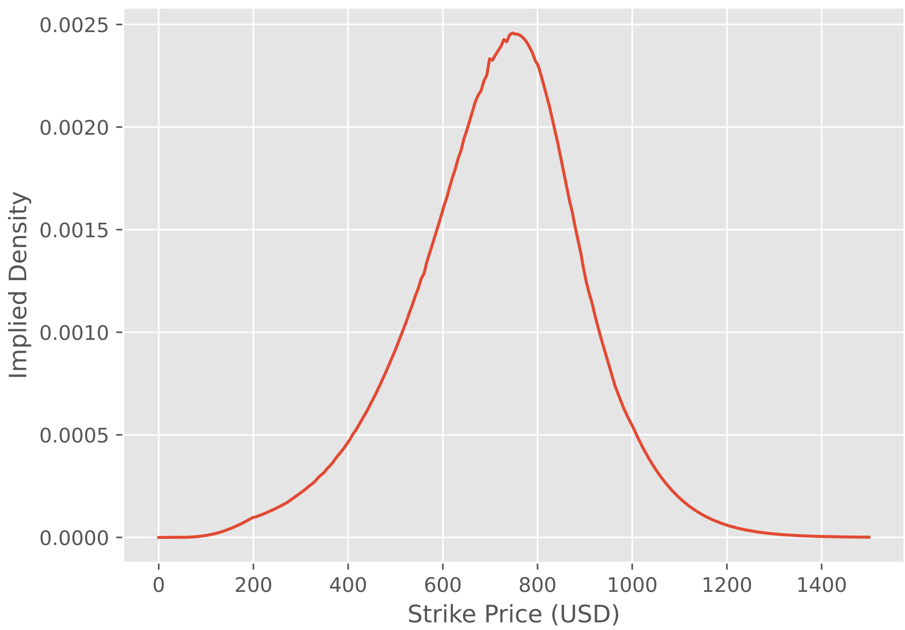

Figure 4: Implied probability density for $\$SPY$ options maturing on 2027-12-17 calculated from the volatility curve in Figure 2. The probability density was normalized over the interval $[0\,,\;1500]$.

$$E(T;\ t\ ,\ S_t) = \int_t^T \int_0^\infty E(\tau;\ \tau\ ,\ K) \frac{\mathcal{N}'(d_+)}{S_t \sigma_I(\tau\ ,\ K;\ t\ ,\ S_t)\sqrt{\tau - t}}\, dK d\tau, \tag{3.9}$$

where $\sigma_I(T\ ,\ K;\ t\ ,\ S_t)$ is the implied volatility calculated from the market price C^* for a call option that matures in period T with strike price K, when the value of the underlying is S_t at period t.

Consider the expression in Equation (3.9). In the expression, the state price for a given state $K \in \mathcal{S}$ (as in Chapter 3.2 $\mathcal{S} = \{x \in \mathbb{R} \mid x \geq 0\}$) is given by the option's Γ. By Equation (3.1) the probability density for the underlying process S_t to have value K at maturity T is

$$\mathbb{P}[S_T = K] = e^{r(T-t)} \frac{\mathcal{N}'(d_+)}{S_t \sigma_I(T\ ,\ K;\ t\ ,\ S_t)\sqrt{T - t}} \propto \Gamma. \tag{3.10}$$

The probability density can be calculated for all possible states $K \in \mathcal{S}$, giving the *option implied probability density*. Unlike probability densities calculated from historical data, the option implied probability density is forward looking and provides the prevailing market estimate for the future value of the underlying. An example of an option implied probability density calculated using the implied volatility curve from Figure 2 is in Figure 4.

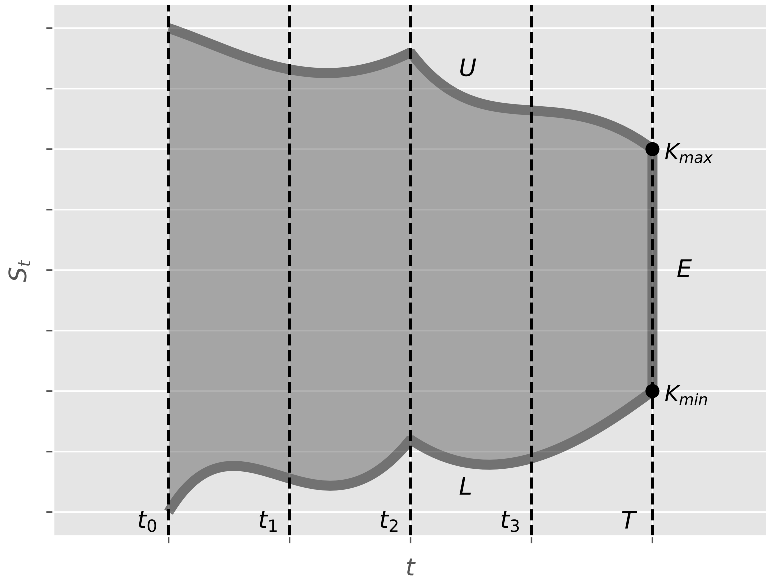

Figure 5: Arbitrary boundary in some $(S_t\ ,\ t)$ plane. In the figure U denotes the upper boundary, L the lower boundary, and E the state-contingent claim at termination $E(T;\ T\ ,\ S_T)$. The points K_{max} and K_{min} represent the maximum and minimum strikes on the termination boundary.

4 Static hedging of exotic options

Chapters 2.1 and 3 described the hedging of vanilla and exotic options based on their sensitivity to the movement of the underlying asset. The chapters also outline the challenges associated with such approaches, namely the need to continuously rehedge the position and the limitations of the possible option pricing models. Thus, many static alternatives have been proposed as an answer to one or both of these limitations.

There are many ways to statically hedge exotic options. The static hedging approach by Carr, Ellis, and Gupta [14] is built on put-call parity and only works for certain path-dependent exotic options. Due to its narrow scope, it will be excluded from this thesis. The static option replication method by Derman, Ergener, and Kani [15] is introduced in Chapter 4.1. The introduction will be very surface level one and the reader is encouraged to read the original paper. The basic idea behind static option replication is applied extensively when deriving the least squares hedge, which in turn will be introduced in Chapter 4.2.

4.1 Static option replication

To understand static option replication, consider that there exists an exotic option defined by the arbitrary boundary in Figure 5. The exotic option is path-dependent, and it's value will change based on where on the boundary the stock price process crosses it. Derman, Ergener and Kani propose that a portfolio of standard options that replicates the exotic option on the boundary, without providing additional cashflows within it, must have the same value as the target exotic option.

Derman, Ergener, and Kani arrive at this conclusion by considering option valuation through binomial backward induction. In the continuous-time limit, this discrete framework converges to the Black–Scholes partial differential equation given in Equation (2.7). The value of the exotic option is then obtained as the solution to this PDE subject to the exotic option's boundary conditions. Hence, it is sufficient that the replicating portfolio matches the exotic option's values along the boundary, since both satisfy the same underlying price dynamics governed by the Black–Scholes equation.

The boundary in Figure 5 is divided into three sub-boundaries: U for the upper boundary, L for the lower boundary, and E for the settlement boundary. The upper boundary can be matched by using out of the money (OTM) call options and the lower boundary by using OTM put options. The settlement boundary can be matched by either call or put options without restrictions on the strike prices.

Assume there are options available for periods t_1, t_2, t_3, T as seen in Figure 5. The static replicating portfolio should match the value of the exotic on the boundary at these periods. First, the settlement boundary E is replicated for period T. Derman, Ergener, and Kani do not provide a way for doing this for any non-trivial payoffs, leaving it up to the reader.

The portfolio of options replicating the settlement boundary will have some theoretical present value on the upper and lower boundary at the earlier periods. Assuming that the present value does not match the boundary value at period t_3, some amount will need to be added or reduced. On the upper boundary, this can be done by buying (or selling) needed amount of call options with strike price greater than the largest possible value of the underlying on the settlement boundary (denoted by K_{max} in Figure 5). Similar can be achieved for the lower boundary by buying (or selling) put options maturing at period T with strike price less than K_{min}.

In general, for an arbitrary period $t_p \neq T$, the boundary values can be replicated by entering a position in OTM call or put options maturing in period t_{p+1} and taking into account the present values of the options maturing in periods $t > t_p$. However, finding the optimal selection of such options can be challenging and Derman, Ergener, and Kani do these calculations very manually. The least squares hedge proposed in this thesis provides a systematic method as an alternative.

4.1.1 Up-and-out call option as an example

Consider an up-and-out call option, with a strike of 100, barrier at 120, 0 rebate and 21 periods (e.g. days) to maturity. The boundaries of this option are shown in Figure 6.

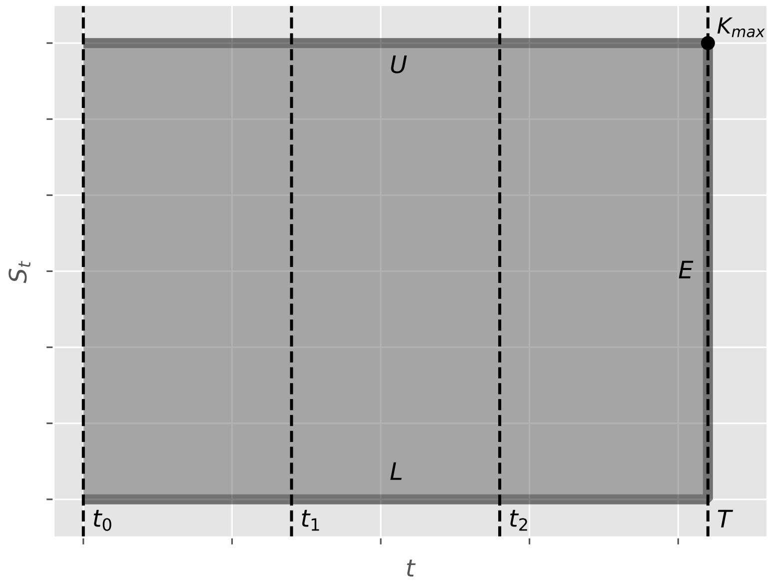

Figure 6: Boundary for an up-and-out call option. The boundary consist on the termination boundary E, which is a call option payoff, and an upper boundary U at which the up-and-out option becomes worthless. The trivial lower boundary for when the underlying company defaults is also included as boundary L, but is not considered in the calculations.

ID	Strike	Expiration period	Value (t=t_0, S_t=100)	Value (t=t_0, S_t=120)	Value (t=t_1, S_t=120)	Value (t=t_2, S_t=120)	Value (t=T, S_t=120)
1	100	$T = 21$	3.032	20.191	20.121	20.056	20
2	120	$T = 21$	0.019	3.638	2.990	2.028	0
3	120	$t_2 = 14$	~ 0	2.990	2.170	0	0
4	120	$t_1 = 7$	~ 0	2.028	0	0	0

Table 1: Options available for hedging the up-and-out call option

On the upper boundary U, the option expires without value. The settlement boundary is defined as the state-contingent claim of a vanilla call option with a strike price of 100. The value of the underlying at period t_0 is 100, the risk-free rate is 4%, dividend yield is 0%, and the volatility is 25%.

Assume that the only available options in the markets are as outlined in Table 1. To replicate the state-contingent claim at the settlement boundary, a single contract of the option with ID 1 needs to be purchased. Denote this by $\#_1 = 1$. However, now the target upper boundary value of 0 at period $t_2 = 14$ is not satisfied. To match it

$$\#_2 = -20.056 \cdot \#_1 / 2.028 = -9.890$$

contracts of option with ID 2 must be purchased (i.e. 9.890 contracts sold). Similarly, in order to match the target upper boundary value at period $t_1 = 7$,

$$\#_3 = -(20.121 \cdot \#_1 + 2.990 \cdot \#_2)/2.170 = 4.355$$

contracts of option with ID 3 must be purchased. Finally, at period $t_0 = 0$

$$\#_4 = -(20.191 \cdot \#_1 + 3.638 \cdot \#_2 + 2.990 \cdot \#_3)/2.028 = 1.365$$

contracts of option with ID 4 must be purchased.

The static hedge found in the above fashion holds exactly in only a very finite number of points. However, if these points are close enough, Derman, Ergener and Kani argue that the deviation from the target value on the rest of the boundary should not be significant. However, they make no claim for the optimality of the static hedge.

4.2 Least squares hedge

4.2.1 Path-independent exotic option

To see how a path-independent exotic option could be hedged by solving a least squares problem, consider an exotic option with state-contingent claim $E(T;\, T,\, S_T)$ on some underlying stock price process S_t. Assume that in the markets there exist n call options maturing in period T on the underlying S_t with strikes in the set $\mathcal{K} = \{K_1,\, K_2,\, \ldots,\, K_n\}$. The objective is to form a portfolio from the available options with weights $\mathbf{x} \in \mathbb{R}^n$ that solves the optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \int_0^\infty \left| \left(\sum_{i=1}^{n} x_i C(K_i,\, T,\, \sigma_I(T,\, K_i);\, T,\, s) \right) - E(T;\, T,\, s) \right| \, ds \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^n. \end{aligned} \tag{4.1}$$

The problem in Equation (4.1) can be interpreted as finding the amounts $\mathbf{x}$ for which the weighted sum of the option payoffs is closest to the target state-contingent claim in all possible future states $\mathcal{S} = \{s \in \mathbb{R} \mid s \geq 0\}$. However, evaluating the functions C and E in infinitely many points is not numerically feasible. A more practical approach arises from discretizing the problem.

Let $\mathbf{E} \in \mathcal{S}^d$ be the discretization of the state-contingent claim E into d points and $\mathbf{C}^{(i)} \in \mathcal{S}^d$ be the discretized payoff of a call option with strike price $K_i \in \mathcal{K}$. How the functions are discretized is up to the reader. For example, if the current value of the underlying is S_0, the interval $[S_0 - S_0/2\,,\ S_0 + S_0/2]$ could be broken down into d evenly spaced points. Alternatively, the extremum of the available strike prices in set $\mathcal{K}$ can be used as the bounds of the interval.

The weights $\mathbf{x}$ can be found by solving the linear system of equations

$$A\mathbf{x} = [\mathbf{C}^{(1)} \quad \mathbf{C}^{(2)} \quad \ldots \quad \mathbf{C}^{(n)}]\mathbf{x} = \mathbf{E}, \tag{4.2}$$

where the matrix $A \in \mathcal{S}^{d\times n}$ has the option payoffs as its columns. However, as there is no guarantee that an exact solution to Equation (4.2) exists, it can be replaced by a least squares approximation as seen in Equation (4.3)

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \|A\mathbf{x} - \mathbf{E}\|_2^2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^n. \end{aligned} \tag{4.3}$$

A linear least squares problem can be solved by solving the system of normal equations [29]

$$A^T A\mathbf{x} = A^T\mathbf{E}. \tag{4.4}$$

The system matrix for normal equations $A^T A$ is positive definite and thus invertible. Hence, the Equation (4.4) can be solved by inverting the system matrix and multiplying both sides by the inverse. However, for large and sparse matrices (as the matrix A can be) this is not necessarily the most efficient approach. Thus, a numerical scheme is introduced in Chapter 4.2.4.

Note that the least squares hedge for path-independent exotic options only depends on the value of the contract at maturity. Thus, it is model-invariant. This is in contrast to the dynamic hedging approaches and is a significant benefit of the approach due to decreased model risk.

4.2.2 Path-dependent exotic option

To extend the least squares hedge from path-independent exotic options as introduced in Chapter 4.2.1 to path-dependent ones, the upper and lower boundary must also be considered. Essentially, the problem then becomes how to represent the static option replication from Chapter 4.1 as a least squares problem.

Static option replication is applicable to exotic options defined by their boundary conditions as discussed in Chapter 4.1. In the used interpretation there can be at most three boundaries (upper, lower, and settlement), and there has to be at least one. An exotic option defined by a single boundary might not be the most intuitive concept, but is apparent via a few examples.

It is clear that an exotic option could be defined just by the settlement boundary as is the case with all path-independent exotic options (although the trivial lower boundary at $S_t = 0$ is also present). Exotic options defined just by either the upper or lower boundary would be perpetuals. Most common example of such exotic options are the perpetual American call (defined by upper boundary) and put (defined by lower boundary) options. Although perpetual American options are not strictly necessary to exercise under any conditions, Merton [30] argues that for any point in time there exists some level for the value of the underlying at which a rational investor would exercise the option.

Each boundary can be treated more or less interchangeably, with the nuance that on the settlement boundary the vanilla option's value is a function of the underlying's value, while on the upper and lower boundary it is a function of time. This time dependency means that the theoretical present values of the options needs to be known, which requires applying an option pricing model.

Knowing that the value of each vanilla option can be represented as a function of a single variable means that they can be discretized with regards to that variable. Assume there exists some set $\mathcal{O}$ of N options containing both calls and puts. The set of options can be assumed to be valid in the context of static option replication - the furthest maturity date is the same as with the settlement boundary and there are arbitrary call and/or put options available for it, while options with shorter time to maturity are either OTM calls or OTM puts with strikes beyond their respective boundaries.

Without loss of generality, assume the options are discretized into d points on each boundary[2]. Denote these discretizations on some put option on i:th index in the set $\mathcal{O}$ by $\mathbf{P}_L^{(i)}$, $\mathbf{P}_U^{(i)}$ and $\mathbf{P}_E^{(i)}$ for the lower, upper, and settlement boundary respectively. The same notation is extended to call options as well. Additionally, let $\mathbf{L} \in \mathbb{R}^d$, $\mathbf{U} \in \mathbb{R}^d$ and $\mathbf{E} \in \mathbb{R}^d$ be the dicretized target boundary values for the lower, upper

[2]The boundaries do not need to be discretized to equal amount of points, but it does make the notation slightly simpler

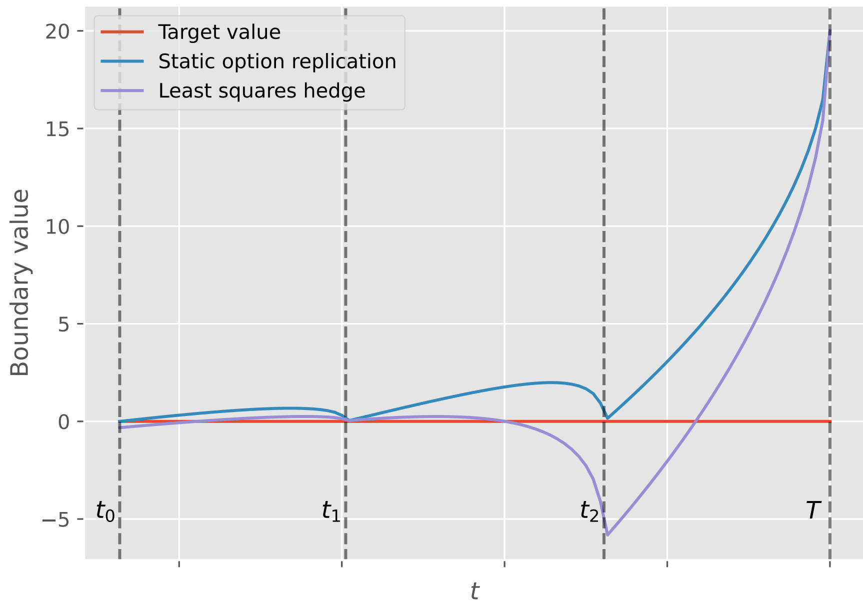

Figure 7: The values of the static option replication and the least squares hedge on the upper boundary. The solution hedges are defined in Chapters 4.1.1 and 4.2.3 for the static option replication and least squares hedge respectively.

and settlement boundaries respectively. The system of equations with these multiple boundaries is then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{C}_E^{(1)} & \mathbf{C}_E^{(2)} & \dots & \mathbf{P}_E^{(N)} \\ \mathbf{C}_L^{(1)} & \mathbf{C}_L^{(2)} & \dots & \mathbf{P}_L^{(N)} \\ \mathbf{C}_U^{(1)} & \mathbf{C}_U^{(2)} & \dots & \mathbf{P}_U^{(N)} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{E} \\ \mathbf{L} \\ \mathbf{U} \end{bmatrix}. \tag{4.5}$$

The system of equations in Equation (4.5) can be solved in the same way as in the path-independent case. First, it needs to be considered as a least squares problem as in Equation (4.3). Then the least squares problem can be solved by solving the system of normal equations seen in Equation (4.4).

4.2.3 Up-and-out call option as an example

Consider the example from Section 4.1.1. To solve it in the way introduced by Derman, Ergener and Kani required first finding the proper portfolio of options that replicates the settlement boundary. Then the upper boundary is replicated by iteratively going backwards and using a subset of the OTM options to match the target value. This is less than optimal as it replicates the upper boundary in only a very few points and requires unspecified insight into replicating the settlement boundary (assuming it is not trivial as it is in this case). Instead, the problem can be represented as a linear system of equations

$$\begin{bmatrix} 20 & 0 & 0 & 0 \\ 20.056 & 2.028 & 0 & 0 \\ 20.121 & 2.990 & 2.170 & 0 \\ 20.191 & 3.638 & 2.990 & 2.028 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where the values from Table 1 are used. Solving this using the normal equations as in Equation (4.4) yields the solution

$$\mathbf{x} = \begin{bmatrix} 1 \\ -9.890 \\ 4.355 \\ 1.365 \end{bmatrix},$$

which is exactly what was found in Section 4.1.1.

Note that, as this example problem could be solved via iterative replication, the system matrix has to be triangular. The triangular form of the matrix means that the system of equations would have been trivial to solve without utilizing the normal equations. However, this is not the case when extending the problem to an arbitrary discretization. Thus, the least squares hedge can be considered as a generalization of the static option replication method.

Instead of replicating the results of static option replication, least squares hedge can be used to find a better hedge for the problem at hand. To see this, discretize the time interval $[t_0, T]$ into 100 uniformly distributed points. Then the methodology from Chapter 4.2.2 can be used to solve this granular least squares problem. The found solution is

$$\mathbf{x}' = \begin{bmatrix} 1 \\ -12.864 \\ 8.465 \\ 0.484 \end{bmatrix}.$$

The boundary values of the static option replication hedge and this more granular least squares hedge are shown in Figure 7. Clearly, the general shape is the same, but the least squares hedge is more evenly distributed around the target value. This observation can be confirmed by the mean squared errors (MSE) of the solutions. For least squares hedge the MSE is 17.0 while for static option replication it is 24.3, meaning that the least squares hedge provides the more accurate solution.

To further illustrate the use of the least squares hedge, a more complex problem is presented in the case study of Chapter 5. This problem is too complex to be solved with just the ideas introduced by Derman, Ergener and Kani. However, as will be shown, the problem does have a static hedge solution that eliminates most of the risk.

4.2.4 Solving linear systems of equations

There are many ways to solve linear systems of equations that are suitable for different types of system matrices. Generally, when discussing these numerical methods, it is usually in the context of sparse matrices. A sparse matrix is a matrix with a large number of zeros. Storing and doing computations with the zeros is a waste of both memory and processing power and thus being able to store the matrix in a format containing just the non-zeros is beneficial. The sparse storage formats are not elaborated further in this thesis. Instead, it is assumed that such format and basic linear algebra operations with it, like matrix-vector multiplication, are available. For an overview of sparse matrices, see e.g. the survey [31].

Linear solvers can be broadly divided into two categories - direct methods and Krylov subspace methods. As their name suggests the direct methods solve the problem without any approximations. This is usually done by decomposing the system matrix into triangular matrices for which the linear systems become trivial to solve. Such methods include e.g. the Cholesky decomposition and the LU decomposition (see, for example [32, 33] and references there in). Krylov subspace methods are a family of linear solvers, which solve the linear system approximately in a subspace of the original problem. For an overview see e.g. the survey [34]. In this thesis a variant of the conjugate gradient (CG) method by Hestenes and Stiefel [35] is utilized to solve the problem in Equation (4.4).

Consider a linear system of equations

$$\hat{A}\mathbf{y} = \mathbf{b}, \tag{4.6}$$

where $\hat{A} \in \mathbb{R}^{m\times m}$ is some positive definite matrix, $\mathbf{b} \in \mathbb{R}^m$ is a given right-hand side vector, and $\mathbf{y} \in \mathbb{R}^m$ the vector to be solved. Hestenes and Stiefel derived the CG method by considering a related quadratic energy function

$$J(\mathbf{u}) = \mathbf{u}^T\hat{A}\mathbf{u} - 2\mathbf{b}^T\mathbf{u}.$$

It can be shown that the global minimizer of J is also the solution to the linear system in Equation (4.6) (for a proof see, the original paper [35]). In the CG method, the minimizer of J is found by applying gradient descent with $\hat{A}$-conjugate search directions. The basic algorithm is outlined in Algorithm 1

Algorithm 1: Conjugate gradient method

Input : System matrix $\hat{A} \in \mathbb{R}^{m \times m}$
Input : Right-hand side vector $\mathbf{b} \in \mathbb{R}^{m}$
Input : Initial guess $\mathbf{y}^{(0)} \in \mathbb{R}^{m}$
Input : Tolerance ε
Output : Solution vector $\mathbf{y} \in \mathbb{R}^{m}$

$\mathbf{r}^{(0)} \leftarrow \mathbf{b} - \hat{A}\mathbf{y}^{(0)}$
$\mathbf{p}^{(0)} \leftarrow \mathbf{r}^{(0)}$
$k \leftarrow 0$
while $\|\mathbf{r}^{(k)}\| > \varepsilon$ **do**
  $\alpha^{(k)} \leftarrow \frac{\mathbf{r}^{(k)T}\mathbf{r}^{(k)}}{\mathbf{p}^{(k)T}\hat{A}\mathbf{p}^{(k)}}$
  $\mathbf{x}^{(k+1)} \leftarrow \mathbf{x}^{(k)} + \alpha^{(k)}\mathbf{p}^{(k)}$
  $\mathbf{r}^{(k+1)} \leftarrow \mathbf{r}^{(k)} + \alpha^{(k)}\hat{A}\mathbf{p}^{(k)}$
  $\beta^{(k)} \leftarrow \frac{\mathbf{r}^{(k+1)T}\mathbf{r}^{(k+1)}}{\mathbf{r}^{(k)T}\mathbf{r}^{(k)}}$
  $\mathbf{p}^{(k+1)} \leftarrow \mathbf{r}^{(k+1)} + \beta^{(k)}\mathbf{p}^{(k)}$
  $k \leftarrow k + 1$
end
return $\mathbf{x}^{(k)}$

The problem in Equation (4.4) can be solved using the CG method by setting $\hat{A} = A^T A$ and $\mathbf{b} = A^T\mathbf{E}$. Using the CG method like this to solve linear system with a non-positive definite system matrix is known as conjugate gradient on normal equations and was also outlined in the original paper by Hestenes and Stiefel [35].

5 A case study on the log-contract

A log-contract is a financial instrument used to speculate on volatility. First presented by Neuberger in 1994 [9], the log-contract was one of the first of its kind in volatility trading. Prior to its publication the most common way to speculate on volatility was to form a delta-hedged position in the underlying. However, the value of such delta-hedged position also depends on parameters other than volatility. Thus, for a pure volatility play, a different instrument was needed.

The log-contract is defined as a futures-style contract with the state-contingent claim

$$E_{log}(T;\ T\ ,\ S_T) = \ln\left(\frac{S_T}{S_0}\right), \tag{5.1}$$

where S_0 is the initial value of the underlying. The value of the contract alone is not purely dependent on the volatility, but Neuberger showed that the present value of the profit from a delta-hedged position in the log-contract is (almost exactly) equal to

$$\frac{1}{2}(\sigma_I^2 - \sigma_T^2)T,$$

where σ_I is the volatility implied by the price of the log-contract and σ_T is the outcome volatility.

The log-contract is itself a useful tool in a trader's toolbox, but it has other theoretical uses as well. For example, it works as a building block in another common volatility instrument - the *volatility swap* [12]. The ability to replicate and hedge the log-contract accurately can be of significant use to equity derivatives desks and other traders. For discussion on contracts with payoffs tied to volatility, see e.g. [36].

5.1 Dataset and source code

This case study uses a dataset consisting of the options available for the SPDR S&P 500 ETF ($SPY) maturing on 2027-12-17. There were a total of 157 such options with strikes varying from 200 USD to 1000 USD. The options were quoted after market close on 2025-08-15. The implied volatility curve calculated from the option prices is shown in Figure 2. The dataset and the source code can be found on GitHub at `https://github.com/krantamaki/least_squares_hedge`.

5.2 Neuberger's model

Neuberger defines the state-contingent claim of the log-contract as outlined in Equation (5.1). Under the Black-Scholes assumptions the fair value of this contract at period $t < T$ is

Sensitivity	Value
Price	−0.0419
Δ	0.0015
ν	−0.4421

Table 2: The price and sensitivities given by the Neuberger model.

$$E_{log}^{(N)}(T;\ t\ ,\ S_t) = \ln\left(\frac{S_t}{S_0}\right) - \frac{1}{2}\sigma^2(T-t).$$

The delta and vega sensitivities of the log-contract under the Neuberger model are

$$\begin{aligned}\Delta_{log}^{(N)} &= \frac{\partial E_{log}}{\partial S_t} = \frac{1}{S_t}\\ \nu_{log}^{(N)} &= \frac{\partial E_{log}}{\partial \sigma} = -\sigma(T-t).\end{aligned}$$

Note that in contrast to the Black-Scholes delta for a vanilla option, as seen in Equation (2.11), the log-contract's delta is independent of the volatility term and the risk-free rate. Assume that a market maker sells a log-contract on some euro-denominated underlying. The market maker can delta hedge it by going long on 1 EUR of the underlying. As shown by Neuberger [9], since this delta hedged position is independent of any assumptions on the volatility term, the profit from the position is determined purely by the realized volatility. The only requirement is that the position is rehedged often enough, as with all dynamic hedges.

To compute the vega sensitivity, a volatility term needs to be chosen. Since only a single value can be used, the at-the-money implied volatility seems a natural choice. The sensitivities and the fair value of the log-contract are outlined in Table 2.

5.3 Breeden-Litzenberger model

Although the Neuberger model discussed in Chapter 5.2 provides an elegant formulation for the log-contract's pricing formula, this elegance comes at the expense of realism. One glaring issue is the imprecision in the selection of the volatility term, as, like discussed in Chapter 2.2, the volatility is in reality a function of the strike price. Taking this into account is all the more important when considering an exotic instrument used to speculate on the volatility like the log-contract. Thus, the Breeden-Litzenberger model applied to the implied volatility curve from Equation (3.9) is used as an alternative pricing method.

The integral in Equation (3.9) was evaluated on the interval for which strikes are available (from 200 USD to 1000 USD) as that is the interval for which the least

Sensitivity	Value
Price	0.0725
Δ	0.0012
ν	−0.2760

Table 3: The price and sensitivities given by the Breeden-Litzenberger model.

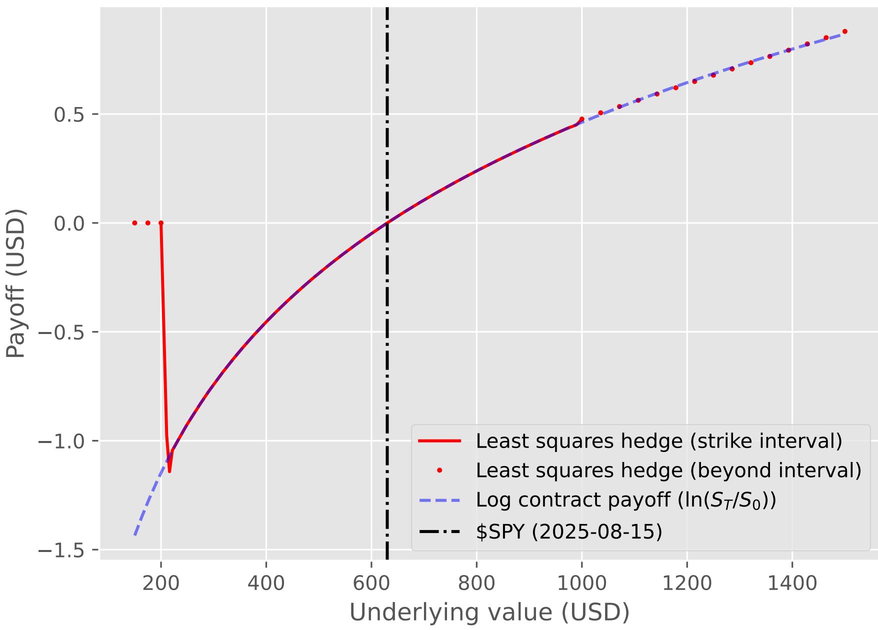

Figure 8: The exact log-contract state-contingent claim and the one for the least squares hedge replicating portfolio.

squares hedge also provides the protection. The sensitivities and the fair value found this way are outlined in Table 3.

5.4 Least squares hedge

Assume a market maker has sold a log-contract to some client. The market maker priced the contract using the Breeden-Litzenberger approach discussed in Chapter 5.3 and is now looking to hedge the position. As delta-hedging does not eliminate the vega-risk, and vega-hedging is quite impractical, the market maker would benefit from finding a static hedge.

The available options can be used to form the system matrix as in Equation (4.2). The least squares solution to the minimization problem in Equation (4.3) can then be solved using the system of normal equations from Equation (4.4). The sensitivities found this way are described in Table 4. Additionally, the state-contingent claim of the

Sensitivity	Value
Price	0.0219
Δ	0.0015
ν	−0.2831

Table 4: The price and sensitivities of the least squares replicating portfolio

least squares hedge is visualized along the exact log-contract state-contingent claim in Figure 8.

Comparing the values in Tables 3 and 4, the hedge appears quite good. While, it slightly overhedges against the delta risk (although the found delta matches the one predicted by Neuberger model, as seen in Table 2), it covers the vega risk almost exactly. Importantly, the hedge is cheaper than the contract itself, making it profitable to sell the contracts. From a relative value perspective, this should not be possible as any securities (or portfolios of securities) that provide the same state-contingent claim should have identical prices. The arbitrage opportunity in this case study would allow the market maker to profit $0.0725 - 0.0219 = 0.0506$ USD (around two thirds of the value of the contract) for each sold log-contract in an almost risk-free fashion.

To further assess the residual risk, the probability that the underlying asset's value falls outside the range of available strike prices at the contract's maturity can be calculated from the option implied probability density introduced in Chapter 3.3. Using the density shown in Figure 4, the probabilities are

$$\begin{aligned}\mathbb{P}[S_T < 200] &= 0.0048\\ \mathbb{P}[S_T > 1000] &= 0.0503.\end{aligned}$$

Thus, there is approximately a 95% probability that the underlying asset will remain within the range of available strike prices. Furthermore, even if the underlying were to move beyond the highest strike, the hedge would not fail catastrophically, as the final option can be used to provide a linear hedge beyond the range of available options. This linear hedge is represented by the dotted line in Figure 8.

6 Conclusions

This thesis set out to compare static and dynamic hedging approaches for exotic options, with a particular emphasis on refining and extending the static replication framework developed by Derman, Ergener, and Kani. As demonstrated in Chapter 4, the problem of constructing a static hedge can be formulated as a least squares problem. The formed problem can then be solved efficiently using methods from numerical linear algebra.

The efficiency of the hedge was evaluated in the case study in Chapter 5 and was found to have near identical sensitivities to those given by the Breeden-Litzenberger pricing model. Additionally, the least squares hedge provided an opportunity for relative value arbitrage with very limited open risk, which would make it an attractive approach to market making desks selling exotic options. Additionally, since the hedged exotic option, the log-contract, is a path-independent exotic option, the least squares hedge has the benefit of being model invariant over the dynamic alternatives.

However, the model invariance is only a property of the hedges on path-independent exotic options. In the case of path-dependent exotic options the present value of the options needs to be evaluated on the upper and lower boundaries, which requires the use of some pricing model. This was the case in the original paper by Derman, Ergener and Kani as well, and seems to be something that cannot be avoided with static hedges. Although model dependency is also a prevalent issue with dynamic hedges, since during rehedging the market assumptions can be reassessed, the effects over time might not be as significant. Thus, the focus of further research should be directed at reducing the impact of market assumptions when hedging path-dependent exotic options.

Static hedging does seem to provide a valid alternative for the more traditional dynamic approaches. However, it comes with many nuances and the success is dependent on both the choice of the hedged exotic option and the liquidity of the underlying. More liquid equities generally have a greater variety of vanilla options available for them, which improves the quality of the static hedge. The SPDR S&P 500 ETF used as the underlying in the case study in Chapter 5 was an ideal choice due to the vast amount of call options available. However, for most equities such cannot be expected. Thus, in practice, a market maker utilizing static hedging methods like the least squares hedge would most likely leave their portfolio open to quite a large amount of tail risk, which financial institutions generally avoid.

In conclusion, static hedging should have it's place in a trader's toolbox. The least squares hedge is an easy-to-form protection, which in the case of path-independent exotic options on highly liquid equities can remove majority of the risk. Importantly, this protection is cheap and allows market making desks to profit from issuing the options. Although static hedges can never fully substitute their dynamic counterparts, the combination of the two can hopefully make the markets that little bit more efficient.

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