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Least squares hedge for exotic options

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0 Overview

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- Conclusions

1 Introduction

1 Introduction

- Hedging is a core part financial risk management
- Institutions that agree to be a counterparty to a contract might not wish carry the associated risk
- This risk is often hedged with a dynamic replicating portfolio
  - Model dependent
  - Labor intensive
- This thesis presents a method for finding a static replicating portfolio as an alternative for a dynamic replicating portfolio

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Dynamic hedging of vanilla options

2 Black-Scholes equation

- Assume the stock price process follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where μ is the drift, σ is the volatility and dB_t is the Brownian motion.

- Black and Scholes[1] show that the call option price process C is governed by the partial differential equation

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} = rC$$

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," Journal of political economy, vol. 81, no. 3, pp. 637–654, 1973.

2 Black-Scholes formula

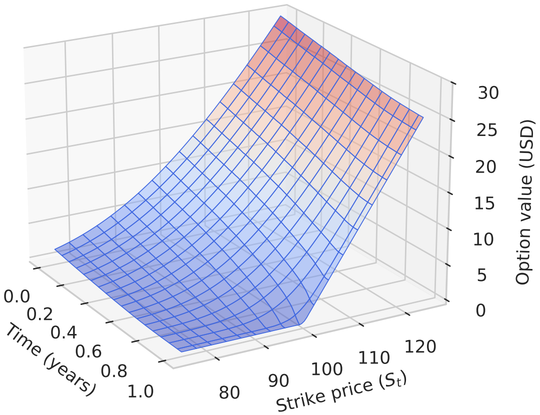
- To find the option price function C the Black-Scholes PDE needs to be solved with appropriate boundary conditions

$$
\begin{aligned}
C(K\,,\ T\,,\ \sigma;\ t\,,\ 0) &= 0 && \forall\ t \\
\lim_{S_t\ \to\ \infty} C(K\,,\ T\,,\ \sigma;\ t\,,\ S_t) &= \lim_{S_t\ \to\ \infty} S_t - K\exp(-r(T-t)) && \forall\ t \\
C(K\,,\ T\,,\ \sigma;\ T\,,\ S_T) &= \max\{0\,,\ S_T - K\} && \forall\ S_T
\end{aligned}
$$

- The solution is known as the Black-Scholes formula

$$
\begin{aligned}
C(K\,,\ T\,,\ \sigma;\ t\,,\ S_t) &= S_t\mathcal{N}(d_+) - K\exp(-r(T-t))\mathcal{N}(d_-) \\
d_\pm &= \frac{\ln(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
\mathcal{N}(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{1}{2}y^2\right)\ dy
\end{aligned}
$$

2 Option price surface



2 Dynamic replicating portfolio

- Consider a portfolio Π consisting of x units of the stock S_t and y units of risk-free bond
$$\Pi = xS_t + ye^{rt}$$
- The portfolio has partial derivatives
$$\frac{\partial \Pi}{\partial t} = yre^{rt}, \quad \frac{\partial \Pi}{\partial S} = x, \quad \frac{\partial^2 \Pi}{\partial S^2} = 0$$
- Placing the partial derivatives into the Black-Scholes equation yields
$$\frac{\partial \Pi}{\partial t} + rS_t \frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$
$$yre^{rt} + rS_t x + 0 = r(xS_t + ye^{rt})$$
- Thus, an issued option can be hedged by forming a *dynamic replicating portfolio* of the underlying asset and the risk-free bond

2 Sensitivities

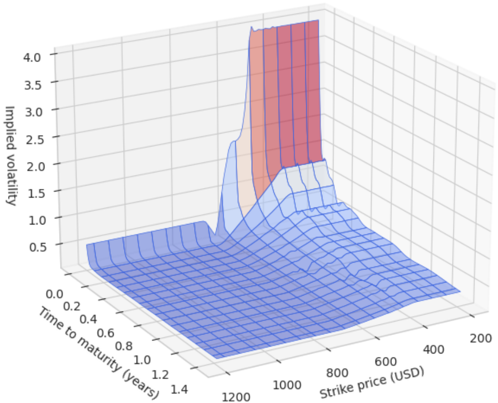
- The quantities x and y are governed by the sensitivity of the option to it's parameters
- The most common hedge is the *delta hedge*, where the sensitivity to the change in the value of the underlying is eliminated
- Assume a market maker has issued a call option. By going long on $\Delta = \frac{\partial C}{\partial S}$ units of the underlying the portfolios value will not change for infinitesimal changes in the underlying asset's value
- Other commonly considered sensitivities are

$$\nu = \frac{\partial C}{\partial \sigma}, \quad \Gamma = \frac{\partial^2 C}{\partial S^2}, \quad \rho = \frac{\partial C}{\partial r}, \quad \text{etc.}$$

2 Implied volatility and model parameters

- The Black-Scholes formula requires two parameters - the risk-free rate r and the volatility σ
- Depending on the trader the risk-free rate could be e.g. some short-term rate (ESTR), interbank offered rate (EURIBOR) or government bond yield (BUND, SCHATZ, BOBL)
- Volatility is less clear. Black and Scholes define it as the standard deviation of the historical stock returns
- Alternatively, it can be calculated based on observed market prices $\rightarrow$ *implied volatility*

2 Implied volatility surface



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Dynamic hedging of exotic options

3 State prices

- Arrow and Debreu[2] define the so called Arrow-Debreu security (elementary claim in Breeden and Litzenberger's work) as a contract that pays unit amount of some numeraire if a particular state occurs at a predefined time
- Let S_T be a random variable giving one of the possible n states in set $\mathcal{S} = \{s_T^{(1)}, s_T^{(2)}, ..., s_T^{(n)}\}$. An Arrow-Debreu security on state $s_T^{(i)}$ has the payoff

$$v^{(i)}(T, S_T) = \begin{cases} 1, & S_T = s_T^{(i)} \\ 0, & otherwise \end{cases}$$

- The present value of $v^{(i)}(t, S_t)$ of the Arrow-Debreu security is known as the *state price*

[2] K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy," Econometrica, vol. 22, no. 3, pp. 265—290, 1954

3 State prices example

- Assume the future states are tied to the results of some sport's game, which can end in one of two ways
  - Home team wins and the away team loses
  - Away team wins and the home team loses
- Denote these states by $s_T^{(H)}$ and $s_T^{(A)}$ respectively
- The random variable S_T associates a probability for both states as $\mathbb{P}[S_T = s_T^{(H)}] = p_H$ and $\mathbb{P}[S_T = s_T^{(A)}] = p_A$
  - States are mutually exclusive and collectively exhaustive

3 State prices example

- Assume an investor buys Arrow-Debreu securities for both states $s_T^{(H)}$ and $s_T^{(A)}$
- This guarantees unit payoff meaning

$$v^{(H)}(t, S_t) + v^{(A)}(t, S_t) = e^{-r(T-t)} = e^{-r(T-t)}(w_H + w_A),$$

  where $w_H > 0$, $w_A > 0$ and $w_H + w_A = 1$
- Arrow and Debreu propose that the state price is directly proportional to the probability of the state occuring. Thus

$$v^{(H)}(t, S_t) = e^{-r(T-t)} p_H \propto p_H \text{ and } v^{(A)}(t, S_t) = e^{-r(T-t)} p_A \propto p_A$$

3 State prices

- More generally, with a set $\mathcal{S}$ of n states

$$v^{(i)}(t, S_t) = e^{-r(T-t)}\mathbb{P}[S_T = s_T^{(i)}]$$

- The present value of a state-contingent claim on the states in the set S is then

$$E(T; t, S_t) = \sum_{i=1}^{n} E(T; T; s_T^{(i)})v^{(i)}(t, S_t)$$

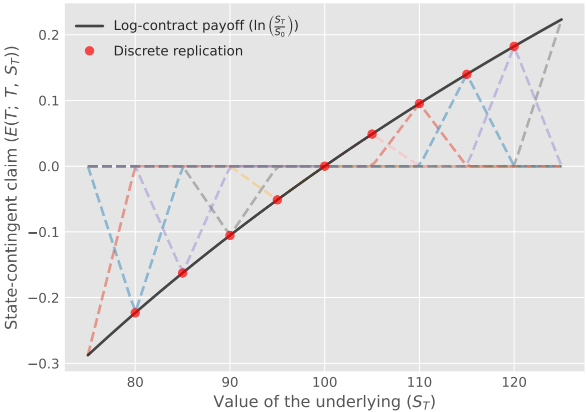
3 Breeden-Litzenberger

- Let the set of possible future states of a random variable S_t be $\mathcal{S} = \{s \in \mathbb{R} | s \geq 0\}$ (non-negative real line)
- Assume there exists call options with strike prices for each of the states in set $\mathcal{S}$
- Breeden and Litzenberger[3] propose using three call options with strikes $K - dS_t$, K and $K + dS_t$ for some small and positive dS_t to form a portfolio

$$e(K, t, \sigma; t, S_t) = \frac{C(K - dS_t, T, \sigma; t, S_t) - 2C(K_t, T, \sigma; t, S_t) + C(K + dS_t, T, \sigma; t, S_t)}{dS_t}$$

[3] D. T. Breeden and R. H. Litzenberger, "Prices of state-contingent claims implicit in option prices," Journal of business, pp. 621-651, 1978

3 Breeden-Litzenberger



3 Breeden-Litzenberger

- In the limit $dS_t \to 0$ the value of the portfolio e approaches an Arrow-Debreu security
- Breeden and Litzenberger observed that in the limit

$$\lim_{dS_t \to 0} \frac{e(K, T, \sigma; t, S_t)}{dS_t} = \frac{\partial^2 C}{\partial S^2} = \Gamma$$

3 Breeden-Litzenberger

- Let $E(T; t, S_t)$ be a function giving the value of an exotic option
- The exotic option has a known state-contingent claim $E(T; T, S_T)$ and potentially some cashflows or boundary conditions for $t < T$
- The value of the option is given by

$$E(T; t, S_t) = \int_t^T \int_0^\infty E(\tau; \tau, K) \frac{\partial^2 C(K, \tau, \sigma; t, S_t)}{\partial S^2} dK d\tau$$

- The sensitivities of the exotic option can be calculated from the above expression and are denoted by

$$\Delta^{(BL)} = \frac{\partial E}{\partial S}, \quad \nu^{(BL)} = \frac{\partial E}{\partial \sigma}, \quad \text{etc.}$$

3 Implied volatilities and option implied probability density

- For the price E to fully reflect the market expectations, the volatility term in the integrand should vary with the maturity date and the strike price.
- This leads to expression

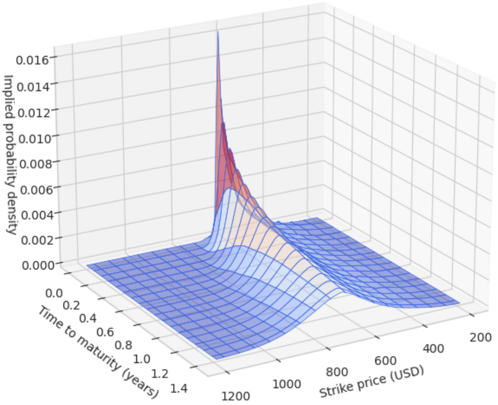
$$E(T; t, S_t) = \int_t^T \int_0^\infty E(\tau; \tau, K) \frac{\partial^2 C(K, \tau, \sigma_I(\tau, K); t, S_t)}{\partial S^2} dK d\tau$$

- Above expression also results in

$$\mathbb{P}[S_T = K] \propto \frac{\partial^2 C(K, \tau, \sigma_I(\tau, K); t, S_t)}{\partial S^2},$$

which is known as the *option implied probability density*

3 Option implied probability density surface



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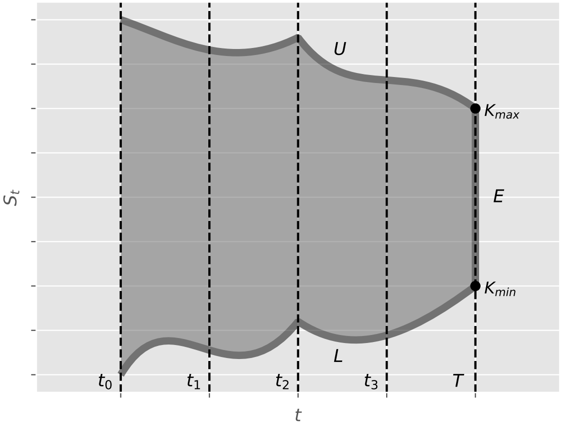
Static hedging of exotic options

4 Static option replication

- Derman, Ergener and Kani[4] propose a static hedge for exotic options defined by boundary conditions
- This hedge is found by replicating the boundaries using vanilla options
  - Since the exotic option is governed by the same dynamic (Black-Scholes PDE) as the vanilla options this provides a sufficient hedge
- The static hedge is found by backward induction and matching the boundary values at the nodes with the target value by buying (selling) OTM call and put options

[4] E. Derman, D. Ergener, and I. Kani, "Static options replication," Journal of Derivatives, vol. 2, no. 4, pp. 1-37, 2000

4 Static option replication



4 Least squares hedge

- Let $\mathcal{K} = \{K_1, K_2, ..., K_n\}$ be a set of strike prices for options $C^{(i)}$ available in the markets
- Goal is to find weights $\mathbf{x} \in \mathbb{R}^n$ that minimize the distance between the boundary values of the replicating portfolio and the target exotic option
  - There are at most three boundaries U (upper), L (lower) and E (termination)

4 Least squares hedge for path-independent exotic options

- Consider a path-independent exotic defined purely by the termination boundary with state-contingent claim E
- The state-contingent claim and available call options can be discretized into d points as $\mathbf{E} \in \mathcal{S}^d$ and $\mathbf{C}^{(i)} \in \mathcal{S}^d$
- The weights $\mathbf{x}$ can be found by solving the optimization problem

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \|A\mathbf{x} - \mathbf{E}\|_2^2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^n, \end{aligned}$$

where $A = [\mathbf{C}^{(1)} \quad \mathbf{C}^{(2)} \quad \ldots \quad \mathbf{C}^{(n)}]$

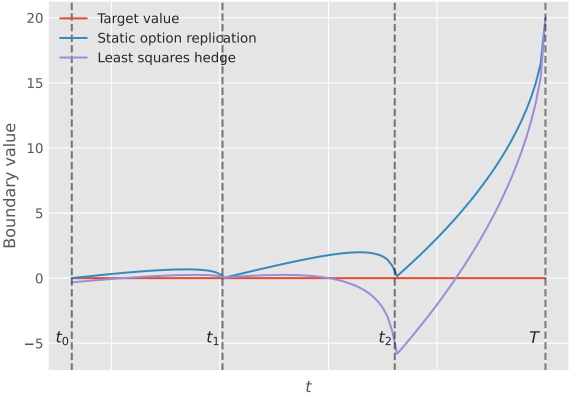
4 Least squares hedge for path-dependent exotic options

- Extending the least squares hedge to path-dependent exotic options requires considering the upper U and lower L boundaries as well
  - Termination boundary discretized with respect to strike price
  - Upper and lower boundaries discretized with respect to time
- Assume there are N options with both calls and puts available. Further assume these are valid in the context of static option replication
- Least squares hedge is found by solving the linear system

$$A\mathbf{x} = \begin{bmatrix} \mathbf{C}_E^{(1)} & \mathbf{C}_E^{(2)} & \dots & \mathbf{P}_E^{(N)} \\ \mathbf{C}_L^{(1)} & \mathbf{C}_L^{(2)} & \dots & \mathbf{P}_L^{(N)} \\ \mathbf{C}_U^{(1)} & \mathbf{C}_U^{(2)} & \dots & \mathbf{P}_U^{(N)} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{E} \\ \mathbf{L} \\ \mathbf{U} \end{bmatrix},$$

via linear least squares

4 Up-and-out call option upper boundary



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Case study on the log-contract

5 Log-contract

- Neuberger[5] presented the *log-contract* as a financial instrument for speculating volatility
- The payoff of the log-contract is defined as

$$E_{log}(T; T, S_T) = \ln\left(S_T/S_0\right)$$

- Neuberger showed that the present value of the profit from delta-hedged position in the log-contract is (almost exactly) equal to

$$0.5(\sigma_I^2 - \sigma_T^2)T,$$

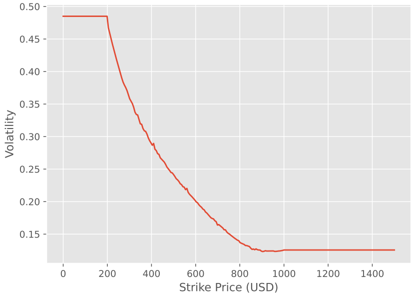
  where σ_I is the volatility implied by the log-contract and σ_T is the outcome volatility

[5] A. Neuberger, "The log contract," Journal of portfolio management, vol. 20, no. 2, pp. 74-80, 1994

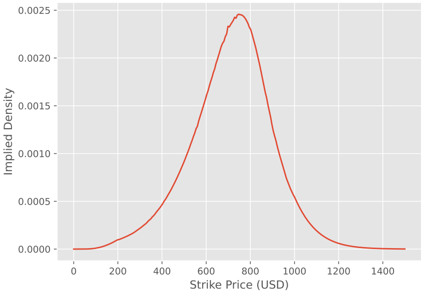
5 Dataset

- Dataset consists of options on SPDR S&P 500 ETF ($SPY) maturing on 2027-12-17
  - 157 different options
  - Strikes varying from 200 USD to 1000 USD
- Option prices were quoted after market close on 2025-08-15

5 Dataset



0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
Volatility
0
200
400
600
800
1000
1200
1400
Strike Price (USD)



0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
Implied Density
0
200
400
600
800
1000
1200
1400
Strike Price (USD)

5 Neuberger

- Under Black-Scholes assumptions the fair value of the log-contract is

$$E_{log}^{(N)}(T; t, S_t) = \ln(S_t/S_0) - 0.5\sigma^2(T - t)$$

- The sensitivities under Neuberger model are defined as

$$\Delta_{log}^{(N)} = \frac{\partial E_{log}}{\partial S_t} = \frac{1}{S_t}$$
$$\nu_{log}^{(N)} = \frac{\partial E_{log}}{\partial \sigma} = -\sigma(T - t)$$

Sensitivity	Price	Δ	ν
Value	-0.0419	0.0015	-0.4421

5 Breeden-Litzenberger

- Breeden-Litzenberger model is used as an alternative pricing method
- Unlike Neuberger model Breeden-Litzenberger model can utilize full volatility curve $\rightarrow$ takes the market expectations better into account

Sensitivity	Price	Δ	ν
Value	0.0725	0.0012	-0.2760

5 Least squares hedge

- Assume a market maker has issued a log-contract
  - Contract was priced with Breeden-Litzenberger model
  - Now the market maker is looking for a hedge. To limit vega-risk static hedge would be optimal

Sensitivity	Price	Δ	ν
Value	0.0219	0.0015	-0.2831

- Most of the risk eliminated
- Importantly, the hedge is cheaper than the contract

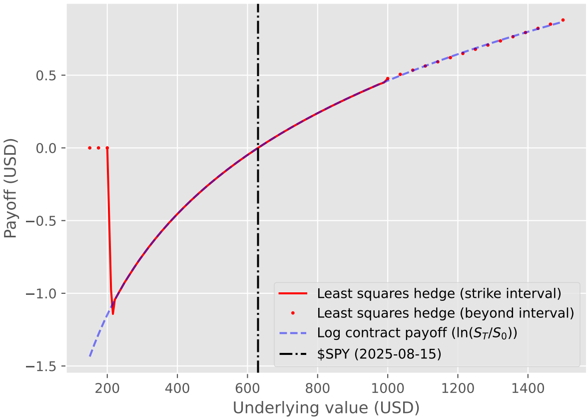
5 Residual risk

- While most of the delta- and vega-risk was eliminated the hedge only holds on a finite interval
- How likely is it that the underlying will go beyond this interval?
- Using the option implied probability density

$$\mathbb{P}[S_T < 200] = 0.0048$$
$$\mathbb{P}[S_T > 1000] = 0.0503$$

- However, the hedge doesn't fail catastrophically if $S_T > 1000$

5 Residual risk



6

Conclusions

6 Conclusions

- The case study shows the effectiveness of least squares hedge
  - Most of the risk eliminated
  - Low cost $\rightarrow$ potential arbitrage?
  - Clear improvement over static option replication
- However, the hedge can be limited
  - Most stocks are more limited with regards to maturity days and strike prices than the case study
  - Model dependency remains an issue with path-dependent exotic options
  - Not all exotic options can be hedged (lookback, Asian, etc.)
- Overall, least squares hedge can be a useful tool in conjunction with the dynamic replicating portfolio