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# Series expansion of call option payoffs to periodic functions

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## Contents

## 1 Introduction

In 1960 physicist Eugene Wigner wrote an article titled *"The Unreasonable Effectiveness of Mathematics in the Natural Sciences"* where he covered some problems in physics where mathematical concepts from completely unrelated topics proved "unreasonably" applicable in solving them. No one has written an equivalent work for mathematical finance, but if such were to be written the topics covered in this paper could certainly contribute an example case.

In comparison with physics mathematical finance studies fundamentally simple phenomena. The simplest, so called "vanilla", products are defined by very simple rules and can usually be modelled with relatively simple dynamics. In financial institutions such vanilla products can be combined to form a more complicated product with more desirable characteristics. These products can be made almost arbitrary complicated. While this can be hugely beneficial for investors it creates challenges with risk modelling and with just finding the correct combination of products for the characteristics. This paper attempts to tackle the second problem in the specific case where the characteristic of interest is the payoff function (more precisely any arbitrary periodic payoff function) and the product is formed by a collection call options.

This paper is structured as follows. In Section 2 we cover the basics of options and some of the most common option trading strategies. These provide the main building blocks for our study of payoff functions. Section 3 presents the main idea in the paper and contains the derivation of the series expansion to periodic functions. Section 4 extends on the work in the preceeding section and presents an optimization for finding the nearest approximation to a periodic function from a finite set of options. Finally, we conclude with discussion in Section 5.

The code used for visualisations can be found in `https://github.com/krantamaki/nonlinear_payoffs`.

## 2 Options and option trading strategies

### 2.1 Options

In finance the simplest derivative is the *futures contract*. When two parties enter a forward contract one of them agrees to buy an asset and the other to sell it at a predefined price at a future time point. Obviously, in such an arrangement if the value of the underlying asset at the contracts expiry differs from the predefined price one of the parties ends up losing money, while the other wins it. Thus, it would be very handy if the parties had only the option to exercise the contract, but no obligation. This is exactly what an *options contract* provides (in this paper by option we always mean an European option).

There are two main types of options: call and put. Call option corresponds to the option to exercise the futures contract for the buyer, while put option provides an equivalent option for the seller of

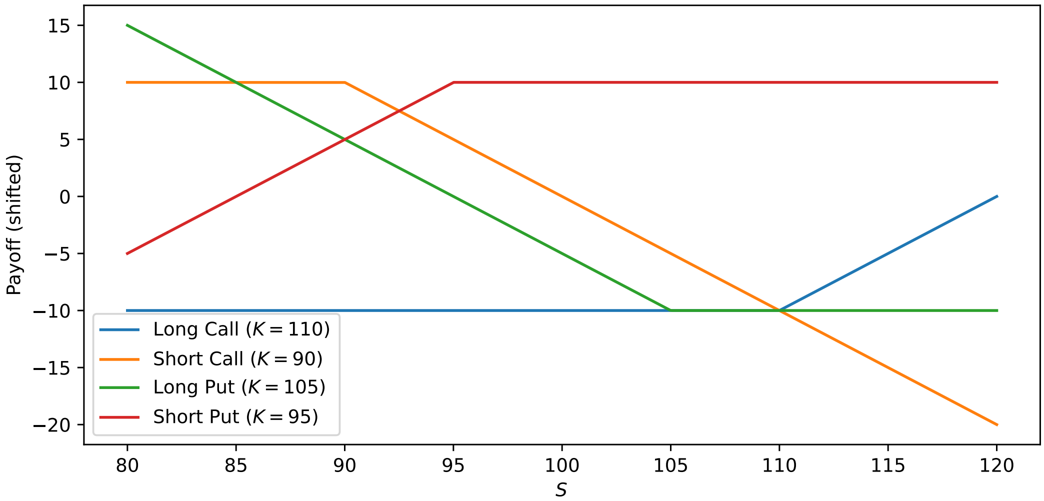


**Figure 1:** Long and short position in call and put options respectively. Note that an option premium has been applied to the options to differentiate them.

the futures contract. These are more precisely defined in Definitions 2.1 and 2.2.

**Definition 2.1** (Call option). *A call option is a contract between two parties where the buyer has the right, but not the obligation to purchase an asset (the underlying) at a specified price (strike price) at a future time point (expiry). Let $S$ be the value of the underlying at expiry and $K$ the strike price. The payoff of a call option would then be*

$$C(K;\, S) = \max\{0,\, S - K\}$$

*The payoff function $C$ is visualised in Figure 1*

**Definition 2.2** (Put option). *A put option is a contract between two parties where the buyer has the right, but not the obligation to sell an asset (the underlying) at a specified price (strike price) at a future time point (expiry). Let $S$ be the value of the underlying at expiry and $K$ the strike price. The payoff of a put option would then be*

$$P(K;\, S) = \max\{K - S,\, 0\}$$

*The payoff function $P$ is visualised in Figure 1*

**Remark 2.3** (Notation). *Many functions discussed in this paper depend not only on the main variable, but also of some parameters. In the function definition the parameters are separated from the variables with a semicolon, with parameters being on the left of it and variables on the right.*

When an options contract is bought it is known as “going long”. Alternatively, the contract can

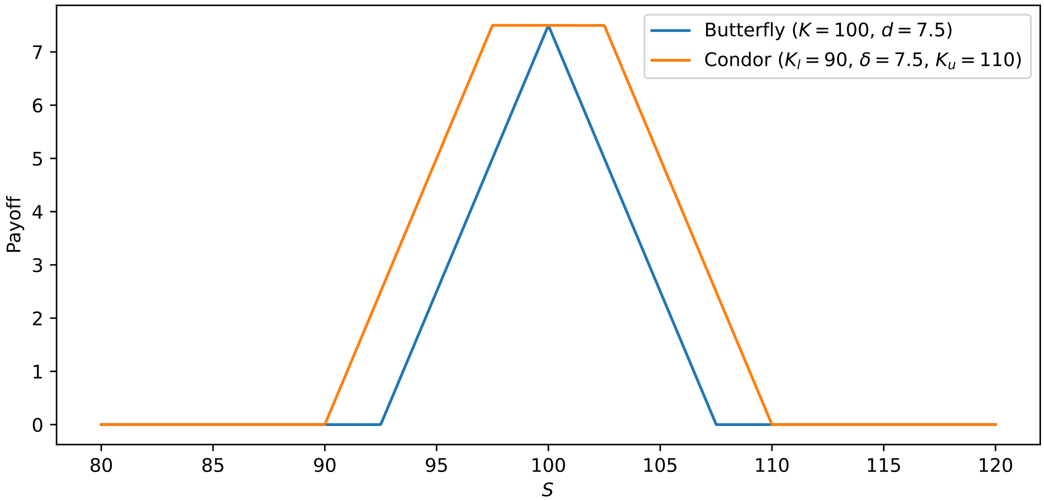


**Figure 2:** Payoff functions for butterfly and condor options strategies

be sold, which is known as "shorting" it. In short positions the payoff function is just the negation of the corresponding long contract.

Of course as option contracts can be used to make a profit, entering such contract can't be free. Thus, it is often common to see the diagrams as in Figure 1 be shifted up or down (depending if the position is long or short) by the cost of entering the contract i.e. the option premium. The option premiums are highly nonlinear with respect to the value of the underlying and thus we will only consider the pure payoff.

### 2.2 Option trading strategies

While vanilla options provide a simple and easily understood payoff, in some cases it might not have the properties that an investor needs. For example if one expects the value of the underlying to remain constant, neither call nor put options can directly accommodate it. However, with a portfolio of options it is possible to form a position benefiting from constant underlying value. An example of it would be the "butterfly" options strategy.

**Definition 2.4** (Butterfly strategy). *Let $S_0$ be the value of the underlying that is assumed to remain near constant. Let $K \approx S_0$ be a strike price of an available call (or put) option and $d > 0$ some value such that $K \pm d$ gives valid strike prices. A (long) butterfly would then consist of going long on call options with strikes $K \pm d$ and shorting two call options with strike $K$. The payoff function would then be*

$$B(K, d; S) = C(K - d; S) - 2C(K; S) + C(K + d; S)$$

*The payoff function $B$ is visualised in Figure 2.*

In the butterfly strategy as given in Definition 2.4 the confidence in the underlying maintaining a constant value is controlled by the value $d$. With greater value $d$ the position will be wider and thus the losses are limited with greater deviations from $K$. However, the potential profit in the case that the value of the underlying remains constant are also limited. If the investor assumes that the value of the underlying might move, but only a limited amount a better alternative could be the "condor" options strategy.

**Definition 2.5** (Condor strategy). *Let $K_l$ and $K_u$ be two strike prices such that $K_u > K_l$ and $\delta > 0$ be some value such that $K_l + \delta$ and $K_u - \delta$ are valid strike prices. A (long) condor would then consist of going long on a call option with strikes $K_l$ and $K_u$ and shorting call options with strikes $K_l + \delta$ and $K_u - \delta$. The payoff would then be*

$$G(K_l, K_u, \delta; S) = C(K_l; S) - C(K_l + \delta; S) - C(K_u - \delta; S) + C(K_u; S) \tag{2.1}$$

*The payoff function $G$ is visualised in Figure 2.*

# 3 Forming nonlinear payoffs

## 3.1 Condor chains and forming of sine

As discussed in Section 2 options on their own are very simple, but by combining them into portfolios, known as strategies, allows for more complicated payoffs. These can be very useful for investors with specific needs. However, finding the option positions for the more complicated strategies can seem somewhat unsystematic. This while less of an issue in real world applications does pose an interesting mathematical problem - what functions can be approximated by option payoffs and how to find such approximations.

To help with the study we make some assumptions. Firstly, we assume that there exists infinite number of options with strike prices spanning the whole real line. Additionally, we allow arbitrary position sizes for each option and allow for arbitrary constant terms. These assumptions are mainly relevant for the mathematical formulation, but can be tightened when moving into more applied problems.

The main tool for us will be the condor strategy as given in Definition 2.5. However, it will be extended in a periodic fashion to the whole real line. The resulting strategy doesn't have a proper name, but for future reference we will call it "condor chain".

**Definition 3.1** (Condor chain). *Let $\lambda$ be the wanted width of the condor strategies, $\varphi$ be some real value used to denote the phase of the period of the chain and $\delta < \lambda/2$ the difference between the outer and inner strikes in the condors. A condor chain is then defined as the infinite series with payoff*

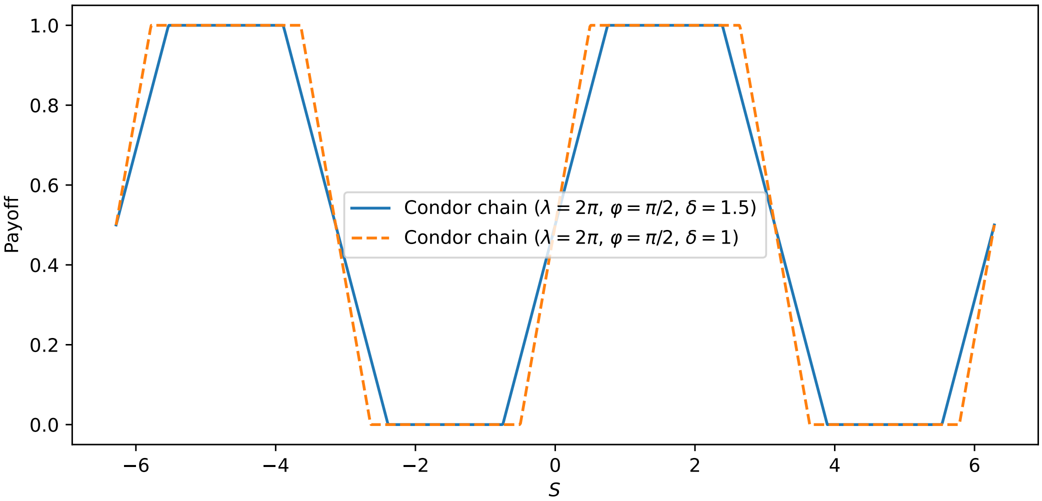


**Figure 3:** Payoff functions for two (normalized) condor chain strategies

$$GC(\lambda, \varphi, \delta; S) = \sum_{i=-\infty}^{\infty} G\left(\varphi - \left(\frac{\lambda}{2} - \delta\right) + \lambda i, \varphi + \left(\frac{\lambda}{2} - \delta\right) + \lambda i, \delta; S\right) \tag{3.1}$$

*The payoff function GC is visualised in Figure 3.*

The condor chain on it's own is already similar to the sine function. So a natural question to ask is would the sine function be formable as an series expansion of condor chains. And indeed this is the case as shown in Theorem 3.2.

**Theorem 3.2** (Cosine as a series expansion of condor chains)**.** *Cosine can be represented as a series expansion of condor chains in form*

$$\begin{aligned} f(A, \lambda, \varphi; S) = A\cos\left(\frac{2\pi}{\lambda}S + \varphi\right) &= \lim_{n\to\infty} AN\sum_{i=1}^{n} \frac{2}{\lambda}\sin\left(\frac{\pi i}{2n}\right) GC\left(\lambda, \varphi, \frac{\lambda i}{2n}; S\right) - 1 \\ N &= \lim_{n\to\infty} \frac{2}{\sum_{i=1}^{n} \frac{i}{n}\sin\left(\frac{\pi i}{2n}\right)} \end{aligned} \tag{3.2}$$

*where $A$ is the amplitude, $\lambda$ the wavelength and $\varphi$ the phase.*

*Proof.* Since a highly rigorous proof of convergence would require tools from analysis that have not been presented, we will instead provide a hand-wavy sketch of a proof to gain some intuition behind the series expansion and to show that trigonometric functions of wanted form arise naturally from it.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued functions

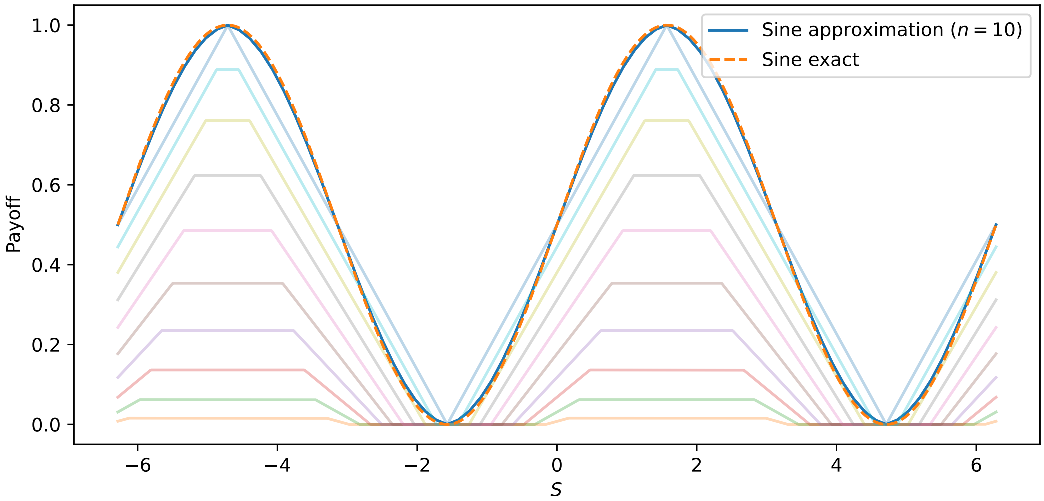


**Figure 4:** Example of a sine approximation and it's constituent condor chains

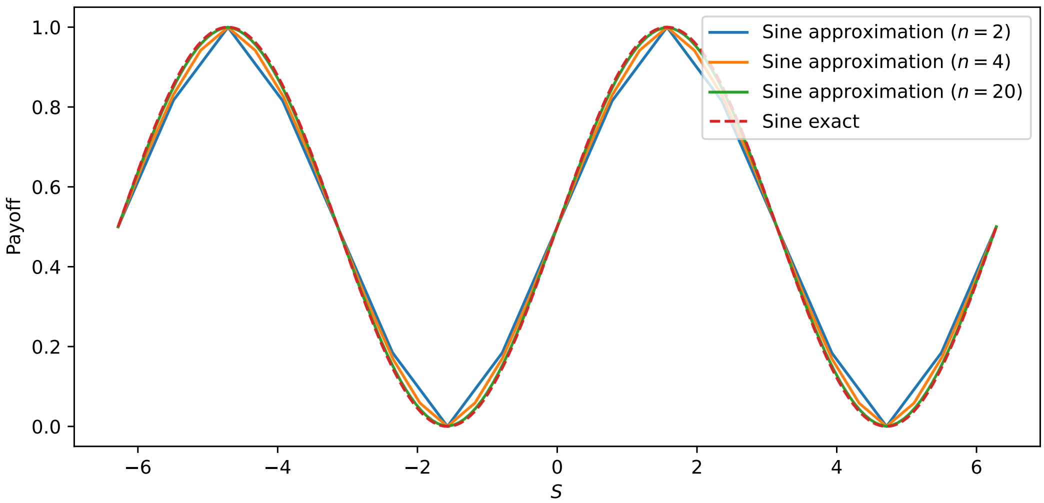


**Figure 5:** Examples of some sine approximations for differing values of $n$

$$
\begin{aligned}
f_n(A,\lambda,\varphi;S) &= AN_n \sum_{i=1}^{n} \frac{2}{\lambda} \sin\left(\frac{\pi i}{2n}\right) GC\left(\lambda,\varphi,\frac{\lambda i}{2n};S\right) - 1 \\
N_n &= \frac{2}{\sum_{i=1}^{n} \frac{i}{n} \sin\left(\frac{\pi i}{2n}\right)}
\end{aligned}
$$

We know that at any given point $S \in \mathbb{R}$ the sum of condor chains can be written as a linear function. That is

$$
f_n(A,\lambda,\varphi;S) = AN_n \left( \left( \sum_{i\in I} \frac{2}{\lambda} \sin\left(\frac{\pi i}{2n}\right) \right) S + \sum_{j\in J} \left( \sin\left(\frac{\pi j}{2n}\right) \frac{j}{n} \right) + c \right)
$$

for some sets $I, J \subseteq \{1, 2, ..., n\}$ and a constant $c$ that are dependent on $S$. For clarity we will exclude the constant $c$ from consideration.

Due to the periodic nature of the function it is enough to consider points from a single wavelength $S \in [\varphi, \varphi + \lambda]$. Additionally, by symmetricity it is enough to consider a single quadrant and w.l.o.g we can set $\varphi = 0$ and look at the first quadrant $S \in [0, \lambda/4]$. By construction the set $I$ grows in a linear fashion and the set $J$ decreases in linear fashion as a function of $S$. Then

$$
f_n(A,\lambda,\varphi;S) = AN \left( \left( \sum_{i=1}^{n} \mathbb{I}_{[1-\frac{S}{\lambda/4},1]} \left(\frac{i}{n}\right) \frac{2}{\lambda} \sin\left(\frac{\pi i}{2n}\right) \right) S + \sum_{j=1}^{n} \mathbb{I}_{[0,1-\frac{S}{\lambda/4}]} \left(\frac{j}{n}\right) \sin\left(\frac{\pi j}{2n}\right) \frac{j}{n} \right)
$$

where $\mathbb{I}$ is used to denote the indicator function. At the limit we can exchange the sum with a Riemann integral

$$
\begin{aligned}
f_\infty(A,\lambda,\varphi;S) &= AN \left( \int_0^1 \mathbb{I}_{[1-\frac{S}{\lambda/4},1]}(x) \frac{2}{\lambda} \sin\left(\frac{\pi x}{2}\right) S \, dx + \int_0^1 \mathbb{I}_{[0,1-\frac{S}{\lambda/4}]}(x) \sin\left(\frac{\pi x}{2}\right) x \, dx \right) \\
&= AN \left( \int_{1-\frac{S}{\lambda/4}}^1 \frac{2}{\lambda} \sin\left(\frac{\pi x}{2}\right) S \, dx + \int_0^{1-\frac{S}{\lambda/4}} \sin\left(\frac{\pi x}{2}\right) x \, dx \right) \\
&= \alpha \cos\left(\frac{2\pi}{\lambda} S\right) + h(S)
\end{aligned}
$$

for some scalar $\alpha$ and some nonlinear function $h : [0, \lambda/4] \to \mathbb{R}$. Thus, we find that a cosine is easy to form. Unfortunately, this derivation doesn't end up cancelling out all other terms. Still as a proof of concept that trigonometric functions arise naturally from simple condor chains and hence by extension from call option payoffs this should be more than adequate.

□

**Remark 3.3** (Pointwise and uniform convergence). *The proof (or sketch of such) in Theorem 3.2 is for pointwise convergence. That is in effect the proof shows that for a sequence of real valued functions* $(f_n)_{n\in\mathbb{N}}$ *holds*

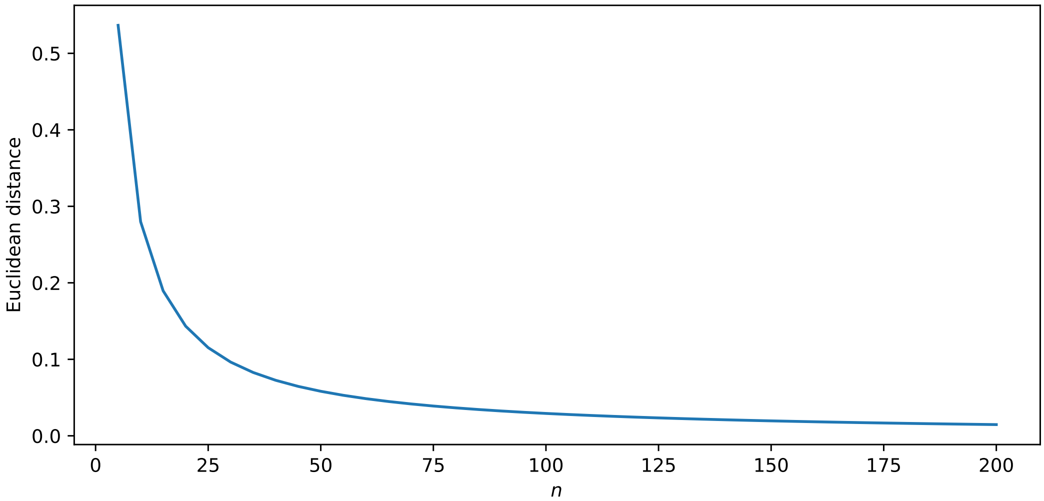


**Figure 6:** The Euclidean distance (between exact sine values and approximations for uniformly distributed points on the given interval) as a function of $n$

$$\forall\, x \in \mathbb{R}\ :\ \lim_{n\to\infty} f_n(x) = f(x)$$

*However, such pointwise convergence doesn't guarantee continuity and thus raises questions about the differentiability of the series expansion. A stronger requirement would be that of uniform convergence stated as*

$$\forall\, \varepsilon > 0\ :\ \exists\, \eta \in \mathbb{N}\ :\ \forall\, n \geq \eta\ :\ \forall\, x \in \mathbb{R}\ :\ |f_n(x) - f(x)| < \varepsilon$$

*Understandably, this is a bit more challenging to prove.*

Figures 4 and 5 illustrate the series expansion in practice.

### 3.2 Numerical analysis

Theorem 3.2 gives us a (sketch of a) proof for the convergence of the series expansion to sine function. However, for practical applications it is beneficial to have some idea on how quickly the series converges. This is easy to test numerically. We can for example discretize some interval of interest and compute the Euclidean distance between the approximation and the exact sine. This is exactly what we did and what is visualised in Figure 6. In the figure the Euclidean distance is computed from a sample of 1000 points and with approximations consisting of $n$ condor strategies. Clearly, the distance seems to approach zero at a relatively rapid rate.

### 3.3 Fourier series

Once we are able to form the sine function from the option payoffs we can extend our study into generic periodic functions. To do this we will apply the Fourier series expansion with sines being replaced by the expansions given in Theorem 3.2. Towards this end we should define the Fourier series.

**Definition 3.4** (Fourier series (sine-cosine form)). *A given real valued continuous and periodic function $g$ can be expressed as a summation of harmonically related sinusoidal functions as*

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(2\pi \frac{n}{P} x\right) + B_n \sin\left(2\pi \frac{n}{P} x\right) \tag{3.3}$$

*where $A_n$ and $B_n$ corresponds with the amplitude $A$ from Equation (3.2) and $P/n$ corresponds with wavelength $\lambda$.*

**Remark 3.5** (Shift from cosine to sine). *As can be remembered from high school mathematics conversion from cosine to sine can be done by shifting the period by one quarter. This gives an identity*

$$\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp \sin(\theta)$$

*Note that this is also the reason why sine and cosine have been used quite interchangeably throughout this paper.*

By applying Remark 3.5 we can place the series expansion presented in Equation (3.2) into the Fourier series seen in Equation (3.3). This gives us

$$\begin{aligned} g(S) = A_0 + \sum_{m=1}^{\infty} A_m N_m & \left( \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{\lambda} \sin\left(\frac{\pi i}{2n}\right) GC\left(\lambda_m, \varphi_m + \frac{\pi}{2}, \frac{\lambda i}{2n}; S\right) - 1 \right) \\ & + B_m N_m \left( \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{\lambda} \sin\left(\frac{\pi i}{2n}\right) GC\left(\lambda_m, \varphi_m, \frac{\lambda i}{2n}; S\right) - 1 \right) \\ N_m = \lim_{n\to\infty} & \frac{2}{\sum_{i=1}^{n} \frac{i}{n} \sin\left(\frac{\pi i}{2n}\right)} \end{aligned} \tag{3.4}$$

where all $\lambda_m$ must be some whole number multiple of a given base frequency and the phase $\varphi_m$ is zero.

# 4 Finding optimal option selection from a finite set

## 4.1 Deriving the optimization problem

The theoretical framework in Section 3 leads to a very clean conclusion in the form of the formula in Equation (3.4). However, as the assumptions used to find this were less than realistic, it isn't directly applicable in practice. So the natural follow up question would be how can one find the closest approximation to a given function from a finite set of options?

For notational simplicity we will exchange the symbol $\lambda$ from Equation (3.4) with $l$. Additionally, the amplitudes $A$ and $B$ will be replaced by the lowercase $a$ and $b$ respectively. Also going forward vectors will be denoted by bolded lowercase letters and the elements of the vectors by corresponding unbolded letters.

The problem of finding the closest approximation is natural to represent as an optimization problem with a general form of

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mu(\{\mathbf{a}_e, \mathbf{b}_e, \mathbf{l}_e\}, \{\mathbf{a}_a, \mathbf{b}_a, \mathbf{l}_a\}) \\ \text{s.t.} \quad & \mathbf{c}^T \mathbf{x} \leq W \\ & x_i \in \mathbb{Z} \quad \forall\, i \in \{1, ..., |\mathbf{x}|\} \end{aligned} \tag{4.1}$$

where $\mu$ is some metric giving the distance between the set of exact parameters (for a finite number of sine waves) $\{\mathbf{a}_e, \mathbf{b}_e, \mathbf{l}_e\}$ computed from the Fourier series and the approximate ones $\{\mathbf{a}_a, \mathbf{b}_a, \mathbf{l}_a\}$ computed from the option selection. Additionally, the vector $\mathbf{c}$ contains option premiums, vector $\mathbf{x}$ the number of each option either purchased or sold and the scalar $W$ gives the budget for the portfolio.

In essence the optimization problem shown in Equation (4.1) is a variant of the unbounded knapsack problem. Main differences would be that the objective function is most likely nonlinear and that the decision variables are not restricted to the set of non-negative integers. Essentially, this makes our problem more challenging than the regular version.

Now we need to determine how the choice of vector $\mathbf{x}$ affects the parameters in the set $\{\mathbf{a}_a, \mathbf{b}_a, \mathbf{l}_a\}$. Once again it is smart to consider the condor chains rather than the options directly. Since we only need to consider sines with some whole number multiple of a given base frequency, we know a priori what wavelengths the sines should have. Thus, we also know what types of condor chains would contribute the most to said sines. Then the consideration should mainly be about the amounts purchased and sold.

Let us denote the new decision variables for the amount of condor chains purchased or sold by $\mathbf{x}^{(c)} \in \mathbb{Z}^k$ for some $k \in \mathbb{N}$ and corresponding position premiums by $\mathbf{c}^{(c)} \in \mathbb{R}^k$. Additionally, denote the condor chain amplitudes corresponding with the sine and cosine amplitudes $A$ and $B$ by $\mathbf{b}^{(c)} \in \mathbb{R}^{k_1}$ and $\mathbf{a}^{(c)} \in \mathbb{R}^{k_2}$ respectively. Note that should hold $k_1 + k_2 = k$. Then we can clearly write the optimization problem as

$$\begin{aligned} \min_{\mathbf{x}^{(c)}}. \quad & \mu([\mathbf{a}^{(c)T}, \mathbf{b}^{(c)T}]^T, \mathbf{x}^{(c)}) \\ \text{s.t.} \quad & \mathbf{c}^{(c)T} \mathbf{x}^{(c)} \leq W \\ & x_i^{(c)} \in \mathbb{Z} \quad \forall\, i \in \{1, ..., k\} \end{aligned} \tag{4.2}$$

for some metric $\mu$. If the metric $\mu$ is chosen to be e.g. the squared Euclidean distance the problem is convex and thus should have at least an unique real-valued extreme point. Obviously, this doesn't guarantee a unique integer solution, but most nonlinear mixed-integer solvers should find a very good solution.

## 5 Discussion

The main theory derived in Section 3 and extended by the optimization problem in Section 4 provide a systematic way of forming the periodic payoffs. However, the main issue with the system is that it by no means always provides the best possible approximation that could be formed with the given set of call options. There are many classic examples for the Fourier series like the sawtooth function or the periodic step function that would be very easy to form directly from the options without the described theory. Still a systematic approach is always more reliable and easier to automate algorithmically and thus can have its merits in some fields of application.

Another pitfall of the theory is that it is limited to periodic functions. In practice, as one can always use it to approximate any function on a finite interval, this is less of an issue, but from a purely theoretical perspective it does raise some questions. For example, since the call option payoffs are not inherently periodic it seems counterintuitive to be limited to just periodic functions. This is in contrast with the Fourier series, which naturally lends itself to periodicity. Thus, we can conjecture that all functions should be formable as a series expansion of call option payoffs. This is more formally stated in Conjecture 5.1.

**Conjecture 5.1** (Arbitrary payoffs)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be any arbitrary function. Function $f$ can be formed as a series expansion of call option payoffs as defined in Definition 2.1 given strike prices on the whole real line, arbitrary position sizes and some arbitrary constant term.*