EE5609 Matrix Theory

Kranthi Kumar P

Download the latex-file codes from

https://github.com/kranthiakssy/AI20RESCH14002_PhD_IITH/tree/master/EE5609_Matrix_Theory/Assignment−12

Assignment-12

Problem:

Linear Transformations:
Let $\mathbb{V}$ be a finite-dimensional vector space and let $\mathbf{T}$ be a linear operator on $\mathbb{V}$. Suppose that $rank(\mathbf{T}^2) = rank(\mathbf{T})$. Prove that the range and null space of $\mathbf{T}$ are disjoint, i.e., have only the zero vector in common.

Solution:

Given	$\mathbb{V}$ is a finite-dimensional vector space, $\mathbf{T} : \mathbb{V} \to \mathbb{V}$ and $rank(\mathbf{T}^2) = rank(\mathbf{T})$
To prove	range and null space of $\mathbf{T}$ are disjoint

Proof	Let $\{\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_n\}$ is the subspace of $\mathbb{V}$. The linear transformation of $\mathbb{V}$ is $\{\mathbf{T}\beta_1, \ldots, \mathbf{T}\beta_k, \mathbf{T}\beta_{k+1}, \ldots, \mathbf{T}\beta_n\}$. Suppose the $rank(\mathbf{T}) = k$, then the basis of $\mathbf{T}$ is $\{\mathbf{T}\beta_1, \ldots, \mathbf{T}\beta_k\}$ and are linearly independent. Now $\mathbf{T}^2 : \mathbb{V} \to \mathbb{V}$ be a linear transformation for any $\alpha \in \mathbb{V}$. $\therefore \mathbf{T}^2(\mathbb{V}) = \mathbf{T}(\mathbf{T}(\alpha))$ and $\{\mathbf{T}^2\beta_1, \ldots, \mathbf{T}^2\beta_k\}$ span the range of $\mathbf{T}^2$ since $rank(\mathbf{T}^2) = rank(\mathbf{T})$ $\implies dim\ range(\mathbf{T}^2) = dim\ range(\mathbf{T})$ $\therefore \{\mathbf{T}^2\beta_1, \ldots, \mathbf{T}^2\beta_k\}$ must be basis for $range(\mathbf{T}^2)$ Now let $\alpha \in range(\mathbf{T})$, then it can be written as linear combinations of vectors in $range(\mathbf{T})$ $\therefore \alpha = C_1\mathbf{T}\beta_1 + C_2\mathbf{T}\beta_2 + \ldots + C_k\mathbf{T}\beta_k$ If $\alpha \in nullspace(\mathbf{T})$ also, then $\mathbf{T}(\mathbb{V}) = 0$ $\implies \mathbf{T}(C_1\mathbf{T}\beta_1 + C_2\mathbf{T}\beta_2 + \ldots + C_k\mathbf{T}\beta_k) = 0$ $\implies C_1\mathbf{T}^2\beta_1 + C_2\mathbf{T}^2\beta_2 + \ldots + C_k\mathbf{T}^2\beta_k = 0$ since $\{\mathbf{T}^2\beta_1, \ldots, \mathbf{T}^2\beta_k\}$ is basis of $\mathbf{T}^2$ $\implies C_1 = C_2 = \ldots = C_k = 0$ $\implies \mathbb{V} = 0$ $\therefore$ if α is in both $range(\mathbf{T}) and nullspace(\mathbf{T})$, then α must be a zero vector. Hence it is proved that range and null space of $\mathbf{T}$ are disjoint.

Eg:	Let $\alpha \in \mathbb{V}$ and $$\alpha = \begin{pmatrix} 1 & 7 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 4 & -2 & 0 & -4 \\ 2 & 3 & 4 & -2 \end{pmatrix}$$ linear transformation of α into $\mathbb{V}$ $\mathbf{T}(\alpha) = c\alpha$ then row reduced echelon form of $\mathbf{T}$ is $$rref(\mathbf{T}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $$\implies rank(\mathbf{T}) = 3,$$ $$nullity(\mathbf{T}) = 1$$ $$\implies range(\mathbf{T}) = \begin{pmatrix} 1 & 7 & -1 \\ -1 & 1 & 2 \\ 4 & -2 & 0 \\ 2 & 3 & 4 \end{pmatrix},$$ $$nullspace(\mathbf{T}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ Now linear transformation $\mathbf{T}(\mathbf{T}(\alpha)) = cd\alpha$. Let $c = d = 1$, then $range(\mathbf{T}^2) = range(\mathbf{T})$ and $rank(\mathbf{T}^2) = rank(\mathbf{T}) = 3$, $nullity(\mathbf{T}^2) = nullity(\mathbf{T}) = 3$ Hence proved that range and null space of $\mathbf{T}$ are disjoint.