EE5609 Matrix Theory

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Download the python code for from

https://github.com/kranthiakssy/AI20RESCH14002_PhD_IITH/tree/master/EE5609_Matrix_Theory/Assignment−9

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Assignment-9

Problem:

SVD:
Find the foot of the perpendicular to the plane

$$2x + 3y - 2z + 4 = 0 \tag{0.0.1}$$

from the point $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ using SVD.

Solution:

The given plane equation is

$$\begin{pmatrix} 2 & 3 & -2 \end{pmatrix}\mathbf{x} = 0 \tag{0.0.2}$$

$$\tag{0.0.3}$$

The equation of plane is

$$\mathbf{n}^T\mathbf{x} = c \tag{0.0.4}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \tag{0.0.5}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{0.0.6}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{0.0.7}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 0 \tag{0.0.8}$$

Let, a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{0.0.9}$$

Let, a=0 and b=1,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{0.0.10}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{0.0.11}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{0.0.12}$$

To solve (0.0.11) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{0.0.13}$$

substituting the value of $\mathbf{M}$ from equation (0.0.13) to (0.0.11),

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{0.0.14}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \tag{0.0.15}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.
First calculating the eigenvectors corresponding to

$\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} \end{pmatrix} \tag{0.0.16}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{0.0.17}$$
$$\implies \begin{pmatrix} 2-\lambda & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4}-\lambda \end{pmatrix} \tag{0.0.18}$$
$$\implies (\lambda - \frac{17}{4})(\lambda - 1) = 0 \tag{0.0.19}$$
$$\therefore \lambda_1 = \frac{17}{4}, \lambda_2 = 1, \tag{0.0.20}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \tag{0.0.21}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{0.0.22}$$
$$\mathbf{v_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{0.0.23}$$
$$\implies \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \tag{0.0.24}$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \tag{0.0.25}$$
$$\implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{13}{4} \end{pmatrix} \tag{0.0.26}$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{0.0.27}$$
$$\implies \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{13}{4}-\lambda \end{pmatrix} \tag{0.0.28}$$
$$\implies \lambda(\lambda-1)(\lambda - \frac{17}{4}) = 0 \tag{0.0.29}$$
$$\therefore \lambda_3 = \frac{17}{4}, \lambda_4 = 1, \lambda_5 = 0 \tag{0.0.30}$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{4}{13} \\ \frac{6}{13} \\ 1 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} \frac{-3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} -1 \\ \frac{-3}{2} \\ 1 \end{pmatrix} \tag{0.0.31}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \begin{pmatrix} \frac{4}{\sqrt{221}} \\ \frac{6}{\sqrt{221}} \\ \frac{13}{\sqrt{221}} \end{pmatrix} \tag{0.0.32}$$
$$\mathbf{v_4} = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{0.0.33}$$
$$\mathbf{v_5} = \begin{pmatrix} \frac{-2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix} \tag{0.0.34}$$
$$\implies \mathbf{U} = \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix} \tag{0.0.35}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{17}{4}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{0.0.36}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{0.0.37}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{17}{4}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}^T \tag{0.0.38}$$

Now substituting the values of (0.0.24), (0.0.37), (0.0.35) and (0.0.12) in (0.0.15),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{0.0.39}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \tag{0.0.40}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \tag{0.0.41}$$

$$\implies \mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \tag{0.0.42}$$

$$\mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \tag{0.0.43}$$

$$\implies \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{0.0.44}$$

$\therefore$ from equation (0.0.15),

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{0.0.45}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{0.0.46}$$

$$\implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{0.0.47}$$

$$\implies \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{0.0.48}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 2 & \frac{3}{2} & 3 \\ \frac{3}{2} & \frac{13}{4} & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} & -2 \end{pmatrix} \tag{0.0.49}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ 0 & \frac{17}{8} & -\frac{17}{4} \end{pmatrix} \tag{0.0.50}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{8}{17}R_2} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ 0 & 1 & -2 \end{pmatrix} \tag{0.0.51}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \tag{0.0.52}$$

$$\implies \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{0.0.53}$$

Hence from equations (0.0.45) and (0.0.53) we conclude that the solution is verified.