EE5609 Matrix Theory

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Download the latex-file codes from

https://github.com/kranthiakssy/AI20RESCH14002_PhD_IITH/tree/master/EE5609_Matrix_Theory/Assignment−13

1 Problem

(ugcdec2014, 78) :
Let $\mathbf{A}$ be a 4 x 7 real matrix and $\mathbf{B}$ be a 7 x 4 real matrix such that $\mathbf{AB} = \mathbf{I}_4$, where $\mathbf{I}_4$ is the 4 x 4 identity matrix. Which of the following is/are always true?

1) $rank(\mathbf{A}) = 4$
2) $rank(\mathbf{B}) = 7$
3) $nullity(\mathbf{B}) = 0$
4) $\mathbf{BA} = \mathbf{I}_7$, where $\mathbf{I}_7$ is the 7 x 7 identity matrix

2 Solution

Given	$\mathbf{A}$ is 4 x 7 real matrix $\mathbf{B}$ is 7 x 4 real matrix $\mathbf{AB} = \mathbf{I}_4$
Option-1	since $\mathbf{I}_4$ is a 4 x 4 identity matrix, $rank(\mathbf{I}_4) = 4 = rank(\mathbf{AB})$ from the properties of matrices $rank(\mathbf{A}) \leq min\{\#cloumns, \#rows\}$ $rank(\mathbf{A}) \leq 4$ and $rank(\mathbf{AB}) \leq rank(\mathbf{A})$ $4 \leq rank(\mathbf{A})$ $\therefore rank(\mathbf{A}) = 4$ Hence Option-1 is True.
Option-2	Similarly from the properties of matrices $rank(\mathbf{B}) \leq min\{\#cloumns, \#rows\}$ $rank(\mathbf{B}) \leq 4$ and

	$rank(\mathbf{AB}) \leq rank(\mathbf{B})$ $4 \leq rank(\mathbf{B})$ $\therefore rank(\mathbf{B}) = 4$ Hence Option-2 is False.
Option-3	Since $rank(\mathbf{B}) = 4$, and $\mathbf{B}$ is a 7 x 4 matrix in finite dimensional vector space $\mathbb{V}$. the column space,$C(\mathbf{B})$ will form the basis. $\implies range(\mathbf{B}) = dim(\mathbb{V}) = 4$ from rank-nullity theorem $rank(\mathbf{B}) + nullity(\mathbf{B}) = dim(\mathbb{V})$ by substituting above values $nullity(\mathbf{B}) = 0$ Hence Option-3 is True.
Option-4	Given $\mathbf{BA} = \mathbf{I}_7$ $rank(\mathbf{I}_7) = 7 = rank(\mathbf{BA})$ from the properties of matrices $rank(\mathbf{BA}) \leq rank(\mathbf{B})$ $7 \leq rank(\mathbf{B})$ the above conditioned can not be satisfied since we know $rank(\mathbf{B}) = 4$. Hence Option-4 is False.
Conclusion	Option-1 and 3 are True Option-2 and 4 are False

TABLE 1: Proof

3 Example

Example	Proving the above results with example in lower dimensions as follows. Let **A** be a 2 x 3 matrix in vector space $\mathbb{V}$ and consider $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -4 \end{pmatrix}$ and **B** be a 3 x 2 matrix in vector space $\mathbb{V}$ and consider $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$ so that $\mathbf{AB} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a 2 x 2 matrix
Option-1	row reduced echelon form of **A** is $rref(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \end{pmatrix}$ $\implies rank(\mathbf{A}) = 2$ Hence Option-1 is True
Option-2	row reduced echelon form of **B** is $rref(\mathbf{B}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\implies rank(\mathbf{B}) = 2$ Hence Option-2 is False
Option-3	from the above rref form of **B** the $range(\mathbf{B}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$ $\implies dim(\mathbb{V}) = 2$ $nullspace(\mathbf{B}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\therefore$ from rank-nullity theorem $nullity(\mathbf{B}) = 0$ Hence Option-3 is True
Option-4	$\mathbf{BA} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$ $\implies \mathbf{BA} \neq \mathbf{I}$ $rank(\mathbf{BA}) = \mathbf{I} = 2$ Hence Option-4 is False

TABLE 2: Example