EE5609 Matrix Theory

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Download the latex-file codes from

https://github.com/kranthiakssy/AI20RESCH14002_PhD_IITH/tree/master/EE5609_Matrix_Theory/Assignment−11

Assignment-11

Problem:

Linear Transformations:
Let $\mathbb{V}$ be a vector space and $\mathbf{T}$ a linear transformation from $\mathbb{V}$ into $\mathbb{V}$. Prove that the following two statements about $\mathbf{T}$ are equivalent.

(a) The intersection of the range of $\mathbf{T}$ and null space of $\mathbf{T}$ is the zero subspace of $\mathbb{V}$.
(b) If $\mathbf{T}(\mathbf{T}\alpha) = 0$, then $\mathbf{T}\alpha = 0$.

Solution:

Given	$\mathbf{T} : \mathbb{V} \to \mathbb{V}$
To prove	a) $range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$ b) If $\mathbf{T}(\mathbf{T}\alpha) = 0$, then $\mathbf{T}\alpha = 0$.

Proof(a)	Let $\mathbf{x} \in \mathbb{V}$ and $\mathbf{x} \in range(\mathbf{T}) \cap nullspace(\mathbf{T})$ then, $\mathbf{x} \in range(\mathbf{T})$ $\mathbf{x} \in nullspace(\mathbf{T})$ Consider $\mathbf{y} \in \mathbb{V}$ whose linear transformation into $\mathbb{V}$ is $\mathbf{x}$. $\implies \mathbf{T}(\mathbf{y}) = \mathbf{x}$ since $\mathbf{x} \in$ null space($\mathbf{T}$) and the sub space is linearly independent $\mathbf{T}(\mathbf{x}) = 0$ from above equations $\mathbf{T}(\mathbf{T}(y)) = 0$ from the definition of linear transformation of independent vector space $\mathbf{T}(y) = 0$ $\implies \mathbf{x} = 0$ $\implies \{0\} \subseteq range(\mathbf{T}) \cap nullspace(\mathbf{T})$ $\therefore range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$ Hence Proved.
Proof(b)	If $\mathbf{T}(\mathbf{T}\alpha) = 0$ then, from the definition of linear transformation, $\mathbf{T}\alpha$ will be in the null space of linear transformation $\mathbf{T}$ and is linearly independent $\therefore \mathbf{T}\alpha = 0$

Eg:	Let $\alpha \in \mathbb{V}$ and $$\alpha = \begin{pmatrix} 1 & 7 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 4 & -2 & 0 & -4 \\ 2 & 3 & 4 & -2 \end{pmatrix}$$ linear transformation of α into $\mathbb{V}$ $\mathbf{T}(\alpha) = c\alpha$ then row reduced echelon form of $\mathbf{T}$ is $$rref(\mathbf{T}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $\implies rank(\mathbf{T}) = 3,$ $nullity(\mathbf{T}) = 1$ $$\implies range(\mathbf{T}) = \begin{pmatrix} 1 & 7 & -1 \\ -1 & 1 & 2 \\ 4 & -2 & 0 \\ 2 & 3 & 4 \end{pmatrix},$$ $$nullspace(\mathbf{T}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ $\therefore range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$ Hence proved that the intersection of the range of $\mathbf{T}$ and null space of $\mathbf{T}$ is the zero subspace of $\mathbb{V}$.