EE5609 Matrix Theory

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Download the latex-file codes from

https://github.com/kranthiakssy/AI20RESCH14002_PhD_IITH/tree/master/EE5609_Matrix_Theory/Assignment−14

1 Problem

(ugcdec2014, 29) :
The determinant of n x n permutation matrix

$$\begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & . & & \\ & & . & & & \\ & . & & & & \\ 1 & & & & & \\ 1 & & & & & \end{pmatrix}$$

1) $(-1)^n$
2) $(-1)^{\lfloor \frac{n}{2} \rfloor}$
3) -1
4) 1

2 Solution

Given	n x n permutation matrix $\begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & . & & \\ & & . & & & \\ & . & & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$
Proof of row exchange	The given n x n permutation matrix can be converted into identity matrix of n x n dimension by doing row exchange operations. Let $\mathbf{A} = \begin{pmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{pmatrix}$

	$\begin{vmatrix} a_1 \\ . \\ . \\ a_i + a_j \\ a_i + a_j \\ . \\ . \\ a_n \end{vmatrix} = 0$ since determinant of a any matrix will be zero, if it has dependent rows. Expanding the above using linear property of determinants $\begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_i \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_i \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_j \\ . \\ . \\ a_n \end{vmatrix} = 0$ $\implies 0 + \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_i \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{vmatrix} + 0 = 0$ $\implies \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_i \\ . \\ . \\ a_n \end{vmatrix} = (-1) \begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{vmatrix}$ Hence it is proved that the exchange of rows a_i *and* a_j changes the sign of the determinant. $\therefore$ for every row exchange in given permutation matrix the determinant gets multiplied by -1.
finding no of exchanges	Let $\mathbf{A} = \begin{pmatrix} a_1 & . & a_i & a_{i+1}. & a_n \end{pmatrix}$ if n is even number then the elements a_1 *to* a_i will be exchanged with a_{i+1} *to* a_n where $i = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$. if n is odd, the center element will be a_{i+1} where $i + 1 = \lceil \frac{n}{2} \rceil$ then $i = \lfloor \frac{n}{2} \rfloor$ and the elements a_1 *to* a_i will be exchanged with a_{i+2} *to* a_n. $\therefore$ The given n x n matrix requires $\lfloor \frac{n}{2} \rfloor$ row exchanges to become identity matrix.

finding determinant	from the above results the determinant of given permutation matrix is $(-1)^{\lfloor \frac{n}{2} \rfloor} \begin{vmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 1 & \\ & & & & & & 1 \end{vmatrix}$ we know that the determinant of identity matrix, $det(\mathbf{I}) = 1$ $\therefore$ the determinant of given n x n permutation matrix = $(-1)^{\lfloor \frac{n}{2} \rfloor}$
Conclusion	Option-2 is the right solution

TABLE 1: Solution

3 Example

Example-1	Let **A** is 5 x 5 permutation matrix, then $det(\mathbf{A}) = \begin{vmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{vmatrix}$ $\xleftrightarrow{R_1 \leftrightarrow R_5} (-1) \begin{vmatrix} 1 & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ & & & & 1 \end{vmatrix}$ $\xleftrightarrow{R_2 \leftrightarrow R_4} (-1)(-1) \begin{vmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{vmatrix}$ $= 1$ substituting n = 5 in the solution $(-1)^{\lfloor \frac{5}{2} \rfloor} = 1$
Example-2	Let **A** is 6 x 6 permutation matrix, then $det(\mathbf{A}) = \begin{vmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{vmatrix}$

	$\xrightarrow[]{R_1 \leftrightarrow R_6} (-1) \begin{vmatrix} 1 & & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & & 1 \end{vmatrix}$ $\xrightarrow[]{R_2 \leftrightarrow R_5} (-1)(-1) \begin{vmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \end{vmatrix}$ $\xrightarrow[]{R_3 \leftrightarrow R_4} (-1)(-1)(-1) \begin{vmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{vmatrix}$ $= -1$ substituting n = 6 in the solution $(-1)^{\lfloor \frac{6}{2} \rfloor} = -1$ Hence the proved that the solution is correct.

TABLE 2: Example