EE5609 Matrix Theory

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Download the latex-file codes from

https://github.com/kranthiakssy/AI20RESCH14002_PhD_IITH/tree/master/EE5609_Matrix_Theory/Assignment−12

1 Problem

(hoffman/page84/11) :
Let $\mathbb{V}$ be a finite-dimensional vector space and let $\mathbf{T}$ be a linear operator on $\mathbb{V}$. Suppose that $rank(\mathbf{T}^2) = rank(\mathbf{T})$. Prove that the range and null space of $\mathbf{T}$ are disjoint, i.e., have only the zero vector in common.

2 Solution

Given	$\mathbb{V}$ is a finite-dimensional vector space, $\mathbf{T} : \mathbb{V} \to \mathbb{V}$ and $rank(\mathbf{T}^2) = rank(\mathbf{T})$
To Prove	range and null space of $\mathbf{T}$ are disjoint
Defining $rank(\mathbf{T})$	Let $\{\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_n\}$ is the span of $\mathbb{V}$. The linear transformation of $\mathbb{V}$ is $\{\mathbf{T}\beta_1, \ldots, \mathbf{T}\beta_k, \mathbf{T}\beta_{k+1}, \ldots, \mathbf{T}\beta_n\}$. Suppose the $rank(\mathbf{T}) = k$, then the basis of $\mathbf{T}$ is $\{\mathbf{T}\beta_1, \ldots, \mathbf{T}\beta_k\}$ and are linearly independent.
Defining $range(\mathbf{T}^2)$	Now $\mathbf{T}^2 : \mathbb{V} \to \mathbb{V}$ be a linear transformation for any $\alpha \in \mathbb{V}$. $\therefore \mathbf{T}^2(\mathbb{V}) = \mathbf{T}(\mathbf{T}(\alpha))$ and $\{\mathbf{T}^2\beta_1, \ldots, \mathbf{T}^2\beta_k\}$ span the range of $\mathbf{T}^2$ since $rank(\mathbf{T}^2) = rank(\mathbf{T})$ $\implies dim\ range(\mathbf{T}^2) = dim\ range(\mathbf{T})$ $\therefore \{\mathbf{T}^2\beta_1, \ldots, \mathbf{T}^2\beta_k\}$ must be basis for $range(\mathbf{T}^2)$

Obtaining *range and nullspace of* $\mathbf{T}$	Now let $\alpha \in range(\mathbf{T})$, then it can be written as linear combinations of vectors in $range(\mathbf{T})$ $\therefore \alpha = C_1\mathbf{T}\beta_1 + C_2\mathbf{T}\beta_2 + \ldots + C_k\mathbf{T}\beta_k$ If $\alpha \in nullspace(\mathbf{T})$ also, then $\mathbf{T}(\mathbb{V}) = 0$ $\implies \mathbf{T}(C_1\mathbf{T}\beta_1 + C_2\mathbf{T}\beta_2 + \ldots + C_k\mathbf{T}\beta_k) = 0$ $\implies C_1\mathbf{T}^2\beta_1 + C_2\mathbf{T}^2\beta_2 + \ldots + C_k\mathbf{T}^2\beta_k = 0$ since $\{\mathbf{T}^2\beta_1, \ldots, \mathbf{T}^2\beta_k\}$ is basis of $\mathbf{T}^2$ $\implies C_1 = C_2 = \ldots = C_k = 0$ $\implies \mathbb{V} = 0$ $\therefore$ if α is in both $range(\mathbf{T})$ *and* $nullspace(\mathbf{T})$, then α must be a zero vector. Hence it is proved that range and null space of $\mathbf{T}$ are disjoint.
Conclusion	The range and null space of $\mathbf{T}$ are disjoint.

TABLE 1: Proof

3 Example

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<td>Example</td>
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Let α be a basis of $\mathbb{V}$ of $\mathbb{R}^{nxn}$ space and

consider, $\alpha = \begin{pmatrix} 1 & 7 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 4 & -2 & 0 & -4 \\ 2 & 3 & 4 & -2 \end{pmatrix}$

linear transformation of α into $\mathbb{V}, \mathbf{T}(\alpha) = c\alpha$, where c is a scalar, then row reduced echelon form of $\mathbf{T}$ is

$rref(\mathbf{T}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\implies rank(\mathbf{T}) = 3,$
$nullity(\mathbf{T}) = 1$

$\implies range(\mathbf{T}) = \begin{pmatrix} 1 & 7 & -1 \\ -1 & 1 & 2 \\ 4 & -2 & 0 \\ 2 & 3 & 4 \end{pmatrix},$
$nullspace(\mathbf{T}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Now linear transformation $\mathbf{T}(\mathbf{T}(\alpha)) = cd\alpha$, where c *and* d are scalars.

Let $c = d = 1$, then $range(\mathbf{T}^2) = range(\mathbf{T})$,
$\implies rank(\mathbf{T}^2) = rank(\mathbf{T}) = 3,$
$\implies nullity(\mathbf{T}^2) = nullity(\mathbf{T}) = 1$
and $range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$

Hence proved that, the range and null space of $\mathbf{T}$ are disjoint.

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TABLE 2: Example