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Perturbations around a Slowly Rotating and Slightly Charged Black Hole*)

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The perturbation equations concerning the gravitational and electromagnetic waves around a Kerr-Newman black hole are separated in the limiting case of a small charge and small angular momentum of the black hole. The connection equations between the ingoing and outgoing parts of the waves are derived. Occurrence of superradiant scattering of gravitational waves off an uncharged rotating black hole is explicitly shown.

§ 1. Introduction

It is several years since the perturbation equations concerning the coupled gravitational and electromagnetic waves around a Kerr-Newman black hole were derived.[1)] However, the apparent lack of separability of variables has hampered further analyses. In the two limiting cases, one with no charge (Kerr) and the other with no rotation (Reissner-Nordström) of the black hole, the equations separate and various authors have treated the scattering problems of gravitational and electromagnetic waves for these limiting cases.[1)~6)] In the Kerr case, we note that the gravitational perturbation does not couple with the electromagnetic perturbation and, while the connections between the ingoing and outgoing parts have been obtained for the electromagnetic waves, they have not been obtained for the gravitational waves in the general case.

In this paper we consider another limiting case where both the charge and the angular momentum of the black hole are small, thus making a/M, Q/M and $a\omega$ small parameters. The terms of second and higher order in these small parameters being ignored, the separation of variables is achieved for the perturbation equations. One can use these separated equations to treat the scattering of gravitational and electromagnetic waves off a slowly rotating and slightly charged black hole. The connection between the ingoing and outgoing parts of the waves is obtained. To get numerical results one will only have to integrate the coupled radial equations from the event horizon to the asymptotically flat region. For the special case of no charge ($Q=0$), making use of a "Wronskian", we explicitly show that the superradiant scattering of gravitational waves occurs when the condition

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$\omega<(ma/4M^2)$ $(\omega>0)$ is satisfied. We adopt the same convention and the same notations as were used in Ref. 1), except some minor changes.

§ 2. Perturbation equations

From now on we consider a/M, Q/M and $a\omega$ as small parameters and ignore the terms of second and higher order in these parameters. Then Eqs. (2·13) ~(2·16) of Ref. 1) become

$$[(D-4\rho-\rho^*-3\varepsilon+\varepsilon^*)(\varDelta-4\gamma+\mu)$$
$$-3\psi_2-(\delta-3\beta-\alpha^*-4\tau+\pi^*)(\delta^*-4\alpha+\pi)]\psi_0{}^B$$
$$=\frac{4\phi_1{}^*}{3\psi_2}[(D-5\rho-3\varepsilon+\varepsilon^*)(\delta-2\beta-5\tau-\pi^*)$$
$$+2\rho^*\pi^*-\pi^*(\varepsilon-\varepsilon^*)-2\rho\tau]\chi_1{}^B\,, \tag{1}$$

$$[(\varDelta-3\gamma-\gamma^*+3\mu+\mu^*)(D-6\rho-2\varepsilon)$$
$$-6\psi_2-(\delta^*+4B^*-\tau^*)(\delta-2\beta-6\tau)]\chi_1{}^B=0\,, \tag{2}$$

$$[(\varDelta+4\mu+\mu^*+3\gamma-\gamma^*)(D+4\varepsilon-\rho)$$
$$-3\psi_2-(\delta^*+3\alpha+\beta^*+4\pi-\tau^*)(\delta+4\beta-\tau)]\psi_4{}^B$$
$$=\frac{4\phi_1{}^*}{3\psi_2}[(\varDelta+5\mu+3\gamma-\gamma^*)(\delta^*+2\alpha+5\pi+\tau^*)$$
$$-\tau^*(\gamma-\gamma^*+\mu+\mu^*)-2\pi\mu]\chi^B_{-1}=0\,, \tag{3}$$

$$[(D+3\varepsilon+\varepsilon^*-3\rho-\rho^*)(\varDelta+6\mu+2\gamma)$$
$$-6\psi_2-(\delta-\alpha^*+3\beta-3\tau+\pi^*)(\delta^*+2\alpha+6\pi)]\chi^B_{-1}=0\,. \tag{4}$$

"ε" which is zero for our present choice of tetrad is kept in the equations because, when we later calculate the connection between $(\psi_0{}^B, \chi_1{}^B)$ and $(\psi_4{}^B, \chi^B_{-1})$ at the event horizon, it is more convenient to take a different choice of tetrad for which "ε" is not zero. Equations (1)~(4) can be separated by writing

$$\psi_0{}^B=e^{-i\omega t}e^{im\phi}\Theta_2{}^{lm}(a\omega;\,\theta)R_2(r)+e^{i\omega t}e^{-im\phi}\Theta_{-2}^{lm}(a\omega;\,\theta)P_2(r),$$

$$\chi_1{}^B=e^{-i\omega t}e^{im\phi}\Theta_1{}^{lm}\frac{R_1}{(r-ia\cos\theta)^3}+e^{i\omega t}e^{-im\phi}\Theta_{-1}^{lm}\frac{P_1}{(r-ia\cos\theta)^3}\,,$$

$$\chi^B_{-1}=e^{-i\omega t}e^{im\phi}\Theta_{-1}^{lm}\frac{\varDelta R_{-1}}{2(r-ia\cos\theta)^5}+e^{i\omega t}e^{-im\phi}\Theta_1{}^{lm}\frac{\varDelta P_{-1}}{2(r-ia\cos\theta)^5}\,,$$

$$\psi_4{}^B=e^{-i\omega t}e^{im\phi}\Theta_{-2}^{lm}\frac{\varDelta^2R_{-2}}{4(r-ia\cos\theta)^4}+e^{i\omega t}e^{-im\phi}\Theta_2{}^{lm}\frac{\varDelta^2P_{-2}}{4(r-ia\cos\theta)^4}\,. \tag{5}$$

Here Δ is equal to r^2-2Mr, while the same notation used in Eqs. (1)~(4) represents the differential operator $n^\mu(\partial/\partial x^\mu)$. The distinction should be quite clear and cause no trouble. The fact that we have two terms for each perturbation, one involving $e^{-i\omega t}$ and the other involving $e^{i\omega t}$, which is not necessary just to separate the equations, is explained when we find the connection between $(\psi_0{}^B, \chi_1{}^B)$ and $(\psi_4{}^B, \chi_{-1}^B)$. The $e^{-i\omega t}$ terms of one group are not only connected with the $e^{-i\omega t}$ terms but also with the $e^{i\omega t}$ terms of the other group. The angular functions $\Theta_s{}^{lm}(a\omega;\theta)$ are the spin-weighted spheroidal harmonics[3] satisfying the following equation:

$$\Big(\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}+a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}$$

$$-2sa\omega\cos\theta-\frac{2sm\cos\theta}{\sin^2\theta}-s^2\cot^2\theta-s^2\Big)\Theta_s{}^{lm}=E_s{}^{lm}\Theta_s{}^{lm}. \tag{6}$$

To first order of $a\omega$, $\Theta_s{}^{lm}$ can be expanded in terms of the spin-weighted spherical harmonics $(Y_s{}^{lm})$ as

$$\Theta_s{}^{lm}=Y_s{}^{lm}+2sa\omega\sum_{l'\neq l}a_s{}^{ll'}Y_s{}^{l'm}, \tag{7}$$

where

$$a_s{}^{ll'}=-a_{-s}^{ll'}=\frac{\langle Y_s{}^{l'm}|\cos\theta|Y_s{}^{lm}\rangle}{l(l+1)-l'(l'+1)},$$

and the eigenvalues are

$$E_s{}^{lm}=-l(l+1)+a\omega\frac{2s^2m}{l(l+1)}. \tag{8}$$

Other properties of $\Theta_s{}^{lm}$ to be used later on are (see Appendix A),

$$\Big(\frac{d}{d\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}\Big)\Big(\frac{d}{d\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}+\cot\theta\Big)\Theta_1{}^{lm}$$

$$=[l(l+1)-2a\omega m]\Theta_{-1}^{lm}, \tag{9}$$

$$\Big(\frac{d}{d\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}-\cot\theta\Big)\Big(\frac{d}{d\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}\Big)$$

$$\times\Big(\frac{d}{d\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}+\cot\theta\Big)\Big(\frac{d}{d\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}-2\cot\theta\Big)\Theta_2{}^{lm}$$

$$=(l-1)(l-2)[l(l+1)-4a\omega m]\Theta_{-2}^{lm}. \tag{10}$$

The symmetry of Eq. (6) and the fact that $Y_s{}^{lm}(\pi-\theta)=(-1)^sY_{-s}^{l-m}(\theta)$ indicate that

$$\Theta_s{}^{lm}(a\omega;\ \pi-\theta)=(-1)^s\Theta_{-s}^{lm}(a\omega;\ \theta)=\Theta_s{}^{l,-m}(-a\omega;\ \theta). \tag{11}$$

The remaining radial equations after the separation of variables are

$$\Big[\Delta\frac{d^2}{dr^2}+6(r-M)\frac{d}{dr}+\omega^2\frac{r^4}{\Delta}+8i\omega r-\frac{4i\omega r^2(r-M)}{\Delta}-\frac{2a\omega mr^2}{\Delta}$$

$$+\frac{4iam(r-M)}{\Delta}-l(l+1)+\frac{8a\omega m}{l(l+1)}+6+2a\omega m\Big]\begin{Bmatrix} R_2 \\ P_2{}^* \\ R^*_{-2} \\ P_{-2}\end{Bmatrix}$$

$$=\frac{-2\sqrt{2}Q\sqrt{(l-1)(l+2)}}{3Mr}\left(\frac{d}{dr}-i\omega\frac{r^2}{\Delta}+\frac{1}{r}\right)\begin{Bmatrix} R_1 \\ P_1{}^* \\ R^*_{-1} \\ P_{-1}\end{Bmatrix}, \tag{12}$$

$$\Big[\Delta\frac{d^2}{dr^2}+4(r-M)\frac{d}{dr}+\omega^2\frac{r^4}{\Delta}+4i\omega r-\frac{2i\omega r^2(r-M)}{\Delta}-\frac{2a\omega mr^2}{\Delta}$$

$$+\frac{2iam(r-M)}{\Delta}-l(l+1)+\frac{2a\omega m}{l(l+1)}+2+2a\omega m\Big]\begin{Bmatrix} R_1 \\ P_1{}^* \\ R^*_{-1} \\ P_{-1}\end{Bmatrix}=0. \tag{13}$$

Each of the pairs (R_2, R_1), $(P_2{}^*, P_1{}^*)$, (R^*_{-2}, R^*_{-1}) and (P_{-2}, P_{-1}) satisfies the same equations.

§ 3. Asymptotic solutions and the connections between the ingoing and outgoing waves

The asymptotic behaviour of the solutions of Eqs. (12) and (13) near infinity $(r\to\infty)$ is

$$R_2,\ P_2{}^*,\ R^*_{-2},\ P_{-2}\sim\frac{e^{-i\omega r'}}{r},\ \frac{e^{-i\omega r'}}{r^3},\ \frac{e^{i\omega r'}}{r^5},\ \frac{e^{i\omega r'}}{r^6},$$

$$R_1,\ P_1{}^*,\ R^*_{-1},\ P_{-1}\sim\frac{e^{-i\omega r'}}{r},\ \frac{e^{i\omega r'}}{r^3}, \tag{14}$$

where

$$dr'=\frac{r^2}{\Delta}dr\,.$$

Making use of these asymptotic solutions, we can expand Eq. (5) in the following way:

$$\psi_0{}^B = e^{-i\omega t}e^{im\phi}\Theta_2\Big[A_2 e^{-i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+B_2 e^{i\omega r'}\Big(\frac{1}{r^5}+\cdots\Big)\Big]$$
$$+e^{i\omega t}e^{-im\phi}\Theta_{-2}\Big[C_2 e^{i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+D_2 e^{-i\omega r'}\Big(\frac{1}{r^5}+\cdots\Big)\Big],$$
$$\chi_1{}^B = e^{-i\omega t}e^{im\phi}\frac{\Theta_1}{(r-ia\cos\theta)^3}\Big[A_1 e^{-i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+B_1 e^{i\omega r'}\Big(\frac{1}{r^3}+\cdots\Big)\Big]$$
$$+e^{i\omega t}e^{-im\phi}\frac{\Theta_{-1}}{(r-ia\cos\theta)^3}\Big[C_1 e^{i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+D_1 e^{-i\omega r'}\Big(\frac{1}{r^3}+\cdots\Big)\Big],$$
$$\chi_{-1}^B = e^{-i\omega t}e^{im\phi}\frac{\Theta_{-1}\Delta}{2(r-ia\cos\theta)^5}\Big[A_{-1} e^{i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+B_{-1} e^{-i\omega r'}\Big(\frac{1}{r^3}+\cdots\Big)\Big]$$
$$+e^{i\omega t}e^{-im\phi}\frac{\Theta_1\Delta}{2(r-ia\cos\theta)^5}\Big[C_{-1} e^{-i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+D_{-1} e^{i\omega r'}\Big(\frac{1}{r^3}+\cdots\Big)\Big],$$
$$\psi_4{}^B = e^{-i\omega t}e^{im\phi}\frac{\Theta_{-2}\Delta^2}{4(r-ia\cos\theta)^4}\Big[A_{-2} e^{i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+B_{-2} e^{-i\omega r'}\Big(\frac{1}{r^5}+\cdots\Big)\Big]$$
$$+e^{i\omega t}e^{-im\phi}\frac{\Theta_2\Delta^2}{4(r-ia\cos\theta)^4}\Big[C_{-2} e^{-i\omega r'}\Big(\frac{1}{r}+\cdots\Big)+D_{-2} e^{i\omega r'}\Big(\frac{1}{r^5}+\cdots\Big)\Big]. \tag{15}$$

Here the indices l and m are omitted from $\Theta_s{}^{lm}$ and only the independent coefficients are explicitly shown. The asymptotic behaviour of the solutions near the event horizon ($\Delta\to 0$) is

$$R_2,\ P_2{}^*,\ R_{-2}^*,\ P_{-2}\sim\Delta^{-2}e^{-ikr'},\ \Delta^{-1}e^{-ikr'},\ e^{ikr},\ \Delta e^{ikr'},$$
$$R_1,\ P_1{}^*,\ R_{-1}^*,\ P_{-1}\sim\Delta^{-1}e^{-ikr'},\ e^{ikr'}, \tag{16}$$

where

$$k\equiv\omega-\frac{am}{4M^2}.$$

The requirement that the solutions be non-singular for a physical observer at the event horizon eliminates $\Delta^{-2}e^{-ikr'}$ and $\Delta^{-1}e^{-ikr'}$ as asymptotic solutions for R_{-2}^* and P_{-2}, and $\Delta^{-1}e^{-ikr'}$ for R_{-1}^* and P_{-1}.[2)] Therefore the physically acceptable solutions should be expanded near the event horizon as follows:

$$\psi_0{}^B = e^{-i\omega t}e^{im\phi}\Theta_2[\alpha_2 e^{-ikr'}(\Delta^{-2}+\cdots)+\beta_2 e^{ikr'}(1+\cdots)]$$
$$+e^{i\omega t}e^{-im\phi}\Theta_{-2}[\gamma_2 e^{ikr'}(\Delta^{-2}+\cdots)+\delta_2 e^{-ikr'}(1+\cdots)],$$

$$\chi_1{}^B = e^{-i\omega t}e^{im\phi}\frac{\Theta_1}{(r-ia\cos\theta)^3}[\alpha_1 e^{-ikr'}(\varDelta^{-1}+\cdots)+\beta_1 e^{ikr'}(1+\cdots)]$$

$$+e^{i\omega t}e^{-im\phi}\frac{\Theta_{-1}}{(r-ia\cos\theta)^3}[\gamma_1 e^{ikr'}(\varDelta^{-1}+\cdots)+\delta_1 e^{-ikr'}(1+\cdots)],$$

$$\chi^B_{-1} = e^{-i\omega t}e^{im\phi}\frac{\Theta_{-1}}{2(r-ia\cos\theta)^5}\beta_{-1}e^{-ikr'}(\varDelta+\cdots)$$

$$+e^{i\omega t}e^{-im\phi}\frac{\Theta_1}{2(r-ia\cos\theta)^5}\delta_{-1}e^{ikr'}(\varDelta+\cdots),$$

$$\psi_4{}^B = e^{-i\omega t}e^{im\phi}\frac{\Theta_{-2}}{4(r-ia\cos\theta)^4}\beta_{-2}e^{-ikr'}(\varDelta^2+\cdots)$$

$$+e^{i\omega t}e^{-im\phi}\frac{\Theta_2}{4(r-ia\cos\theta)^4}\delta_{-2}e^{ikr'}(\varDelta^2+\cdots). \tag{17}$$

Here again only the independent coefficients are explicitly shown. We note however that they are arbitrary and independent of each other only in the sense that they satisfy Eqs. (1)~(4). Since they also have to satisfy other members of the Bianchi identity which are not involved in the derivation of Eqs. (1)~(4), these coefficients are not completely independent of each other after all. The details of the calculation of the connections among the coefficients are given in Appendix B and the results are

$$\beta_1 = \delta_1 = 0\,,$$

$$\beta_{-1} = \frac{l(l+1)-2a\omega m}{64M^4(ik)\left(ik-\dfrac{1}{4M}\right)}\alpha_1\,,$$

$$\delta_{-1} = \frac{l(l+1)-2a\omega m}{64M^4(ik)\left(ik+\dfrac{1}{4M}\right)}\gamma_1\,, \tag{18}$$

$$\beta_2 = \delta_2 = 0\,,$$

$$64M^8\left(ik-\frac{1}{2M}\right)\beta_{-2} = \frac{-(l-1)(l+2)[l(l+1)-4a\omega m]}{64(ik)\left|ik-\dfrac{1}{4M}\right|^2}\alpha_2$$

$$-\frac{3\omega M}{16k\left|ik-\dfrac{1}{4M}\right|^2}\gamma_2{}^* + \frac{Q\sqrt{(l-1)(l+2)}[l(l+1)-a\omega m]}{6\sqrt{2}(ik)\left(ik-\dfrac{1}{4M}\right)}\alpha_1$$

$$+\frac{Q\sqrt{(l-1)(l+2)}M}{3\sqrt{2}\left(ik-\dfrac{1}{4M}\right)}\gamma_1{}^*\,,$$

$$64M^8\left(ik+\frac{1}{2M}\right)\delta_{-2}=\frac{-(l-1)(l+2)[l(l+1)-4a\omega m]}{64(ik)\left|ik+\frac{1}{4M}\right|^2}\gamma_2$$

$$+\frac{3\omega M}{16k\left|ik+\frac{1}{4M}\right|^2}\alpha_2^*-\frac{Q\sqrt{(l-1)(l+2)}[l(l+1)-a\omega m]}{6\sqrt{2}(ik)\left(ik+\frac{1}{4M}\right)}\gamma_1$$

$$+\frac{Q\sqrt{(l-1)(l+2)}M}{3\sqrt{2}\left(ik+\frac{1}{4M}\right)}\alpha_1^*. \tag{19}$$

The number of remaining arbitrary coefficients is four representing two possible polarizations of the ingoing gravitational wave plus two possible polarizations of the ingoing electromagnetic wave.[7] The outgoing waves are completely determined once the magnitudes, phases and polarizations of the ingoing waves are given. The task of finding connections among the coefficients in the asymptotic solutions near infinity (Eq. (15)) to reduce the number of arbitrary coefficients to four is to be done by numerically integrating the coupled radial equations, Eqs. (12) and (13), from the event horizon to infinity. Once it is done, one can consider the scattering problem and compare the outgoing energy flux with the ingoing energy flux. One half of the task (reducing the number of arbitrary coefficients from sixteen to eight) can also be achieved, without computer work, by following a procedure similar to the one explained in Appendix B, this time near infinity instead of near the event horizon. We list the results below:

$$\begin{aligned}
&B_{-1}=\Gamma_1 A_1\,,\\
&D_{-1}=\Gamma_1 C_1\,,\\
&B_{-2}=\Gamma_2 A_2+\Gamma_3 C_2^*-\Gamma_4 C_1^*\,,\\
&D_{-2}=\Gamma_2 C_2+\Gamma_3^* A_2^*+\Gamma_4 A_1^*\,,\\
&B_1=\Gamma_1 A_{-1}\,,\\
&D_1=\Gamma_1 C_{-1}\,,\\
&B_2=\Gamma_2 A_{-2}+\Gamma_3^* C_{-2}^*-\Gamma_4 C_{-1}^*\,,\\
&D_2=\Gamma_2 C_{-2}+\Gamma_3 A_{-2}^*+\Gamma_4 A_{-1}^*\,,
\end{aligned} \tag{20}$$

where

$$\Gamma_1\equiv\frac{-l(l+1)+2a\omega m}{4\omega^2}\,,\qquad \Gamma_2\equiv\frac{(l-1)(l+2)[l(l+1)-4a\omega m]}{16\omega^4}\,,$$

$$\Gamma_3 \equiv \frac{3iM}{4\omega^3}\,, \qquad \Gamma_4 \equiv \frac{\sqrt{2}Q\sqrt{(l-1)(l+2)}}{6\omega^2 M}$$

The expressions given in Ref. 1) for the energy flux at infinity of the ingoing and outgoing gravitational and electromagnetic waves are still valid for our present case, namely,

$$\frac{dE^{\text{in}}_{\text{gr.}}}{dt} = \frac{1}{64\pi\omega^2}(|A_2|^2 + |C_2|^2),$$

$$\frac{dE^{\text{out}}_{\text{gr.}}}{dt} = \frac{1}{64\pi\omega^2}(|A_{-2}|^2 + |C_{-2}|^2),$$

$$\frac{dE^{\text{in}}_{\text{e.m.}}}{dt} = \frac{1}{72\pi M^2}\left(\left|A_1 + \frac{iQ\sqrt{(l-1)(l+2)}}{2\sqrt{2}\omega}A_2\right|^2 + \left|C_1 + \frac{iQ\sqrt{(l-1)(l+2)}}{2\sqrt{2}\omega}C_2\right|^2\right),$$

$$\frac{dE^{\text{out}}_{\text{e.m.}}}{dt} = \frac{1}{72\pi M^2}\left(\left|A_{-1} + \frac{iQ\sqrt{(l-1)(l+2)}}{2\sqrt{2}\omega}A_{-2}\right|^2 + \left|C_{-1} + \frac{iQ\sqrt{(l-1)(l+2)}}{2\sqrt{2}\omega}C_{-2}\right|^2\right). \tag{21}$$

§ 4. Superradiant scattering

Superradiance has been discussed by many authors.[2),8)] Here we explicitly show, for the case of no charge ($Q=0$), that a superradiant scattering indeed occurs for the gravitational wave if a certain condition is satisfied. A superradiant scattering is surely expected for the general case as long as $a \neq 0$. However we have not been able to construct an appropriate "Wronskian" which is to be used to show it. With $Q=0$, Eq. (12) can be written as

$$\mathcal{O}\begin{pmatrix} X_2 \\ T_2^* \\ X_{-2}^* \\ T_{-2} \end{pmatrix} = 0\,, \tag{22}$$

where $\mathcal{O} \equiv (d^2/dr'^2) + f$ with f being a function of r with no differential operator in it and

$$X_2 \equiv r\Delta R_2, \quad T_2 = r\Delta P_2, \quad X_{-2} = r\Delta R_{-2}, \quad T_{-2} = r\Delta P_{-2}.$$

From Eq. (22) we take

$$T_{-2}\mathcal{O}X_2 - X_2\mathcal{O}T_{-2} + X_{-2}\mathcal{O}^*T_2 - T_2\mathcal{O}^*X_{-2} = 0$$

or explicitly

$$\frac{d}{dr'}W \equiv \frac{d}{dr'}\left[T_{-2}\frac{dX_2}{dr'} - X_2\frac{dT_{-2}}{dr'} + X_{-2}\frac{dT_2}{dr'} - T_2\frac{dX_{-2}}{dr'}\right] = 0\,. \tag{23}$$

Hence we obtain a conserved quantity W. The value of W evaluated at infinity using the asymptotic solutions near infinity and the connection equations (Eq. (20)) is

$$W = \frac{3M}{2\omega^2}[|A_{-2}|^2 + |C_{-2}|^2 - |A_2|^2 - |C_2|^2]. \tag{24}$$

Comparing this value with Eq. (21) one can see that

$$\frac{dE_{\text{gr.}}}{dt} \equiv \frac{dE^{\text{out}}_{\text{gr.}}}{dt} - \frac{dE^{\text{in}}_{\text{gr.}}}{dt} = \frac{W}{96\pi M}\,. \tag{25}$$

Now we evaluate W at the event horizon using the asymptotic solutions near the horizon and the connection equations (Eq. (19)) to find

$$\tilde{W} = \frac{-3\omega(|\alpha_2|^2 + |\gamma^2|^2)}{256M^6\left(\omega - \dfrac{ma}{4M^2}\right)\left|ik - \dfrac{1}{4M}\right|^2}\,. \tag{26}$$

Comparing Eqs. (25) and (26) one concludes that the net outgoing energy flux is positive (superradiant scattering) when the following condition is satisfied:

$$\omega < \frac{am}{4M^2} \qquad \text{if} \qquad \omega > 0$$

or

$$\omega > \frac{am}{4M^2} \qquad \text{if} \qquad \omega < 0\,. \tag{27}$$

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Appendix A

——Angular Equations——

The operator of Eq. (6) can be written as

$$\mathcal{O}_s = \mathcal{L}^-_{(s+1)}\mathcal{L}^+_{(s)} - (4s+2)a\omega\cos\theta + a^2\omega^2 - 2ma\omega - s(s+1), \tag{A1}$$

where

$$\mathcal{L}^{\pm}_{(s)} \equiv \frac{d}{d\theta} \pm a\omega \cos\theta \mp \frac{m}{\sin\theta} \mp s\cos\theta \,.$$

Also it can easily be shown that

$$\mathcal{L}^{-}_{(s+1)}\mathcal{L}^{+}_{(s)} = \mathcal{L}^{+}_{(s-1)}\mathcal{L}^{-}_{(s)} + 4a\omega\cos\theta + 2s \,. \tag{A2}$$

Operating $\mathcal{O}_{-1}$ on the left-hand side of Eq. (9) and using Eq. (A2) repeatedly, one obtains

$$\mathcal{O}_{-1}\mathcal{L}^{-}_{(0)}\mathcal{L}^{-}_{(1)}\Theta_1^{lm} = E_{-1}^{lm}\mathcal{L}^{-}_{(0)}\mathcal{L}^{-}_{(1)}\Theta_1^{lm} \tag{A3}$$

which proves that

$$\mathcal{L}^{-}_{(0)}\mathcal{L}^{-}_{(1)}\Theta_1{}^{lm} = A\Theta_{-1}^{lm} \,, \tag{A4}$$

where A is a constant. To first order of $a\omega$, Eq. (A4) becomes

$$\begin{aligned}
&\left[\left(\frac{d}{d\theta}+\frac{m}{\sin\theta}\right)\left(\frac{d}{d\theta}+\frac{m}{\sin\theta}+\cot\theta\right)\right.\\
&\qquad\left. -2a\omega\sin\theta\left(\frac{d}{d\theta}+\frac{m}{\sin\theta}+\cot\theta\right)\right][Y_1{}^{lm}+2a\omega\sum_{l'\neq l}a_1{}^{ll'}Y_1{}^{l'm}]\\
&\quad = l(l+1)Y_{-1}^{lm}+2a\omega\sum_{l'\neq l}l'(l'+1)Y_{-1}^{l'm}-2a\omega\sqrt{l(l+1)}\sin\theta Y_0{}^{lm}\\
&\quad = A[Y_{-1}^{lm}-2a\omega\sum_{l'\neq l}a_{-1}^{ll'}Y_{-1}^{l'm}].
\end{aligned}\tag{A5}$$

Taking the inner product of both sides of Eq. (A5) with Y_{-1}^{lm} gives the value of A:

$$\begin{aligned}
A &= l(l+1)-2a\omega\sqrt{l(l+1)}\int_0^\pi Y_{-1}^{lm}Y_0{}^{lm}\sin^2\theta d\theta\\
&= l(l+1)-2a\omega\left[\int_0^\pi \frac{dY_0{}^{lm}}{d\theta}Y_0{}^{lm}\sin^2\theta d\theta + m\int_0^\pi (Y_0{}^{lm})^2\sin\theta d\theta\right]\\
&= l(l+1)-2a\omega\left[-\int_0^\pi (Y_0{}^{lm})^2\cos\theta\sin\theta d\theta + m\right]\\
&= l(l+1)-2a\omega m \,.
\end{aligned}\tag{A6}$$

A similar procedure as above, although requiring many more steps, proves Eq. (10).

Appendix B

——Derivation of Connections near the Event Horizon——

To find the connection among the coefficients in the asymptotic solutions near the event horizon, it is more convenient to use a different coordinate system which is non-singular at the horizon, and also a different tetrad which is non-singular in this non-singular coordinate system. If one makes the transformations[2)]

$$dv = dt + \frac{r^2+a^2}{\Delta}dr, \qquad d\tilde{\phi} = d\phi + \frac{a}{\Delta}dr\,, \tag{B1}$$

one can see that the background metric is non-singular in this v, r, θ, $\tilde{\phi}$ coordinate system. Also the transformations

$$\tilde{l} = \frac{\Delta}{2\Sigma}l, \qquad \tilde{n} = \frac{2\Sigma}{\Delta}n, \qquad \tilde{m} = m \tag{B2}$$

give a non-singular background tetrad. The background values of our new tetrad and spin coefficients are, ignoring the second and higher order terms in a,

$$\begin{aligned}
&\tilde{l}^\mu = \delta_0{}^\mu + \frac{\Delta}{2r^2}\delta_1{}^\mu + \frac{a}{r^2}\delta_3{}^\mu\,,\\
&\tilde{n}^\mu = -\delta_1{}^\mu\,,\\
&\tilde{m}^\mu = \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta\delta_0{}^\mu + \delta_2{}^\mu + \frac{i}{\sin\theta}\delta_3{}^\mu\right),\\
&\kappa = \sigma = \nu = \lambda = 0\,,\\
&\varepsilon = \frac{r-M}{2r^2} - \frac{\Delta}{2r^3}, \qquad \rho = \frac{-\Delta}{2r^2(r-ia\cos\theta)}\,,\\
&\pi = \frac{ia\sin\theta}{\sqrt{2}r^2}\,, \qquad \tau = \frac{-ia\sin\theta}{\sqrt{2}r^2}\,,\\
&\beta = \frac{\cot\theta}{2\sqrt{2}(r+ia\cos\theta)}, \qquad \alpha = \pi - \beta^*\,,\\
&\mu = \frac{-1}{r-ia\cos\theta}, \qquad \gamma = \frac{-ia\cos\theta}{r^2}\,.
\end{aligned} \tag{B3}$$

The transformations (B2) cause the following transformations of the perturbations:

$$\tilde{\psi}_0{}^B = \frac{\Delta^2}{4r^4}\psi_0{}^B, \quad \tilde{\chi}_1{}^B = \frac{\Delta}{2r^2}\chi_1{}^B, \quad \tilde{\chi}^B_{-1} = \frac{2r^2}{\Delta}\chi^B_{-1}, \quad \tilde{\psi}_4{}^B = \frac{4r^4}{\Delta^2}\psi_4{}^B\,. \tag{B4}$$

The asymptotic forms of these new quantities near the event horizon are, from Eqs. (17) and (B4),

$$\begin{aligned}
\tilde{\psi}_0{}^B = {} & e^{-i\omega v}e^{im\tilde{\phi}}\Theta_2\frac{1}{64M^4}[\alpha_2(1+\cdots)+\beta_2 e^{2ikr'}(\Delta^2+\cdots)]\\
& + e^{i\omega v}e^{-im\tilde{\phi}}\Theta_{-2}\frac{1}{64M^4}[\gamma_2(1+\cdots)+\delta_2 e^{-2ikr'}(\Delta^2+\cdots)],
\end{aligned}$$

$$\tilde{\chi}_1{}^B = e^{-i\omega v} e^{im\tilde{\phi}} \frac{\Theta_1}{8M^2(r-ia\cos\theta)^3}[\alpha_1(1+\cdots)+\beta_1 e^{2ikr'}(\Delta+\cdots)]$$

$$+ e^{i\omega v} e^{-im\tilde{\phi}} \frac{\Theta_{-1}}{8M^2(r-ia\cos\theta)^3}[\gamma_1(1+\cdots)+\delta_1 e^{-2ikr'}(\Delta+\cdots)],$$

$$\tilde{\chi}^B_{-1} = e^{-i\omega v} e^{im\tilde{\phi}} \Theta_{-1} \frac{4M^2}{(r-ia\cos\theta)^5}\beta_{-1}(1+\cdots)$$

$$+ e^{i\omega v} e^{-im\tilde{\phi}} \Theta_1 \frac{4M^2}{(r-ia\cos\theta)^5}\delta_{-1}(1+\cdots),$$

$$\tilde{\psi}_4{}^B = e^{-i\omega v} e^{im\tilde{\phi}} \Theta_{-2} \frac{16M^4}{(r-ia\cos\theta)^4}\beta_{-2}(1+\cdots)$$

$$+ e^{i\omega v} e^{-im\tilde{\phi}} \Theta_2 \frac{16M^4}{(r-ia\cos\theta)^4}\delta_{-2}(1+\cdots). \tag{B5}$$

The procedure for obtaining Eq. (18) starts with the following equations:

$$(D-2\rho)\phi_1-(\delta^*+\pi-2\alpha)\phi_0=0\,, \tag{B6}$$

$$(\delta^*+2\pi)\phi_1-(D-\rho+2\varepsilon)\phi_2=0\,, \tag{B7}$$

$$3(D-3\rho)\psi_2-3(\delta^*-2\alpha+2\pi)\psi_1$$
$$=2(D-2\rho^*+\rho)\phi_{11}-2(\delta+\tau-2\alpha^*+\pi^*)\phi_{10}+(\delta^*-2\alpha-2\pi-2\tau^*)\phi_{01}\,, \tag{B8}$$

$$3(\delta^*+3\pi)\psi_2-3(D+2\varepsilon-2\rho)\psi_3$$
$$=-2(\delta^*-\pi-2\tau^*)\phi_{11}+2(\Delta+\mu^*-\mu-2\gamma^*)\phi_{10}-(D-2\rho^*+2\rho+2\varepsilon)\phi_{21}\,. \tag{B9}$$

Operating by $(\delta^*+3\pi-\beta^*-\alpha)$ on Eq. (B6) and by $(D-3\rho+\varepsilon-\varepsilon^*)$ on Eq. (B7) and subtracting one from the other results

$$(\delta^*+3\pi-\beta^*-\alpha)(\delta^*+\pi-2\alpha)\phi_0-(D-3\rho+\varepsilon-\varepsilon^*)(D-\rho+2\varepsilon)\phi_2=-2\phi_1 F\,, \tag{B10}$$

where

$$F\equiv(\delta^*\rho)+(D\pi)-\rho(\alpha+\beta^*)+\rho(\varepsilon-\varepsilon^*)+\mu\kappa^*+\tau\sigma^*$$

Now operate $(\delta^*+4\pi-\beta^*-\alpha)$ on Eq. (B8) and $(D-4\rho+\varepsilon-\varepsilon^*)$ on Eq. (B9) to obtain

$$3\psi_2 F=-3(\delta^*+4\pi-\beta^*-\alpha)(\delta^*-2\alpha+2\pi)\psi_1$$

$$+3(D-4\rho+\varepsilon-\varepsilon^*)(D+2\varepsilon-2\rho)\psi_3$$

$$-2[(\delta^*+4\pi-\beta^*-\alpha)(D-2\rho^*+\rho)$$
$$+(D-4\rho+\varepsilon-\varepsilon^*)(\delta^*-\pi-2\tau^*)]\phi_{11}$$
$$+2[(\delta^*+4\pi-\beta^*-\alpha)(\delta+\tau-2\alpha^*+\pi^*)$$
$$+(D-4\rho+\varepsilon-\varepsilon^*)(\Delta+\mu^*-\mu-2\gamma^*)]\phi_{10}$$
$$-(\delta^*+4\pi-\beta^*-\alpha)(\delta^*-2\alpha-2\pi-2\tau^*)\phi_{01}$$
$$-(D-4\rho+\varepsilon-\varepsilon^*)(D-2\rho^*+2\rho+2\varepsilon)\phi_{21}\,. \tag{B11}$$

From Eq. (B11), one can see that the background value of F is of second order in Q. Using the freedom of tetrad to make the perturbations of ψ_1 and ψ_3 to be zero, one also sees from Eq. (B11) that the perturbation of F can be made to be of first order in Q. Therefore the perturbation of Eq. (B10) becomes

$$(\delta^*+3\pi-\beta^*-\alpha)(\delta^*+\pi-2\alpha)\frac{\overset{B}{\tilde{\chi}_1}}{3\psi_2}=(D-3\rho+\varepsilon-\varepsilon^*)(D-\rho+2\varepsilon)\frac{\overset{B}{\tilde{\chi}_{-1}}}{3\psi_2}\,. \tag{B12}$$

Now Eqs. (B3), (B5) and (9) being used in Eq. (B12), a straightforward calculation results in the connection equations, Eq. (18).

To find the rest of the connections, Eq. (19), we need some preparations. First, we write the tetrad perturbations in terms of the background tetrad in the following way:

$$\overset{B}{\tilde{l}}{}^{\mu}=b_1\tilde{n}^{\mu}\,,$$
$$\overset{B}{\tilde{n}}{}^{\mu}=a_2\,\tilde{l}^{\mu}+b_2\tilde{n}^{\mu}\,,$$
$$\overset{B}{\tilde{m}}{}^{\mu}=a_3\,\tilde{l}^{\mu}+b_3\tilde{n}^{\mu}+c_3\tilde{m}^{\mu}+d_3{}^*\tilde{m}^{*\mu}\,. \tag{B13}$$

Here b_1, a_2, b_2 and c_3 are real; a_3, b_3 and d_3 are complex. Equation (B13) is the result after exhausting all the tetrad freedom to minimize the number of small coefficients (b_1, a_2, etc.). The perturbations of the covariant components of the tetrad are found (through the orthonormality conditions of the tetrad) to be

$$\overset{B}{\tilde{l}}{}_{\mu}=-b_2\,\tilde{l}_{\mu}-b_1\tilde{n}_{\mu}+b_3{}^*\tilde{m}_{\mu}+b_3\tilde{m}_{\mu}{}^*\,,$$
$$\overset{B}{\tilde{n}}{}_{\mu}=-a_2\,\tilde{l}_{\mu}+a_3{}^*\tilde{m}_{\mu}+a_3\tilde{m}_{\mu}{}^*\,,$$
$$\overset{B}{\tilde{m}}{}_{\mu}=-c_3\tilde{m}_{\mu}-d_3{}^*\tilde{m}_{\mu}{}^*\,. \tag{B14}$$

The perturbations of the spin coefficients and the differential operators can now be given in terms of these newly introduced quantities. Particularly, the ones that will be used later on are,

$$D^B = b_1 \Delta \;, \tag{B15}$$

$$\delta^{*B} = a_3^* D + b_3^* \Delta + d_3 \delta + c_3 \delta^* \;, \tag{B16}$$

$$\kappa^B = -(D - 2\varepsilon - \rho^*) b_3 + (\delta + \tau - \pi^*) b_1 \;, \tag{B17}$$

$$\sigma^P = (D - 2\varepsilon + 2\varepsilon^* + \rho - \rho^*) d_3^* + (\pi^* + \tau) b_3 \;, \tag{B18}$$

$$\begin{aligned} \alpha^B = & -\frac{1}{4}(D - 2\varepsilon - \rho - 2\rho^*) a_3^* + \frac{1}{4}(\Delta + 4\gamma - 2\gamma^* - 2\mu + \mu^*) b_3^* \\ & -\frac{1}{4}(\delta^* - 2\pi) b_2 + \frac{1}{2}(\delta^* - 2\beta^*) c_3 - \frac{1}{2}(\delta + 2\beta) d_3 \;, \end{aligned} \tag{B19}$$

$$\begin{aligned} \beta^B = & \frac{1}{4}(\Delta + 2\gamma + \mu + 2\mu^*) b_3 - \frac{1}{4}(D - 4\varepsilon + 2\varepsilon^* + 2\rho - \rho^*) a_3 \\ & -\frac{1}{4}(\delta + 2\pi) b_2 - \frac{1}{2}(\delta - 2\beta - 2\pi) c_3 + \frac{1}{2}(\delta^* + 2\beta^* - 2\pi) d_3^* \;, \end{aligned} \tag{B20}$$

$$\begin{aligned} \rho^B = & -\frac{1}{2}(\delta^* - 2\alpha - \pi) b_3 + \frac{1}{2}(\delta - 2\alpha^* + \pi^* + 2\tau) b_3^* \\ & + (D + \rho - \rho^*) c_3 - \mu b_1 - \frac{1}{2}(\rho - \rho^*) b_2 \;. \end{aligned} \tag{B21}$$

Now we are prepared to proceed and start with the following perturbation equations (from now on we ignore the terms which approach zero as $r \to 2M$):

$$(\delta^* + 4\beta^* - 3\pi) \tilde{\psi}_0{}^B = (D - 2\varepsilon) \tilde{\psi}_1{}^B + 3\psi_2 \kappa^B - 2\phi_1^* (D - 2\varepsilon) \tilde{\phi}_0{}^B \;, \tag{B22}$$

$$D\psi_2{}^B = (\delta^* + 2\beta^*) \tilde{\psi}_1{}^B - D^B \psi_2 + 3\psi_2 \rho^B + 2\phi_1^* (\delta^* + 2\beta^*) \tilde{\phi}_0{}^B \;, \tag{B23}$$

$$\begin{aligned} (\delta^* + 3\pi) \psi_2{}^B = & (D + 2\varepsilon) \tilde{\psi}_3{}^B - \delta^{*B} \psi_2 \\ & - 3\psi_2 \pi^B - 2\phi_1^* (D + 2\varepsilon) \tilde{\phi}_2{}^B - 4\mu \phi_1 \tilde{\phi}_0{}^{*B} \;, \end{aligned} \tag{B24}$$

$$(D + 4\varepsilon) \tilde{\psi}_4{}^B = (\delta^* - 2\beta^* + 6\pi) \tilde{\psi}_3{}^B - 3\psi_2 \lambda^B + 2\phi_1^* (\delta^* - 2\beta^*) \tilde{\phi}_2{}^B \;, \tag{B25}$$

$$(\delta^* + 2\beta^* - 2\pi) \kappa^B - (D - 2\varepsilon) \rho^B = D^B \rho + \tau \kappa^{*B} \;, \tag{B26}$$

$$(D + 2\varepsilon) \lambda^B - (\delta^* - 2\beta^* + 3\pi) \pi^B = \delta^{*B} \pi + \pi(\alpha^B - \beta^{*B}) + \mu \sigma^{*B} \;, \tag{B27}$$

$$(D - 2\varepsilon) \sigma^{*B} - (\delta^* - 2\beta^* + \pi) \kappa^{*B} = \tilde{\psi}_0{}^{*B} \;. \tag{B28}$$

Equations (B2) and (B28) can also be written in terms of the newly introduced quantities as

$$\begin{aligned} (D - 2\varepsilon) D c_3 = & [\mu D + (\delta^* + 2\beta^* - 2\pi) \delta + \pi \delta^*] b_1 \\ & -\frac{1}{2}(D - 2\varepsilon)[(\delta + 2\beta + \pi) b_3^* + (\delta^* + 2\beta^* - \pi) b_3], \end{aligned} \tag{B26$'$}$$

$$(D-2\varepsilon)Dd_3=(\delta^*-2\beta^*-\pi)\delta^* b_1-(\delta^*-2\beta^*+\pi)(D-2\varepsilon)b_3^*+\tilde{\psi}_0{}^{*B} . \tag{B28$'$}$$

Operating $(\delta^*+2\beta^*)$ on Eq. (B22) and $(D-2\varepsilon)$ on Eq. (B23), subtracting one from the other and then using Eqs. (B15), (B17) and (B26) one obtains

$$\begin{aligned}(\delta^*+2\beta^*)(\delta^*+4\beta^*-3\pi)\tilde{\psi}_0{}^B-(D-2\varepsilon)D\psi_2{}^B\\ =-3\psi_2[\mu Db_1+\pi\delta b_1+\pi\delta^* b_1-\pi(D-2\varepsilon)(b_3+b_3^*)]\\ -4\phi_1^*(D-2\varepsilon)(\delta^*+2\beta^*)\tilde{\phi}_0{}^B .\end{aligned} \tag{B29}$$

Similarly, operating $(\delta^*-2\beta^*+6\pi)$ on Eq. (B24) and $(D+2\varepsilon)$ on Eq. (B25), subtracting one from the other and using Eqs. (B16), (B18), (B19), (B20) and (B27) one obtains

$$\begin{aligned}(D+2\varepsilon)(D+4\varepsilon)\tilde{\psi}_4{}^B-(\delta^*-2\beta^*+6\pi)(\delta^*+2\pi)\psi_2{}^B\\ =-3\psi_2[2\pi\delta^* c_3+\pi\delta^* d_3-\pi\delta d_3+\mu Dd_3+\mu(\delta^*-2\beta^*+\pi)b_3^*]\\ +4\phi_1^*(D+2\varepsilon)(\delta^*-2\beta^*)\tilde{\phi}_2{}^B+4\mu\phi_1(\delta^*-2\beta^*)\tilde{\phi}_0{}^{*B} .\end{aligned} \tag{B30}$$

Finally, operate $(\delta^*-2\beta^*+6\pi)(\delta^*+3\pi)$ on Eq. (B29) and $(D-2\varepsilon)D$ on Eq. (B30), subtract one from the other, make use of Eqs. (B26)$'$ and (B28)$'$ and manipulate the terms a little to obtain

$$\begin{aligned}(D-2\varepsilon)D(D+2\varepsilon)(D+4\varepsilon)\tilde{\psi}_4{}^B\\ =(\delta^*-2\beta^*+6\pi)(\delta^*+3\pi)(\delta^*+2\beta^*)(\delta^*+4\beta^*-3\pi)\tilde{\psi}_0{}^B\\ -3\psi_2[\mu D\tilde{\psi}_0{}^{*B}+\pi(\delta^*-\delta)\tilde{\psi}_0{}^{*B}]\\ +4\phi_1^*(D-2\varepsilon)\Big[(\delta^*-2\varepsilon^*)\delta^*(\delta^*+2\beta^*)\frac{\tilde{\chi}_1{}^B}{3\psi_2}\\ +\mu D(\delta^*-2\beta^*)\frac{\tilde{\chi}_1{}^{*B}}{3\psi_2}+D(D+2\varepsilon)(\delta^*-2\beta^*)\frac{\tilde{\chi}_{-1}^B}{3\psi_2}\Big].\end{aligned} \tag{B31}$$

Now we put Eqs. (B3) and (B5) into Eq. (B31) and make use of Eqs. (10) and (18) to obtain Eq. (19).

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