Magnetically charged black hole

Nada Khaled[1], Mohamed Elashri[2]

[1]Physics department - Ain shams University, Abbasia, Cairo, Egypt
[2]Physics and Astronomy department - University of Minnesota, Duluth, USA

[1]Corresponding Author E-mail: nadakhaled0102021@sci.asu.edu.eg

Abstract. In this study, the concept of magnetically charged black hole is discussed through calculating the angular momentum (L) of its interaction with an electric test charge. Results confirm that the angular momentum form will agree with the prediction of no-hair theorem and we will show that L will depend on the distance between the charge and the black hole.

1. Introduction.

In 1796, Laplace proposed in his "Exposition du Systeme du Monde" some speculations concerning astronomical observations like that a luminous star having a density as the Earth, larger than the sun with two hundred and fifty times in diameter, and because of its very high attraction, it doesn't allow its light rays to reach the Earth. Thus, he supposed that the largest luminous body could be found in our universe may be invisible. After that, in the nineteenth century, black holes became one of the greatest astronomical concerns and this allowed the scientists at this time to find strong evidence regarding black holes production through the collapse of the stars after their death. In 1905, with the appearance of Einstein's special relativity, it was found that the escape velocity is equal to the speed of light in the black holes. Few years later, Schwarzschild put the first solution describes the simplest black hole, as it describes the gravitational field out of a spherical mass, assuming that the angular momentum, electric charge, and universal cosmological constant are equal to zero. Then, comes Reissner-Nordstrm metric solved by Einstein-Maxwell equations could describe an electrically or magnetically charged black hole, and this is the metric we used in our derivation for the angular momentum of the magnetically charged black hole in order to be able to understand its concept. At first, we will derive the general form of magnetic black hole metric using Einstein field equation, and then we discuss the main features of this space-time, after that we will review the concept of classical Dirac's magnetic monopole with a discussion about quantization of angular momentum and its relation to Dirac quantization condition. The No-hair theorem states that black hole could be fully described and characterized by three parameters, namely mass, charge and angular momentum. There are many attempts to derive the equations for calculating these parameters. We will stick in the present work to the Komar integral method, in which the calculation depends on the properties of a perturbed metric. Therefore, we will calculate the angular momentum of the black hole system using Komar integral formula and then compare it to the case of magnetic monopole point particle. There, corrections for this quantity will be considered due to gravity and electromagnetism.

The paper falls in 4 sections, where in section 2, we start our derivation of Reissner-Nordstorm metric using Einstein field equation. In section 3, we use Komar integral formula to calculate the angular momentum of a black hole. Section 4 contains the discussion and conclusion of our work.

2. Reissner-Nordstorm

We first aim to derive the general form of magnetic black hole spacetime [1,2,3], assuming that we have general spherically symmetric metric carrying a charge

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{1}$$

Then we use it to solve Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2}$$

with the $T_{\mu\nu}$ is the energy momentum tensor written in terms of electromagnetic field tensor.

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{3}$$

Since we have R=0, then equation (2) will reduce to

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{4}$$

Maxwell's equations can be written in tensor form as

$$g^{\mu\nu}\nabla_{\mu}F_{\mu\sigma} = 0$$

$$\nabla[\mu F_{\nu\sigma}] = 0. \tag{5}$$

We will use equation (3) to get the components of the $F_{\mu\nu}$, and show that it is spherically symmetric. Thus, we will have only non-zero radial component, and the electromagnetic field strength can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & f(r,t) & 0 & 0 \\ 0 & -f(r,t) & 0 & 0 \\ 0 & 0 & 0 & g(r,t)r^2\sin(\theta) \\ 0 & 0 & -g(r,t)r^2\sin(\theta) & 0 \end{pmatrix} \tag{6}$$

Consequently, we evaluate Ricci tensor, the result for spherical symmetric solution of Einstein's field equation can be written as [1].

IOP Conf. Series: Journal of Physics: Conf. Series **1253** (2019) 012008 doi:10.1088/1742-6596/1253/1/012008

$$R_{tt} = [\partial_t^2\beta + (\partial\beta)^2 - \partial_t\alpha\partial_t\beta] + e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha] \tag{7}$$

$$R_{rr} = -\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] + e^{2(\beta-\alpha)}[\partial_t^2 + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\alpha\partial_t\beta] \tag{8}$$

$$R_{\theta\theta} = e^{2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \tag{9}$$

$$R_{\varphi\varphi} = R_{\theta\theta}sin^2(\theta) \tag{10}$$

For the components of the energy momentum tensor using equations (5) and (6) we get:

$$T_{tt} = \frac{f(r,t)^2}{2}e^{-2(\beta,t)} + \frac{g(r,t)^2}{2}e^{2(\alpha,t)} \tag{11}$$

$$T_{rr} = -\frac{f(r,t)^2}{2}e^{-2\alpha(r,t)} - \frac{g(r,t)^2}{2}e^{2\beta(r,t)} \tag{12}$$

$$T_{\theta\theta} = \frac{r^2g(r,t)^2}{2} + \frac{r^2f(r,t)^2}{2}e^{-2(\alpha(r,t)+\beta(r,t))} \tag{13}$$

$$T_{\varphi\varphi} = T_{\theta\theta}sin^2(\theta) \tag{14}$$

Then, by solving the R components with the T components we get that: $R_{tr} = 0$ and β = β(r), and since the electric field will be in the radial direction only, therefore,

$$\alpha(r,t) + \beta(r) = 0 \tag{15}$$

Now, let's solve Maxwell's equations for the form in equation (6), using the fact that F_{tr} is time independent because $\partial t F_{tr} = 0$. Then:

$$F_{tr} = f(r) \tag{16}$$

Further evaluation of (16) gives us the relation

$$f(r) = \frac{const.}{r^2} \tag{17}$$

and the case is similar for the magnetic charge g using the fact that it is also time independent.

$$g(r,t) = \frac{const.}{r^2} \tag{18}$$

Gauss flux relation can be calculated for magnetic field and for the electric field to get the constant in equations (17), (18). If we use g as total magnetic charge and Q as total electric charge, Then:

$$F_{\mu\nu} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 0 & Qr^{-2} & 0 & 0 \\ -Qr^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & P\sin\theta \\ 0 & 0 & -P\sin\theta & 0 \end{pmatrix} \tag{19}$$

From the $\theta\theta$ component. $R_{\theta\theta} = 8\pi T_{\theta\theta}$ component equation we get:

$$e^{2\alpha} = 1 + \frac{2GM}{r^2} + \frac{G}{r}(g^2 + Q^2) \tag{20}$$

On using the last equation and equation (15) we can get the metric for the magnetically charged black holes as follows:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{G}{r^2}(g^2)\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{G}{r^2}g^2\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{21}$$

If an electric charge (e) is setting at the origin and a magnetic charge g are separated by distance d, this assumed to result in electric and magnetic fields as follows:

$$E = \frac{1}{4\pi\varepsilon_o}\frac{e}{r^2}\hat{r} \tag{22}$$

$$B = \frac{1}{4\pi}\frac{g}{r_b^2}\hat{r}_b$$

Now, put magnetic charge g on z-axis on distance d from electric charge e, then its location is $d\hat{z}$, $r_b = r - d\hat{z}$.. Then:

$$\hat{r}_b = \sqrt{(x+d)^2 + y^2 + z^2} \tag{23}$$

$$\approx r - \frac{xd}{r}$$

The total angular momentum of the system is: [4,5,6]

$$L = \frac{1}{c^2}\int r \times (E \times B) d^3x \tag{24}$$

By inserting eq (23) in eq (25), we will get the same value as confirmed in [7]

$$L = eg \tag{25}$$

3. Conserved quantities and Komar integrals

We can use the virtue of both Noether's theorem and Einstein's field equation to write an equation that defines the conserved quantities in stationary space-time. That work was originally done by Komar [8]

and Smarr [9]. In 1950, Komar introduced a method to write covariant conserved quantities. We will discuss this method and present the method that will be used to calculate the angular momentum of the black holes.
let ζ be a killing vector of some general space-time, then we can use the killing vector equation to write:

$$\nabla_a \nabla_b \xi_c = R_{abcd} \xi^d \tag{26}$$

This will give us:

$$\nabla_a \nabla^{[a} \xi^{b]} = -R_a^b \xi^a \tag{27}$$

Komar conserved quantity for general killing vector field ζ is usually defined as integral on 2 surfaces (2-sphere for example) [ ].

$$K[\xi] = -\frac{1}{8\pi} \int_s K_\xi \tag{28}$$

where S is any 2-surface and that integration should not depend on any particular selection of the surface if it contains the specific volume. Now, using Einstein's equation and integrate on generic space-like hypersurface ε with two boundaries S_f and S_i.

$$\int_\varepsilon dK_\xi = 16\pi \int_\varepsilon \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) \xi^\mu n^\nu d\varepsilon_{\mu\nu} \tag{29}$$

where n_μ satisfies the equation $n^\mu n_\mu = -1$, and it is the normal vector of ε, and $d\varepsilon_{\mu\nu}$ is the volume element of the surface ε and if we use Stokes' theorem with the boundaries we defined earlier, we get:

$$K_{S_f}[\xi] - K_{S_i}[\xi] = \int_\varepsilon dK_\xi = 16\pi \int_\varepsilon (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \xi^\mu n^\nu d\varepsilon_{\mu\nu} \tag{30}$$

From this we can deduce that for stationary axisymmetric solution of Einstein's field equation, if we assume φ^a to be a killing vector related to the rotation symmetry. we can then define angular momentum as follows:

$$J = K_s[\phi] \tag{31}$$

If we adjust the normalization of the killing vector to require it to generate rotation at spatial infinity. Then, by using Stokes' theorem, we will write the total angular momentum as follows:

$$J = \frac{1}{8\pi G} \int_\varepsilon \nabla^\mu \xi^\nu d\varepsilon_{\mu\nu} \tag{32}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1253** (2019) 012008 doi:10.1088/1742-6596/1253/1/012008

The last equation would allow us to evaluate the angular momentum of the black holes, which we will do in the next section.

3.1. Angular momentum

The goal is to study the perturbation of Reissner-Nordstrm metric based on work of Chandrasekhar [10,11] due to point charge q at point r = b on polar axis θ = 0. This perturbation will induce the black hole-charge system to rotate. A more detailed analysis of this perturbation can be found in [12] and [13] that we depend on.

If we do so, then the only non-vanishing components are:

$$T_{00}^{per} = \frac{m}{2\pi b^2} f(b)^{\frac{3}{2}} \partial(r-b)\partial(cos\theta - 1) \tag{33}$$

$$J_{per}^{o} = \frac{9}{2\pi b^2} \partial(r-b)\partial(cos\theta - 1) \tag{34}$$

where $f(r) = 1 - \frac{2GM}{r} + \frac{g^2}{r^2}$

This will enter Einstein-Maxwell field equation as follows:

$$\tilde{G}_{\mu\nu} = 8\pi\left({T_{\mu\nu}}^{per} + {T_{\mu\nu}'}^{en}\right) \tag{35}$$

$$\nabla_\nu \tilde{F}^{\mu\nu} = 4\pi j_{per}^{\mu} \tag{36}$$

and the fields will be up to first order of perturbation.

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

$$\tilde{F}_{\mu\nu} = g_{\mu\nu} + j_{\mu\nu} \tag{37}$$

$$\tilde{T}_{\mu\nu}^{em} = \frac{1}{4\pi}\left[\tilde{g}^{\rho\sigma} F_{\rho\mu}\tilde{F}_{\sigma r} - \frac{1}{4}\breve{g}_{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}\right] \tag{38}$$

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} \tag{39}$$

Solving equations (35), (36) with the perturbed fields gives us:

$$V = \frac{q}{br}\frac{(r-M)(b-M) - r^2 cos\theta}{\alpha} + \frac{qM}{br} \tag{40}$$

Where V is the electrostatic potential.

$r = \sqrt{M^2 - Q^2}$

IOP Conf. Series: Journal of Physics: Conf. Series **1253** (2019) 012008 doi:10.1088/1742-6596/1253/1/012008

$$\alpha = \sqrt{(r-M)^2 + (b-M)^2 - 2(r-M)(b-M)cos\theta - r^2 sin^2\theta}$$

The electric field due to perturbation is:

$$\tilde{E}_r = \frac{q}{br^2}\left[M - \frac{M(b-M)+r^2cos\theta}{\alpha}\right] + \frac{r((r-M)(b-M)-r^2cos\theta)((r-M)-(b-M)cos\theta)}{\alpha^3} \tag{41}$$

$$g_{t\varphi} = 2gE_r\left(r - \frac{g^2}{r} + \frac{g^4}{2Mr^2}\right) \tag{42}$$

Using the perturbed $F_{\mu\nu}$ and t-$p\varphi$ component of the perturbed metric, we can calculate Komar integral as follows:

$$J = \frac{1}{\rho\sigma\pi G}\int_s \ \epsilon_{abcd}\nabla^c\xi^d \tag{43}$$

Or we can use L(r) as angular momentum within two spheres S(r) of constants t and r

$$L(r) = \frac{1}{16\pi\sigma}\int_{s(r)}\epsilon_{abcd}\nabla^c\xi^d \tag{44}$$

$$J = \lim_{r\to\infty} L(r)$$

L(r) can be expressed in terms of $g_{t\phi}$ and unperturbed metrics, so straight forward calculating eq (45), We get:

$$L(r) = \frac{1}{\sigma^6}R^4\frac{d}{dr}\left(\frac{N}{R^2}\right) \tag{45}$$

where R^2 is radial metric component and,

$$N = \frac{3}{4}\int_0^\pi g_{t\varphi}\ sin\theta\ d\theta \tag{46}$$

and because $g_{t\phi}$ and A_t is always mixed in Einstein's field equations; we define another quantity as:

$$\beta = -\frac{3}{2}\int_0^\pi A_t\ sin\theta\ cos\theta\ d\theta \tag{47}$$

We take $t-\phi$ component of

$$R_{ab} = F_{ac}F_b^c - \frac{1}{2}g_{ab}F_{ab}F^{ab} \tag{48}$$

and t component of

$$\nabla_a F^{ab} = 0 \tag{49}$$

Then, calculate all quantities to first order $g_{t\phi}$ and A_t by using the new defined quantities, we get:

$$\frac{d}{dr}\left[\frac{R^4}{r^2}\frac{d}{dr}\left(\frac{f^2r^2}{R^2}\frac{dL}{dr}\right) - 2\left(1+\frac{2g^2}{r^2}\right)\right] = \frac{-4\pi gR^2}{f^2}\int_0^{\pi} j_t sin\theta\, cos\theta\, d\theta \qquad (50)$$

where the right hand-side of the equation is equal to −2qgδ(r−b). After doing some algebraic calculations; this equation can be solved and by taking the limit we get the proposed angular momentum as follows:

$$J = eg\left[1 - \frac{\beta(r_H)}{\beta(b)}\left(\frac{r_H}{b}\frac{3Mb-2g^2}{3Mr_H-2g^2}\right) - 2\gamma(b)f(b)\left(\frac{GMb-Gg^2}{2r_H-GM}\right)\right] \qquad (51)$$

$$\beta(r) = r^2 - 3Gg^2 + \frac{2Gg^4}{Mr}$$

$$\gamma(r) = 3X(r)\int_r^{inf}\frac{r'}{f(r')\beta^2(r')} \qquad (52)$$

$$r_H = GM + \sqrt{G^2M^2 - Gg^2}$$

where we define r_H as the event horizon radius of black hole.

We have got the angular momentum of our system, it shows an expected dependencies. Discussing these dependencies and further discussing this result with the statement of no-hair theorem will be in the next section.

4. Discussion

The main equation that presents the paper result that we are going to discuss is equation (51).This equation gives us an expression for the rotation effect of magnetically charged black hole in presence of electric charge on a finite distance. It is obvious that this equation can be separated to two different contribution to angular momentum, the first is due to the electromagnetic field caused by magnetic/electric charge and the second is due to the different contribution of gravitational interaction that is zero in case of flat space-time.

If we cancel out gravity, this is equivalent to say that $G \rightarrow 0$ then, this confirms the claim that the second term corresponds to the gravity effect that will vanish in that case. This gives hint about the possible reason for the difference between angular momentum in flat space-time and in our black-hole system is that the black hole geometry has an effect on the expression that is not presented in the flat case. There is also a dependence on the value of distance b, simply we can see equation (51) as monotonic function of variable b as it is equal to eg at infinity and decreases until it becomes zero at r_H . The infinity limit corresponds to the case that is equivalent exactly to the flat case. From the gravity contribution we can suggest that it is the cause of that b dependence, this is supported by the fact that in flat space-time the angular momentum is simply eg independent on b distance. We can rule out this idea by discussing the basic assumption we have used when deriving equation $L = eg$ which is that both magnetic and electric charges are point particles.

If we assumed that we have a nontrivial magnetic charge distribution instead of a point particle monopole, then if we try to calculate the angular momentum of this system it would give us a distance dependence. To give a clear example to this, we are going to assume that we have a electric charge interacting with a non-conducting spherical shell that carries a magnetic charge. we will assume that they are separated with distance b from the shell center. Then, if we tried to calculate the electromagnetic angular momentum using the electric and magnetic fields of these objects, we will get this example which is a modified idea to what is done in [14] for slowly rotating charged black hole case.

$$L = eg\left(1 - \frac{r^2}{3b^2}\right) \quad for\; b \geq r \qquad (53)$$

$$L = eg\frac{2b}{3r} \quad for\, b\, < r$$

It is obvious then that the separation b dependence comes from the nontrivial geometry of the object not from the gravity effect. This tells us that the nontrivial geometry of the black hole is responsible. This can be explained as if charge was at infinity distance from the shell, then there is no electric flux. All angular momentum stored outside the horizon and it is eg. If we make the charge closer to the shell then there is an electric flux crossing, this means that there is less angular momentum. When it arrives to the horizon it will eventually go to zero. This angular momentum is conserved so the black hole will start to rotate. This analogy between this shell example and charged black holes confirms our result. This means that the horizon of the black hole has a role of the shell even if there is actually no material in this region.

When the electric charge reaches the horizon limit, we can see that all multipoles fade away except the monopole which confirms the conjecture of the no-hair theorem [15]. At this point the angular momentum stored in these fields goes to zero and the transfer of the angular momentum to the black hole which will have maximum angular momentum now is completed. If we want to describe this process, then we will have to consider the case of slowly rotating charged black hole as our new metric would be. These results don't contradict with the no-hair theorem and although this system cannot be observed in nature as it is impossible to have astrophysical charged black-hole but it is of great interest to study this angular momentum transfer and how magnetically charged black hole interacting with electric charge would turn into slowly rotating charged black-hole which angular momentum is similar with the limiting case of [16].

References

[1] Chandrasekhar S 2009. *Oxford: Clarendon Press.* The mathematical theory of black holes.
[2] Reissner H 1916 *Ann. der Phys* **50** 106.
[3] Nordstrom L 1918 *Proc. Amsterdam Acad.Wet* **20** 1238.
[4] Jackson J D, 1975 *John Wiley Sons* Classical Electrodynamics 2nd Edition 251
[5] Dirac P A M 1931 Proc. Roy. Soc. **133** 60.
[6] Dirac P A M, 1948 A *Phys. Rev.***74** 817.
[7] Lipkin H, Weisberger W and Peshkin M 1969 *Annals of Physics.* Magnetic charge quantization and angular momentum. **53** 203.
[8] Komar A 1963 *Physical Review.* Positive-Definite Energy Density and Global Consequences for General Relativity **129** 18731876.
[9] Smarr L 1973 *Phys. Rev.* Lett. **30** 7173.
[10] Chandrasekhar S and Xanthopoulos B 1979. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*. On the Metric Perturbations of the Reissner-Nordstrm Black Hole **367** 1728
[11] Chandrasekhar S. 1979. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences.* On the Equations Governing the Perturbations of the Reissner-Nordstrm Black Hole **365** 1723.
[12] Bini D, Geralico A and Ruffini R 2007. *Physical Review D.* Charged massive particle at rest in the field of a Reissner-Nordstrm black hole. **75** 4.
[13] Bini D, Geralico A and Ruffini R 2008. *The Eleventh Marcel Grossmann Meeting.* Static Perturbations Of A Reissner-Nordstrm Black Hole By A Charged Massive Particle.
[14] Will C M 1974. *The Astrophysical Journal*, Perturbation of a Slowly Rotating Black Hole by a Stationary Axisymmetric Ring of Matter. I. Equilibrium Configurations **191** 521.
[15] Misner, Charles W, Thorne, Kip S, Wheeler and John A 1973. *San Francisco: W. H.* Freeman. Gravitation. 875876
[16] Lee C H 1981. *Progress of Theoretical Physics*, Perturbations around a Slowly Rotating and Slightly Charged Black Hole. **66** 180.