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An Electroweak Monopole, Dirac Quantization and the Weak Mixing Angle

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Abstract

We consider an extension of the Standard Model that was proposed recently by one of the current authors (PQH), which admits magnetic monopoles with a mass of order of a few TeV. We impose, in addition to topological quantization in the SU(2) sector of the model, the Dirac Quantization Condition (DQC) required for consistency of the quantum theory of a charged electron in the presence of the monopole. This leads to the prediction $\sin^2\theta_W = 1/4$, where θ_W is the weak mixing angle at the energy scale set by the monopole mass. A leading-order renormalization-group analysis yields the value of $\sin^2\theta_W \simeq 0.231$ at the Z-boson mass, as measured by experiment, under suitable conditions on the spectrum of the extra particles in the model.

The electroweak mixing angle θ_W is a free parameter within the Standard Model (SM) of particle physics. However, it becomes possible to predict its value within extensions of the SM, e.g., by embedding the SM in a Grand Unified Theory (GUT), where the magnitude of θ_W is controlled by the details of unification [1–3], or in string theory [4]. In SU(5) GUT theories, for instance, there is a characteristic tree-level prediction that

$$\sin^2\theta_W = 3/8 \tag{1}$$

at the GUT scale. This is renormalized by quantum loop effects in the SM that yield the prediction $\sin^2\theta_W \simeq 0.20$ at the Z-boson mass [2], which is close to, but different from, the experimental value $\sin^2\theta_W \simeq 0.231$ in the $\overline{MS}$ prescription [5]. The experimental value can be recovered by including the quantum corrections due to new particle degrees of freedom in the renomalization calculation. For example, including the supersymmetric partners of SM particles in the SU(5) GUT calculation reproduces very accurately the experimental value [6].

In this Letter we make a different prediction for $\sin^2\theta_W$ in an extension of the SM that is not a high-scale GUT, but rather a theory, proposed by one of the current authors (PQH) in [7], that includes a topologically non-trivial magnetic monopole with a mass of a few TeV. This magnetic monopole is associated with a real scalar triplet of the SU(2) group, in a spirit similar to the Georgi-Glashow model [8], and obeys a topological quantization condition that stems from the known non-trivial homotopy properties of the SU(2) group.

We show that this condition is not sufficient by itself to guarantee satisfaction of the Dirac Quantization Condition (DQC), which is required for consistency of the quantum theory of a charged particle such as the electron in the monopole's magnetic field. In the model in [7] the DQC must be imposed as an extra condition [9], which leads to the prediction

$$\sin^2\theta_W = 1/4 \tag{2}$$

at the monopole mass scale in the model. This value is renormalized by extra particles with masses between m_Z and the monopole mass that appear in the model, and the experimental value $\sin^2\theta_W \simeq 0.231$ [5] is recovered under suitable conditions on the spectrum of these particles.

We now review briefly the main features of the model proposed in [7]. It involves *non-sterile* right-handed neutrinos with masses of the order of the electroweak scale, which participate in a seesaw mechanism for light neutrinos that is testable in principle at colliders, e.g., by searching for like-sign dileptons with displaced vertices. For brevity, in what follows we term this model the EW-ν_R model. The central reason why the EW-ν_R model admits monopoles with masses at the electroweak scale, Λ_{EW}, is that its right-handed neutrinos acquire [10] electroweak-scale Majorana masses $M_R \propto \Lambda_{EW} \sim 246$ GeV through their coupling to a *complex* Higgs triplet $\tilde{\chi}$.

Because the ν_Rs are not sterile, consistency with the measured width of the Z-boson requires $M_R \geq 46$ GeV, which implies $\langle\tilde{\chi}\rangle = v_M \propto \Lambda_{EW}$. Such non-sterile neutrinos would seriously affect the experimentally-verified relationship between the W- and Z-boson masses $M_W =$

$M_Z \cos\theta_W$ in the SM, in the absence of an additional real triplet of (Higgs-like) scalar fields ξ with $\langle\xi\rangle = \langle\tilde{\chi}\rangle = v_M$ [11], which realize a custodial symmetry in the EW-ν_R model [10]. The ξ triplet is hypercharge-neutral and gives rise to a *finite-energy* electroweak monopole solution of the classical Euler-Lagrange equations of the model [7], following the pattern of the SO(3) monopole in the Georgi-Glashow model [8], discovered by 't Hooft [12] and Polyakov [13] (the 'tHP monopole). In that model the electromagnetic $\mathrm{U_{em}}(1)$ gauge group is embedded into the SO(3) gauge group, whose algebra is isomorphic to that of the SU(2) appearing in the model of [7, 10]. However, in the model of [7] $\mathrm{U_{em}}(1)$ is a combination of the SU(2) and $\mathrm{U_Y}(1)$ of the SM, parametrized by $\sin^2\theta_W$ in the usual way.

We now review the topological arguments [7] for the existence of the monopole, clarifying the independence of the topological quantization condition from the DQC that we explore subsequently. The EW-ν_R model contains [7, 10]. in addition to the real triplet ξ and the complex triplet $\tilde{\chi}$, four complex Higgs doublets, ϕ_i^{SM} (which couple to SM fermions only), and ϕ_i^M(which couple to mirror fermions (MF) [14], each with $i = 1, 2$ and some Higgs singlets ϕ_S that are not relevant to the magnetic monopole solution. The vacuum alignment that guarantees the custodial electroweak symmetry has $\langle\tilde{\chi}\rangle = \langle\xi\rangle = v_M$ [10]. The vacuum manifold of the Higgs sector is

$$S_{vac} = S^2 \times S^5 \times \sum_{i=1,2} S^3_{SM_i} \times \sum_{i=1,2} S^3_{M_i}\,. \qquad (3)$$

where an n-sphere S^n is described by the equation $x_1^2 + x_2^2 + .. + x_{n+1}^2 =$ constant. Here, the x_i denote the various scalar field values, and the constant radii of the various spheres correspond to the vacuum expectation values of the various Higgs field components. The second homotopy group of the vacuum manifold of the EW-ν_R model (3) is therefore [7]

$$\begin{aligned} \pi_2(S_{vac}) &= \pi_2(S^2) \oplus \pi_2(S^5) \oplus_{i=1,2} \pi_2(S^3_{SM_i,M_i}) \quad (4) \\ &= \pi_2(S^2) = Z\,, \end{aligned}$$

which is the standard topological argument for the existence of an 'tHP monopole [12, 13]. We see that the EW-ν_R model has a topologically-stable monopole solution thanks to the real SU(2) triplet ξ, corresponding to the sphere S^2, for which $\pi_2(S^2) = Z$. Thus the EW-ν_R model makes an interesting connection between the light neutrino masses and the existence of magnetic monopole solutions.

It was noted in [7] that, since S^2 is associated with the vacuum manifold of the real triplet ξ, topological quantization would involve the SU(2) coupling g, rather than the electromagnetic coupling e, leading to the following quantization condition for the magnetic charge $\tilde{g}$ of the monopole:

$$\frac{g\tilde{g}}{\hbar\, c} = \frac{n}{2}\,, \quad n \in \mathbb{Z}\,. \qquad (5)$$

From now on we work in units with $\hbar = c = 1$. The fact that the quantization condition (5) is in terms of the monopole charge $\tilde{g}$ and the weak charge g instead of the electric charge e appearing in the standard DQC is a characteristic feature of the model of [7]. It distinguishes the monopole in the model of [7] from the 'tHP magnetic monopole or the Cho-Maison monopole [15] and its finite energy extensions [16], to which the standard DQC applies.

Including the full SM gauge group structure, $\mathrm{SU(2)}\times\mathrm{U_Y}(1)$, which is broken down to the electromagnetic $\mathrm{U_{em}}(1)$ by the complex Higgs doublets and the triplet $\tilde{\chi}$ of the EW-ν_R model [7], one sees that the W^3_μ gauge field of the SU(2) subgroup is a mixture of the Z-boson and photon fields, parametrized as usual by the weak mixing angle θ_W: $W^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$, with $\sin\theta_W = g'/\sqrt{g^2+g'^2}$ where g' is the $\mathrm{U_Y}(1)$ coupling. The corresponding field strengths are

$$W^3_{ij} = \cos\theta_W Z_{ij} + \sin\theta_W F_{ij}\,, \qquad (6)$$

where F_{ij} is the usual electromagnetic field-strength tensor and Z_{ij} is the Z field-strength tensor. This mixing between the photon and the Z-boson is the reason why the terminology "γ-Z magnetic monopole" was used in [7] to describe the magnetic monopole solution. As discussed in [7], the magnetic monopole has a mass

$$M_M = \frac{4\pi v_M}{g} f(\lambda/g^2)\,, \qquad (7)$$

where the function $f(\lambda/g^2)$ varies between 1 for $\lambda = 0$ (the Prasad-Sommerfield limit [17]) and 1.78 for $\lambda = \infty$.

The phenomenological analysis of Ref. [10] shows that the value of v_M is bounded from below by the Z width (assuming only three light neutrinos): $v_M > M_Z/2 \sim 45.5$ GeV, and from above by the sum of the squared scalar fields VEVs in the model: $(\sum_{i=1,2} v_i^2 + v_i^{M,2}) + 8v_M^2 = (246\ \mathrm{GeV})^2$. The monopole mass range given by (7) is then obtained by saturating the bounds on v_M:

$$M_M \;\sim\; 890\ \mathrm{GeV} - 3.0\ \mathrm{TeV}\,, \qquad (8)$$

The corresponding magnetic field intensity, defined by $\frac{1}{2}\epsilon^{ijk}\, W^3_{jk}$ with ϵ^{ijk} $(i, j, k = 1, 2, 3)$ the totally antisymmetric symbol in three Euclidean (spatial) dimensions, is [7]

$$\begin{aligned} B_i^{\gamma Z} &= \frac{1}{gr^2}\hat{r}_i(\cos\theta_W e^{-M_Z r} + \sin\theta_W) \\ &= \frac{\sin\theta_W}{er^2}\hat{r}_i(\cos\theta_W e^{-M_Z r} + \sin\theta_W)\,, \qquad (9) \end{aligned}$$

where

$$e = g\sin\theta_W \qquad (10)$$

denotes the usual electromagnetic coupling, as in the SM.

We note the exponential damping factor $\propto \exp(-M_Z r)$ in the expression (9) for the magnetic field strength, due to the finite Z-boson mass, $M_Z \neq 0$. The short- and long-range parts of $B_i^{\gamma Z}$ become comparable in strength at a distance $r = \frac{1}{M_Z}\ln(\cot\theta_W) \sim 0.6/M_Z$ from the centre of the monopole, which is well inside its core. At large distances compared to the monopole core radius, $r \gg R_c \sim (gv_M)^{-1}$, the magnetic field differs in strength from that of a point-like Dirac monopole by a factor $\sin^2\theta_W$.

At these large distances, the $\gamma - Z$ magnetic field is

$$B_i^{\gamma - Z} \approx \frac{\sin^2\theta_W}{er^2}\hat{r}_i\,. \tag{11}$$

The true magnetic field, B_i, $i = 1, 2, 3$, is defined in terms of the electromagnetic tensor F_{ij}, which is seen from (6) to be related to $B_i^{\gamma - Z}$ by a factor of $1/\sin\theta_W$, so that at large distances compared to the core monopole $R_c \sim (gv_M)^{-1}$:

$$B_i \approx \frac{\sin\theta_W}{er^2}\hat{r}_i \quad i = 1, 2, 3\,, \quad r \gg R_c \tag{12}$$

Comparing this magnetic field [7] with the conventional definition of the magnetic charge of the monopole [18], we see that

$$g_M = \frac{\sin\theta_W}{e}\,. \tag{13}$$

In general, *the DQC is violated by the weak mixing-angle factor in* g_M, and thus the electron wave function would *not* be single-valued along a loop that surrounds the monopole at large distances away [18].

This is the central point of this article: in the model of [7], unlike the 'tHP monopole, *the topological quantisation rule* (5) stemming from the homotopy properties of the SU(2) group is *not sufficient for the quantum consistency* of the electron wave function in the presence of the magnetic field induced by the $\gamma - Z$ monopole.

In a similar spirit to the Kalb-Ramond monopole of [9], one must impose the DQC as an *additional* constraint:

$$e\,g_M = \frac{m}{2}, \quad m \in \mathbb{Z}\,. \tag{14}$$

We then obtain from (14) and (13) a consistency condiion for the weak mixing angle, and the prediction

$$\sin\theta_W = \frac{m}{2} \;\Rightarrow\; \sin^2\theta_W = \frac{m^2}{4}\,,\, m \in \mathbb{Z}\,, \tag{15}$$

where $\sin^2\theta_W$ is the quantity that it is usually quoted in experimental measurements [19]. Since $\sin\theta_W \leq 1$, the condition (15) allows only two topological sectors, namely $|m| = 1$ and $|m| = 2$. The case $m = 2$ would imply $\sin\theta_W = 1$, which corresponds to the limit $g/g' \to 0$ and a massless W boson. In the allowed case $m = 1$ we have the prediction

$$\sin^2\theta_W = \frac{1}{4}\,, \tag{16}$$

which is close to the experimental value $\sin^2\theta_W \simeq 0.231$.

The DQC (14) is a discrete consistency condition that should be understood as applying to the electric charge in the large-distance (IR) limit and the monopole charge measured at the monopole mass (7). There is no renormalization of the monopole charge below this scale, as there are no magnetically-charged objects with masses below (7). On the other hand, as $\sin^2\theta_W$ is related to the SU(2) and $\mathrm{U(1)_Y}$ couplings, it is in general subject to scale-dependent renormalization in the non-magnetic sector where experiments are performed. This is a well-understood effect that has been studied in detail in many GUT models such as SU(5) [2, 6].

We have made leading-logarithmic one-loop calculations of the renormalization of $\sin^2\theta_W$ from the monopole mass scale M_M down to the Z-boson mass M_Z for different values of M_M, the numbers of light families F (including both SM and mirror fermions), light Higgs doublets n_H, real triplets n_3 and complex triplets $\bar{n}_3$ with masses below M_M that enter the evolution. We use the notation $x_W \equiv \sin^2\theta_W(M_Z^2)$ and assume that, at M_M, $\sin^2\theta_W(M_M^2) = g'^2/[g^2 + g'^2] = 1/4$ giving $\alpha' = (1/3)\alpha_2$, and the following one-loop renormalization formula

$$\begin{aligned} x_W &\approx \frac{\alpha'}{\alpha' + \alpha_2}[1 + \frac{4\pi\alpha_2}{\alpha' + \alpha_2}(-\alpha' b' + \alpha_2 b_2)\ln(M_Z^2/M_M^2)] \\ &\approx (1/4)[1 + 4\pi\alpha_2(-\frac{1}{4}b' + \frac{3}{4}b_2)\ln(M_Z^2/M_M^2)]\,, \end{aligned} \tag{17}$$

where

$$b_2 = (1/16\pi^2)[\frac{22}{3} - \frac{4}{3}F - \frac{1}{6}n_H - \frac{2}{3}n_3] \tag{18}$$

and

$$b' = -(1/16\pi^2)[\frac{20}{9}F + \frac{1}{6}n_H + \bar{n}_3]\,. \tag{19}$$

The scalar contributions to Eqs. (18,19) come from $-(1/3)T_S$ with $T_S = 1/2, 2$ (doublets and triplets) for b_2 and $(1/3)\sum(Y_S/2)^2$ with $Y_S/2 = 1/2, 1$ (doublet and complex triplet) for b'. Tabulated below are some examples of spectra with M_M in the range (8) that yield $0.230 < x_W < 0.233$, to be compared with the experimental central value $x_W = 0.23121$ in the $\overline{MS}$ prescription [5] (one should allow for higher-order uncertainties in the renormalization calculation).

M_M (TeV)	F	n_H	n_3	$\bar{n}_3$	x_W
2.3	3	1	0	0	0.232
3	3	3	0	0	0.2314
3	3	1	1	1	0.2318
3	4	1	0	0	0.2328

We note that cases with $F = 5, 6$ are disfavoured experimentally, as they yield $\sin^2\theta_W(M_Z^2) > 0.233$. Also disfavoured are scenarios such as $n_H = 2$, $n_3 = 1$, $\bar{n}_3 = 1$ light Higgs fields below M_M.

The EW-ν_R model we have studied here has many interesting properties. In addition to containing a seesaw scenario for neutrino masses that predicts several possibilities for new particles that could be detected at the LHC, it also predicts the existence of an electroweak magnetic monopole with mass $\lesssim 3$ TeV, light enough to be detected in principle by the MoEDAL experiment [20, 21].

Remarkably, as we have shown in this Letter, the Dirac Quantization Condition needed for the quantum consistency of the EW-ν_R model imposes a specific value of the weak mixing parameter $\sin^2\theta_W = 1/4$ at the monopole mass scale. Plausible choices of the monopole mass and the numbers of fermions and Higgs bosons with masses below that of the monopole yield predictions for the renormalized weak mixing parameter $\sin^2\theta_W(M_Z^2)$ that are consistent with experimental measurements, within the theoretical uncertainties. The success of this prediction has interesting implications on the Majorana masses of right-handed neutrinos, since both quantities depend on the Higgs triplet VEV v_M, as well as the spectra of light new particles. With Majorana masses of the EW-ν_R model being $M_R = g_{\nu_R} v_M$, Eq. (7) gives an interesting relation between the monopole and right-handed neutrino Majorana masses $M_M = \frac{4\pi}{g\, g_{\nu_R}} f(\lambda/g^2) M_R$. Electroweak-scale non-sterile ν_R could be discovered via like-sign dilepton events with displaced vertices, and give a range for the monopole mass: $19\, M_R \lesssim M_M \lesssim 34\, M_R$ for $f(\lambda/g^2) = 1, 1.78$, with $g \sim 0.65$ and assuming $g_{\nu_R} \sim 1$. The search for charged mirror quarks and leptons, which are long-lived particles in this model, have been discussed in [10]. A detailed analysis of this and other aspects of the model are beyond the scope of this Letter and will be given elsewhere.

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