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1.5 UNION-FIND

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

Dynamic connectivity problem

Given a set of N objects, support two operation:

- Connect two objects.
- Is there a path connecting the two objects?

connect 4 and 3

connect 3 and 8

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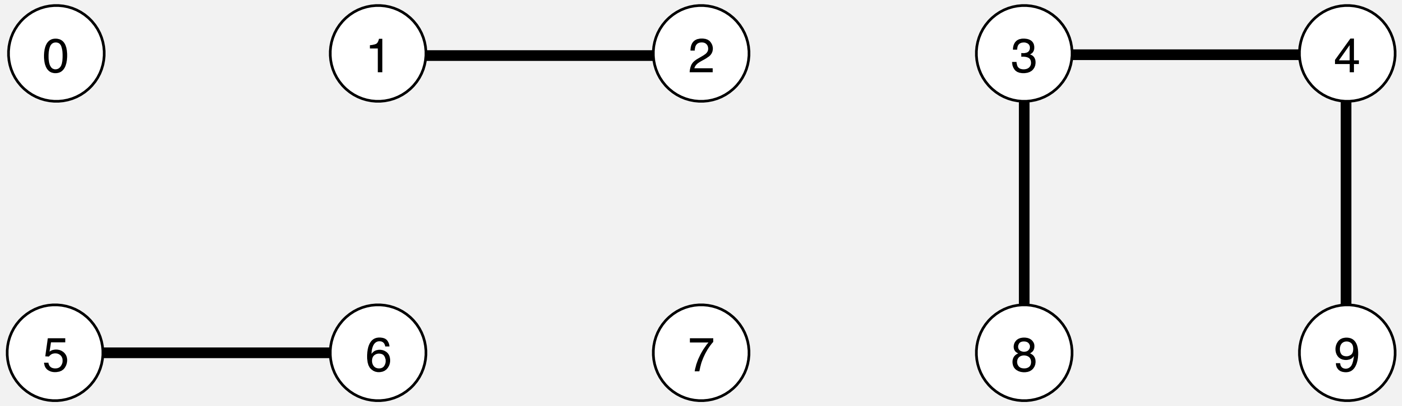

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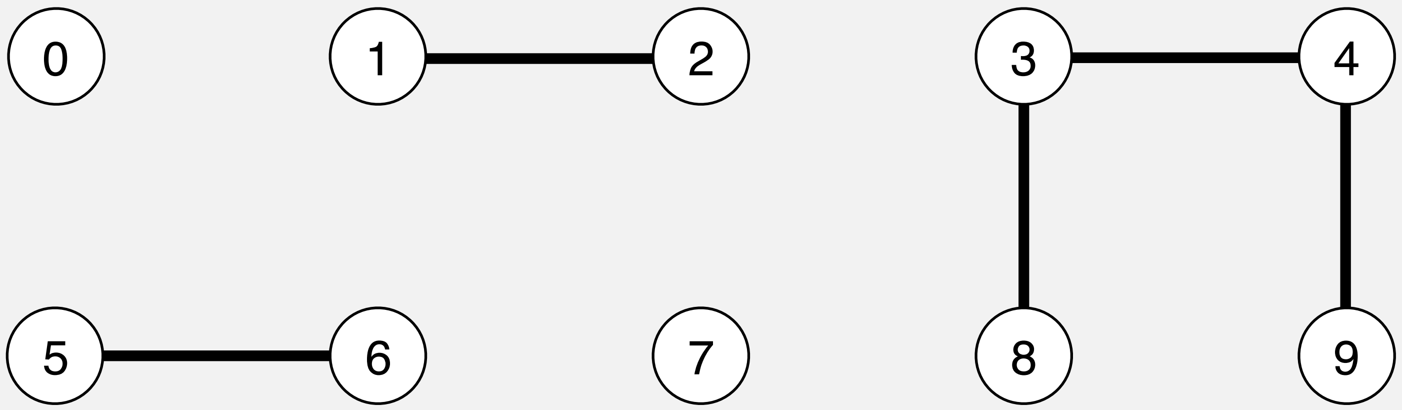

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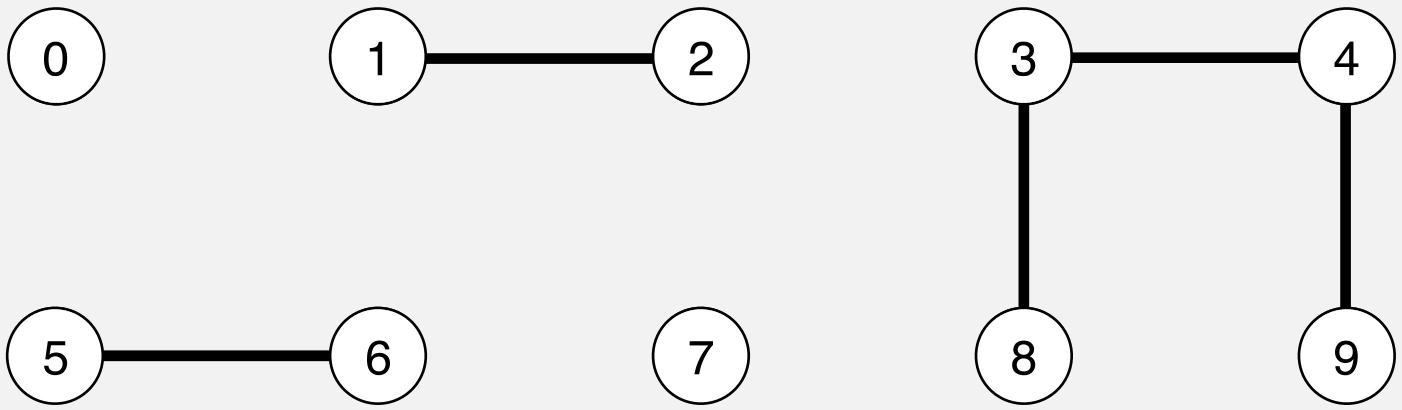

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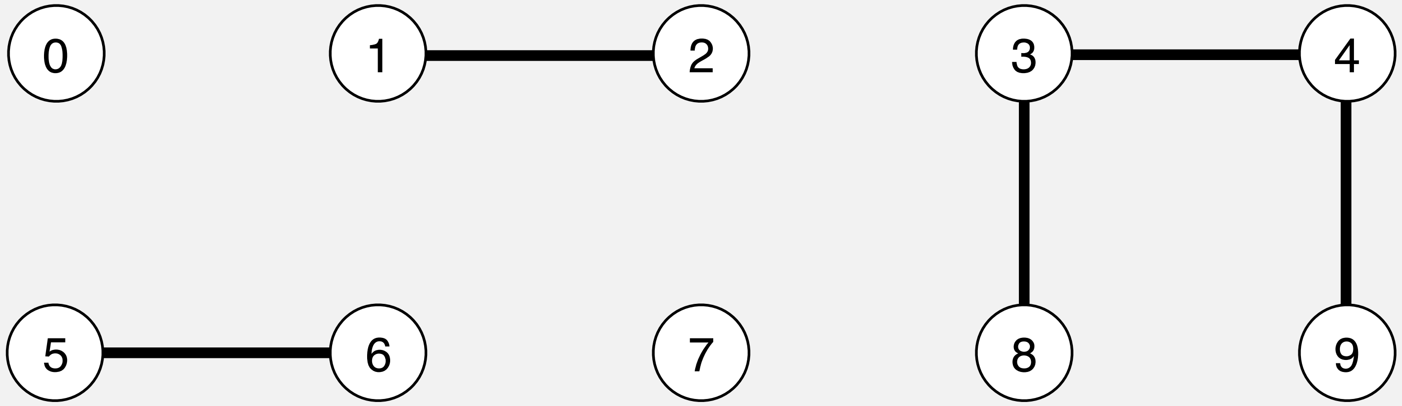

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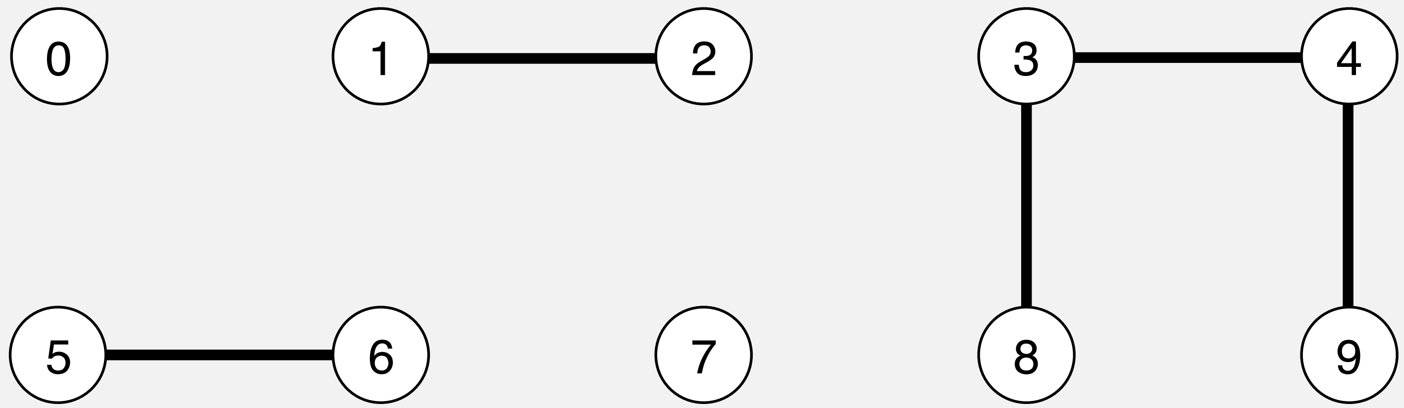

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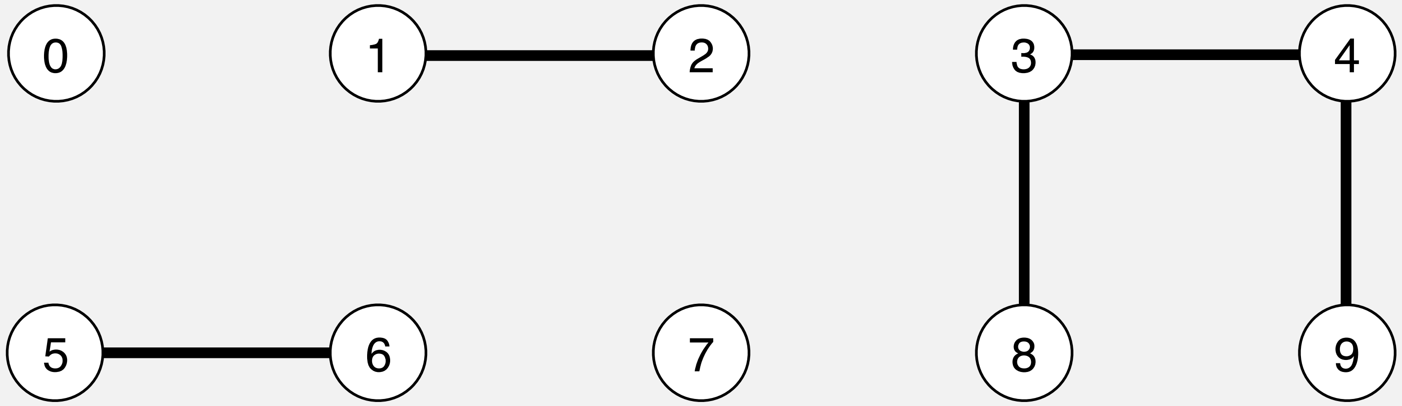

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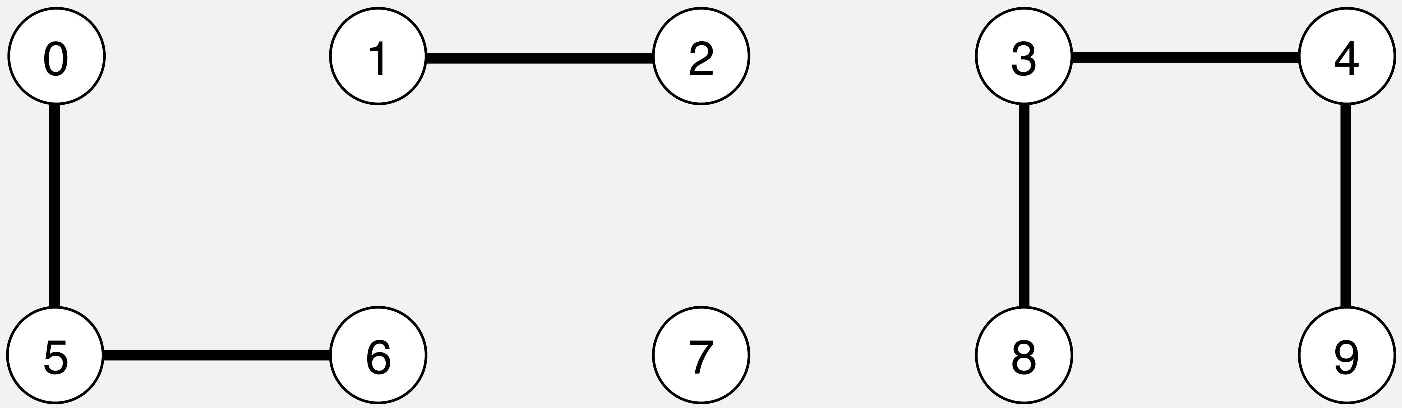

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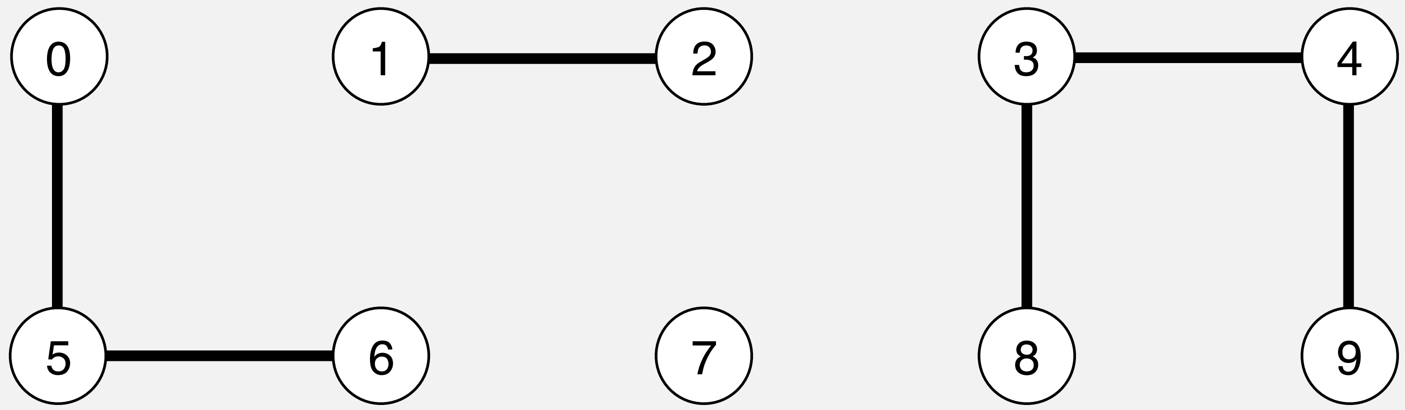

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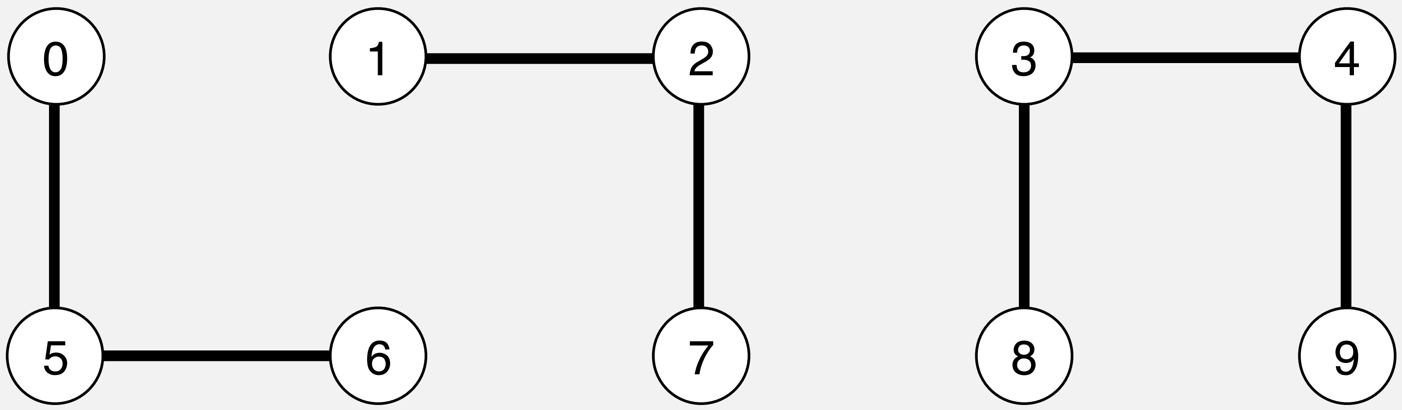

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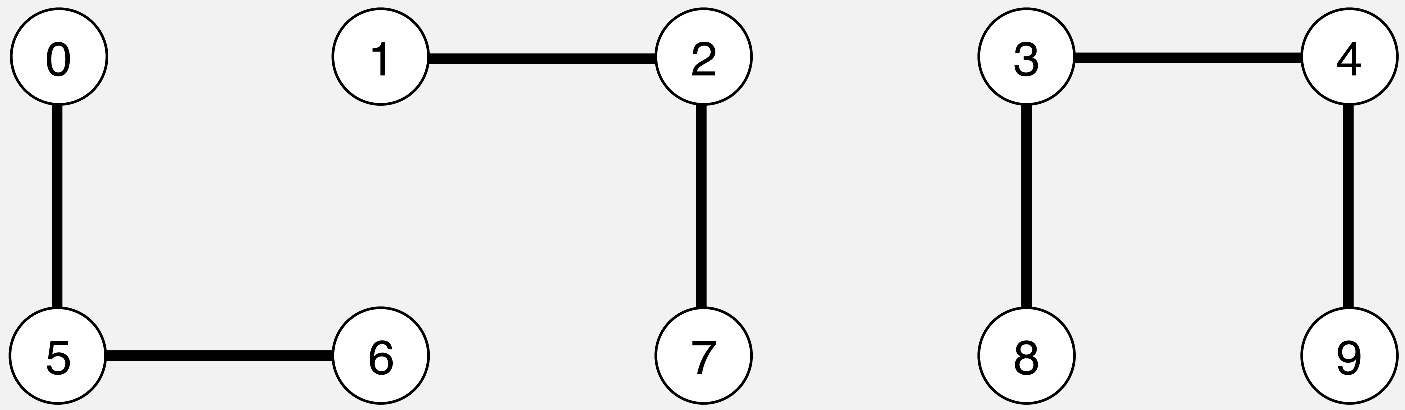

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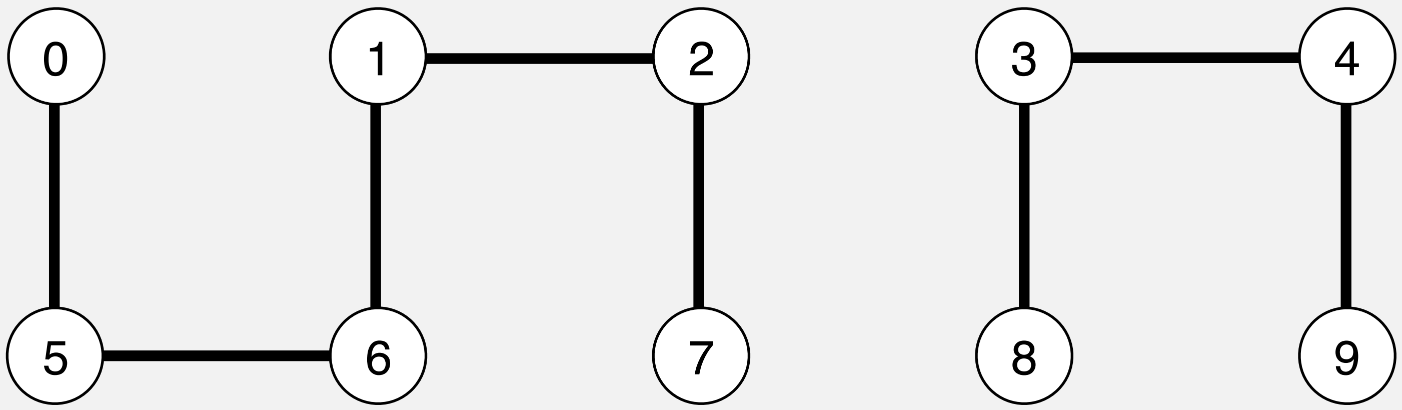

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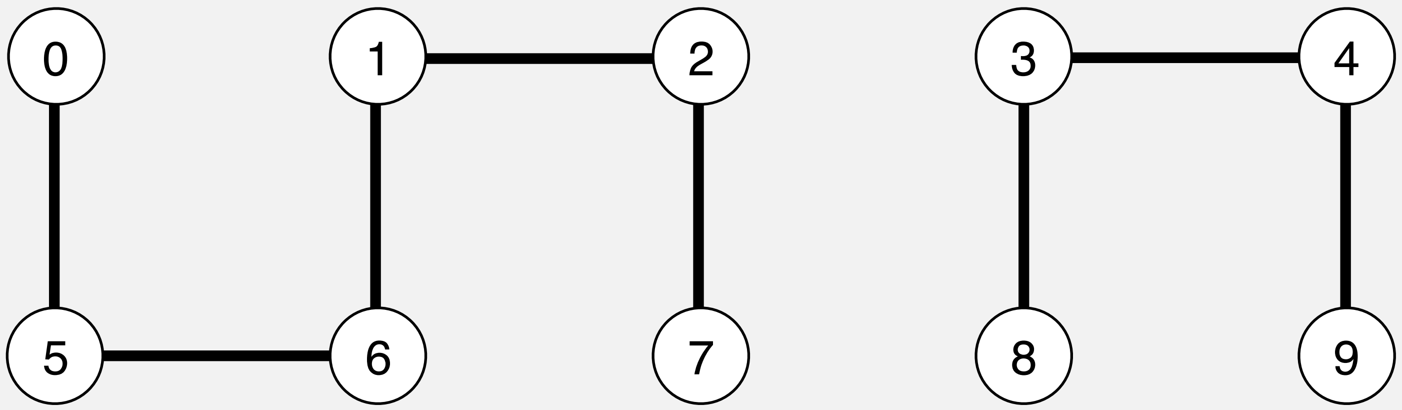

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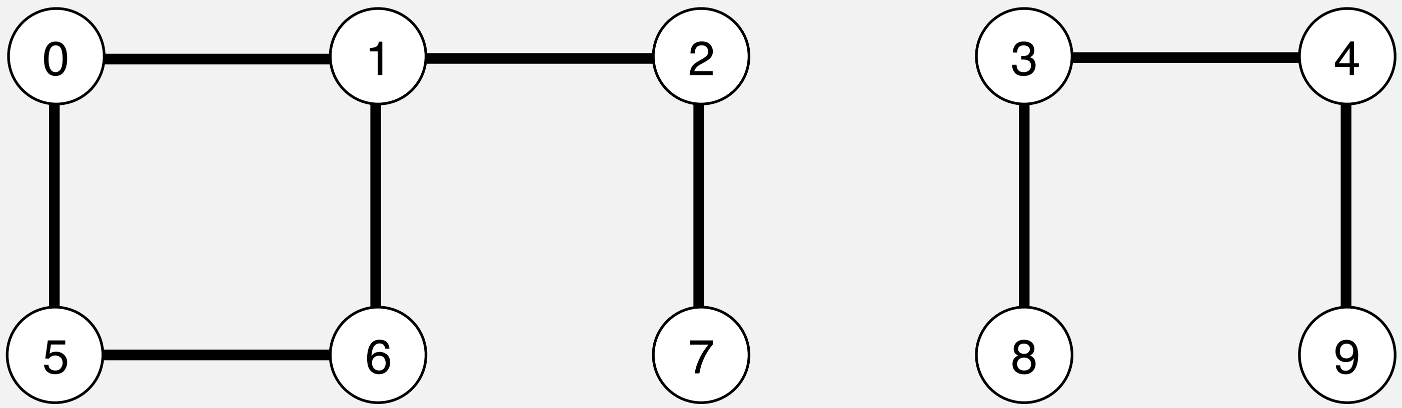

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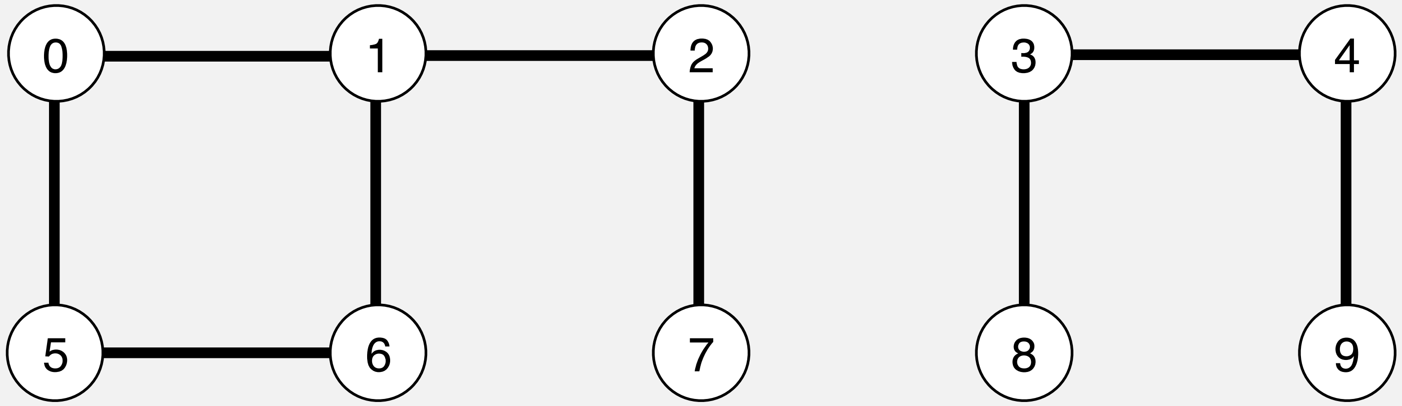

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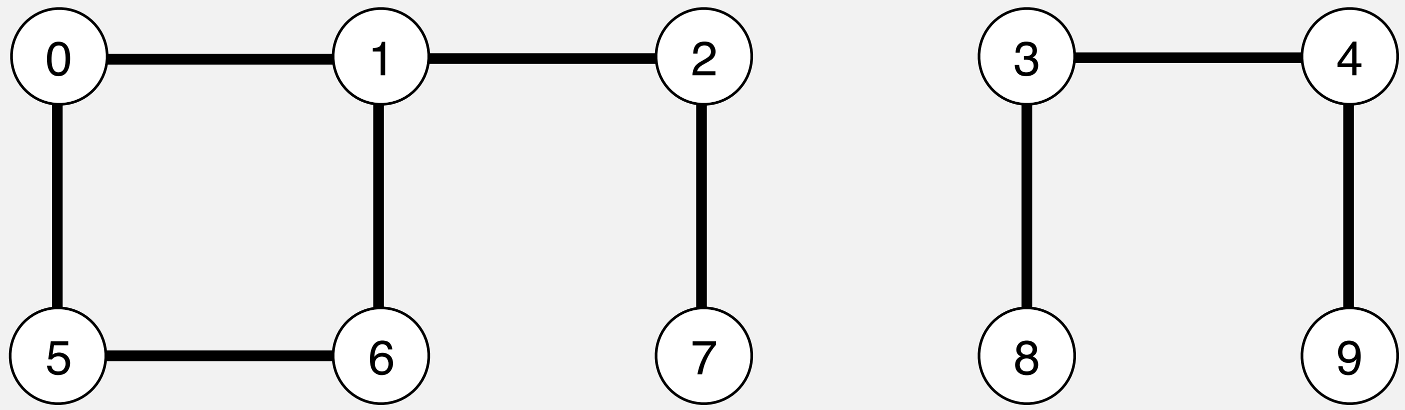
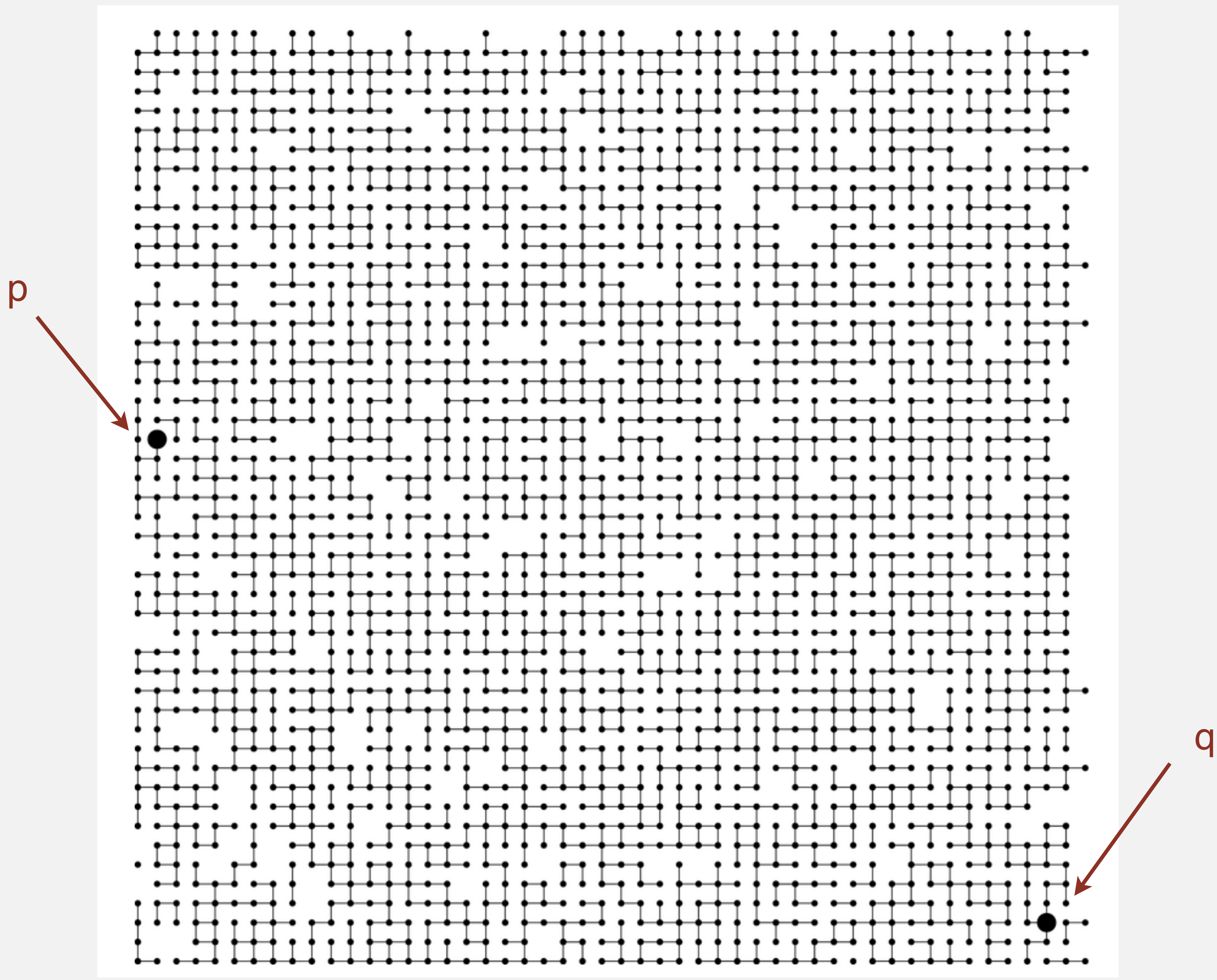
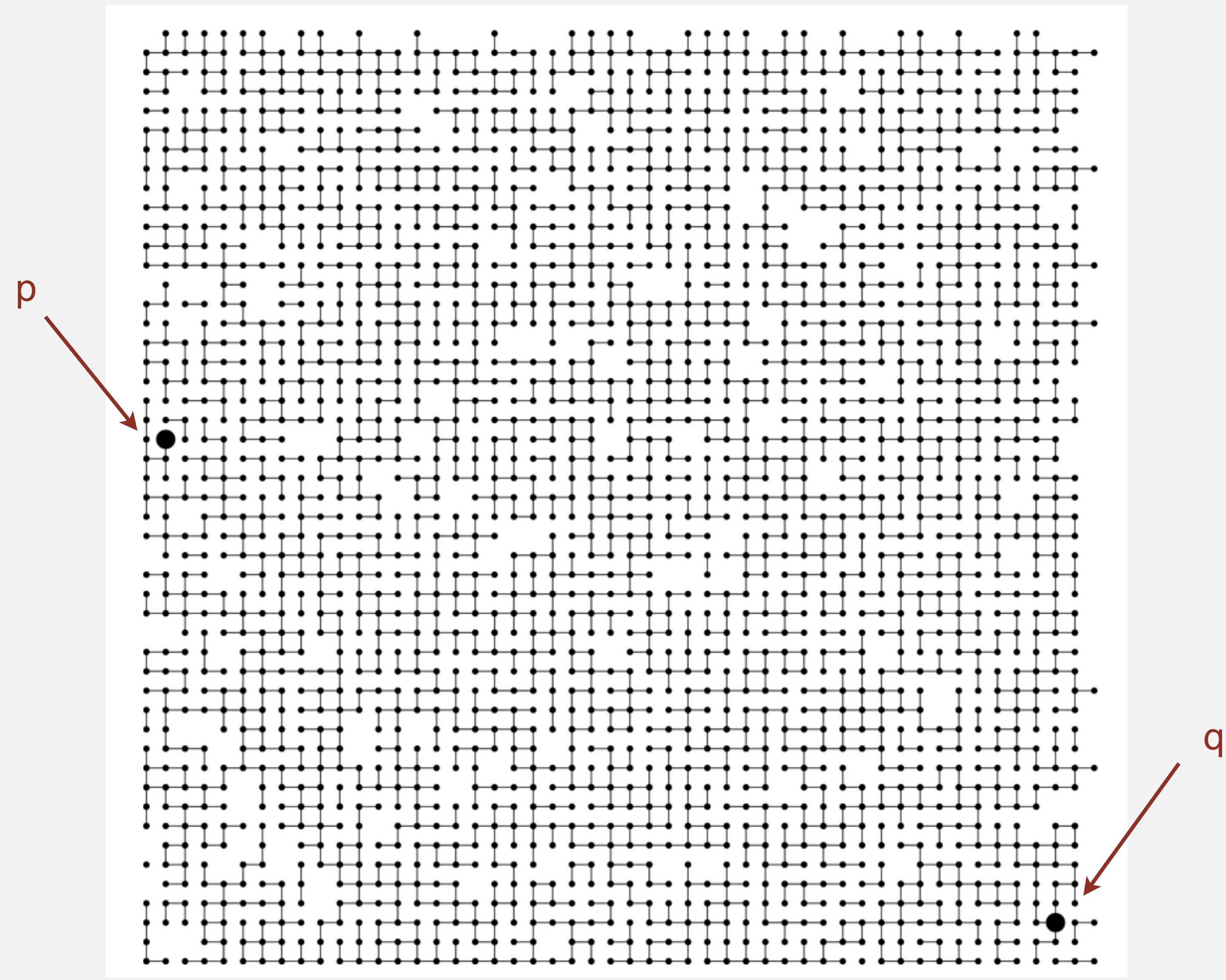


A larger connectivity example



A larger connectivity example

Q. Is there a path connecting p and q ?



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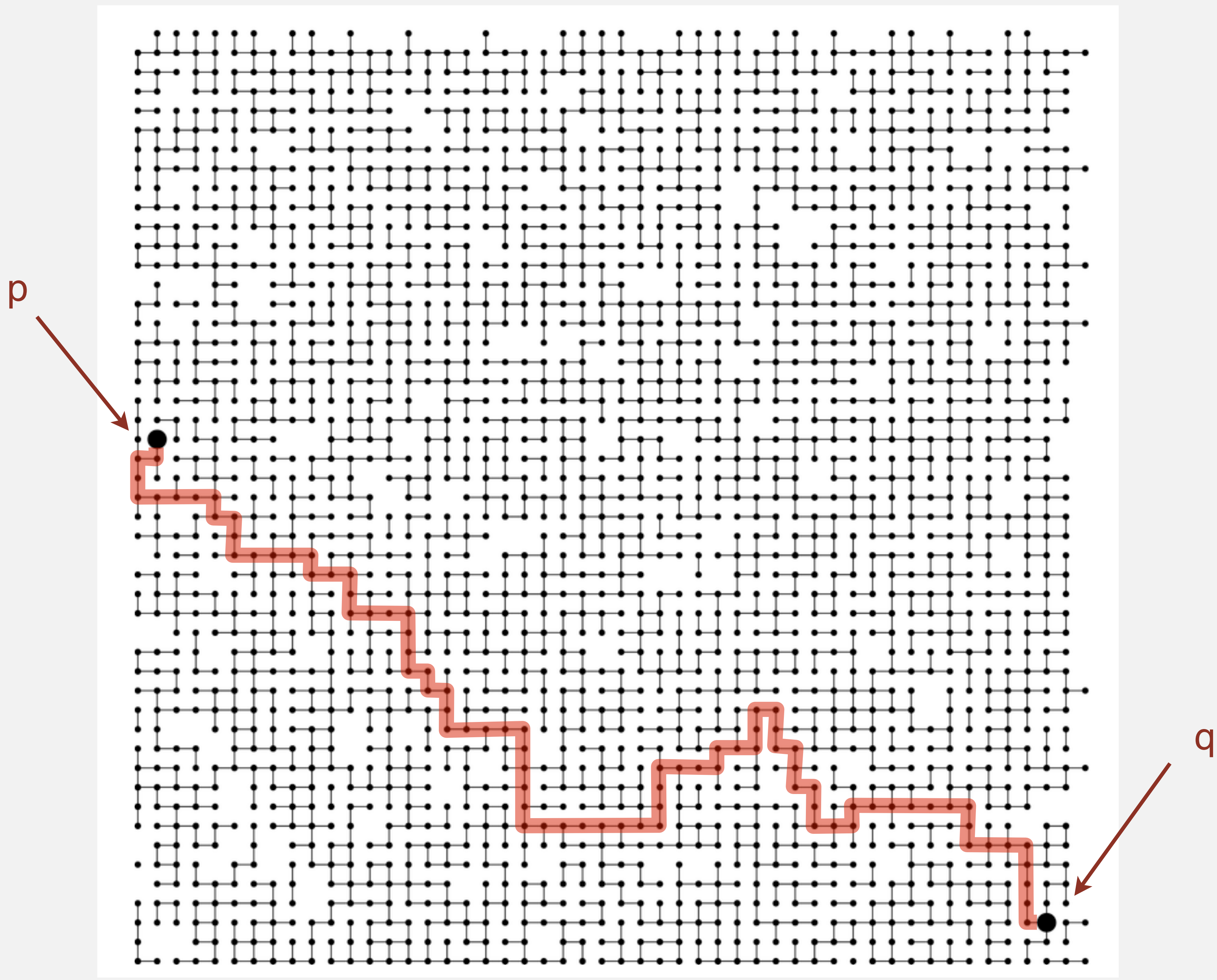

A. Yes.

Modeling the objects

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in a Fortran program.
- Metallic sites in a composite system.

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When programming, convenient to name objects 0 to N – 1.

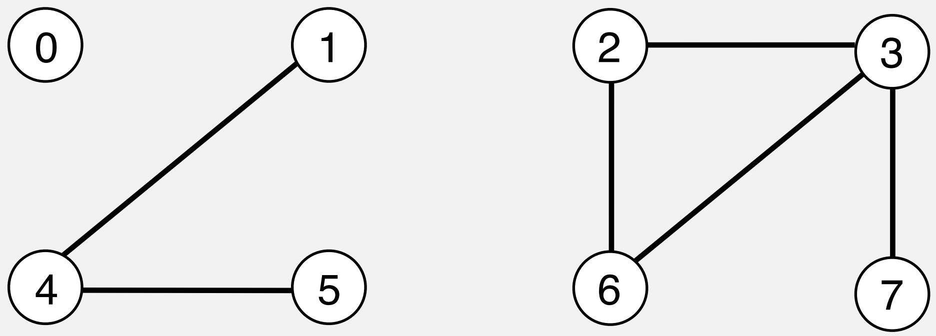
- Use integers as array index.
- Suppress details not relevant to union-find.

can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: p is connected to p.
- Symmetric: if p is connected to q, then q is connected to p.
- Transitive: if p is connected to q and q is connected to r, then p is connected to r.



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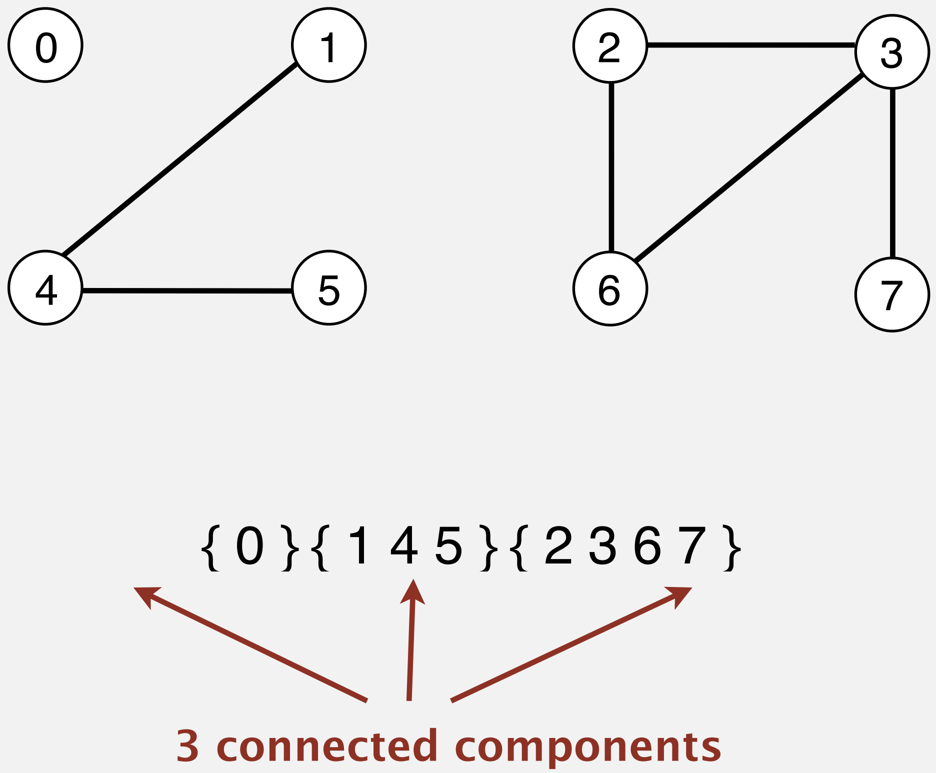
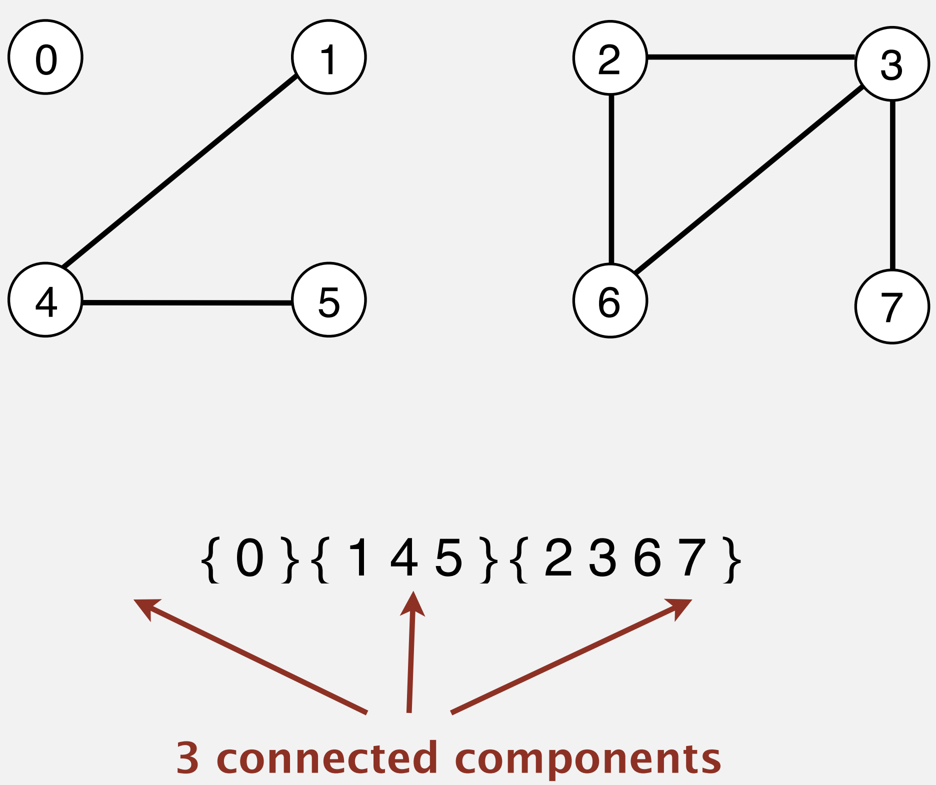
Connected component. Maximal set of objects that are mutually connected.



Implementing the operations

Find. In which component is object p ?

Connected. Are objects p and q in the same component?



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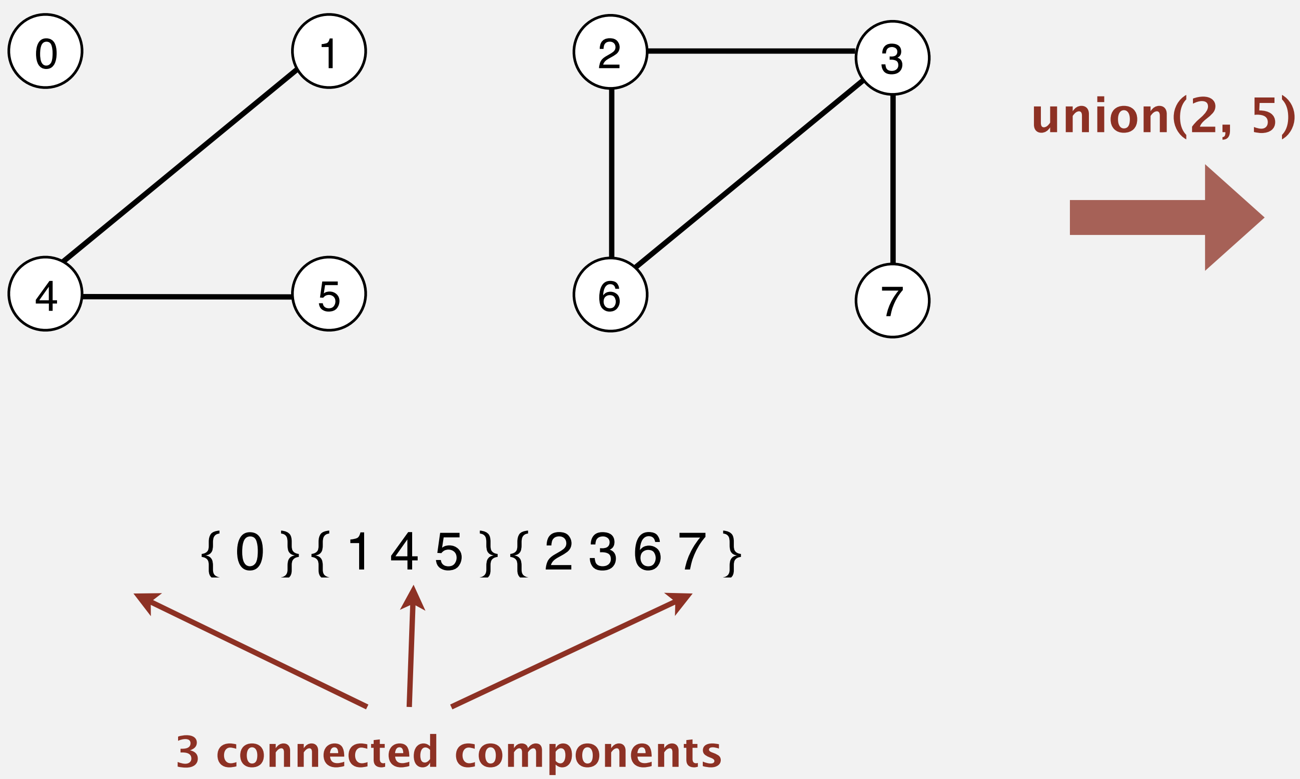
Union. Replace components containing objects p and q with their union.



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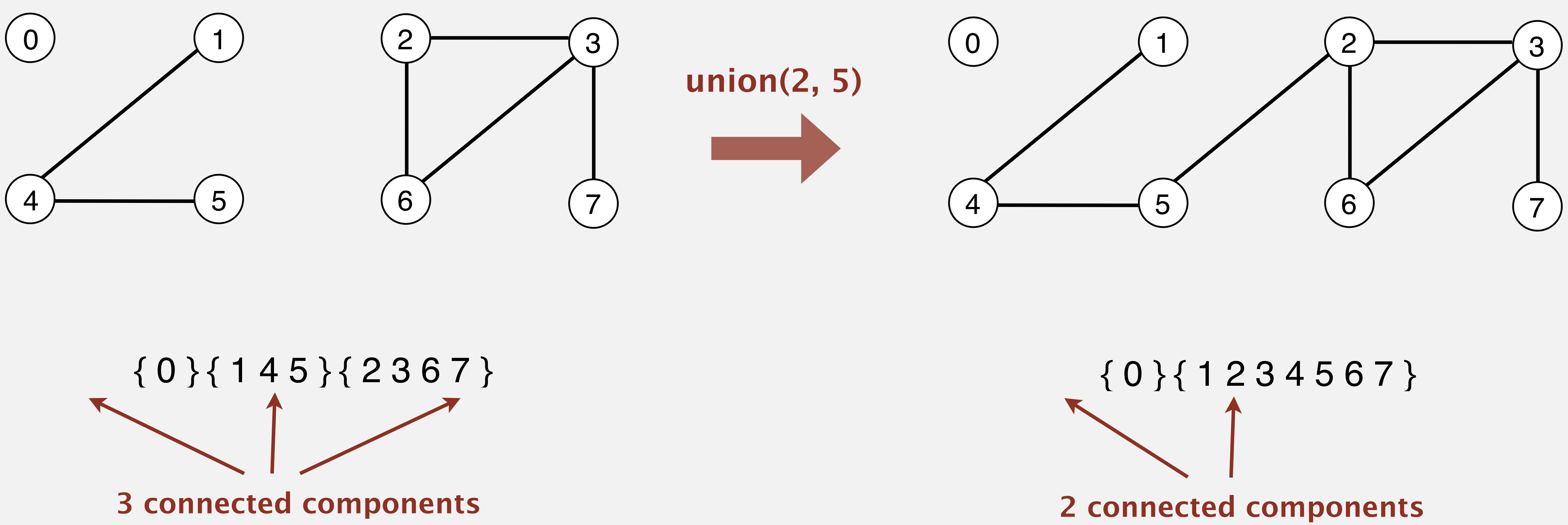

Union-find data type (API)

Goal. Design efficient data structure for union-find.

- Number of objects N can be huge.
- Number of operations M can be huge.
- Union and find operations may be intermixed.

public class UF		
	UF(int N)	*initialize union-find data structure with N singleton objects (0 to $N-1$)*
void	union(int p, int q)	*add connection between p and q*
int	find(int p)	*component identifier for p (0 to $N-1$)*
boolean	connected(int p, int q)	*are p and q in the same component?*

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```
public boolean connected(int p, int q)
{  return find(p) == find(q);  }
```

1-line implementation of connected()

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1.5 Union-Find

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Quick-find [eager approach]

Data structure.

if and only if

- Integer array id[] of length N.
- Interpretation: id[p] is the id of the component containing p.

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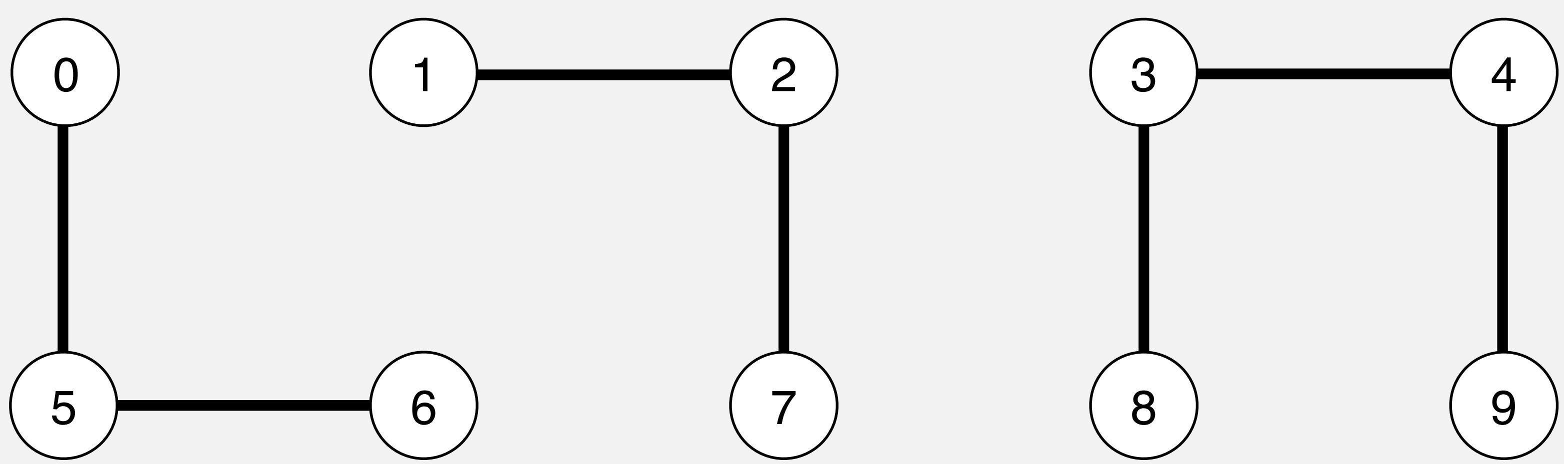
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected

1, 2, and 7 are connected

3, 4, 8, and 9 are connected



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id[]	0	1	1	8	8	0	0	1	8	8

Find. What is the id of p?

Connected. Do p and q have the same id?

id[6] = 0; id[1] = 1

6 and 1 are not connected

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6 and 1 are not connected

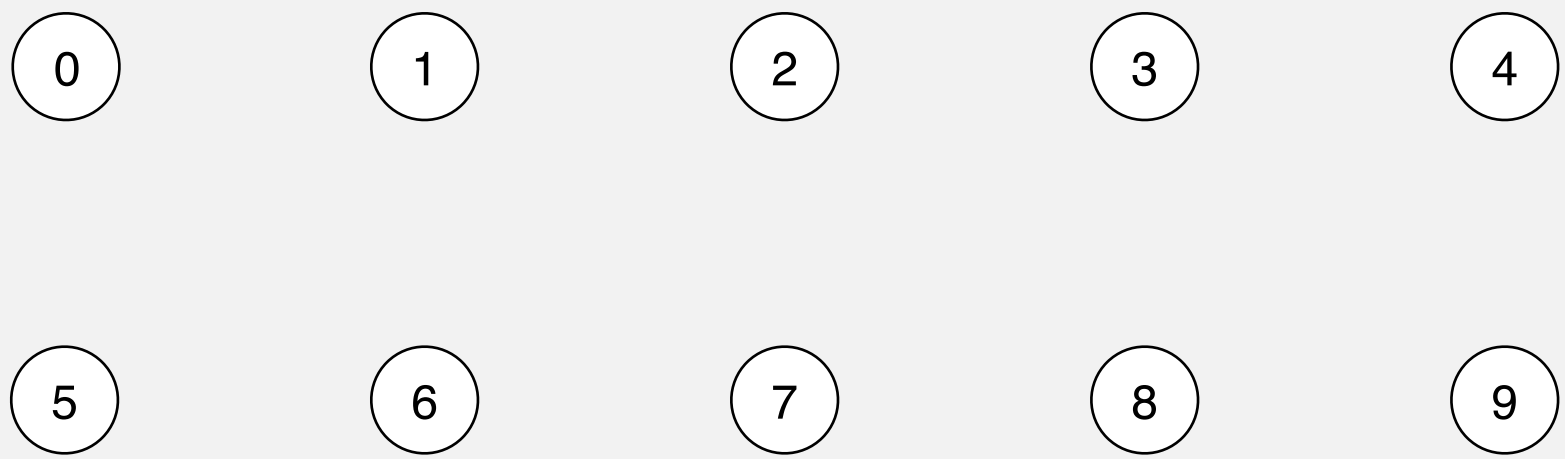
Union. To merge components containing p and q, change all entries whose id equals id[p] to id[q].

	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

problem: many values can change

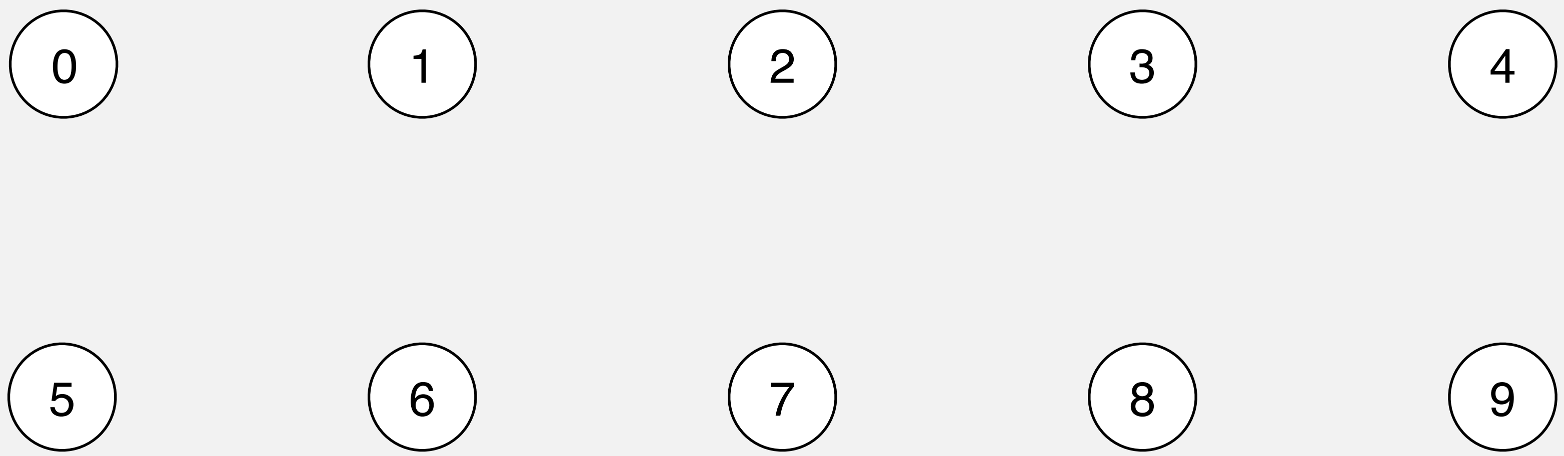
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-find demo

union(4, 3)



	0	1	2	3	4	5	6	7	8	9
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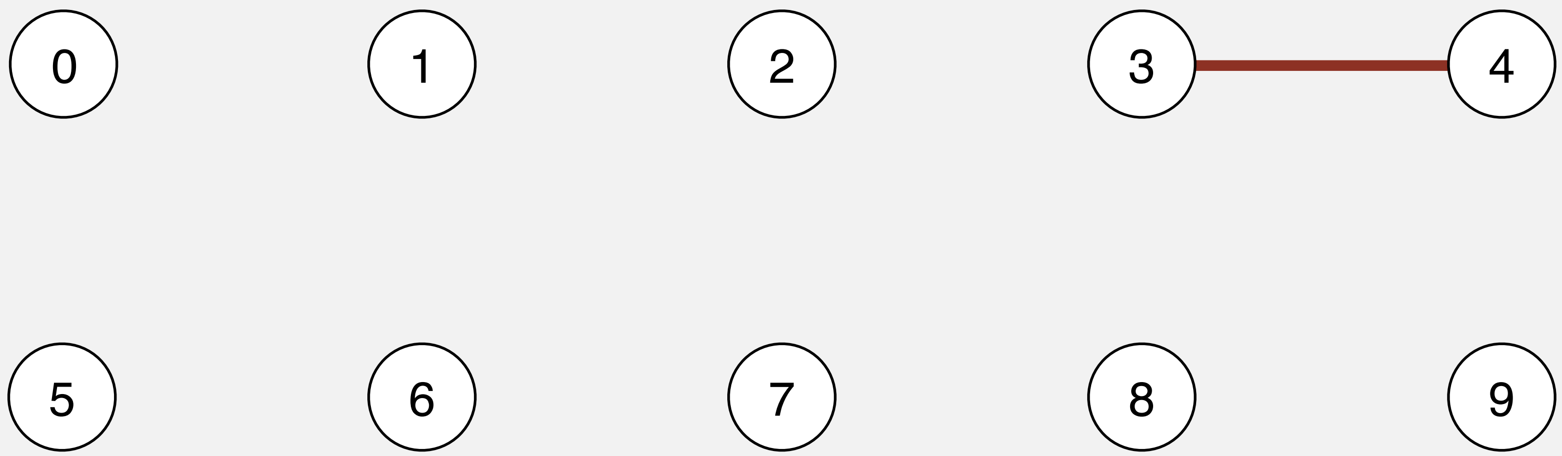

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id[]	0	1	2	3	4	5	6	7	8	9

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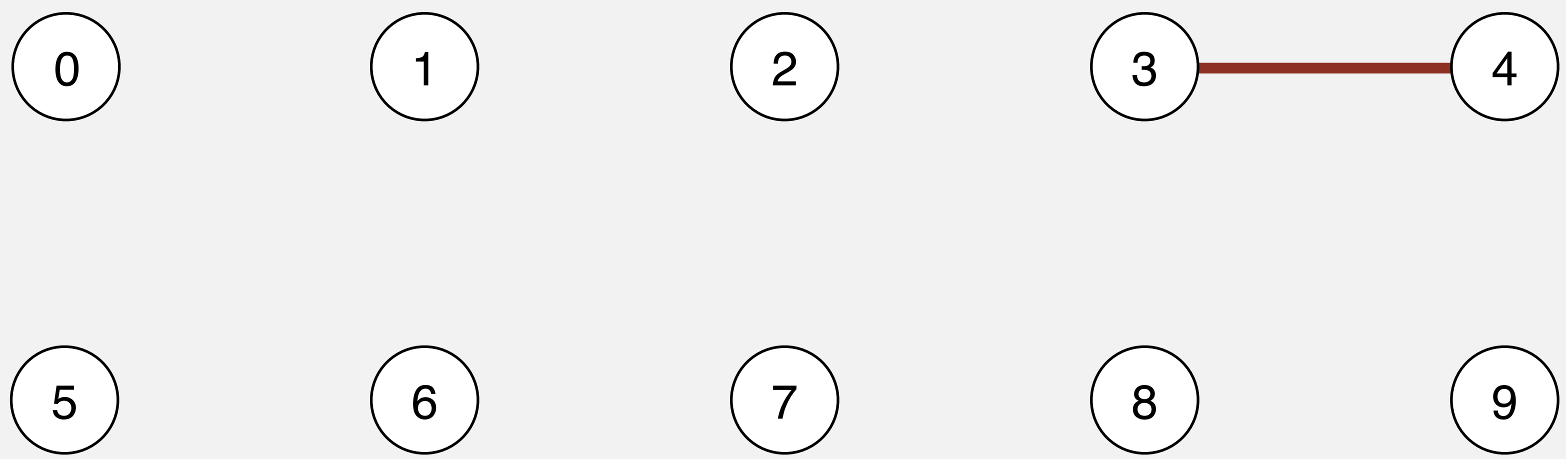

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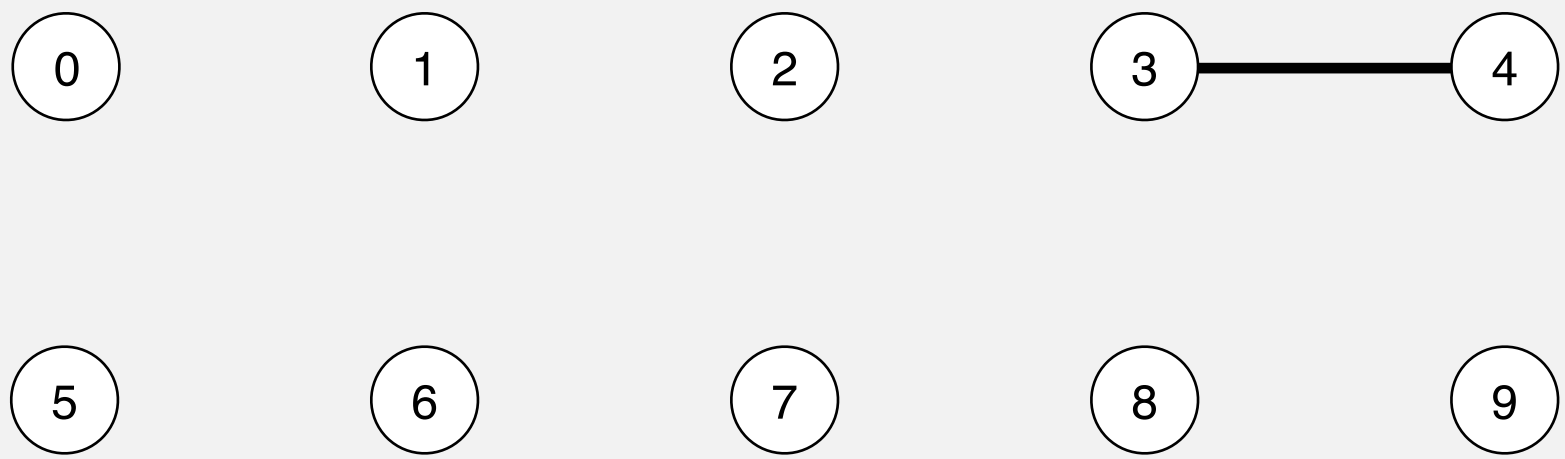

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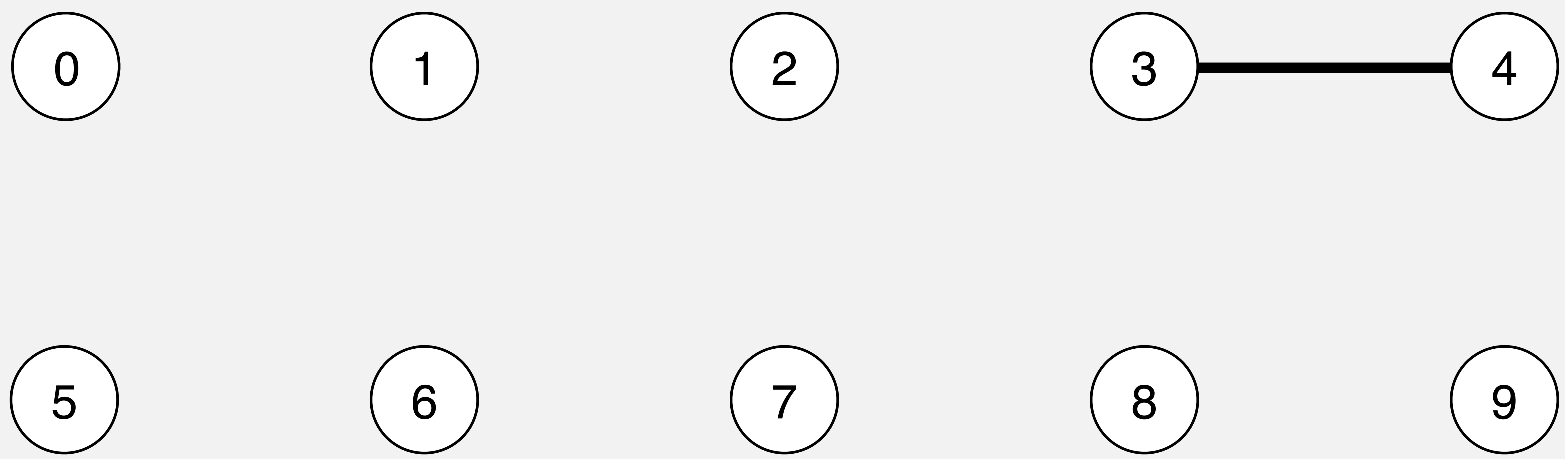
Quick-find demo

union(3, 8)



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Quick-find demo

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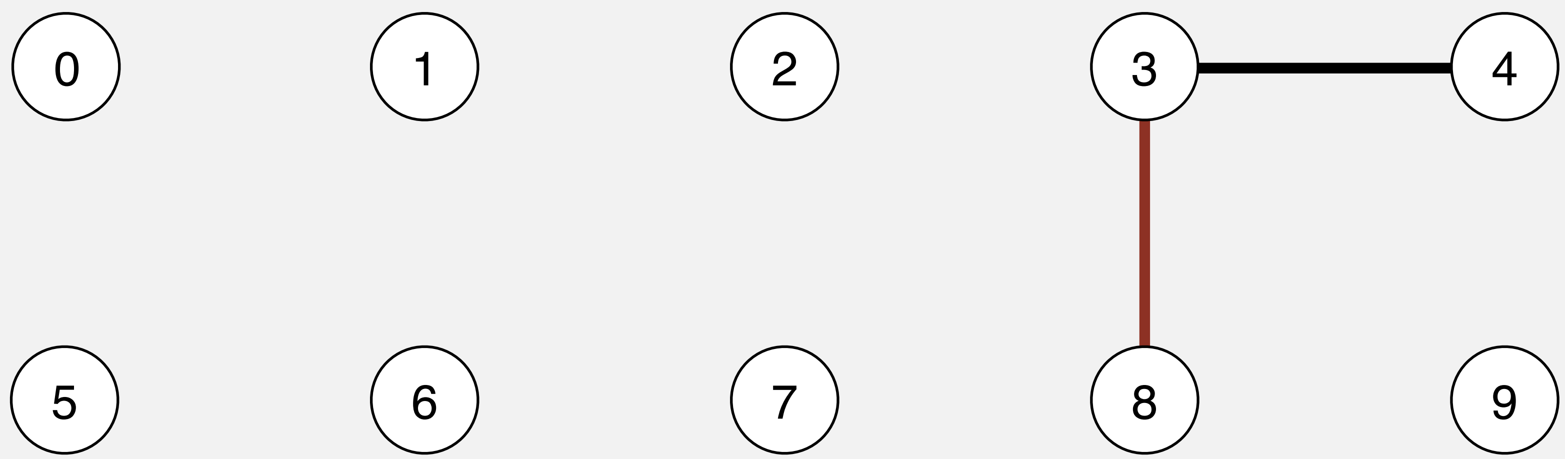

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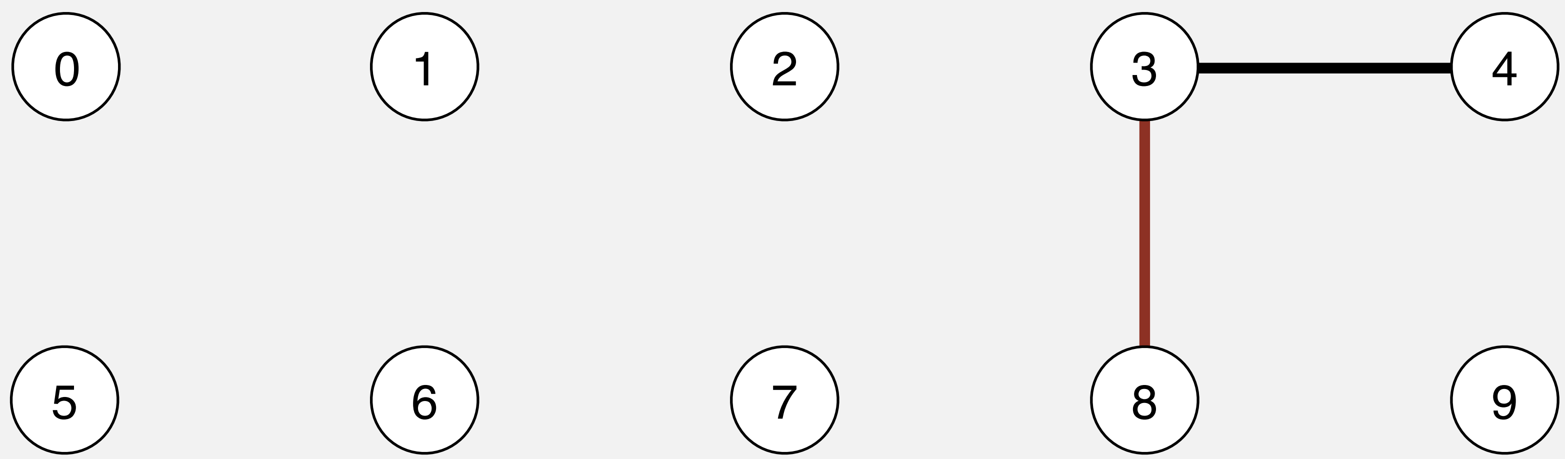

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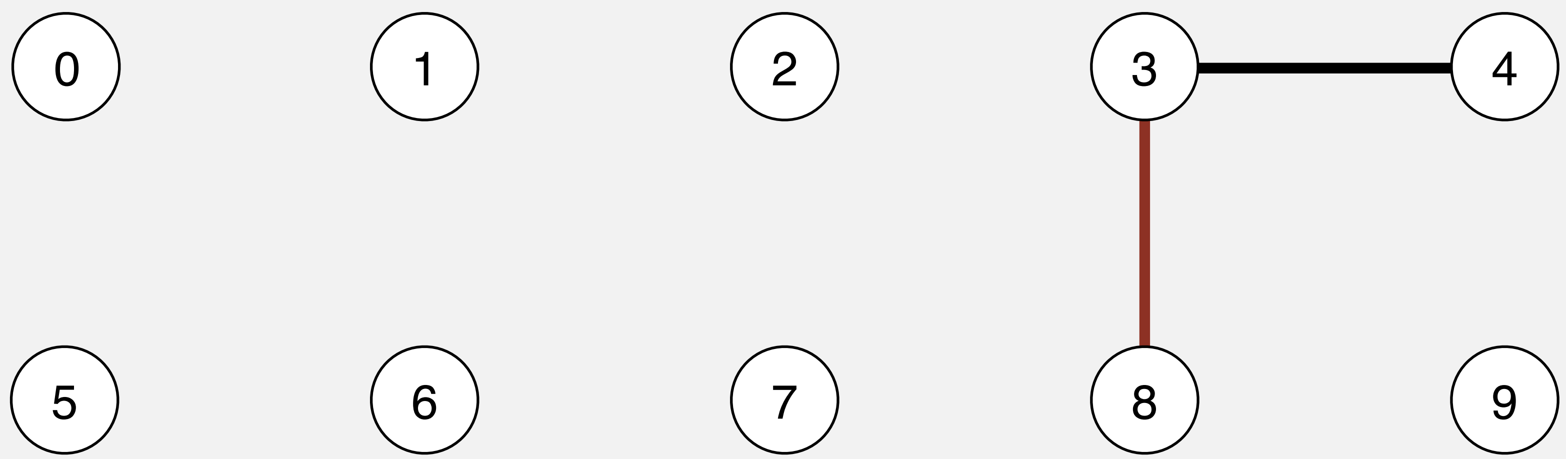

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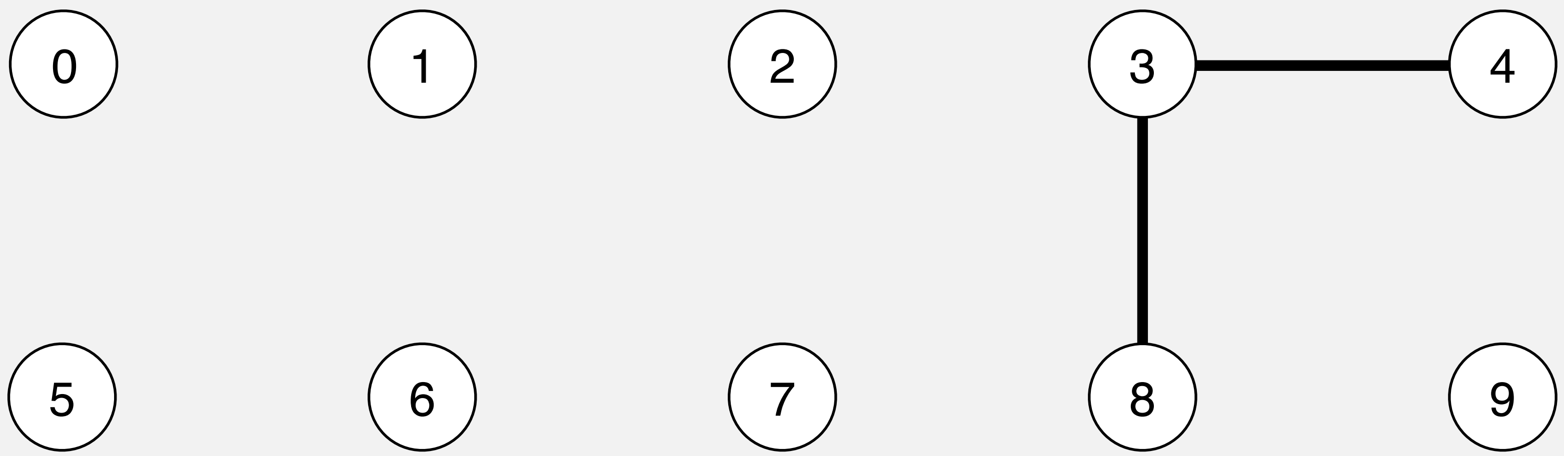
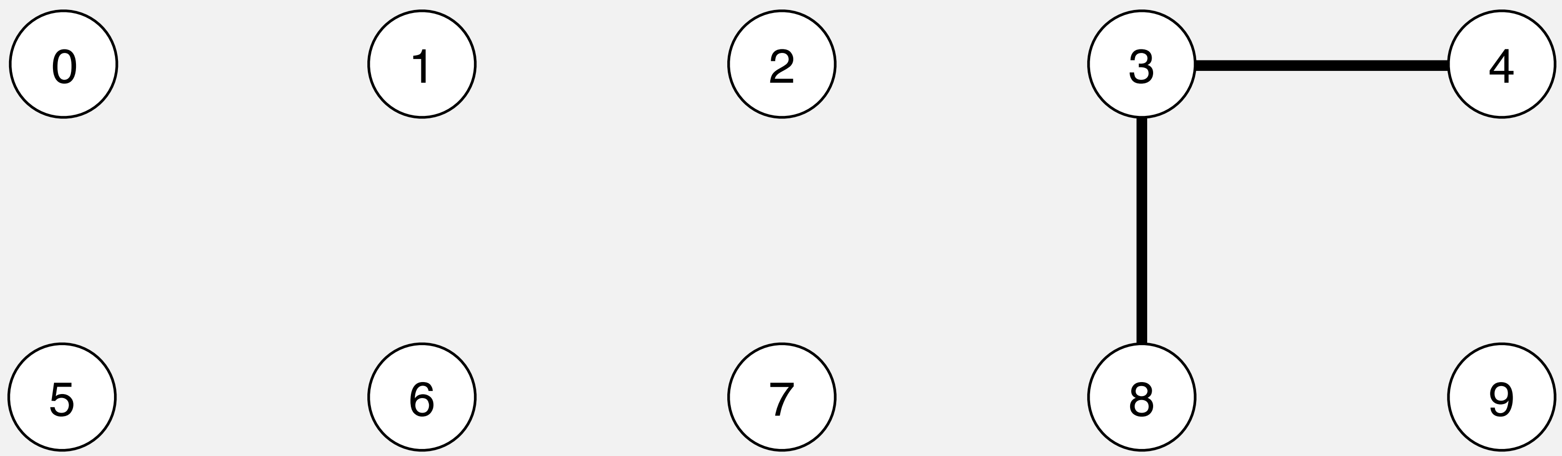


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	6	7	8	9

Quick-find demo

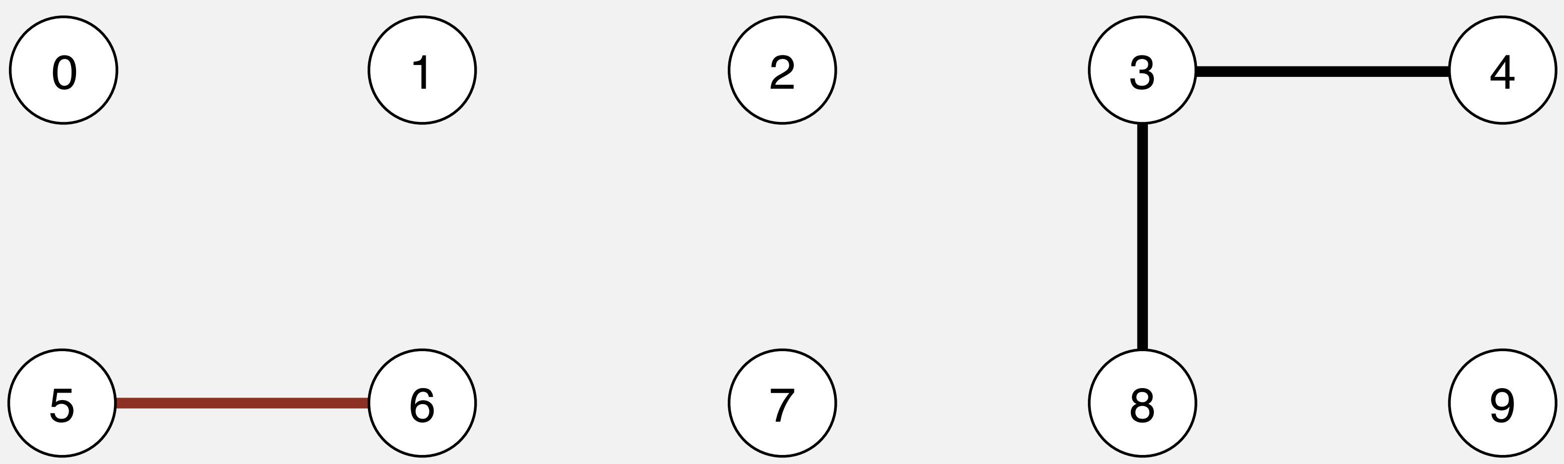
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	6	7	8	9

Quick-find demo

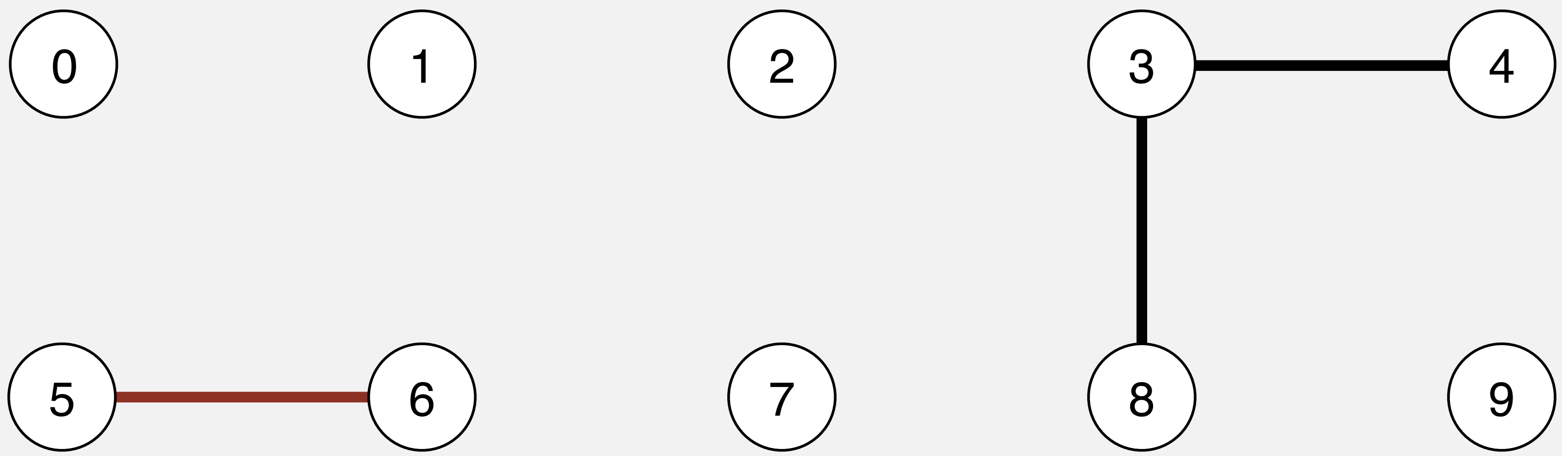
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	6	7	8	9

Quick-find demo

union(6, 5)



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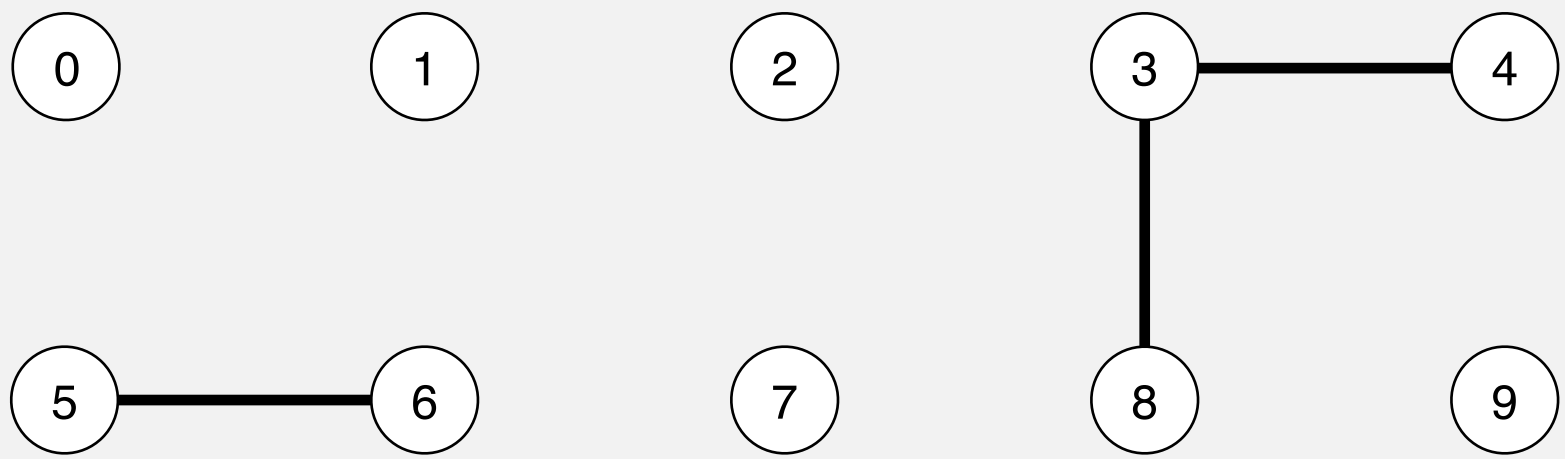
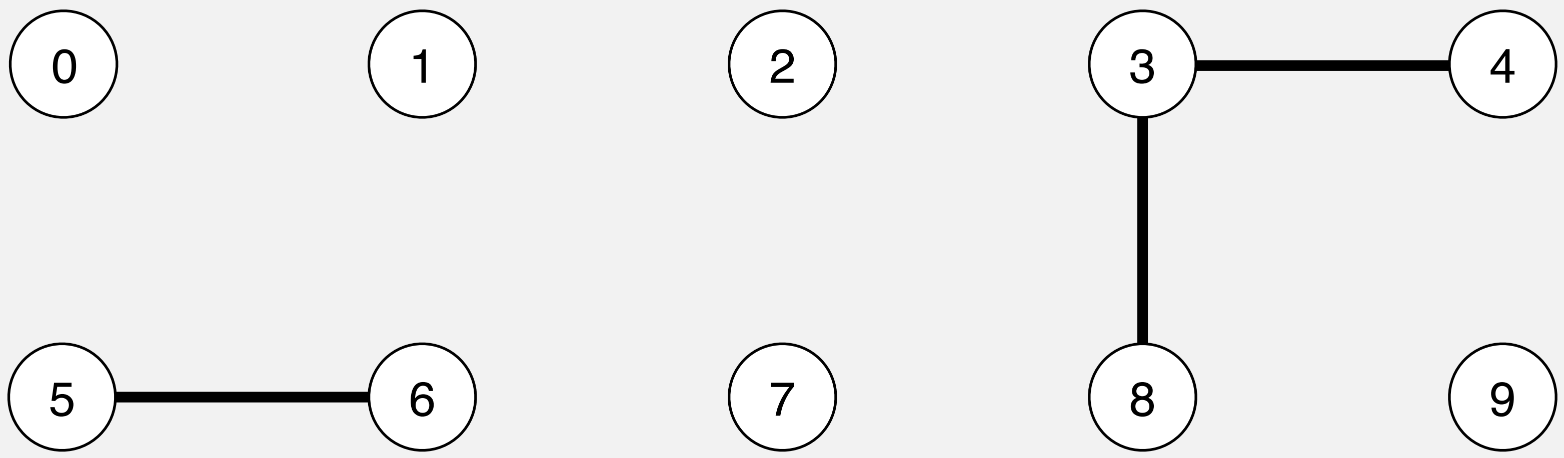
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	9

Quick-find demo

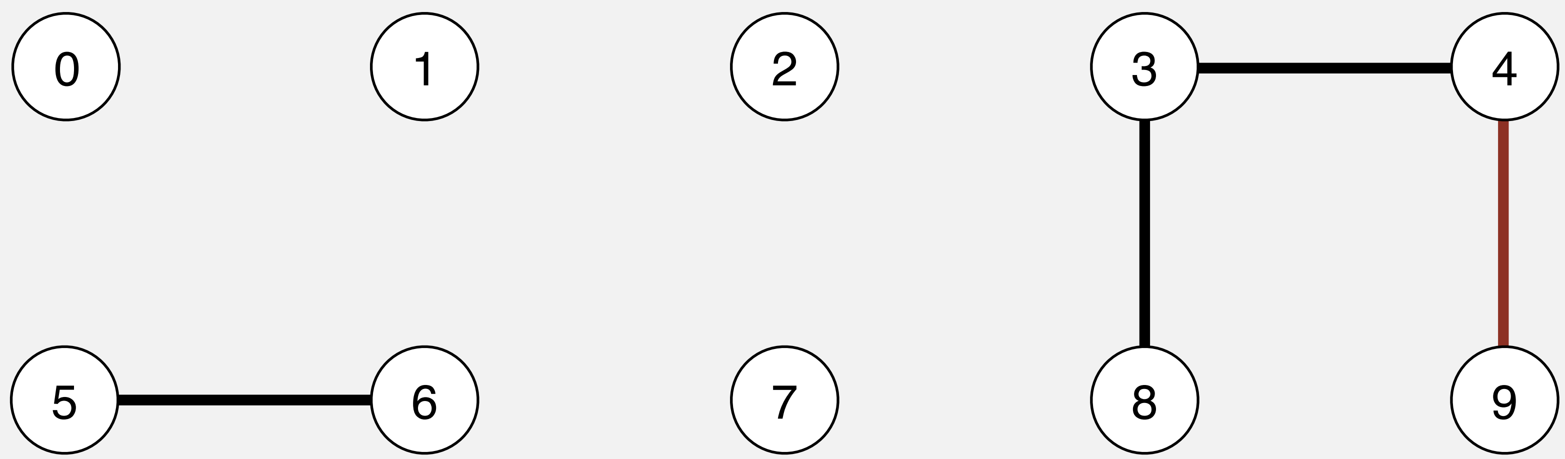
union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	9

Quick-find demo

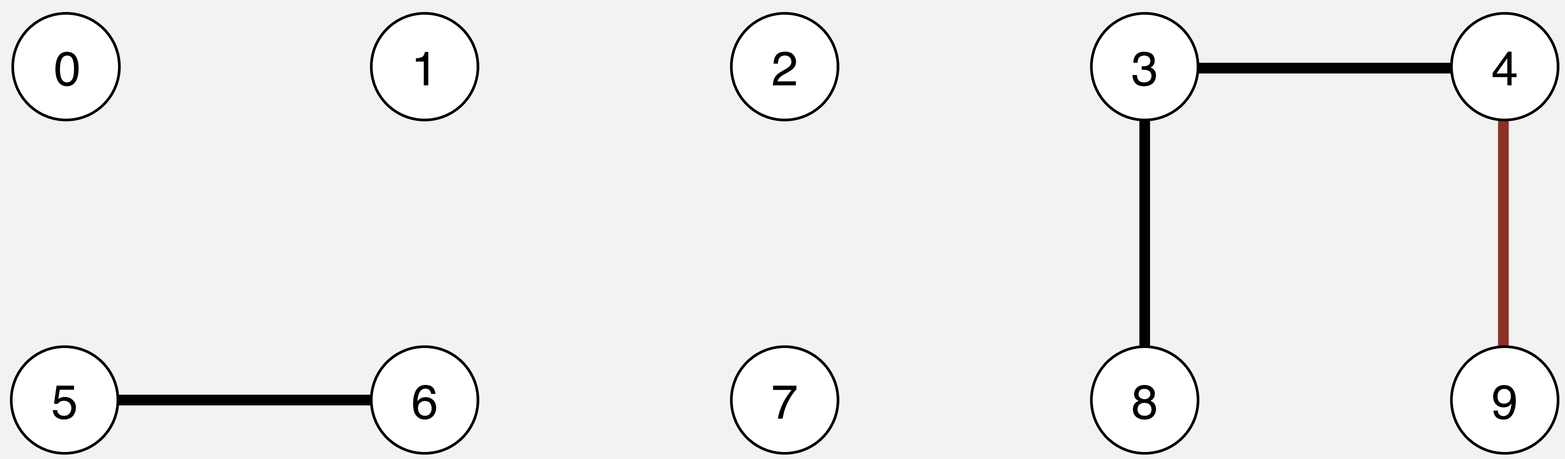
union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	9

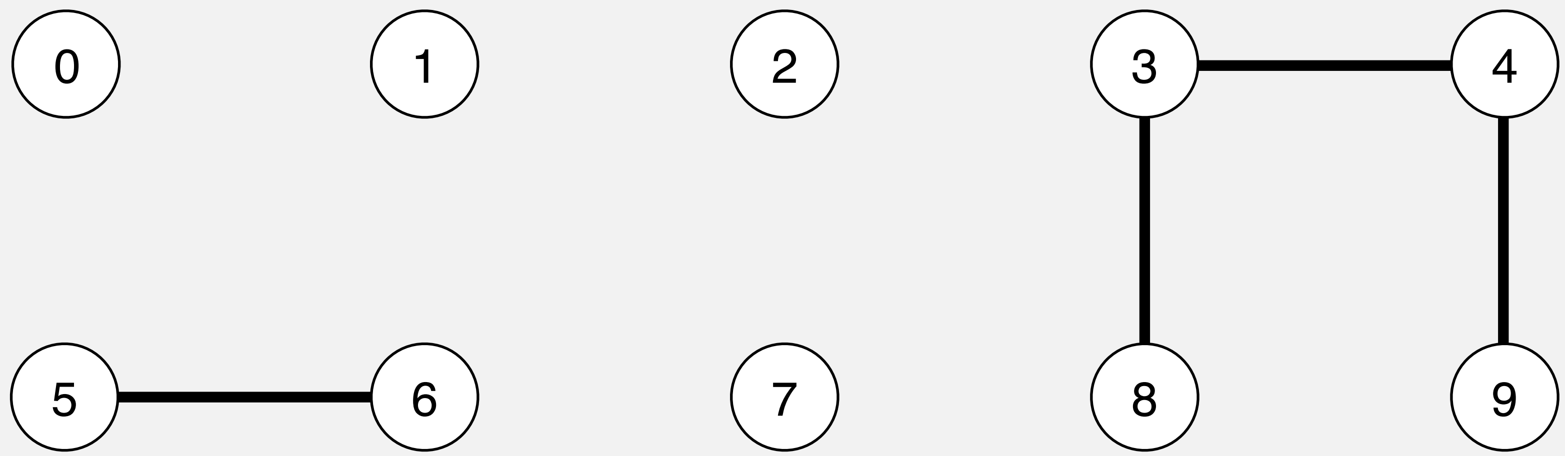
Quick-find demo

union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	8

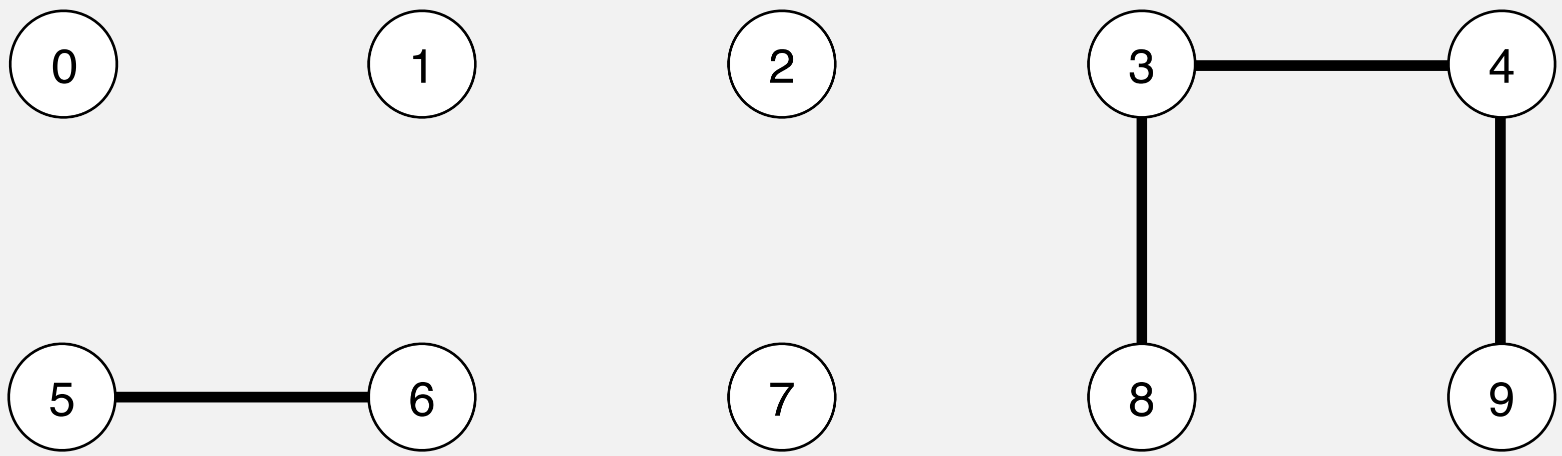
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	8

Quick-find demo

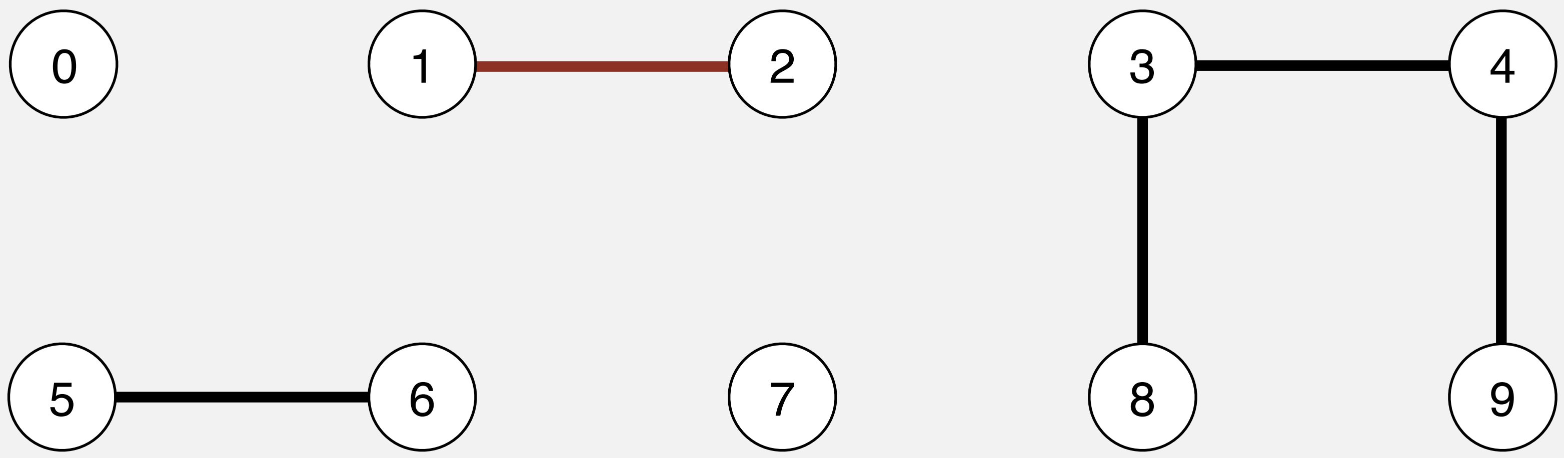
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	8

Quick-find demo

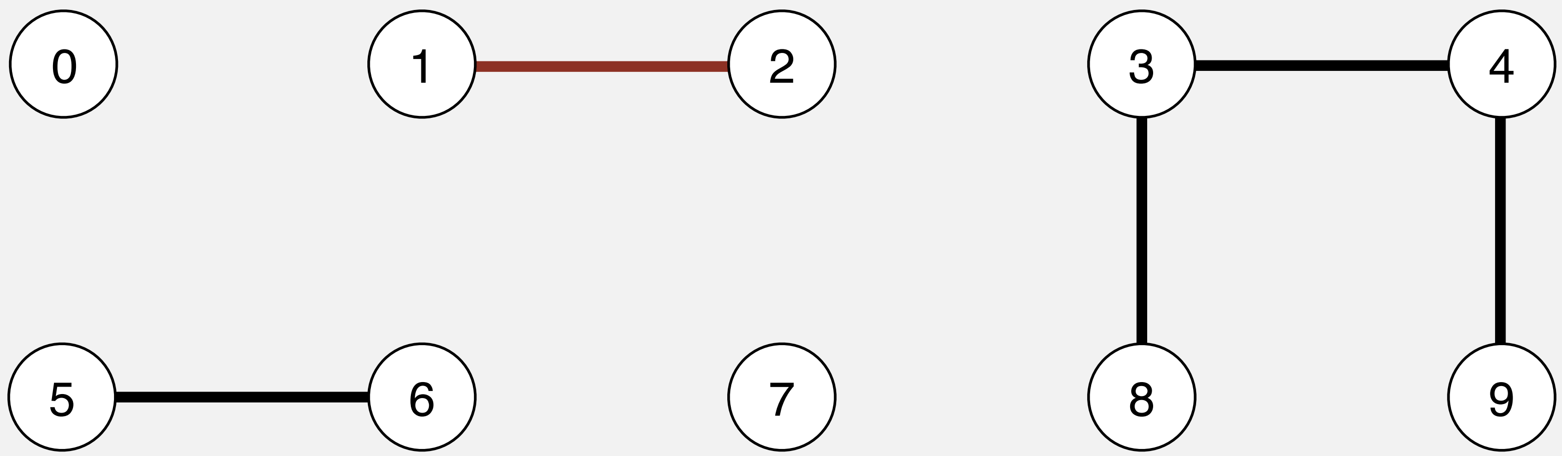
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	8

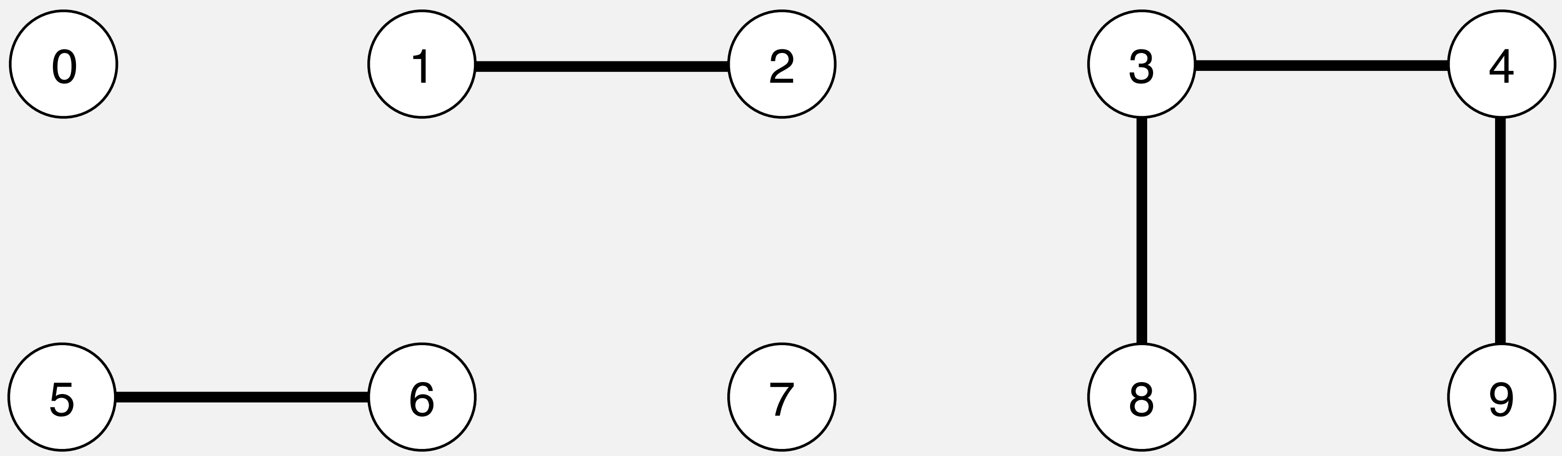
Quick-find demo

union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

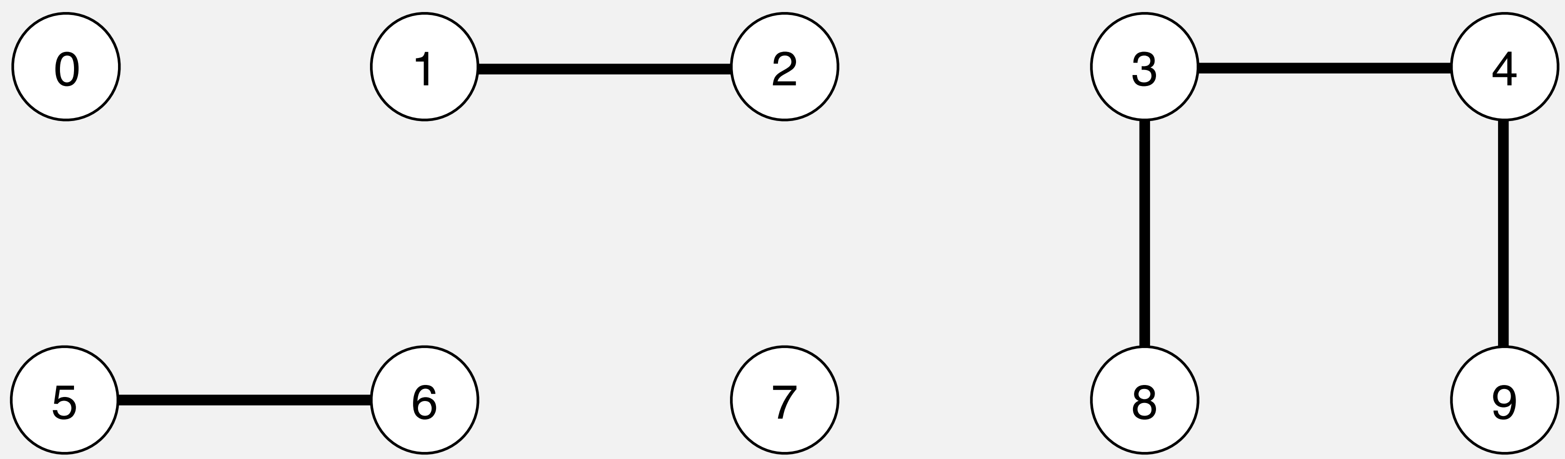
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

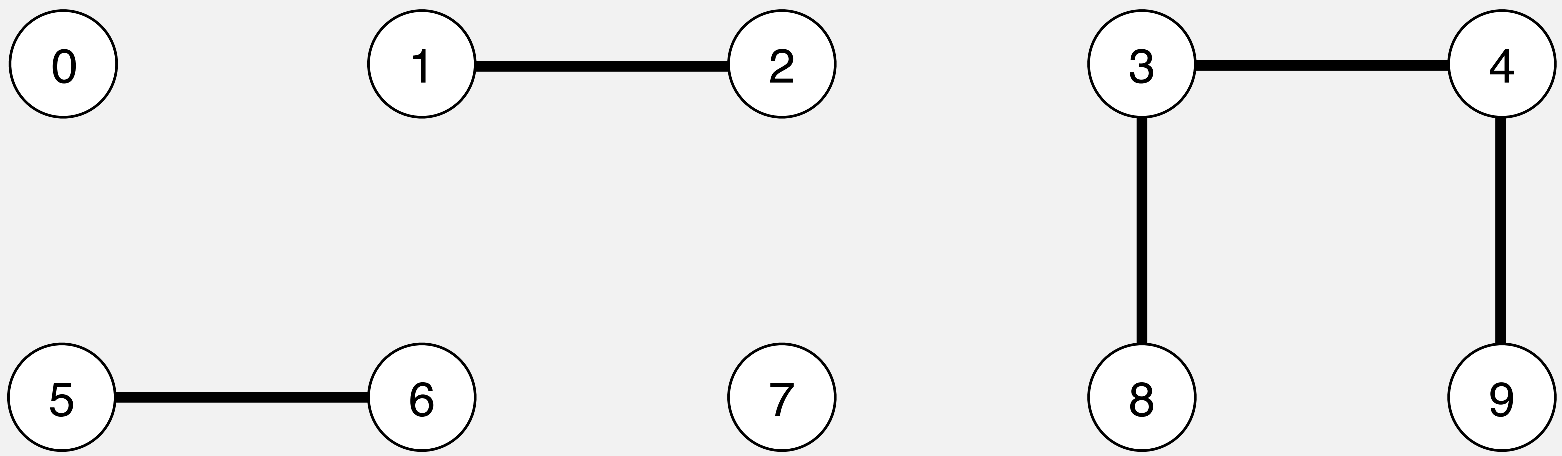
connected(8, 9)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

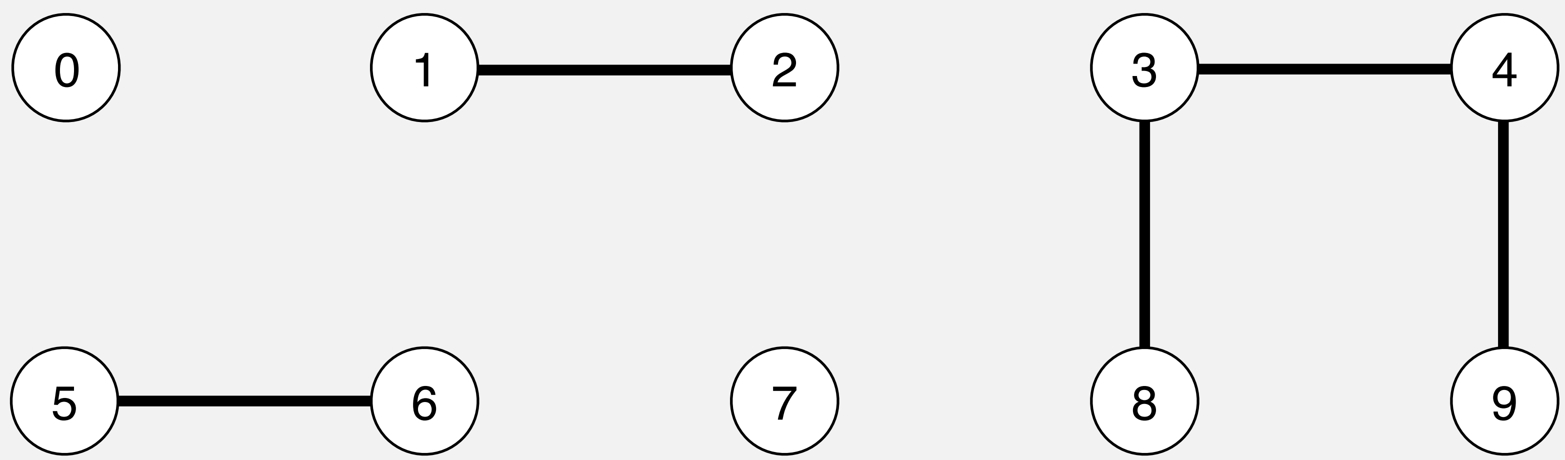
connected(8, 9)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

already connected

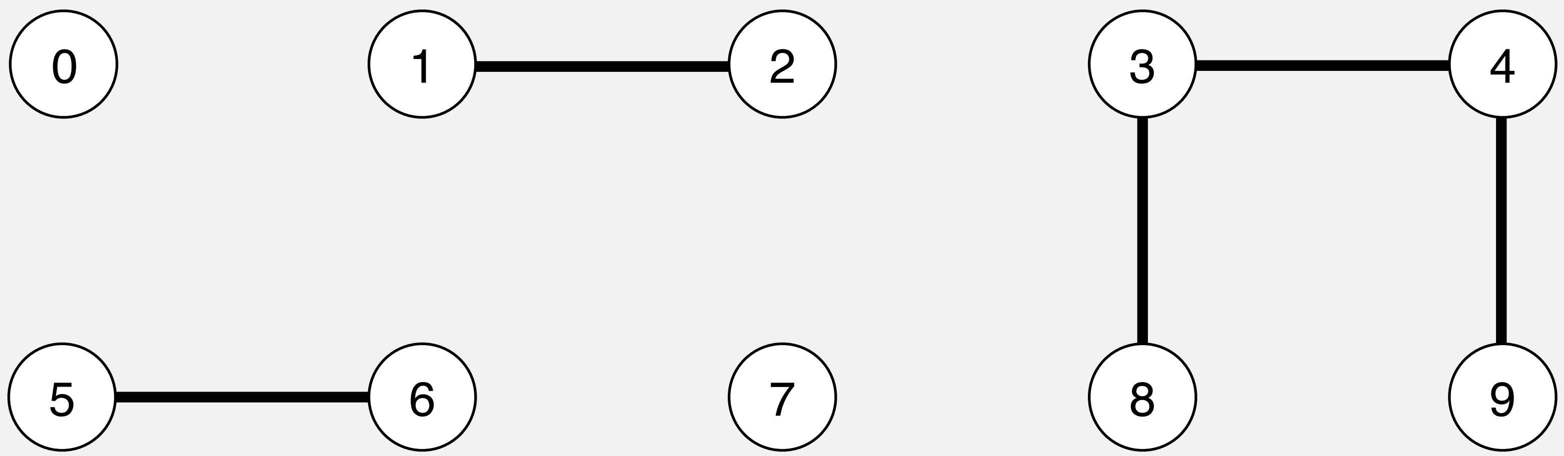
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

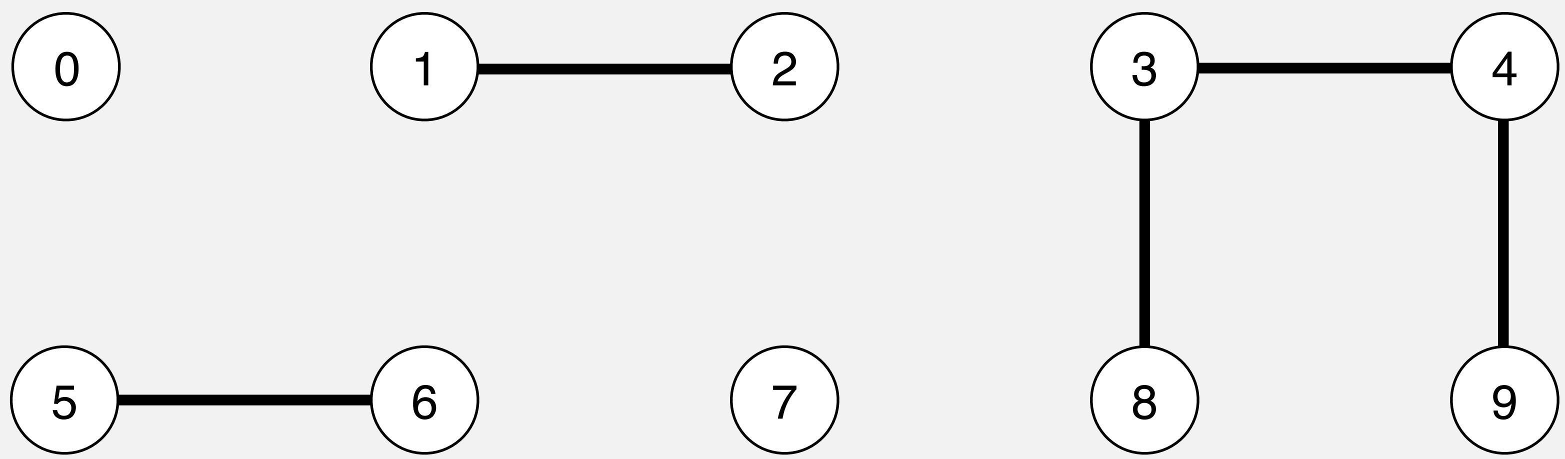
connected(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

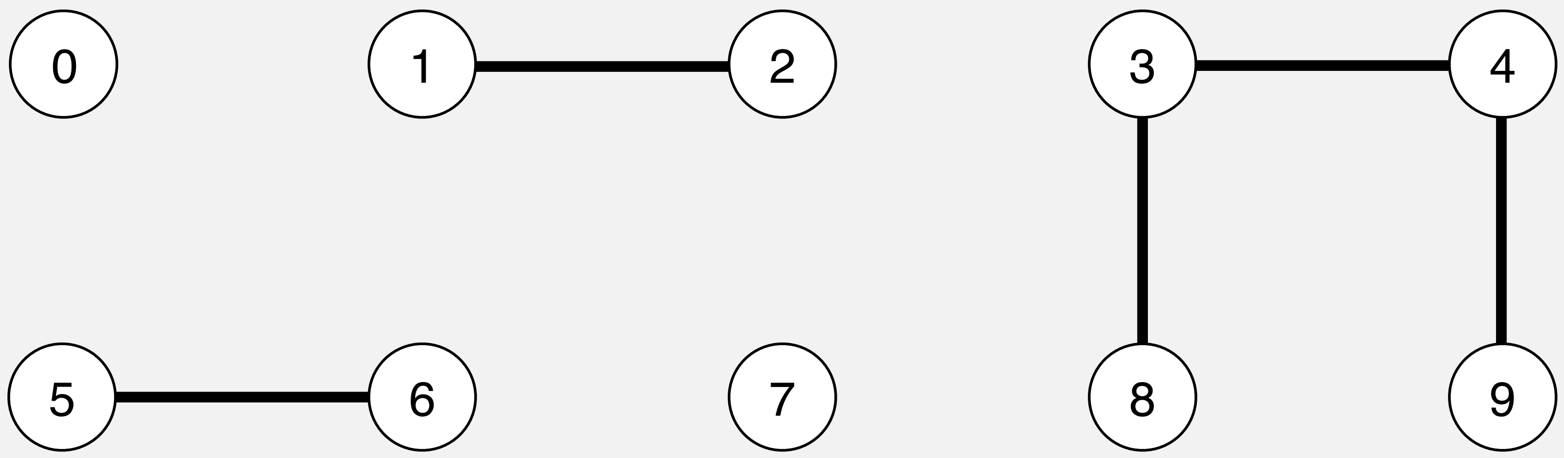
connected(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

not connected

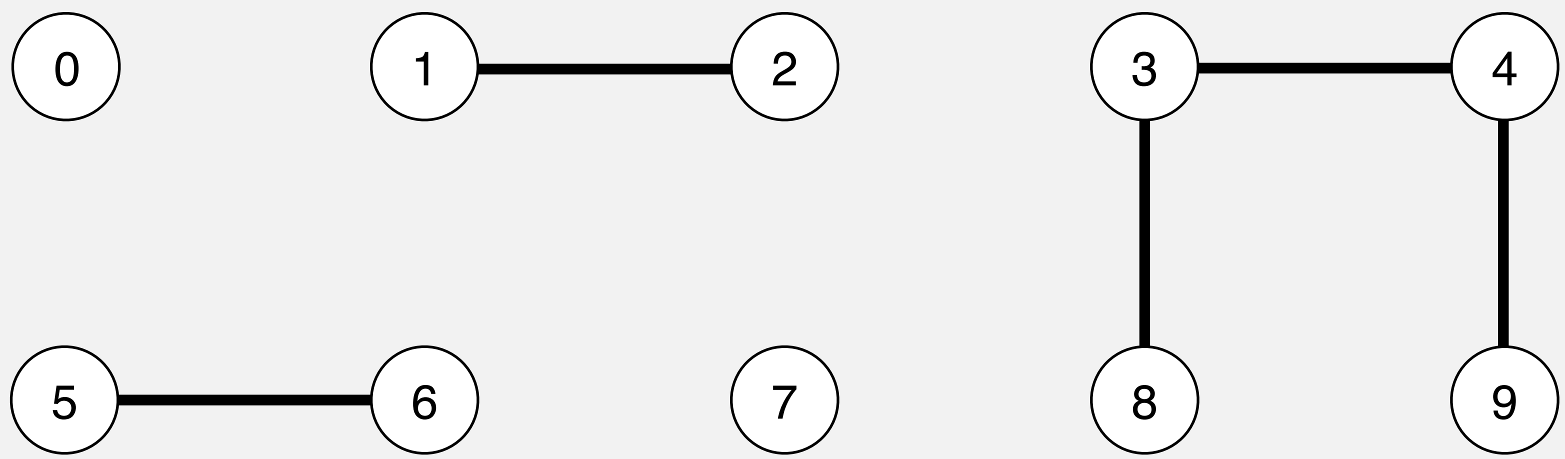
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

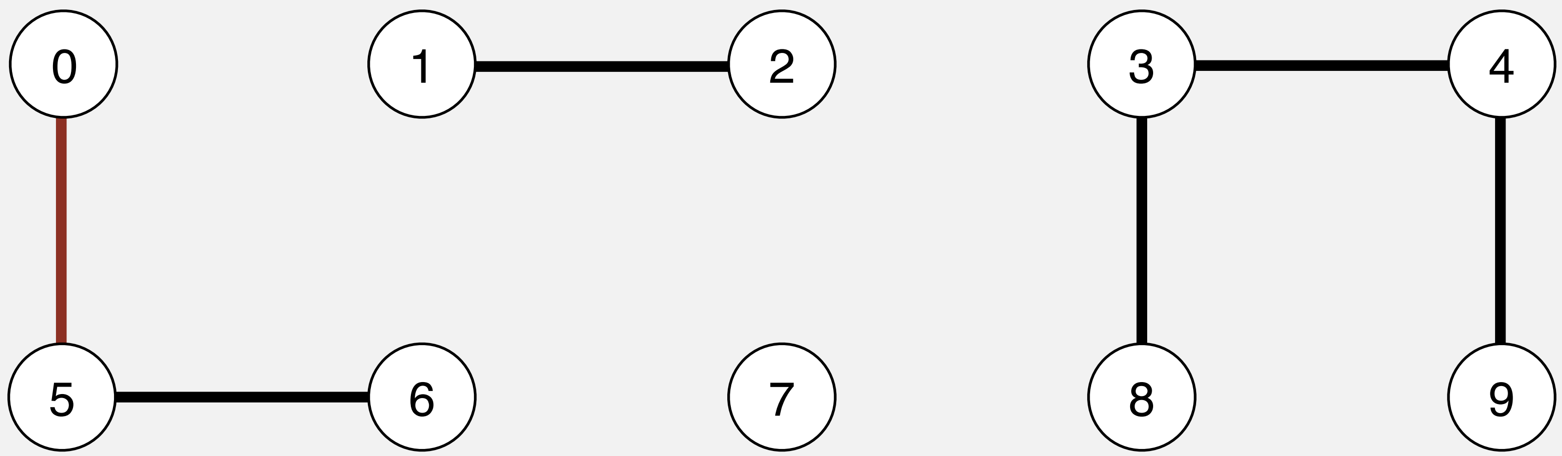
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

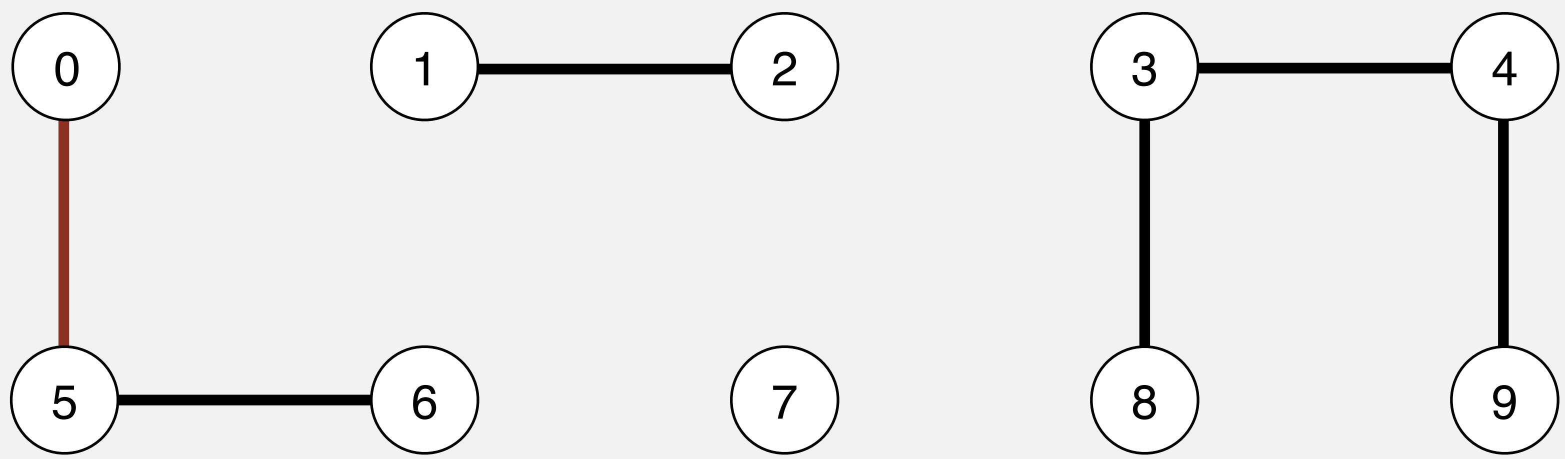
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

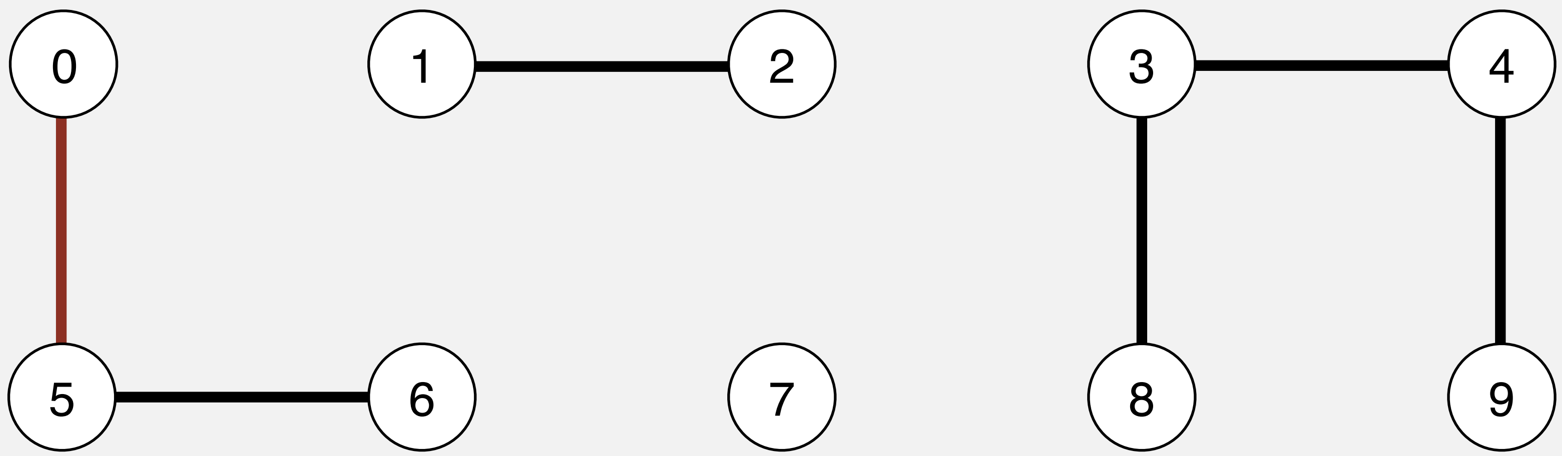
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	5	7	8	8

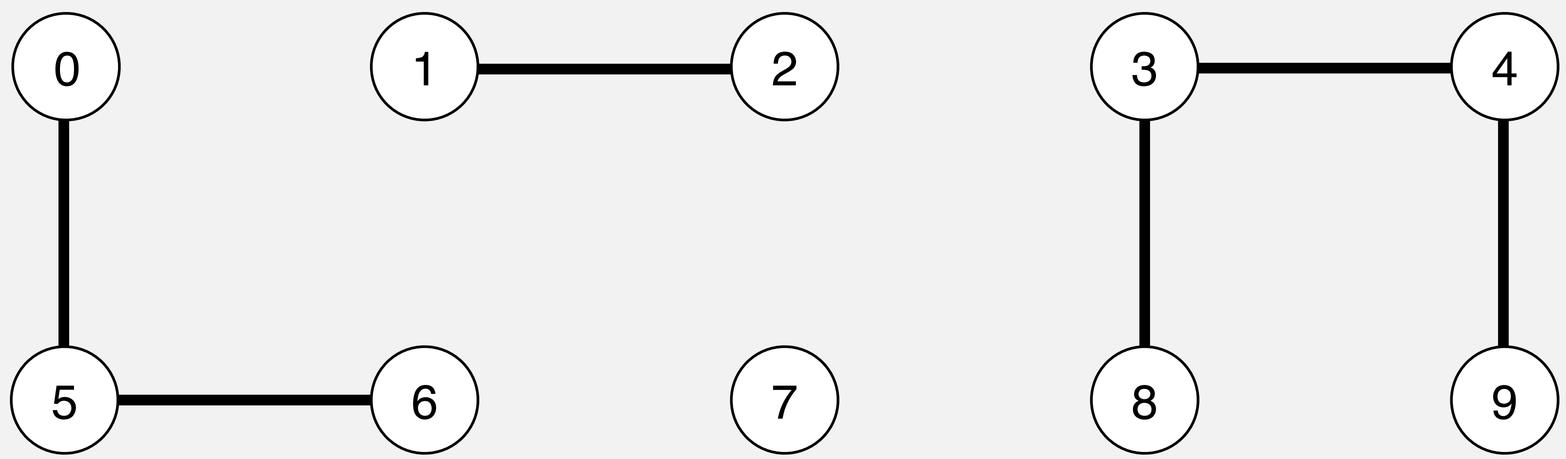
Quick-find demo

union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	7	8	8

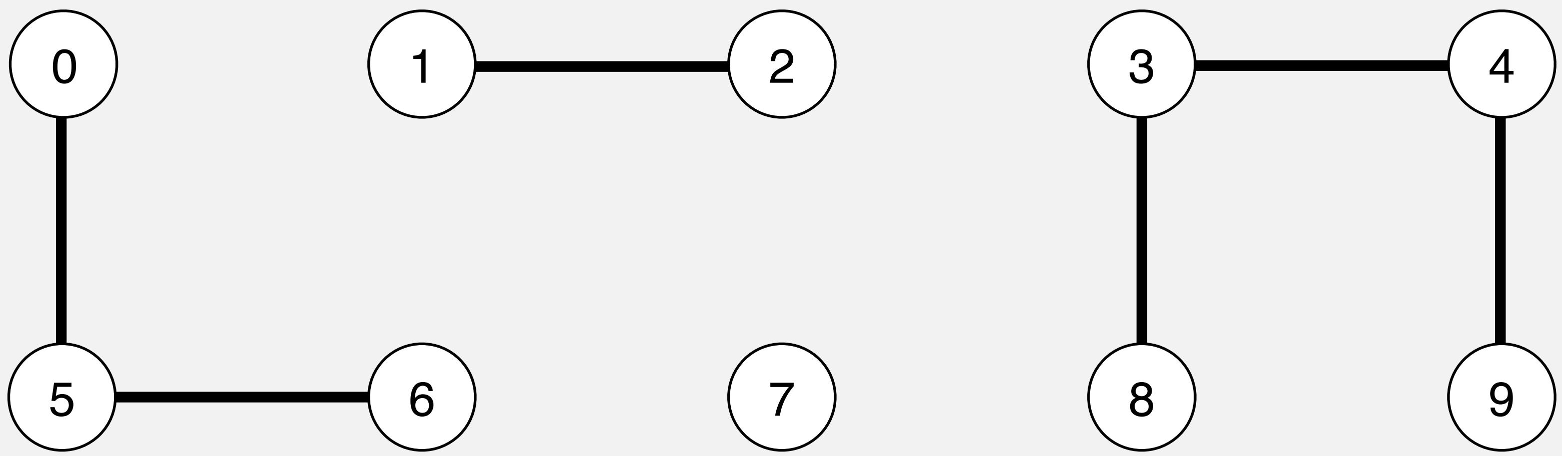
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	7	8	8

Quick-find demo

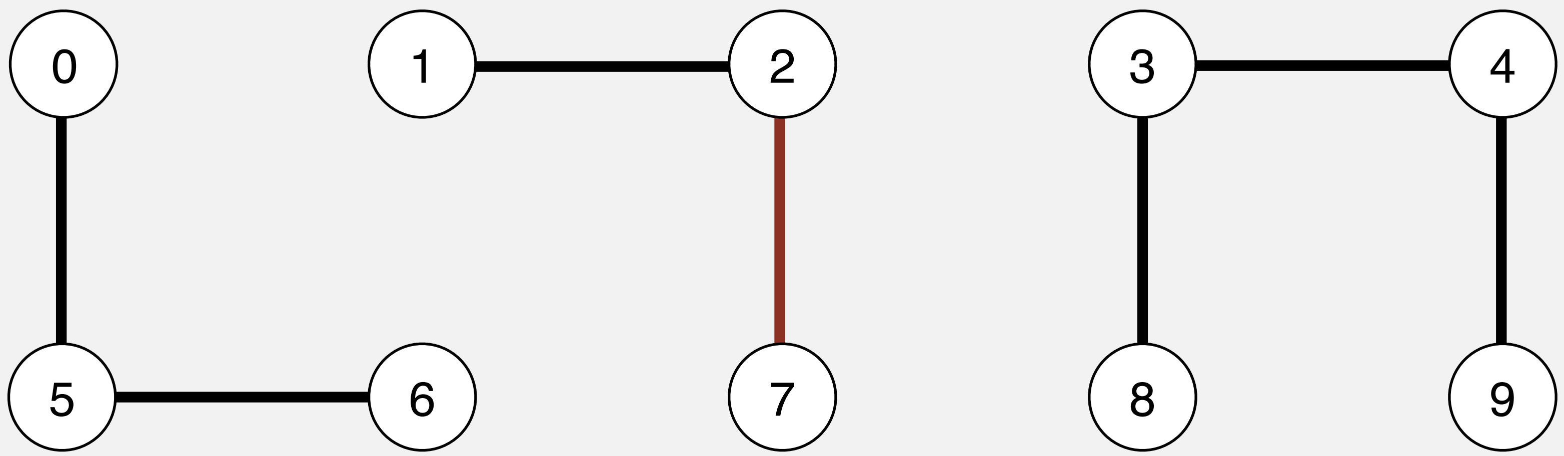
union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	7	8	8

Quick-find demo

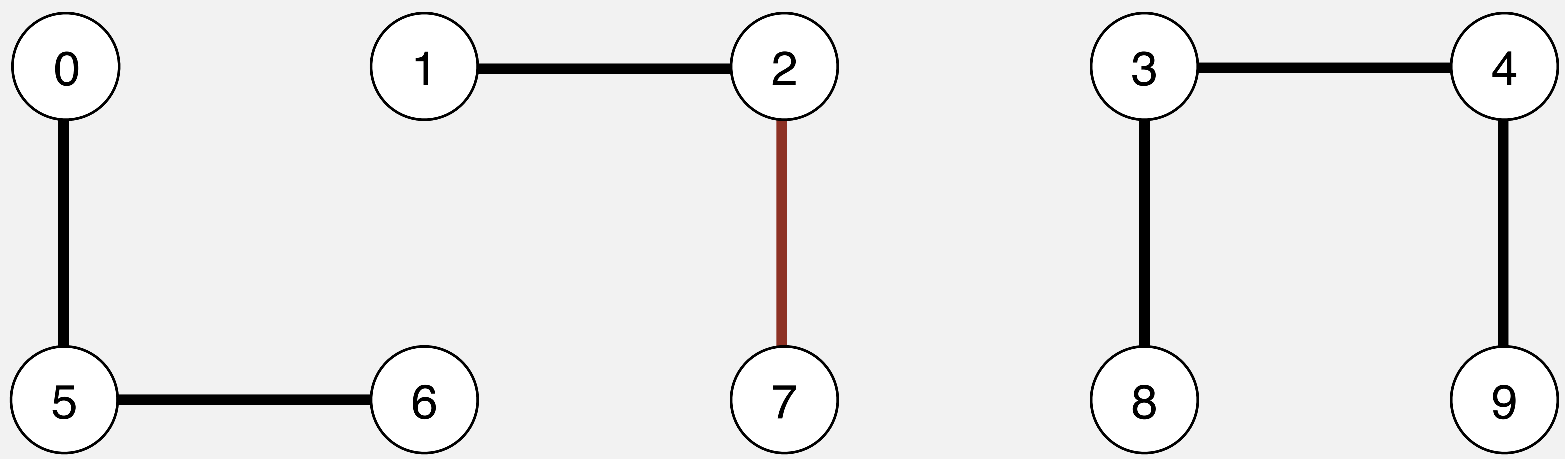
union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	7	8	8

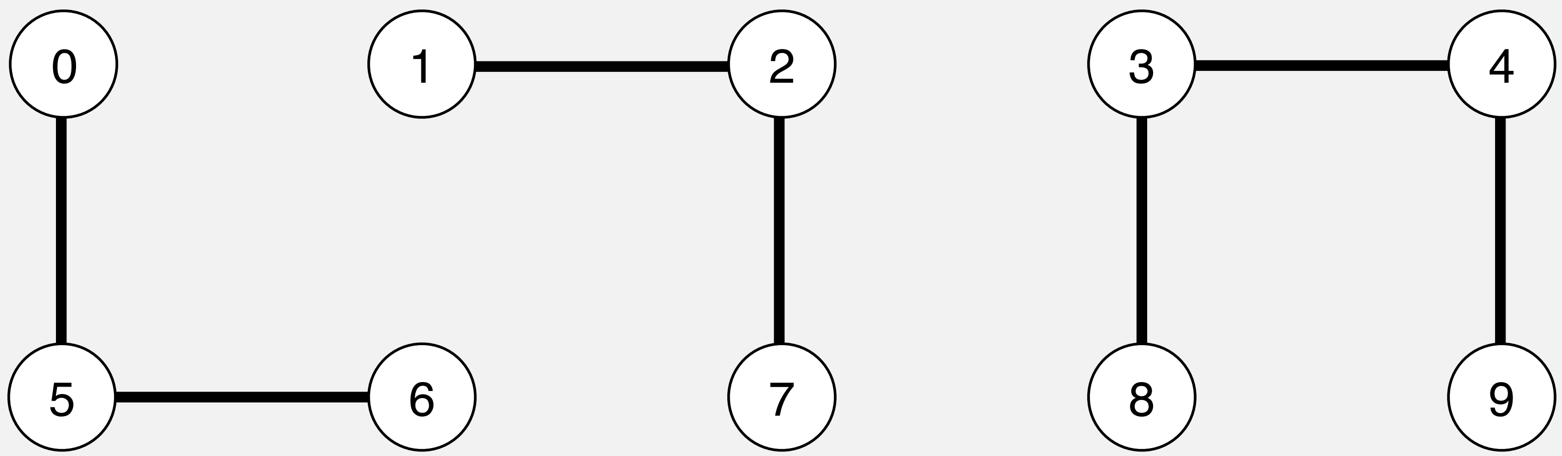
Quick-find demo

union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

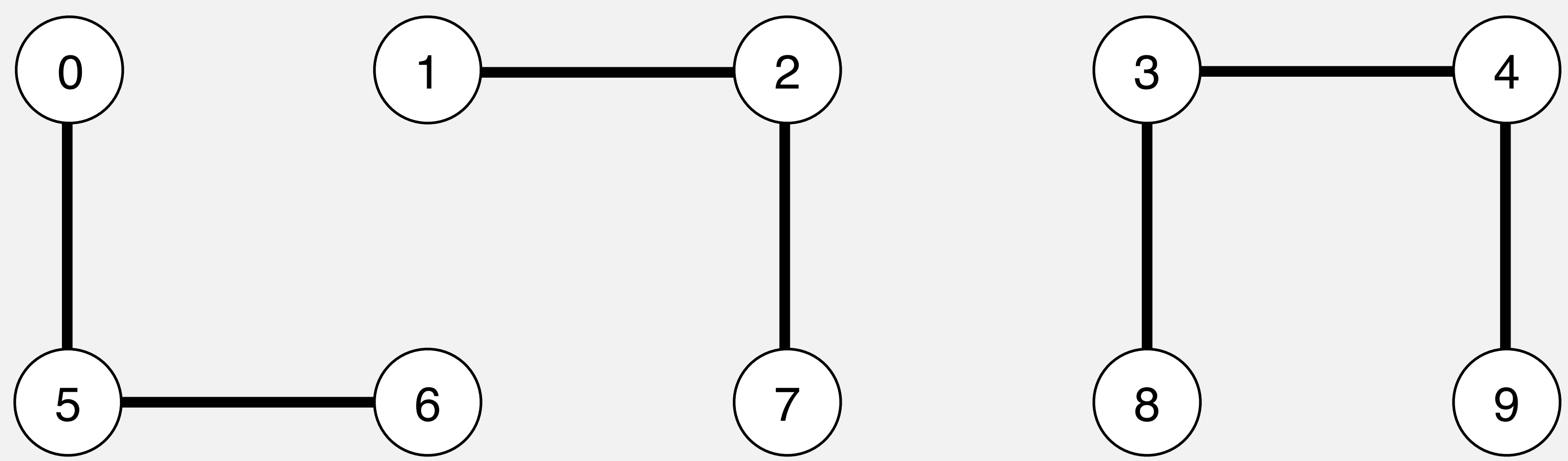
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Quick-find demo

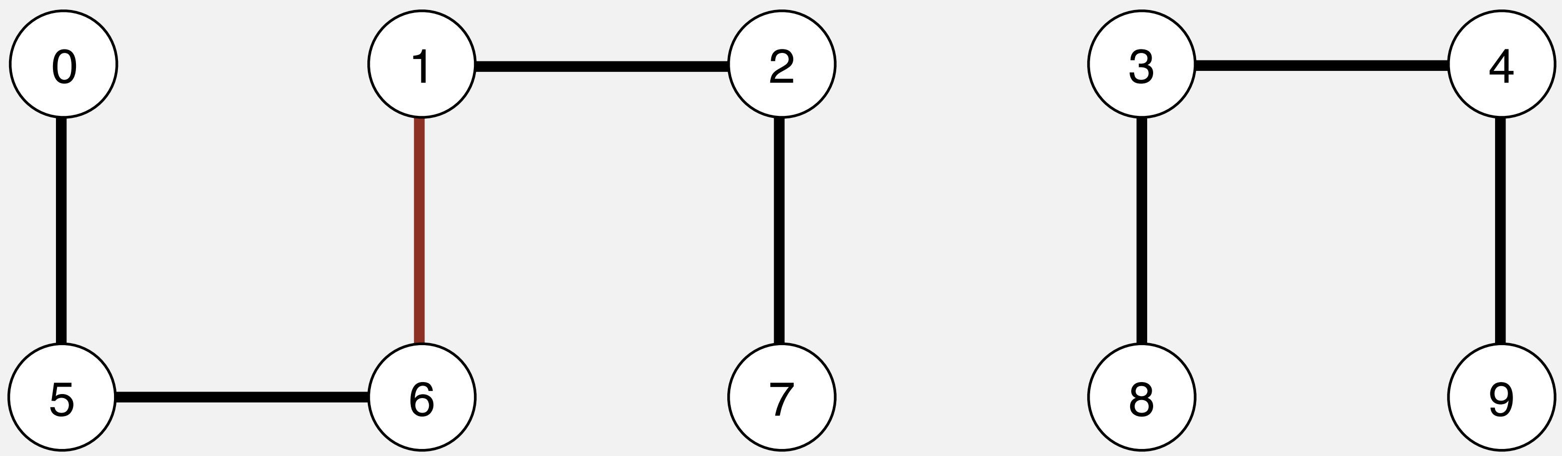
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Quick-find demo

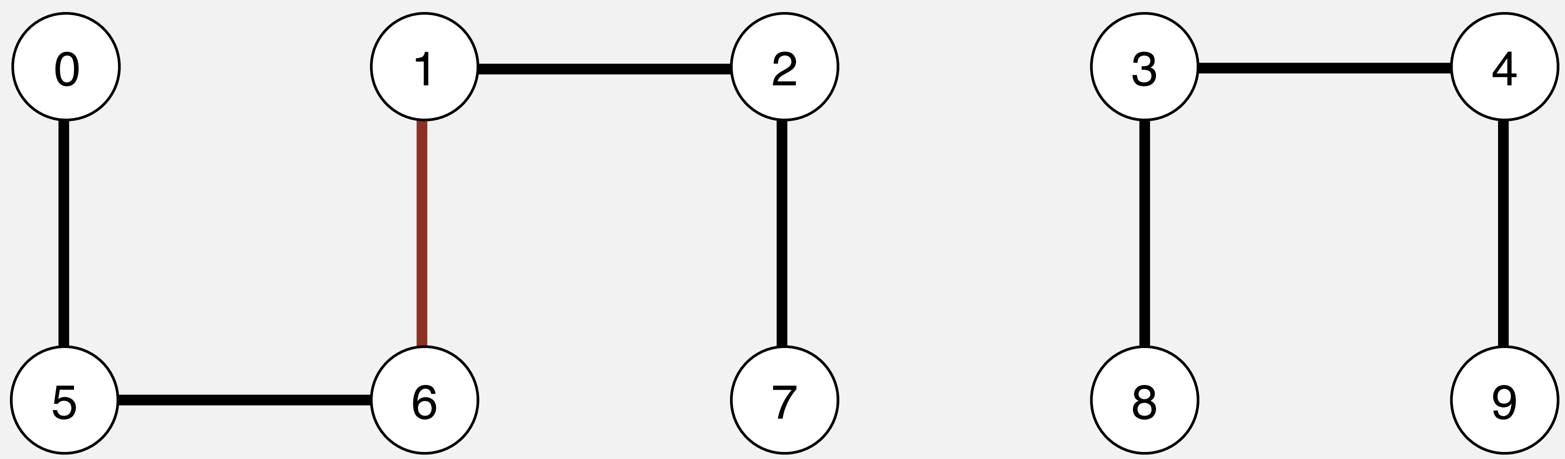
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Quick-find demo

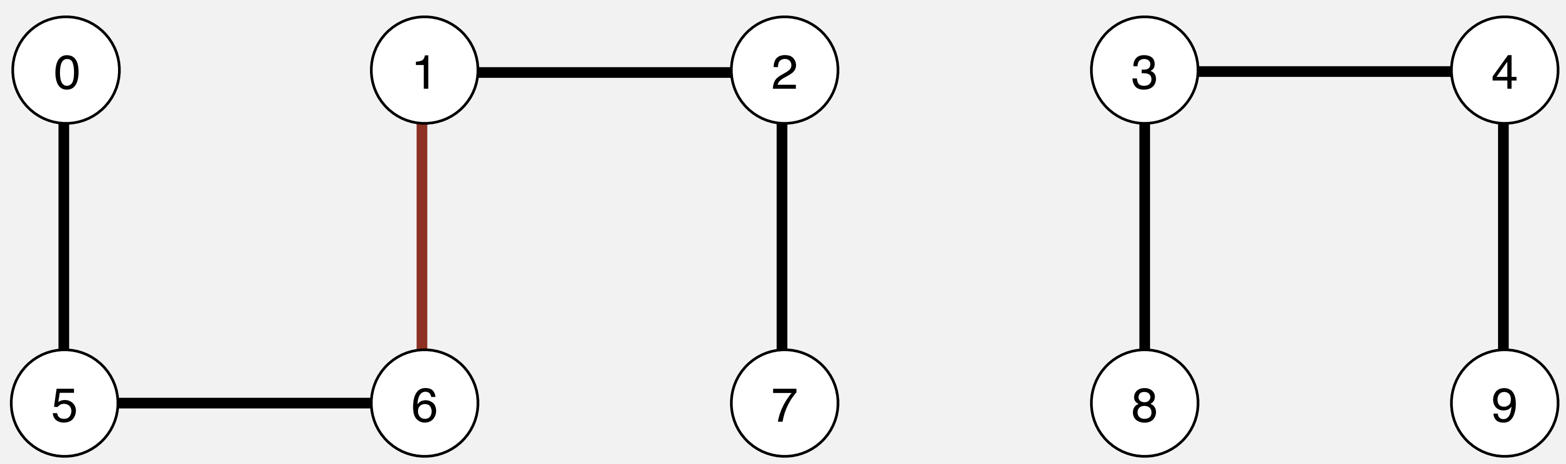
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	0	0	1	8	8

Quick-find demo

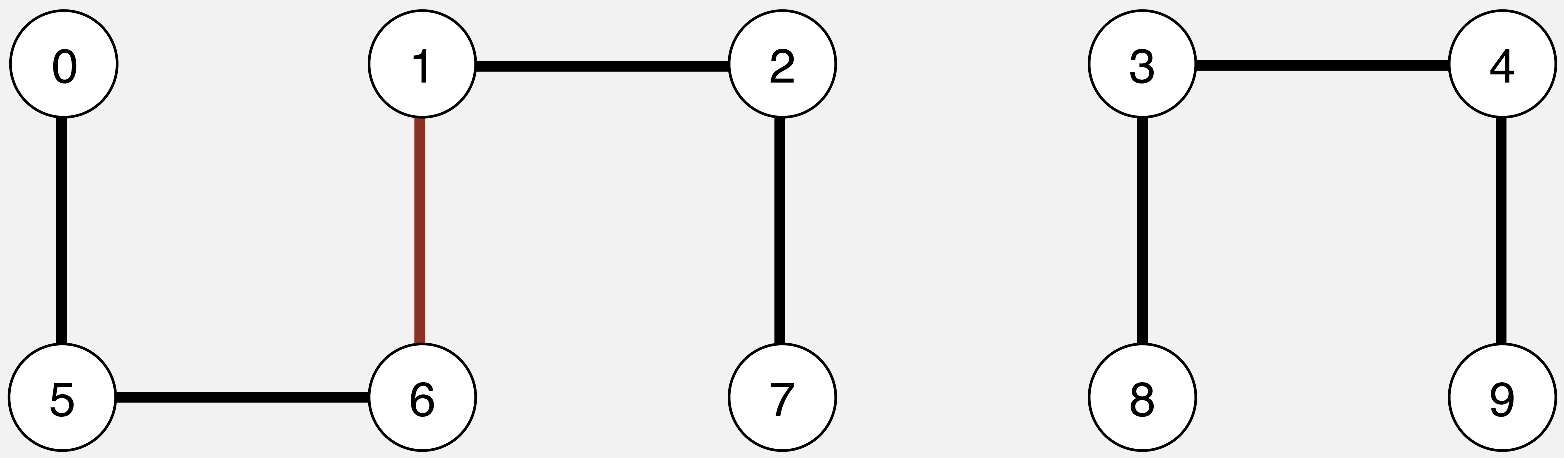
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	0	1	8	8

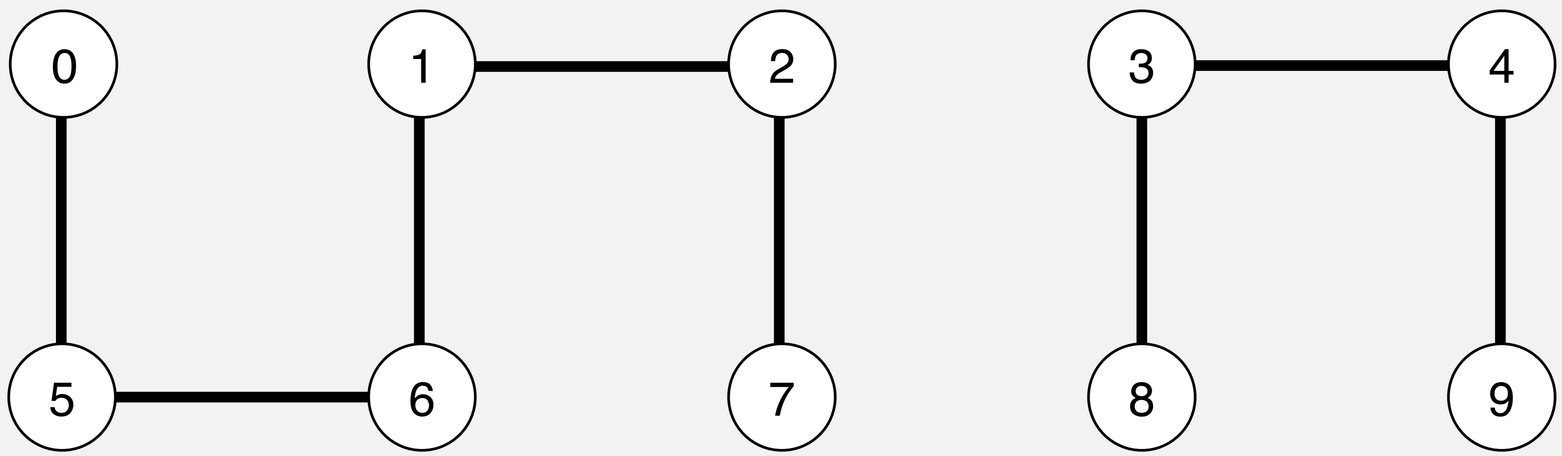
Quick-find demo

union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

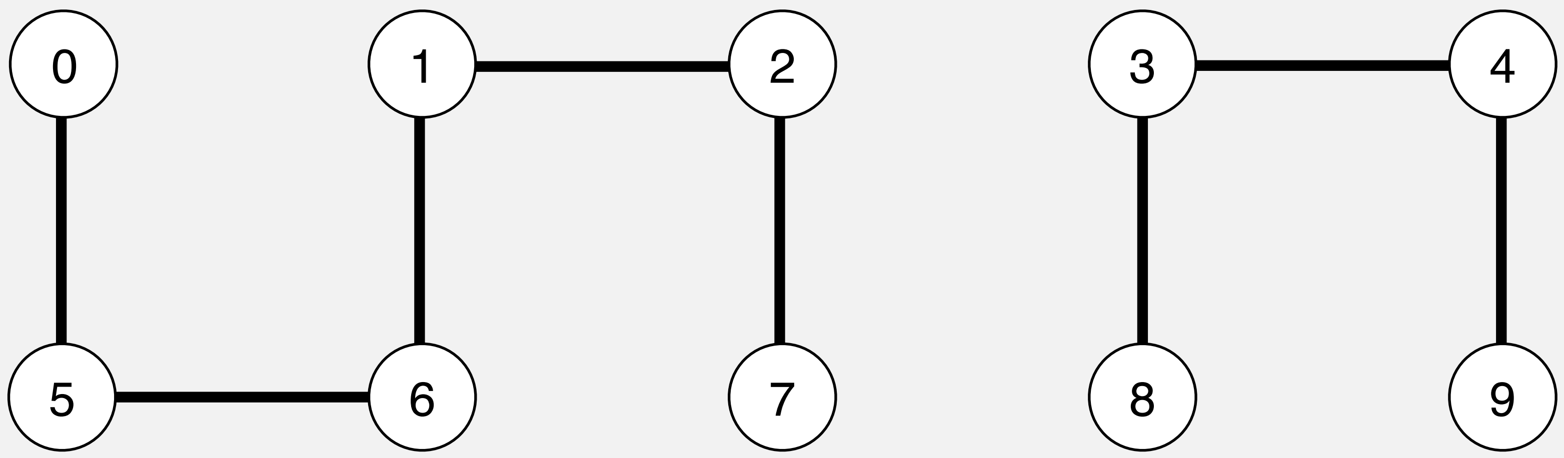
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

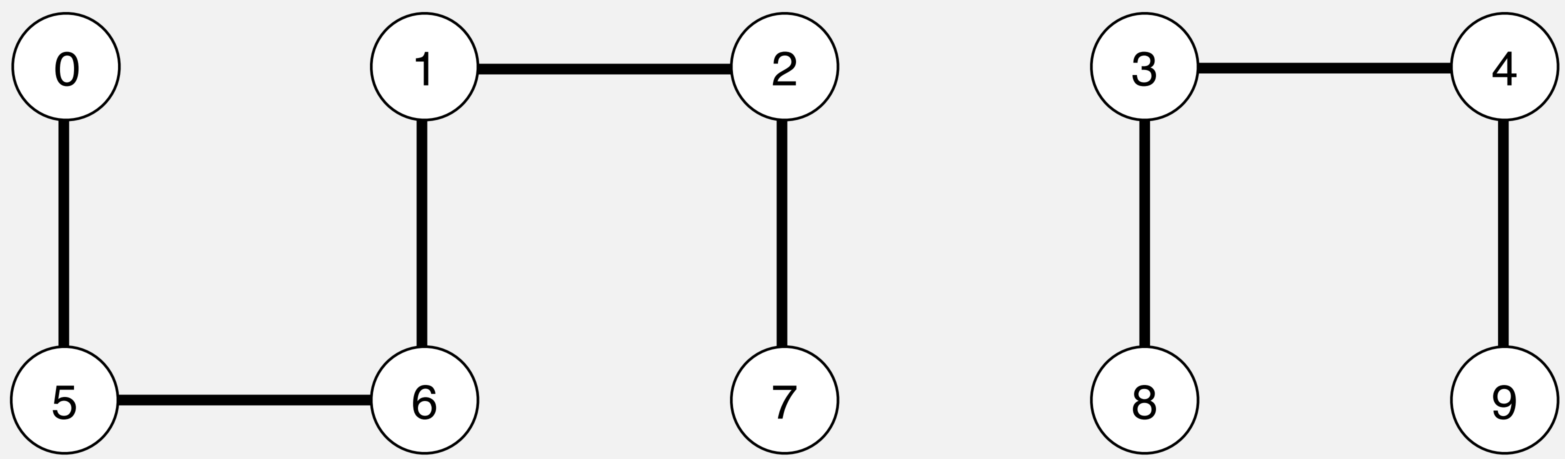
Quick-find demo

connected(1, 0)



Quick-find demo

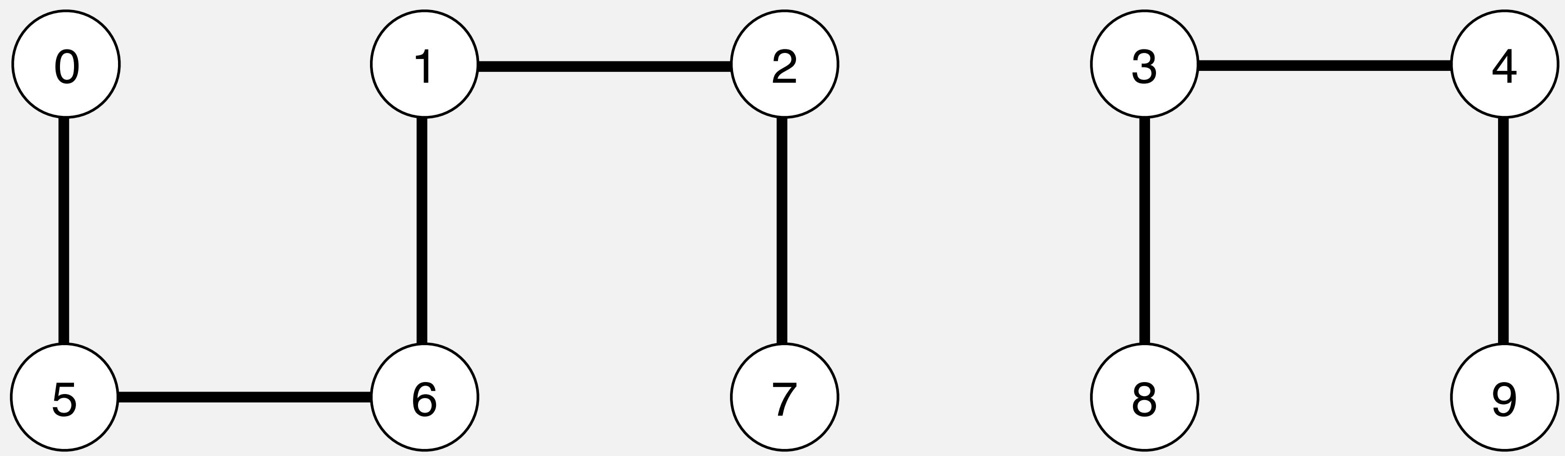
connected(1, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

already connected

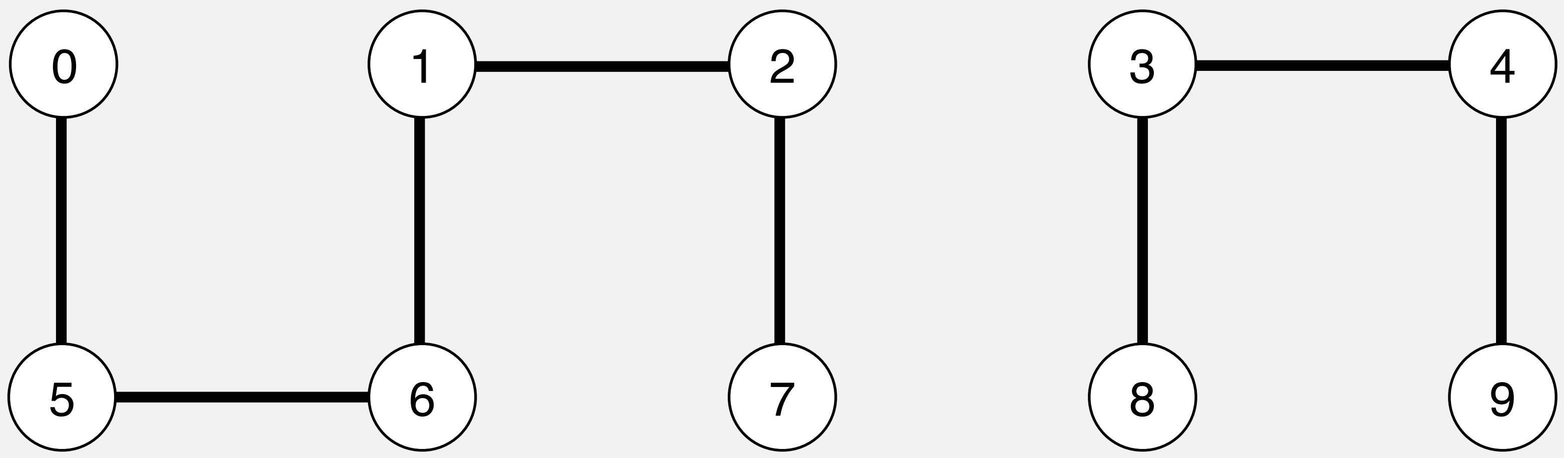
Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find demo

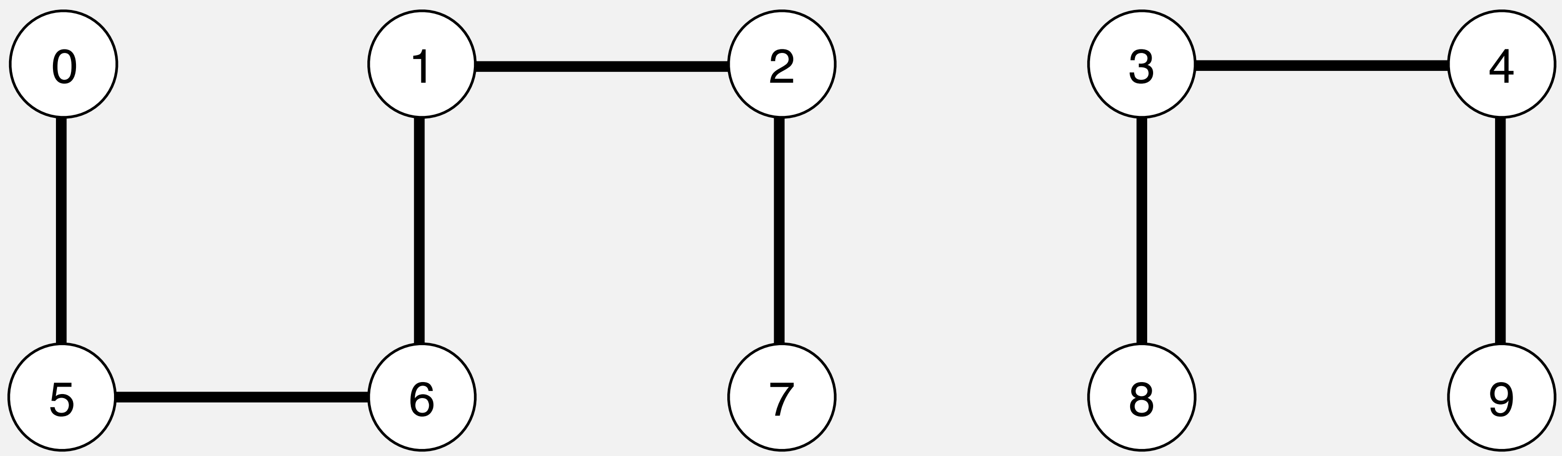
connected(6, 7)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find demo

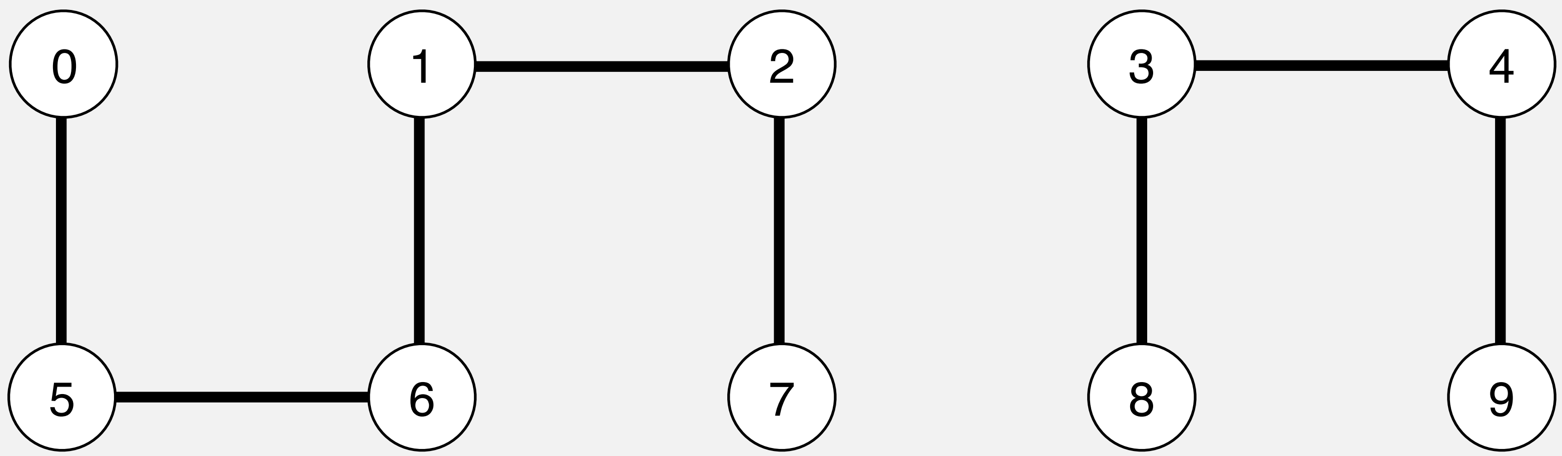
connected(6, 7)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

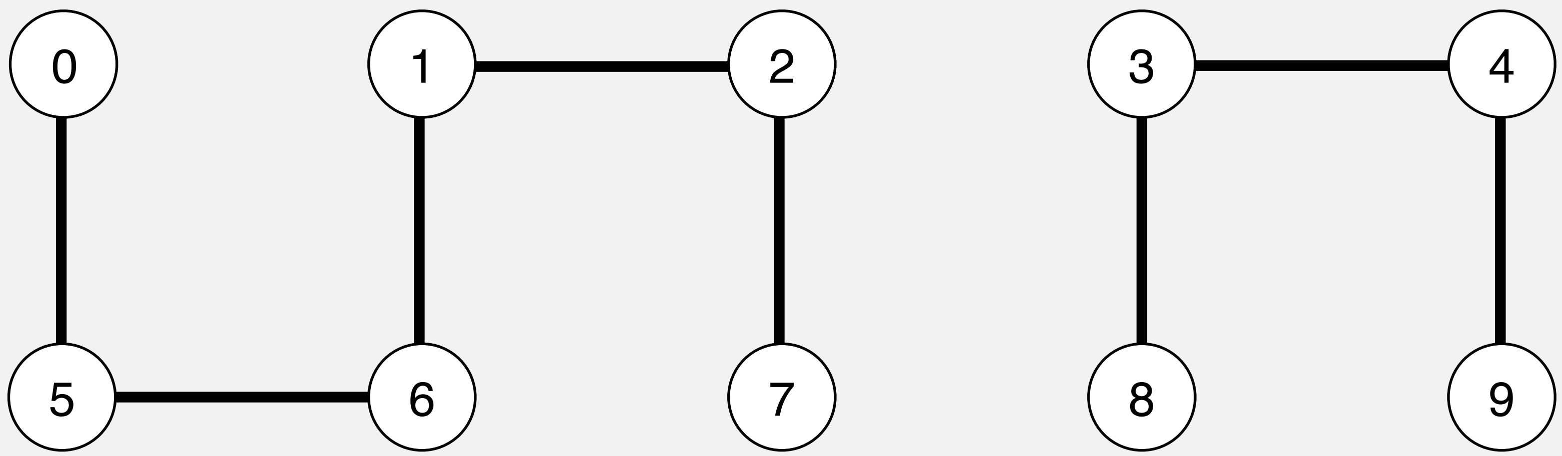
already connected

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find: Java implementation

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```
public class QuickFindUF {
   private int[] id;
```

Quick-find: Java implementation

```
public class QuickFindUF {
   private int[] id;

   public QuickFindUF(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }
```

set id of each object to itself

(N array accesses)

Quick-find: Java implementation

```
public class QuickFindUF {
   private int[] id;

   public QuickFindUF(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }

   public boolean find(int p)
   {  return id[p];  }
```

set id of each object to itself
(N array accesses)

return the id of p
(1 array access)

Quick-find: Java implementation

```
public class QuickFindUF {
   private int[] id;

   public QuickFindUF(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }

   public boolean find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

return the id of p
(1 array access)

change all entries with id[p] to id[q]
(at most 2N + 2 array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1

order of growth of number of array accesses

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1

order of growth of number of array accesses

quadratic

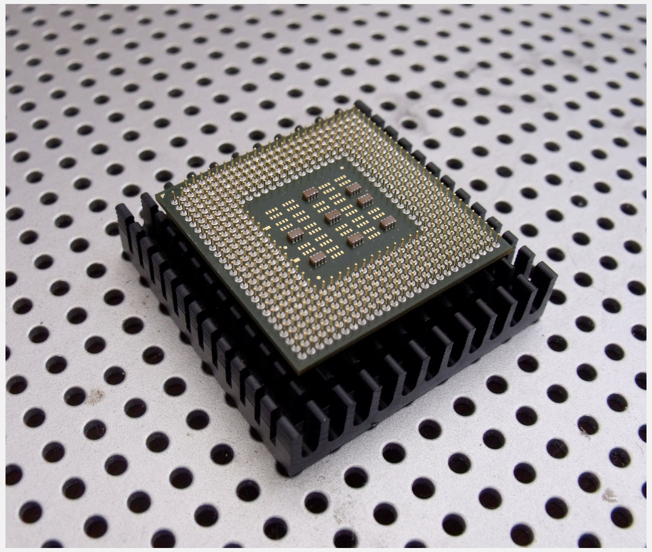
Union is too expensive. It takes N^2 array accesses to process a sequence of N union operations on N objects.

Quadratic algorithms do not scale

Rough standard (for now).

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950!



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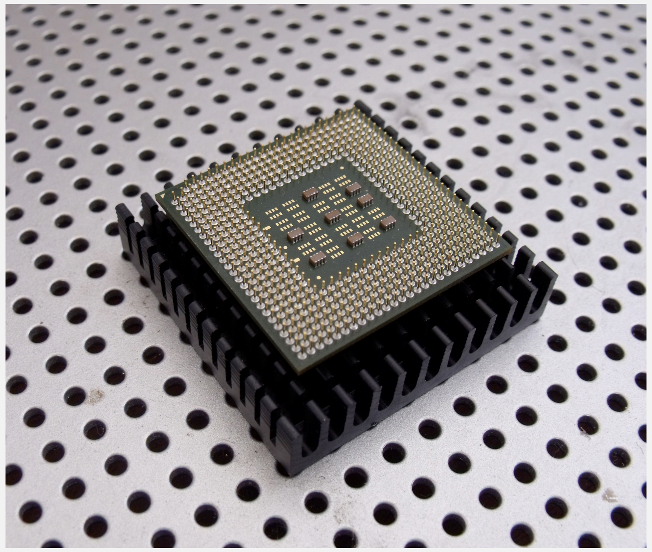

Ex. Huge problem for quick-find.

- 10^9 union commands on 10^9 objects.
- Quick-find takes more than 10^{18} operations.
- 30+ years of computer time!

Quadratic algorithms do not scale

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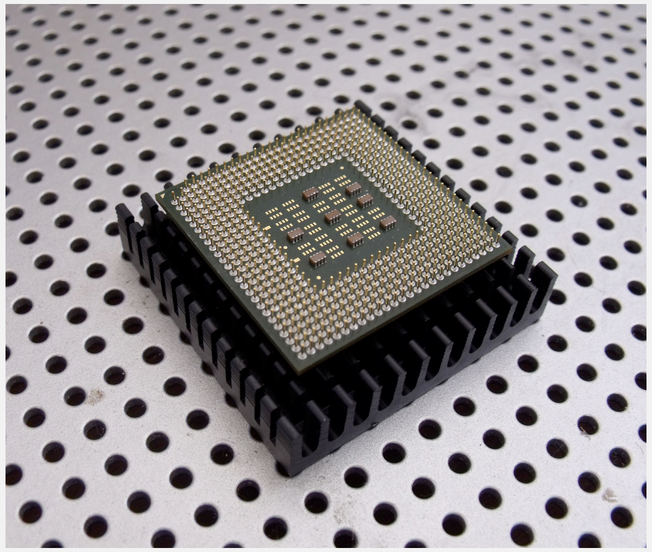
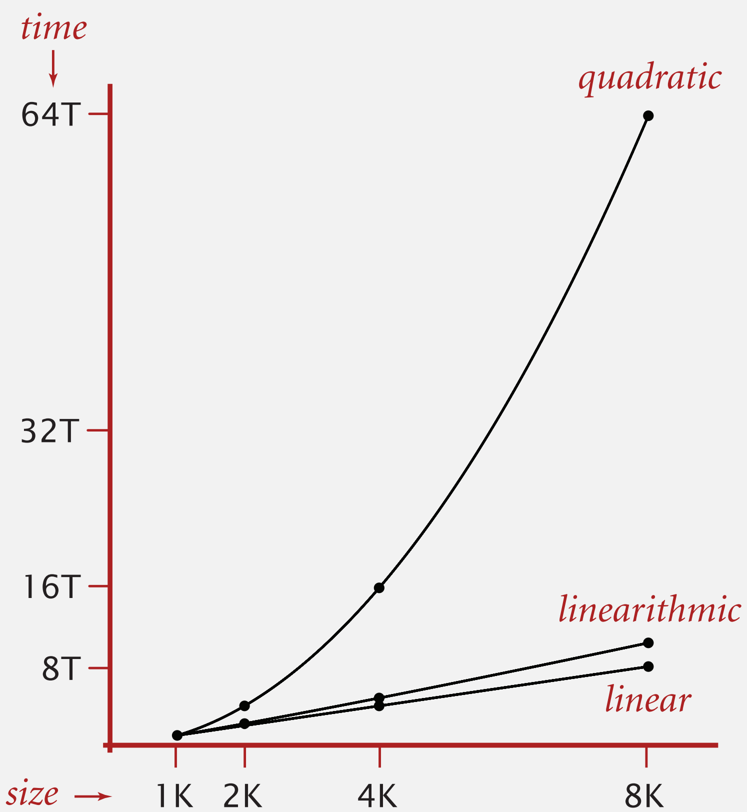


Ex. Huge problem for quick-find.

- 10^9 union commands on 10^9 objects.
- Quick-find takes more than 10^{18} operations.
- 30+ years of computer time!

Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory ⇒ want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!



http://algs4.cs.princeton.edu

1.5 Union-Find

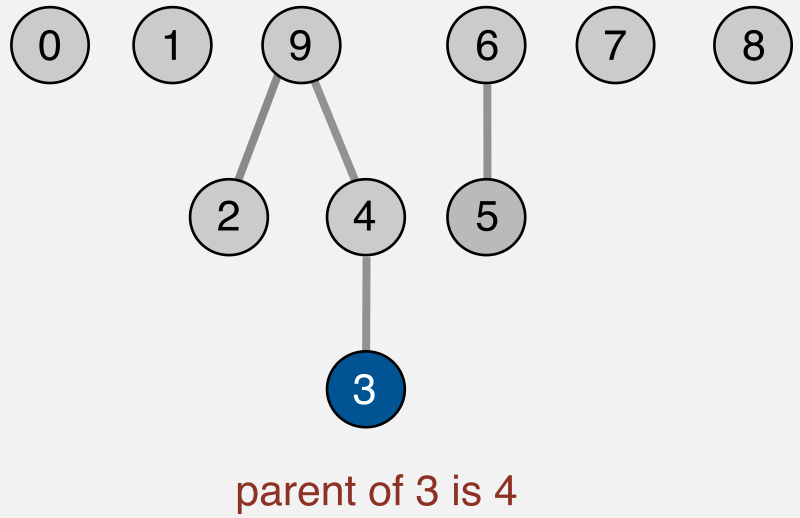
- dynamic connectivity
- quick find
- quick union
- improvements
- applications

Quick-union [lazy approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



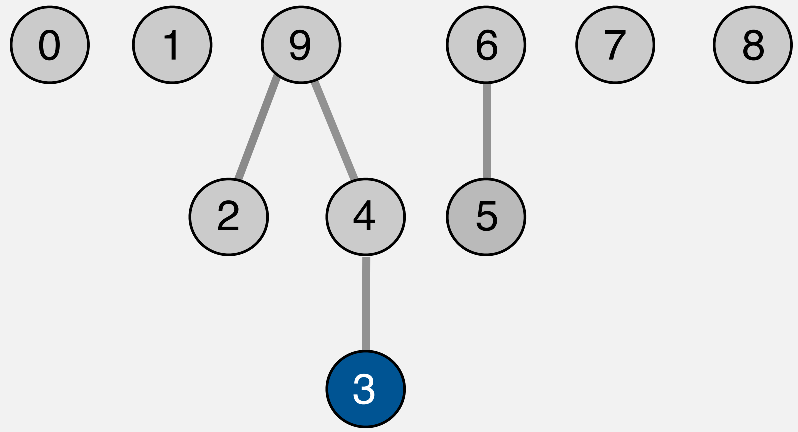
Quick-union [lazy approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

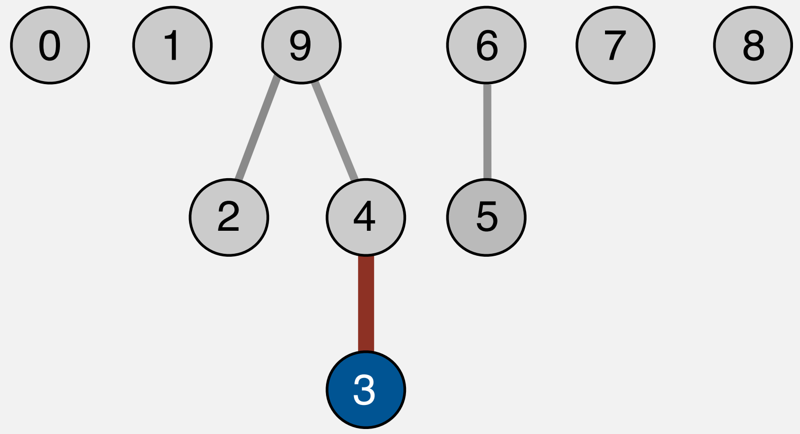
Quick-union [lazy approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

Quick-union [lazy approach]

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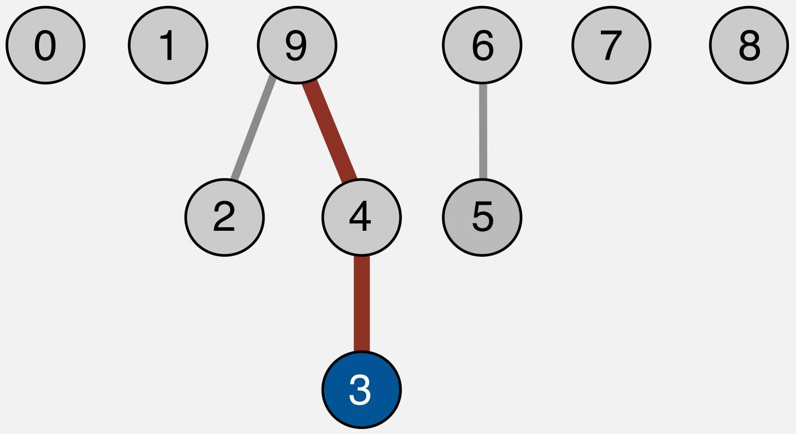
- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

Quick-union [lazy approach]

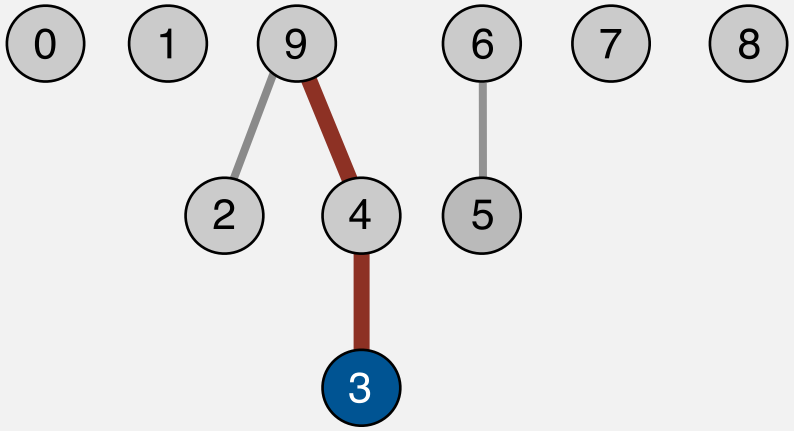
Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

keep going until it doesn't change

(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

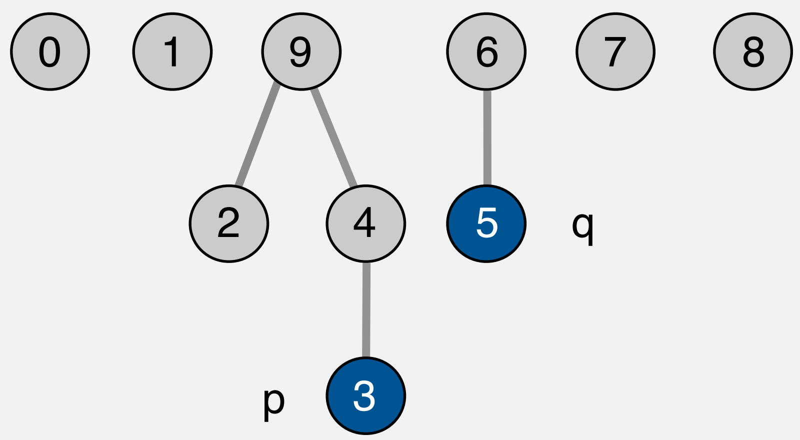
root of 3 is 9

Quick-union [lazy approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



root of 3 is 9

root of 5 is 6

3 and 5 are not connected

Find. What is the root of p?

Connected. Do p and q have the same root?

Quick-union [lazy approach]

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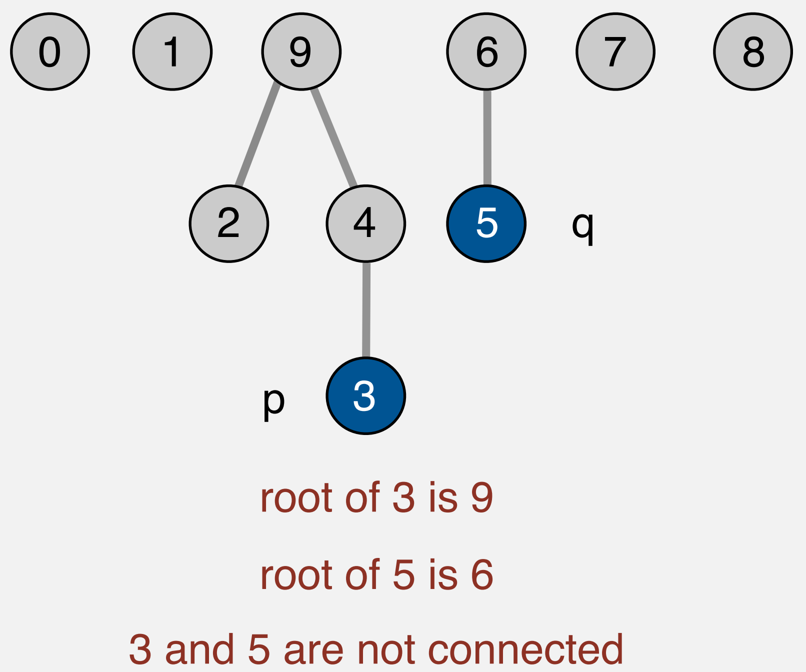
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



Find. What is the root of p?

Connected. Do p and q have the same root?

Quick-union [lazy approach]

Data structure.

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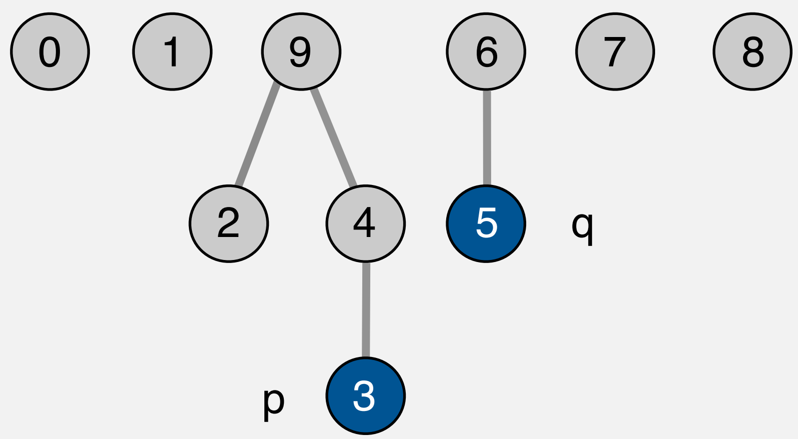
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



root of 3 is 9

root of 5 is 6

3 and 5 are not connected

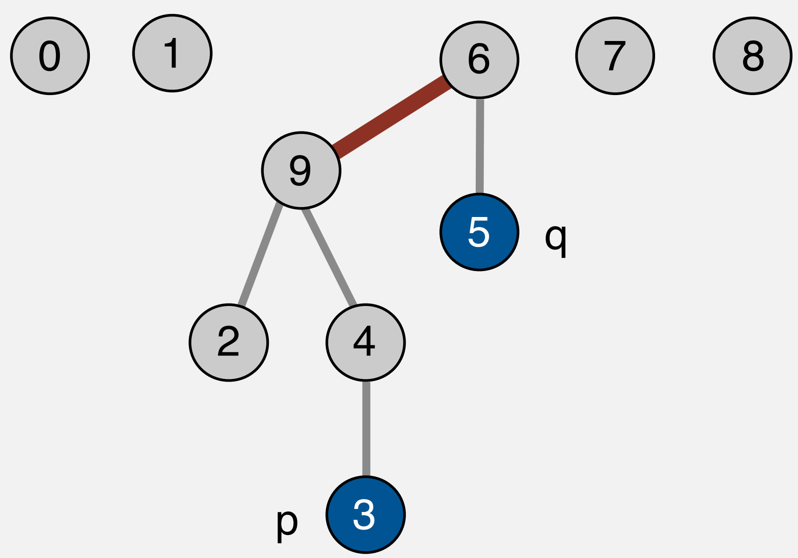
Find. What is the root of p?

Connected. Do p and q have the same root?

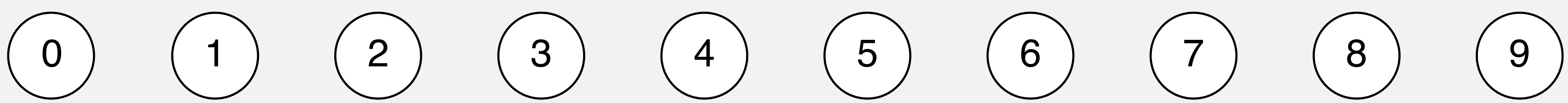
Union. To merge components containing p and q, set the id of p's root to the id of q's root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



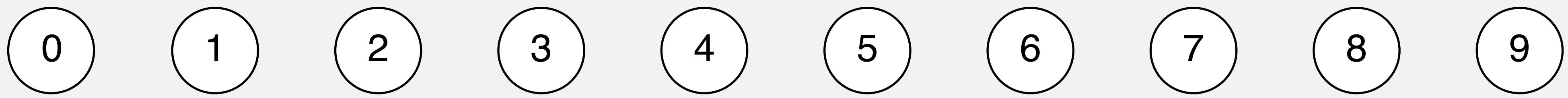
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-union demo

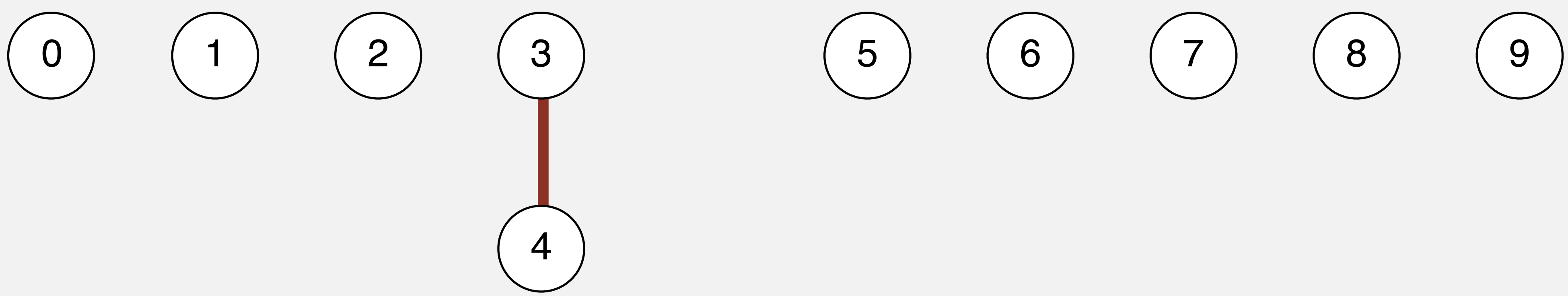
union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

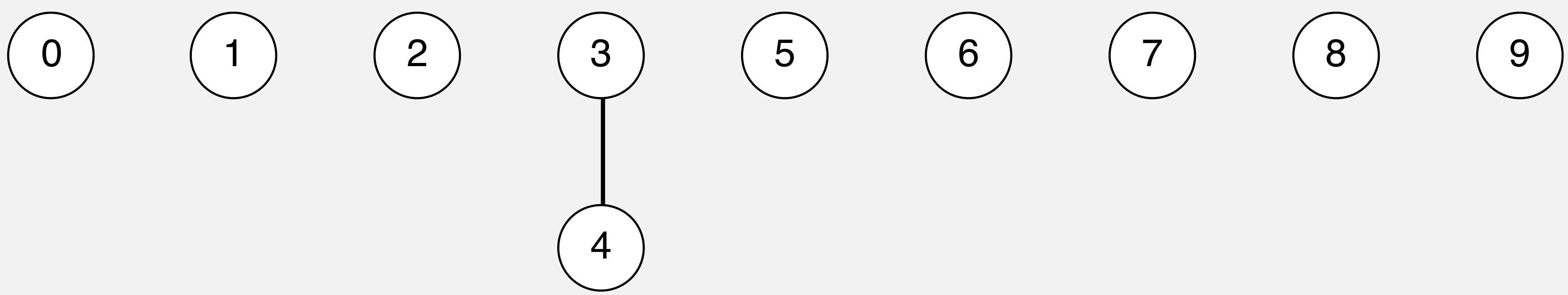
Quick-union demo

union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

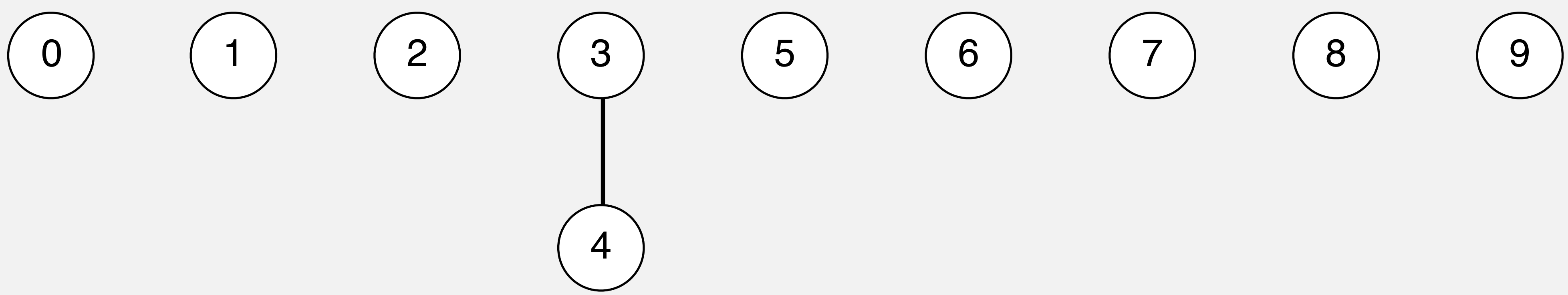
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

Quick-union demo

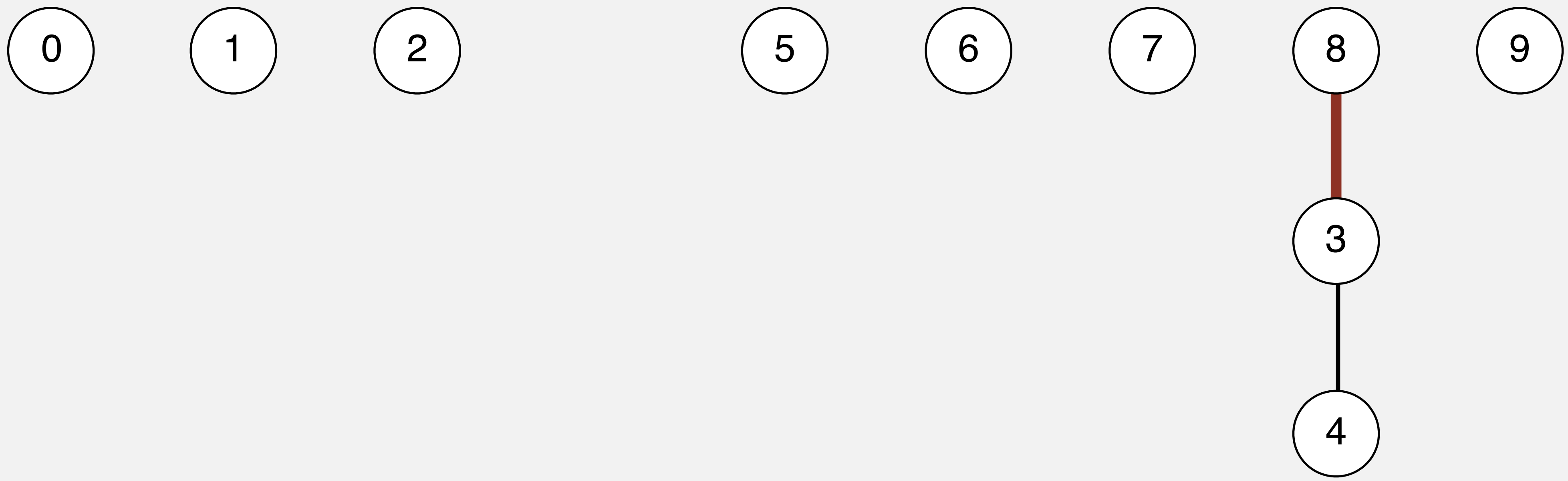
union(3, 8)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

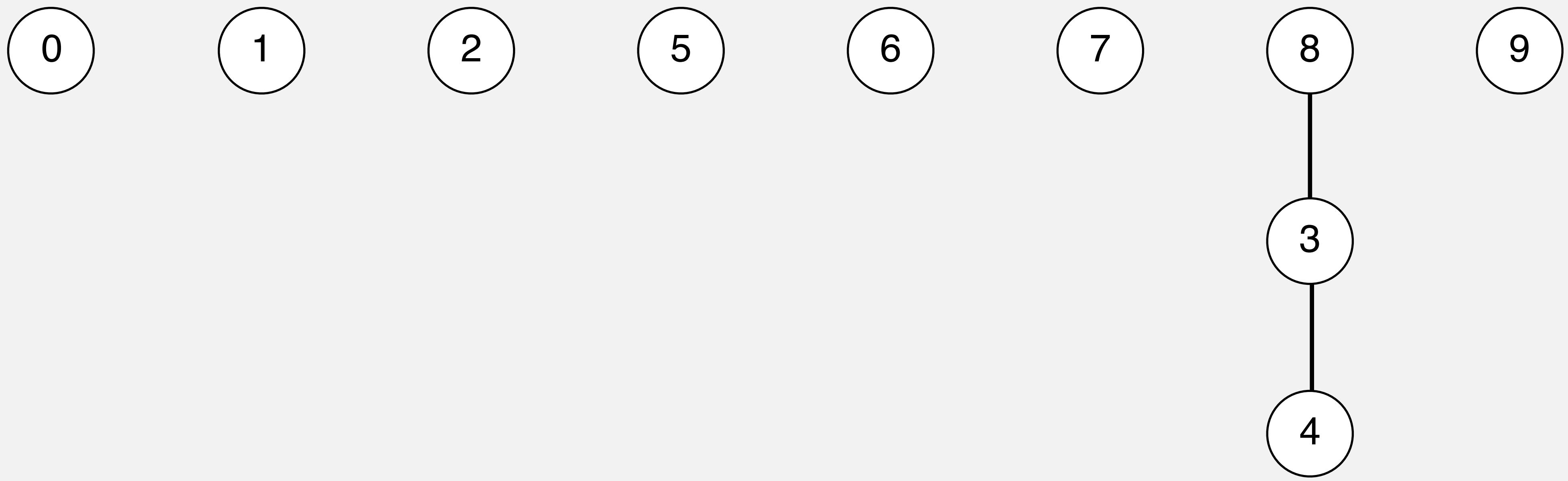
Quick-union demo

union(3, 8)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	6	7	8	9

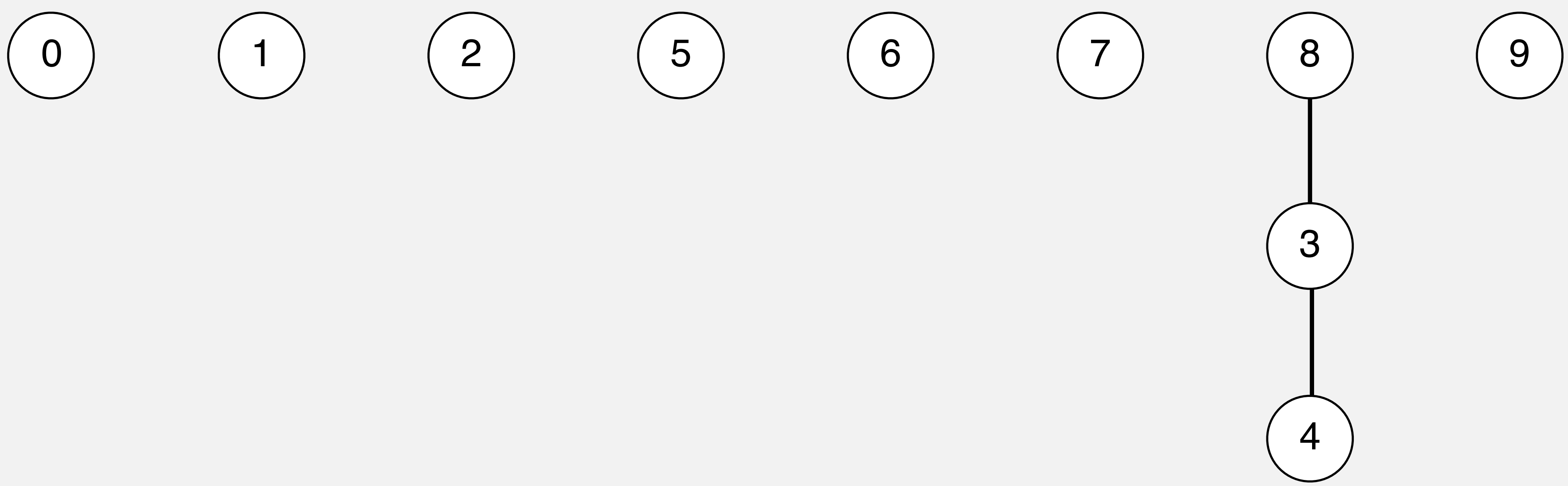
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	6	7	8	9

Quick-union demo

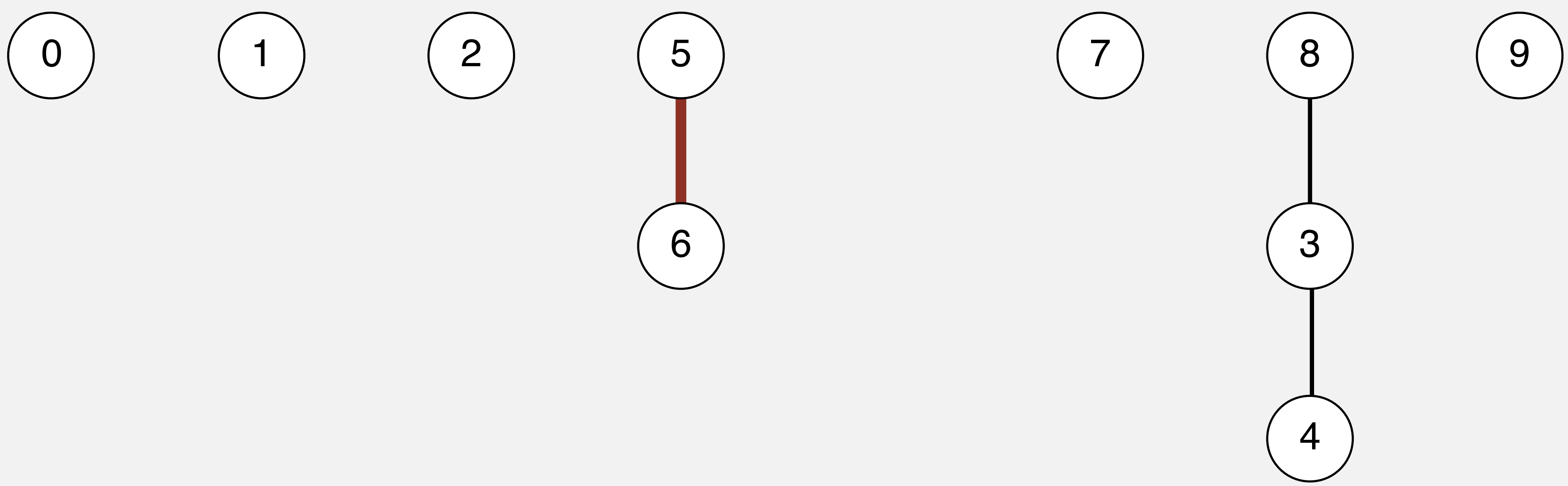
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	6	7	8	9

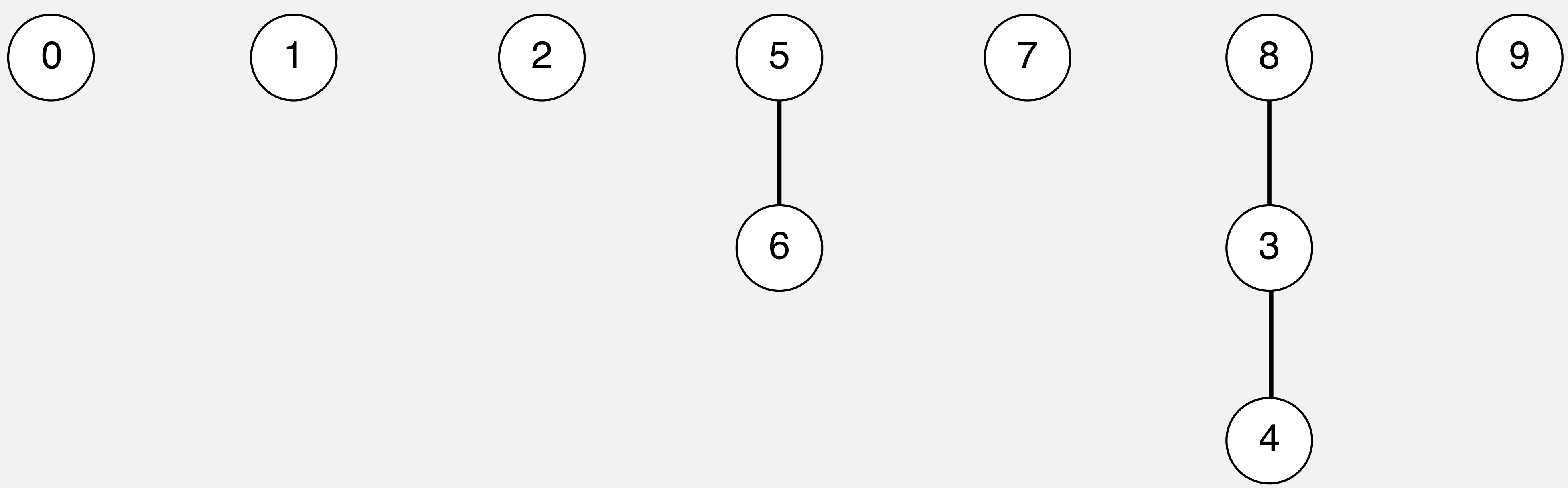
Quick-union demo

union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	9

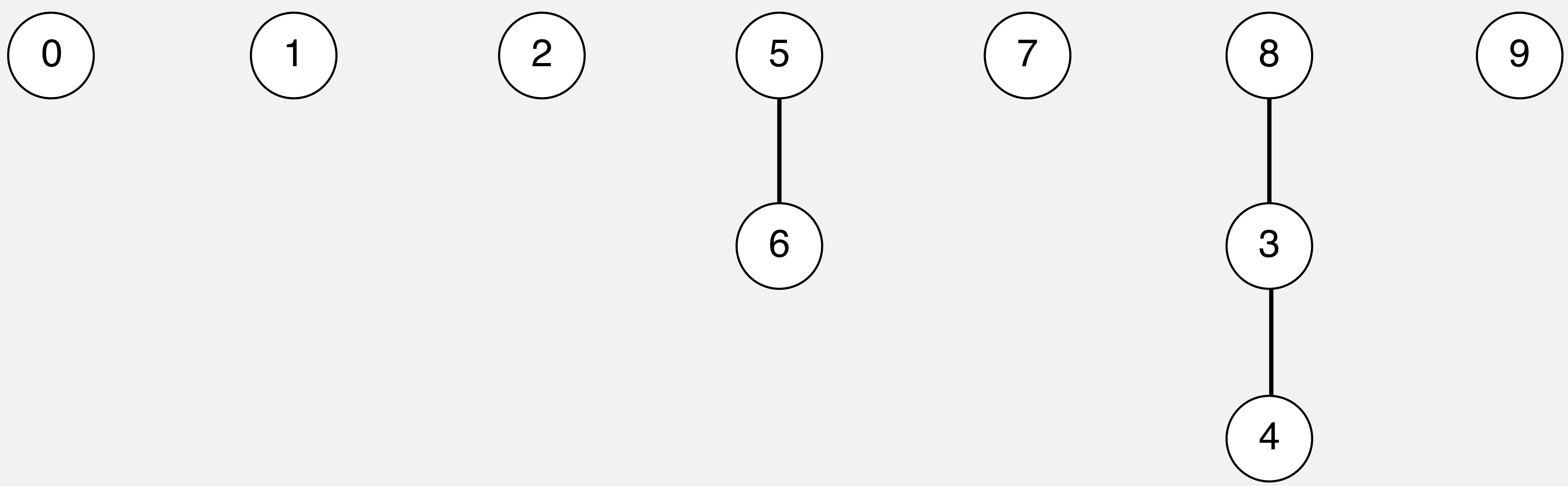
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	9

Quick-union demo

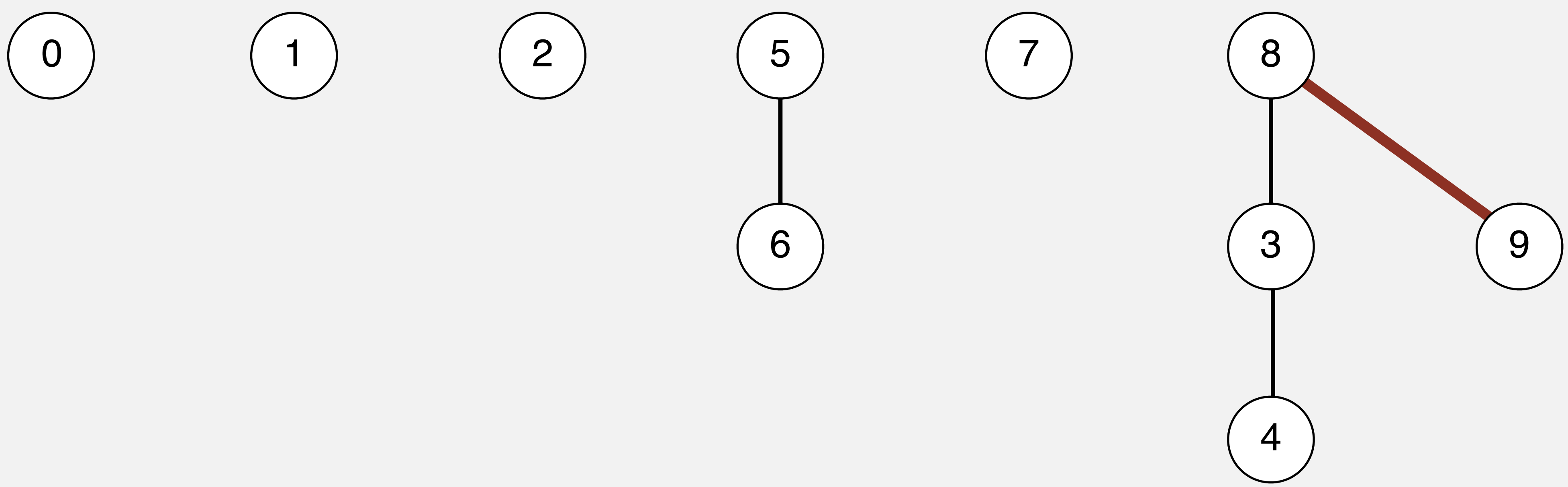
union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	9

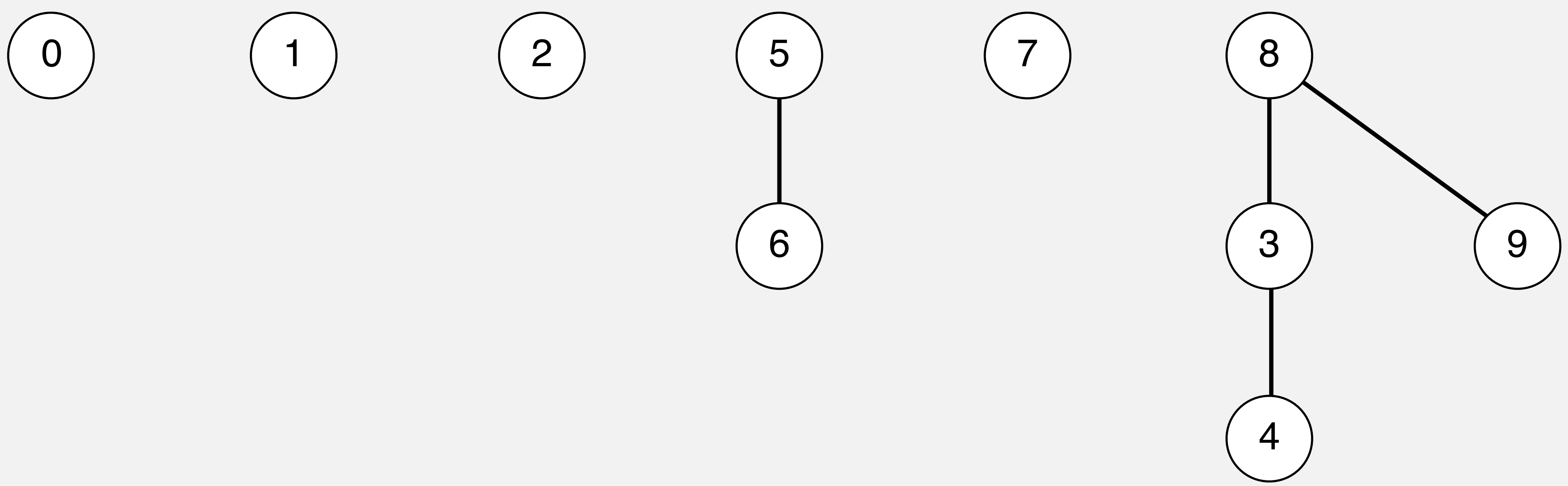
Quick-union demo

union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	8

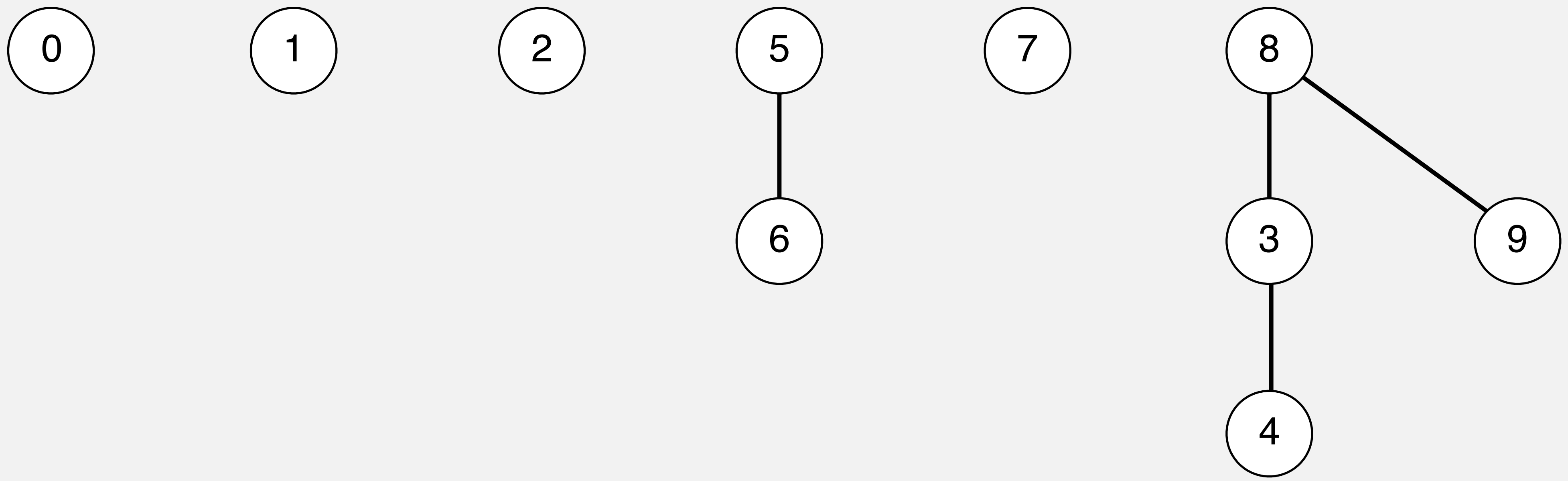
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	8

Quick-union demo

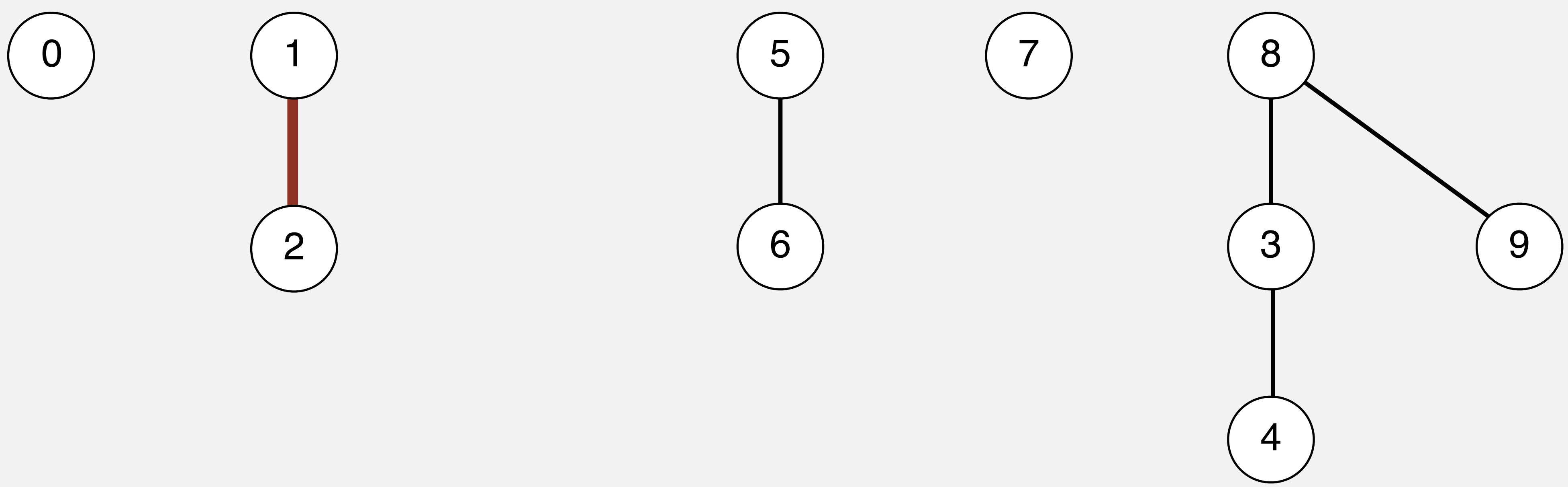
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	8

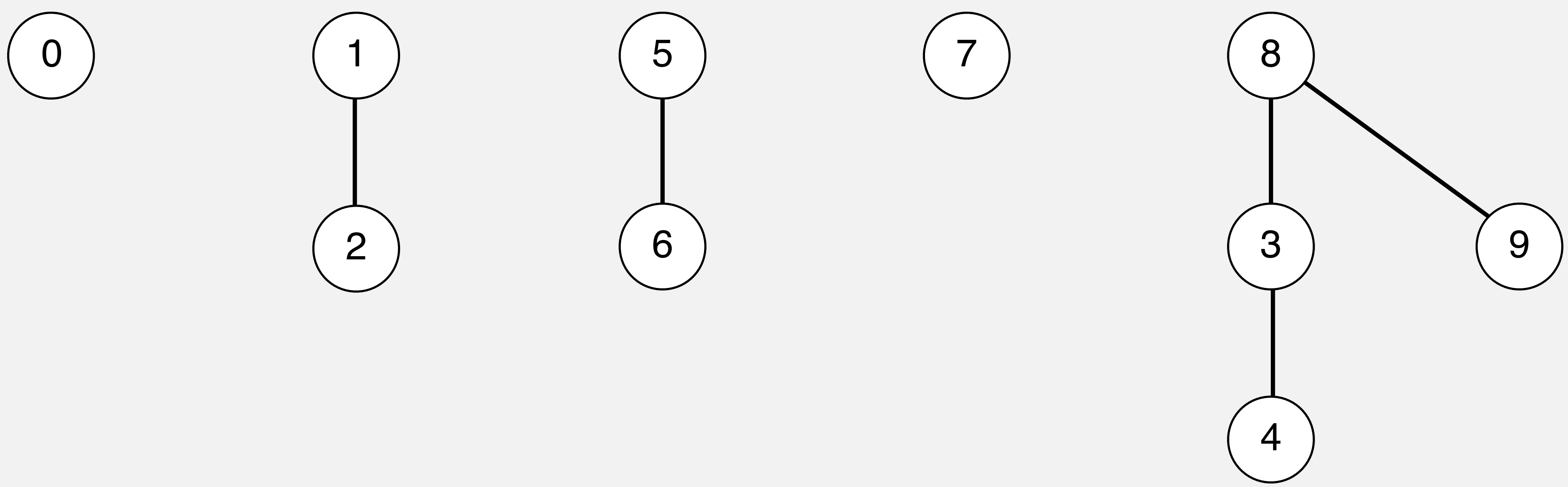
Quick-union demo

union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

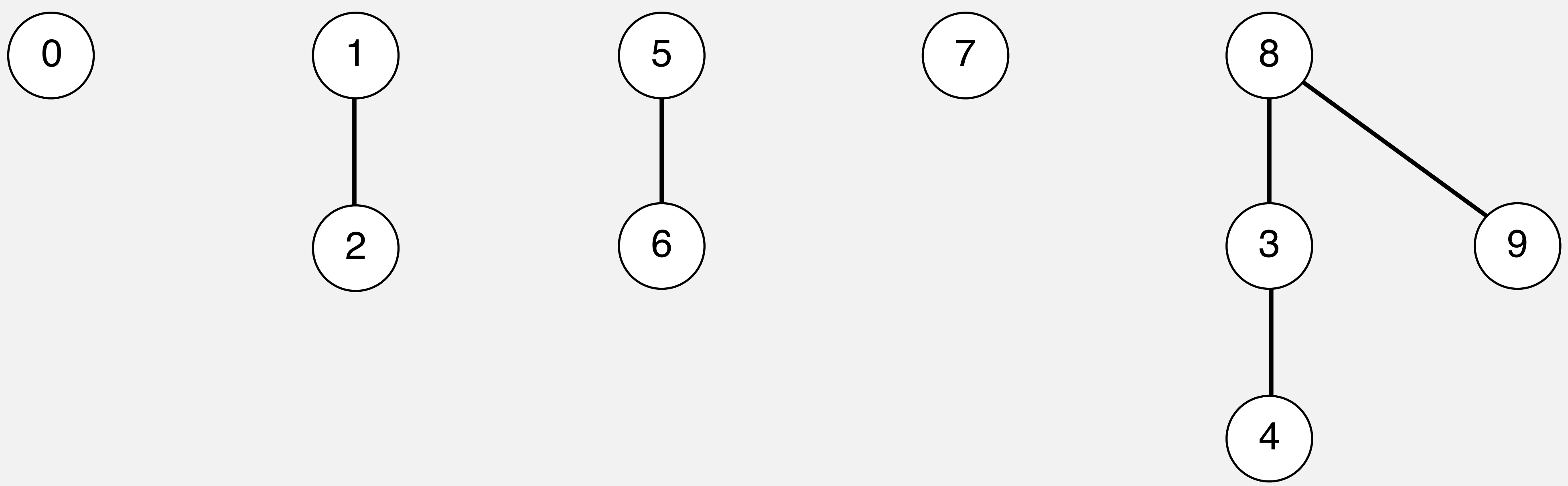
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

Quick-union demo

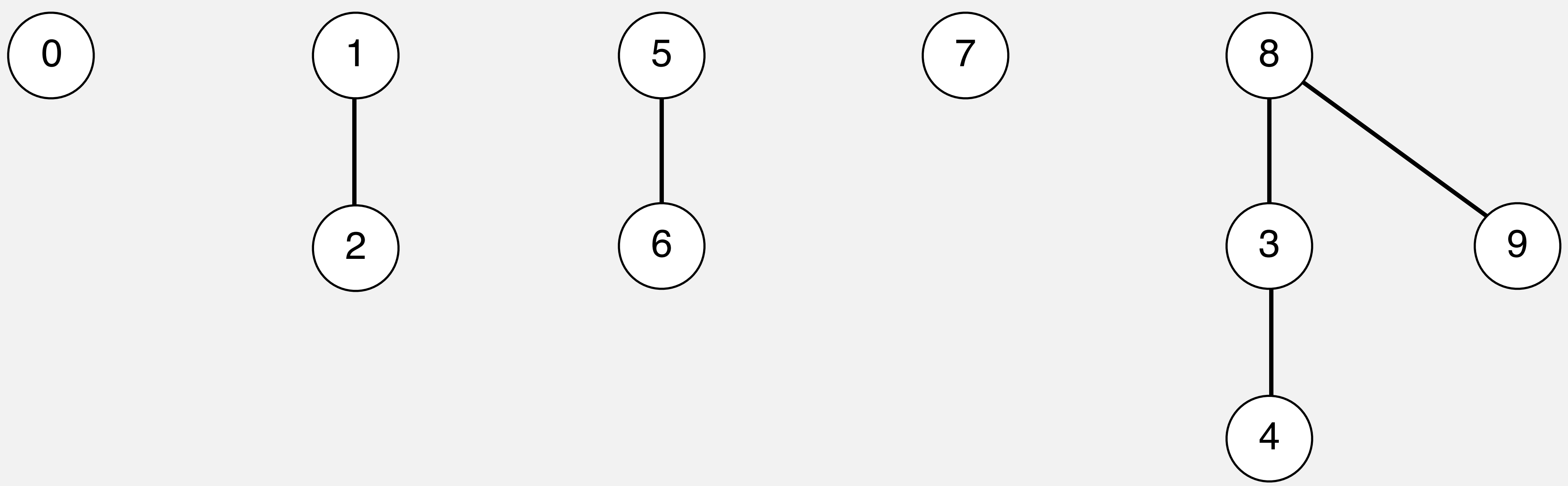
connected(8, 9) ✔



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

Quick-union demo

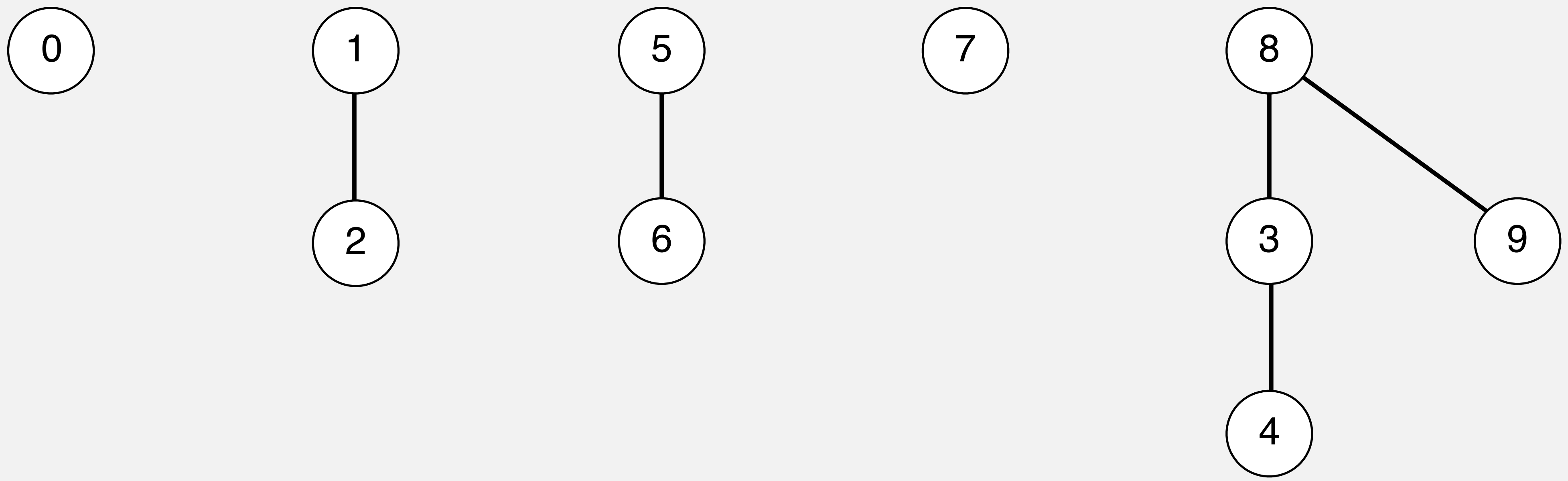
connected(5, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

Quick-union demo

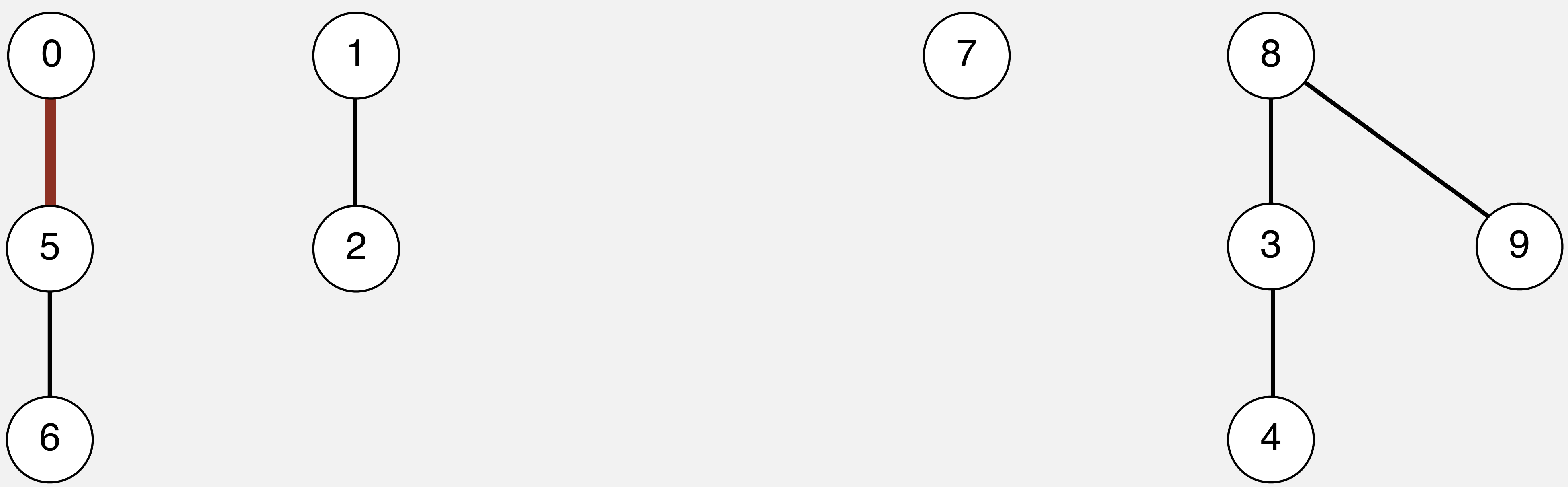
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

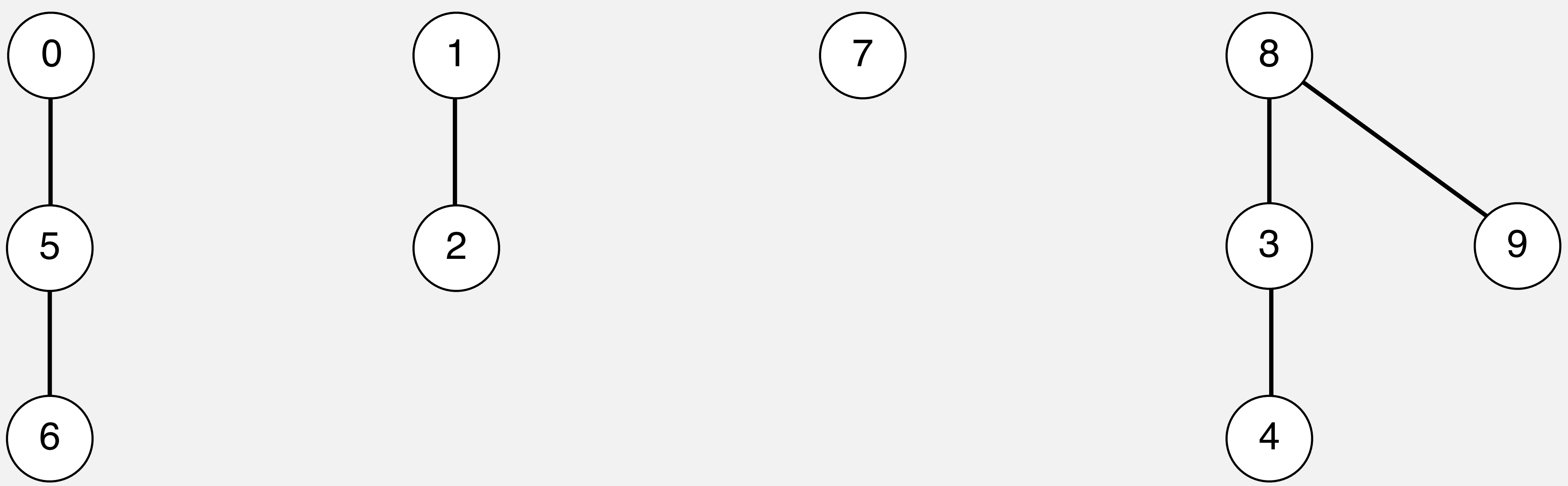
Quick-union demo

union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	7	8	8

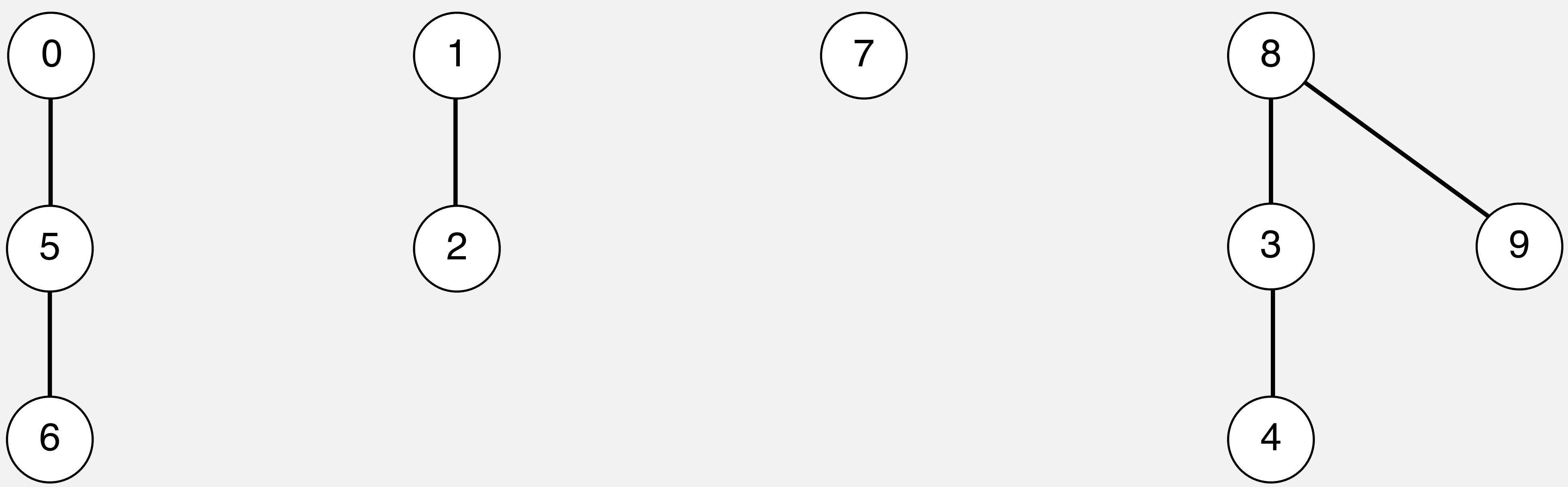
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	7	8	8

Quick-union demo

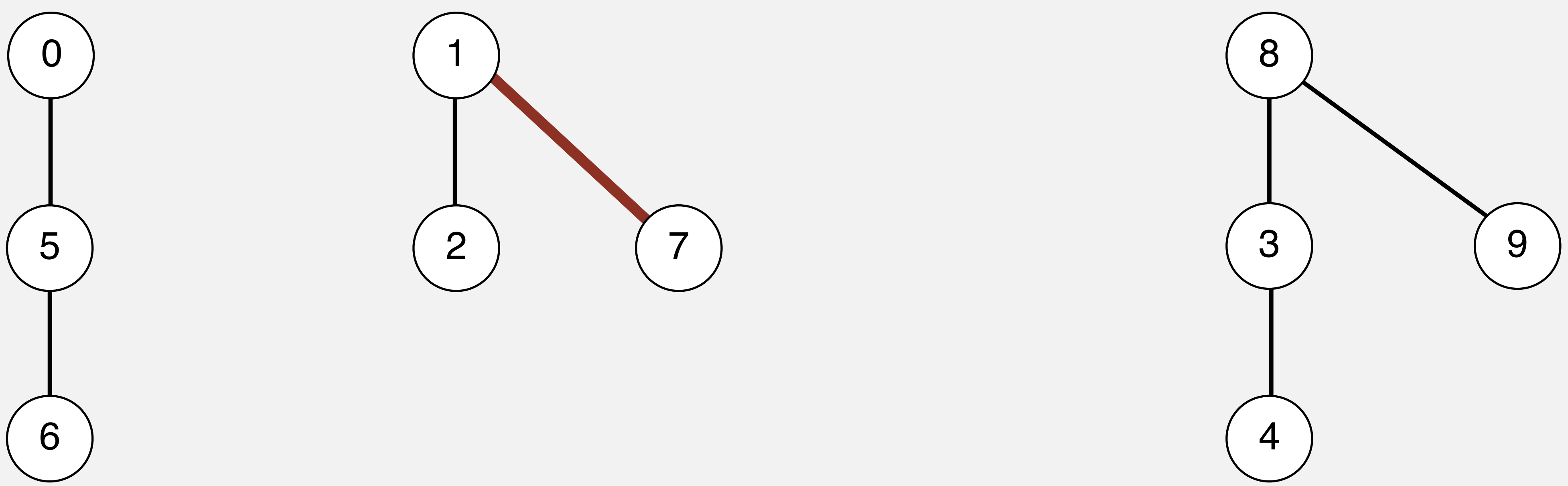
union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	7	8	8

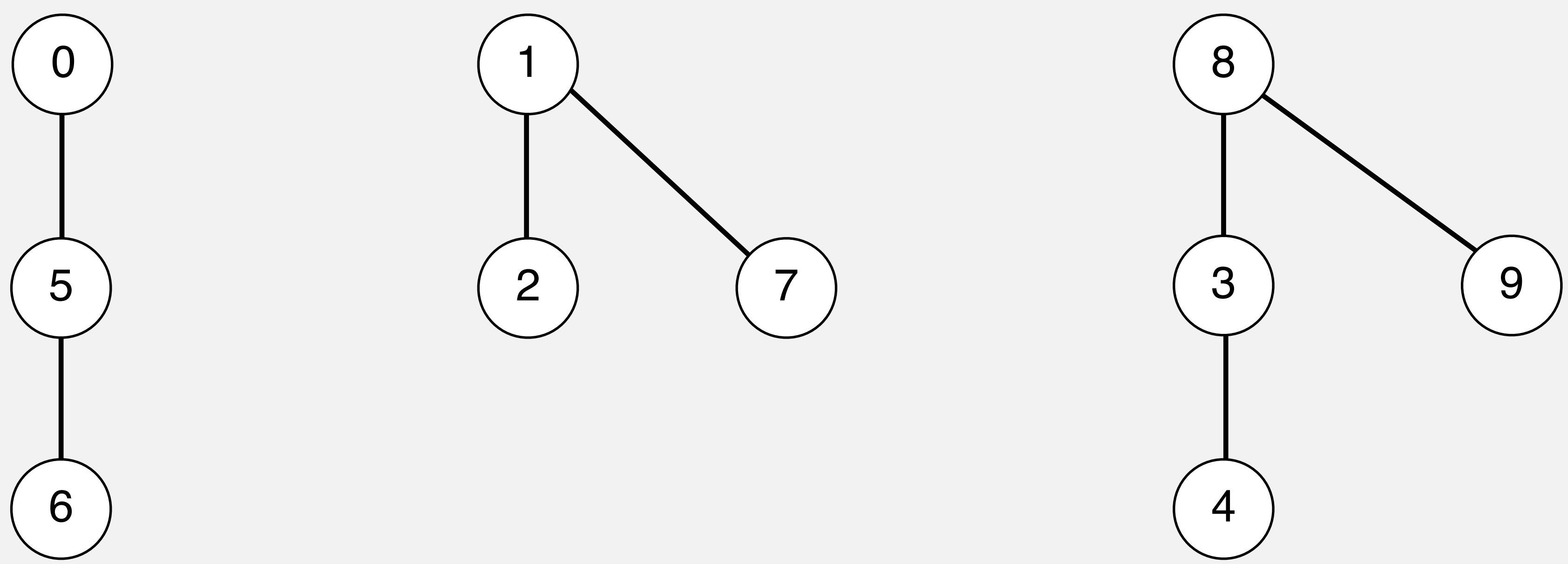
Quick-union demo

union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	1	8	8

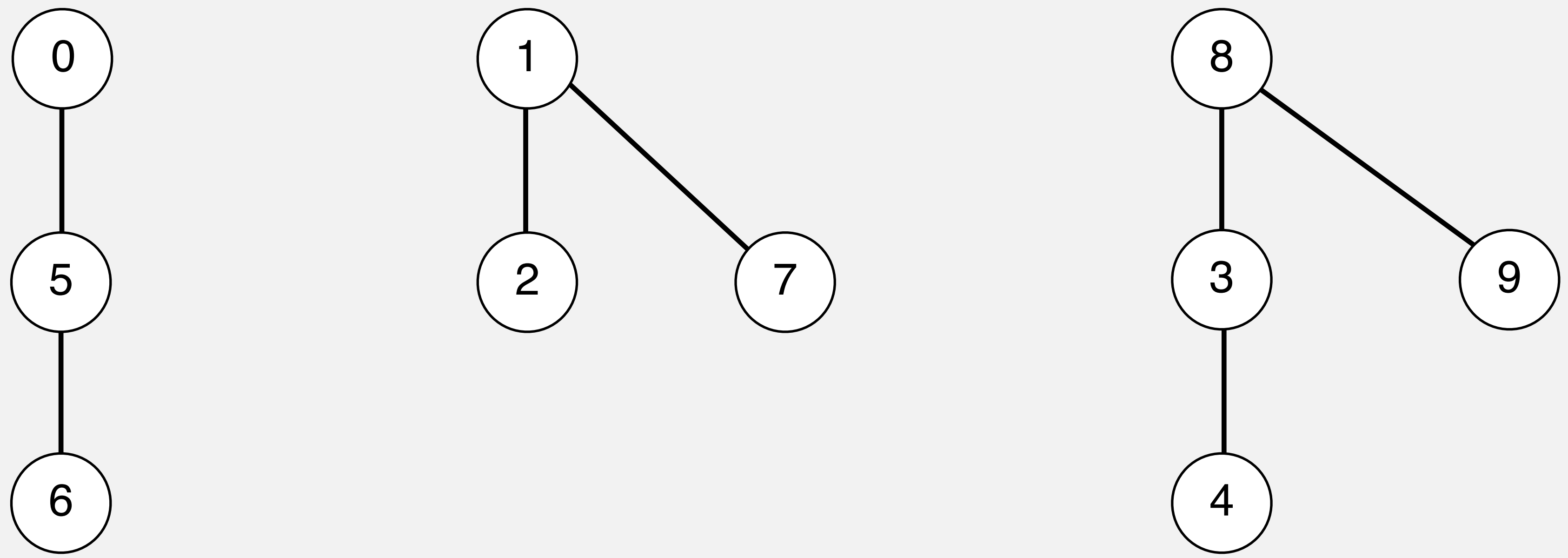
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	1	8	8

Quick-union demo

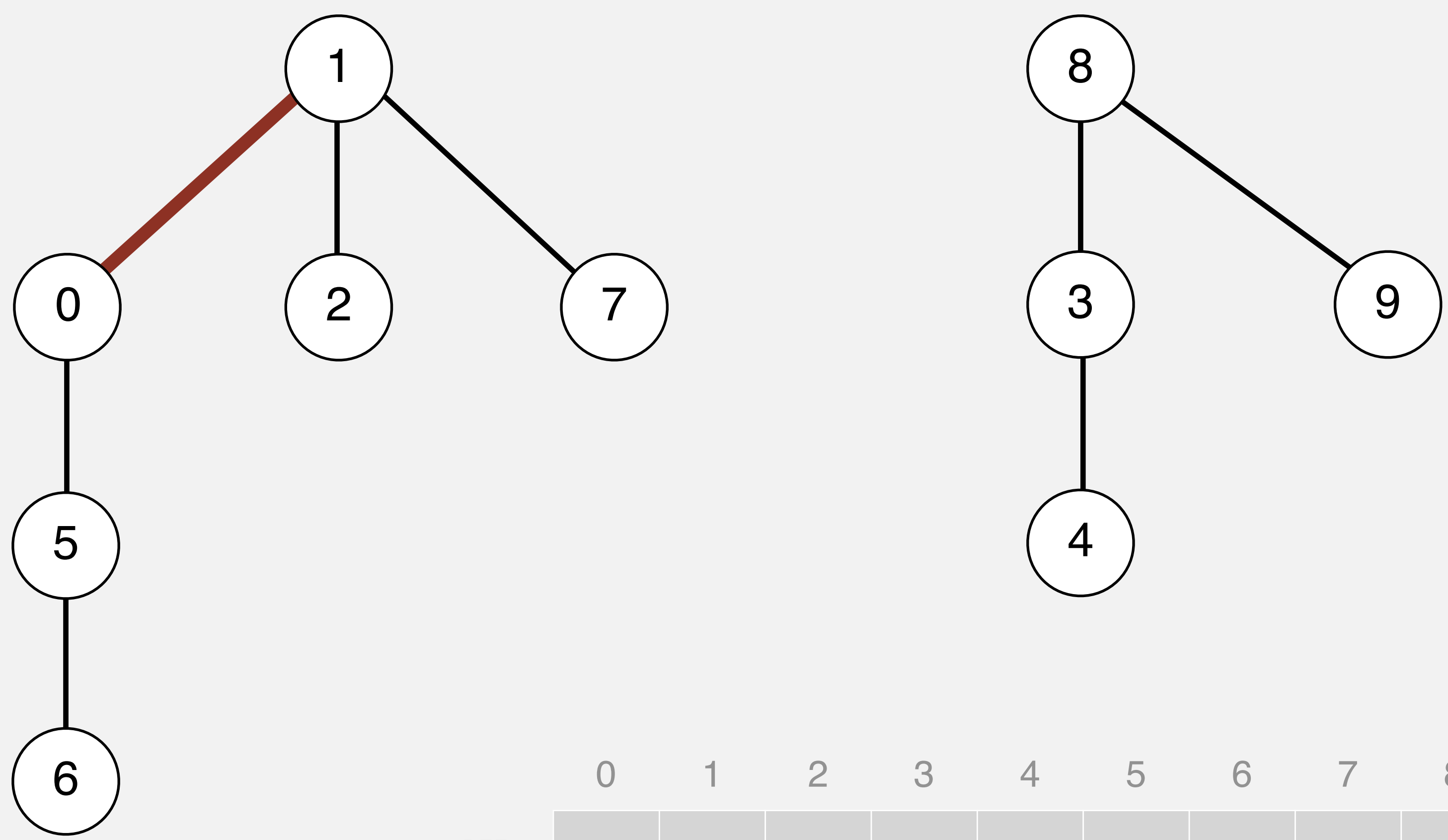
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	1	8	8

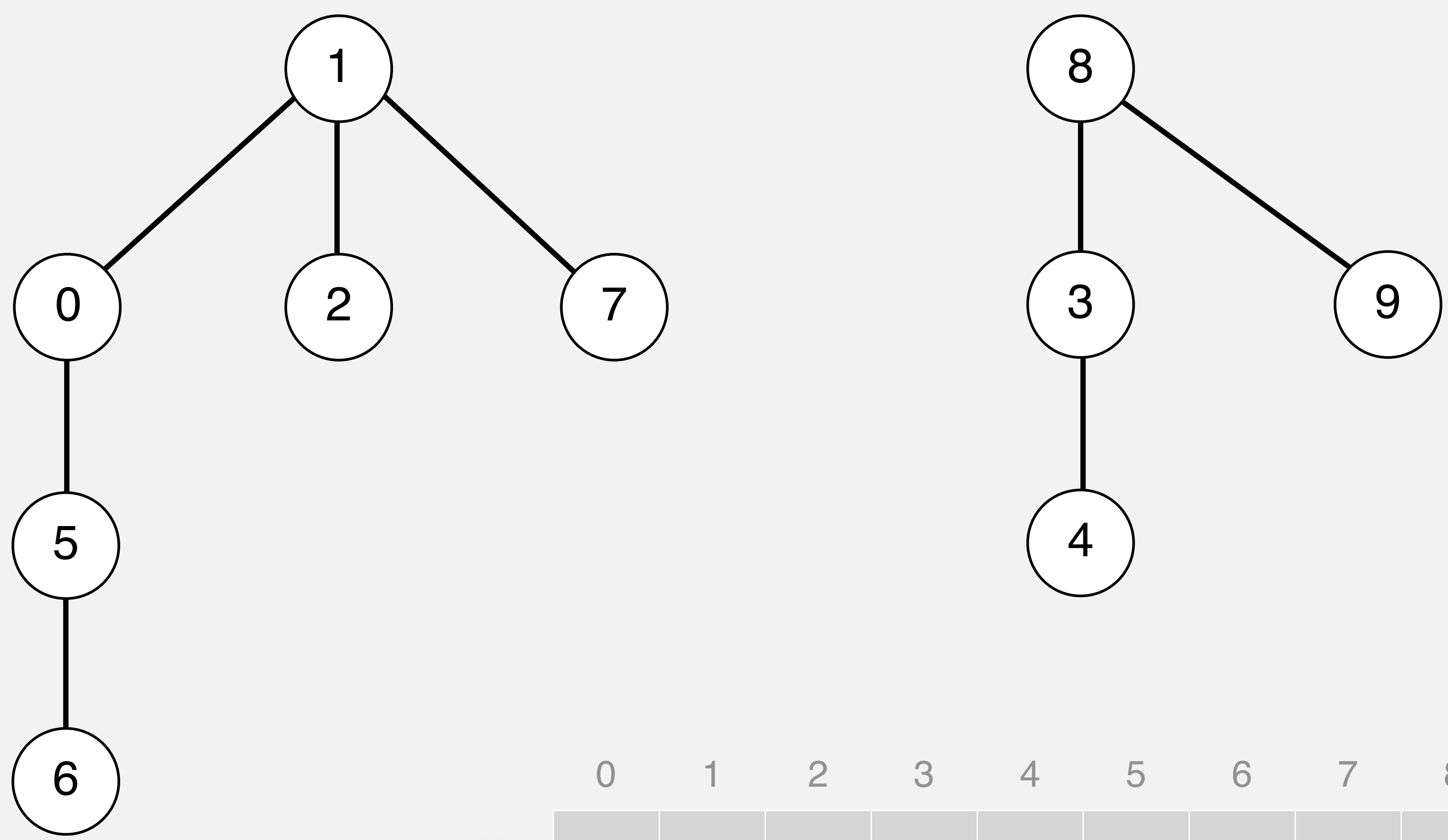
Quick-union demo

union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	3	0	5	1	8	8

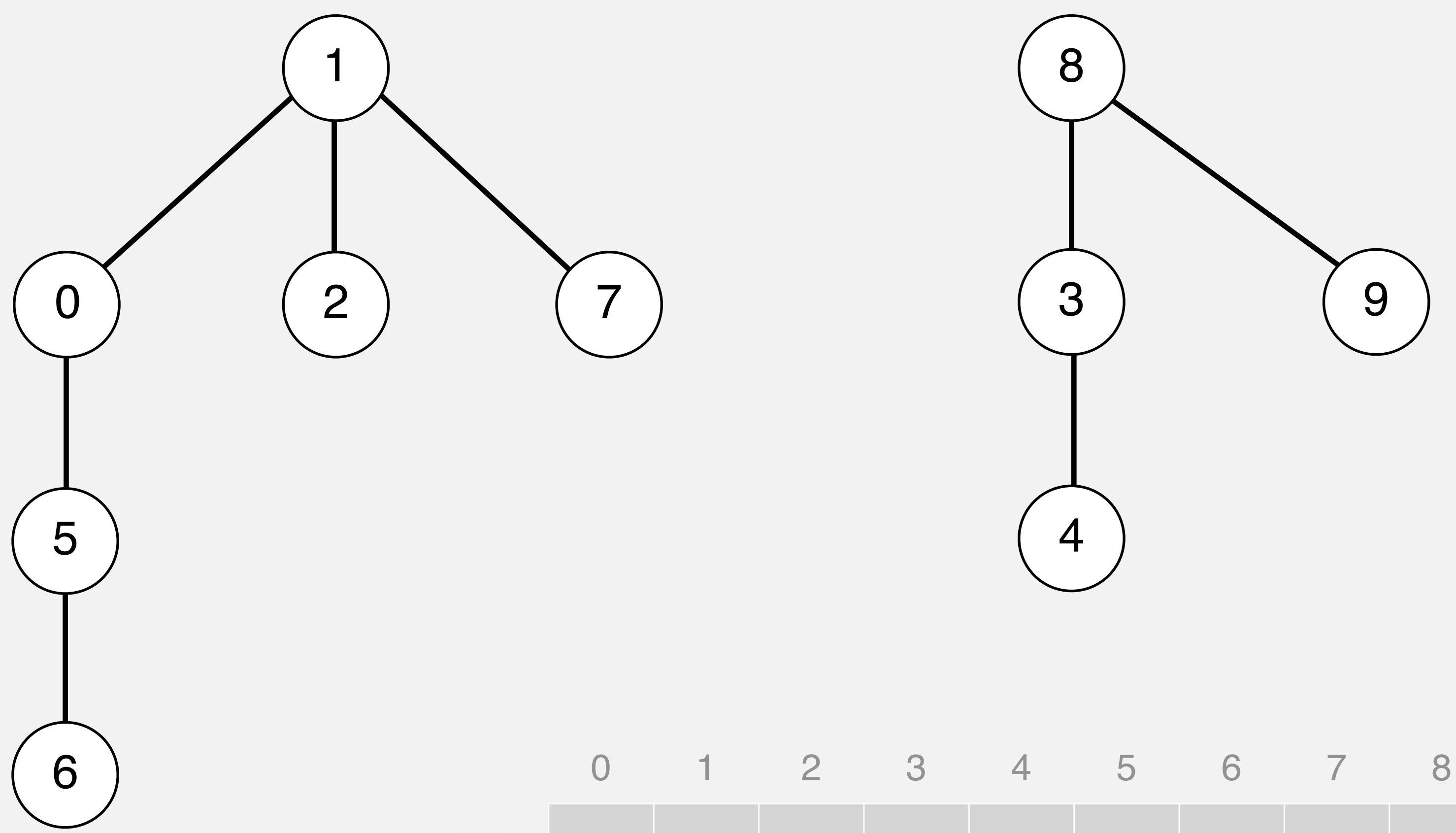
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	3	0	5	1	8	8

Quick-union demo

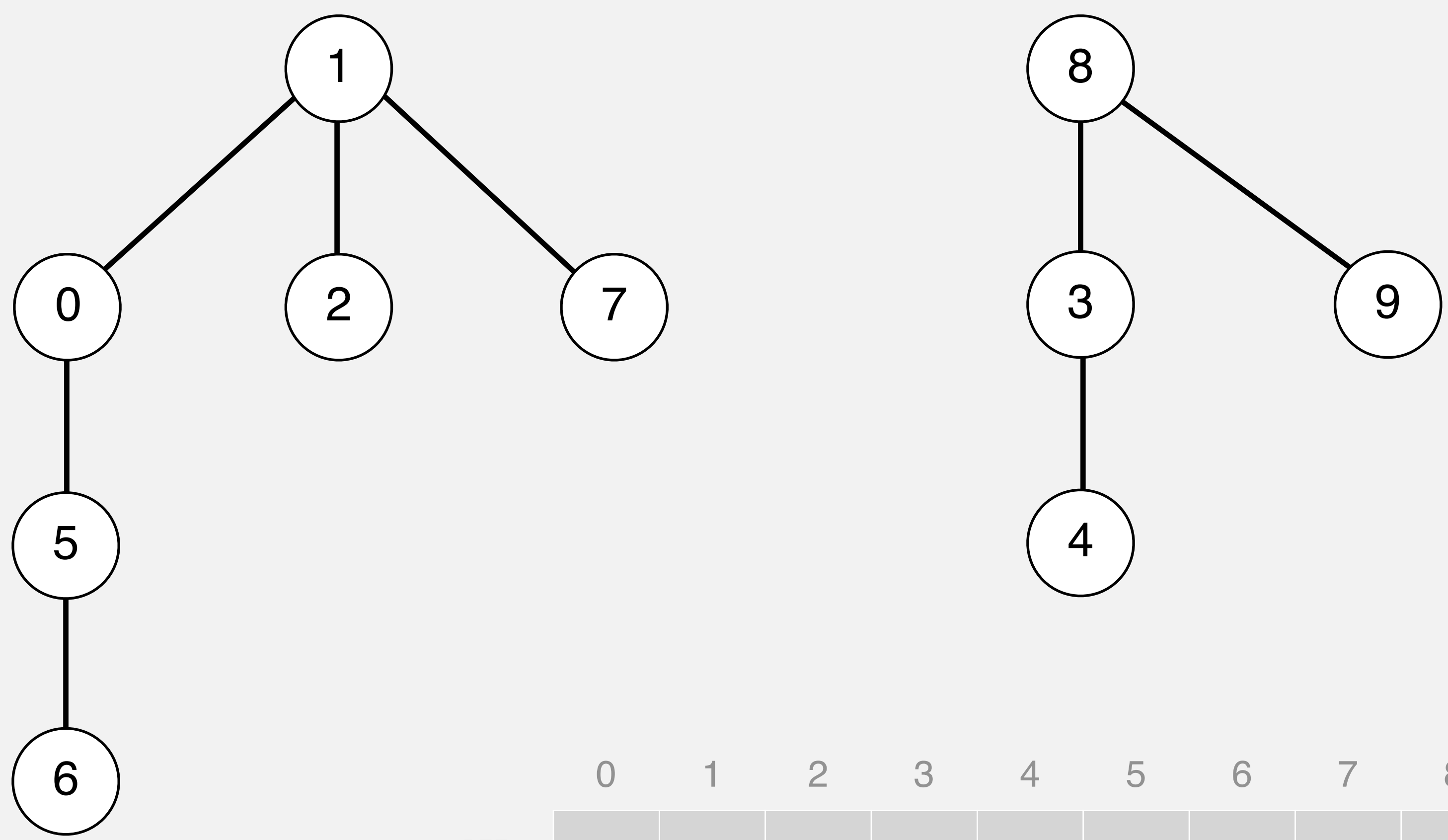
connected(1, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	3	0	5	1	8	8

Quick-union demo

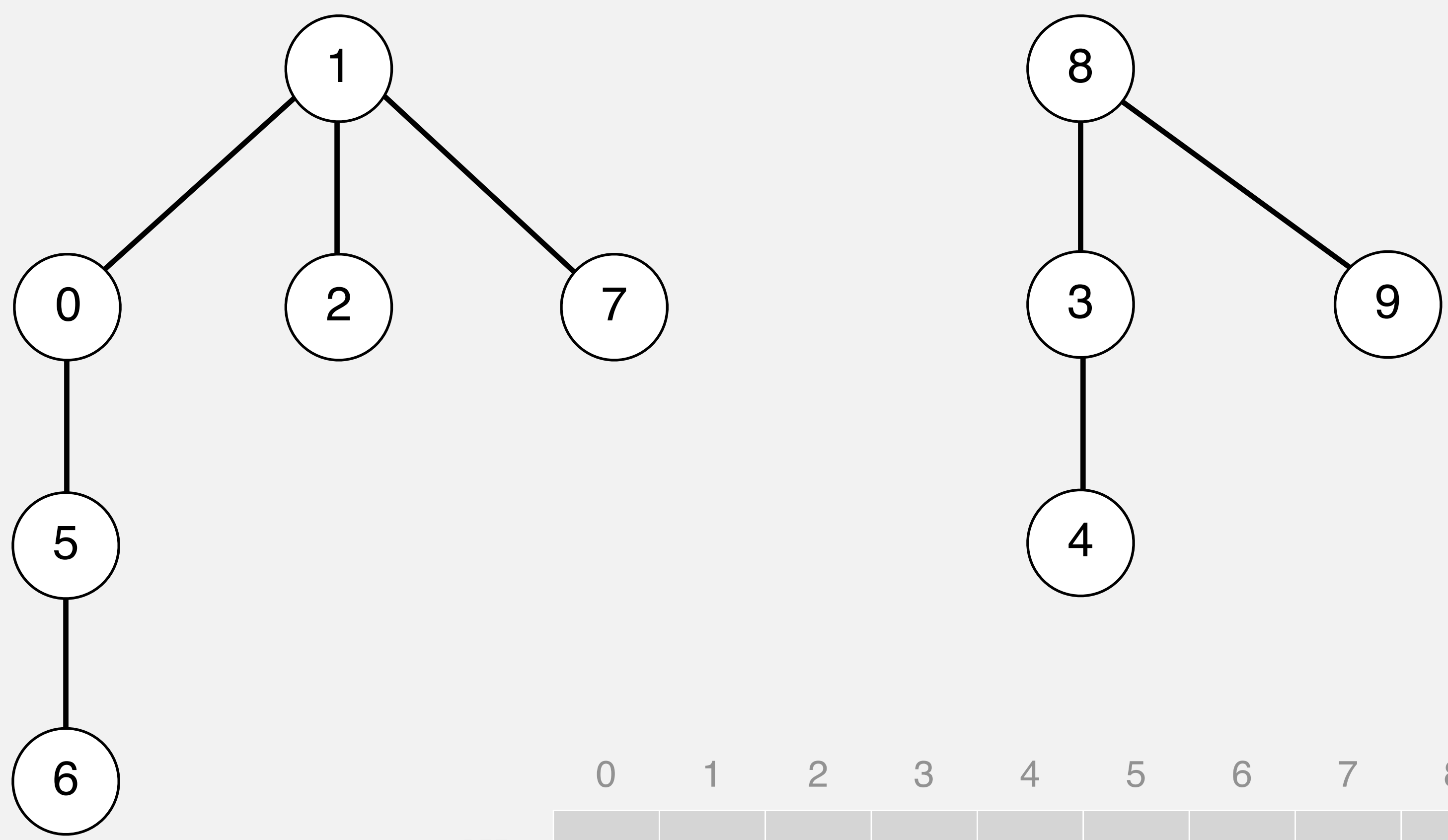
connected(6, 7)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	3	0	5	1	8	8

Quick-union demo

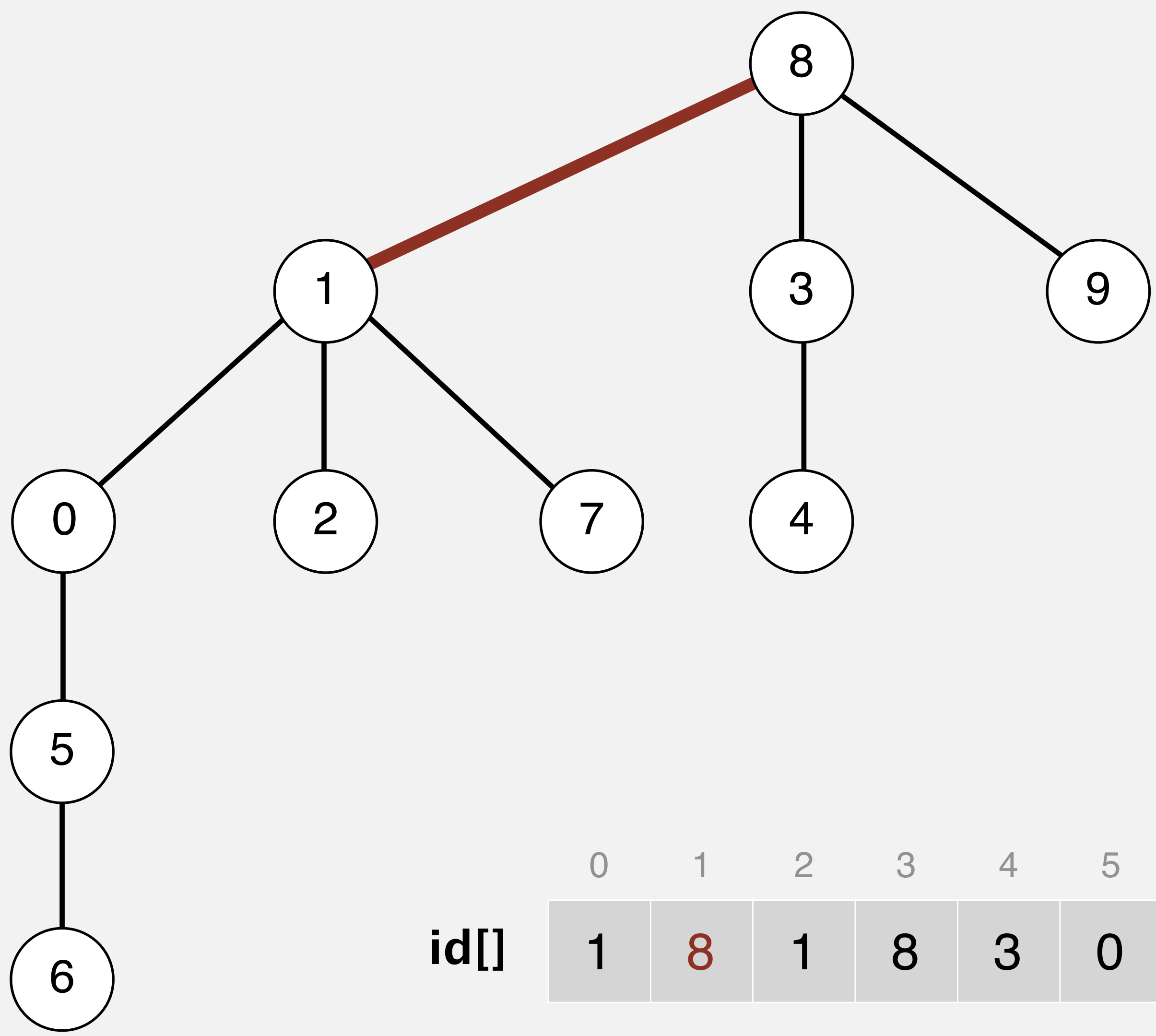
union(7, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	3	0	5	1	8	8

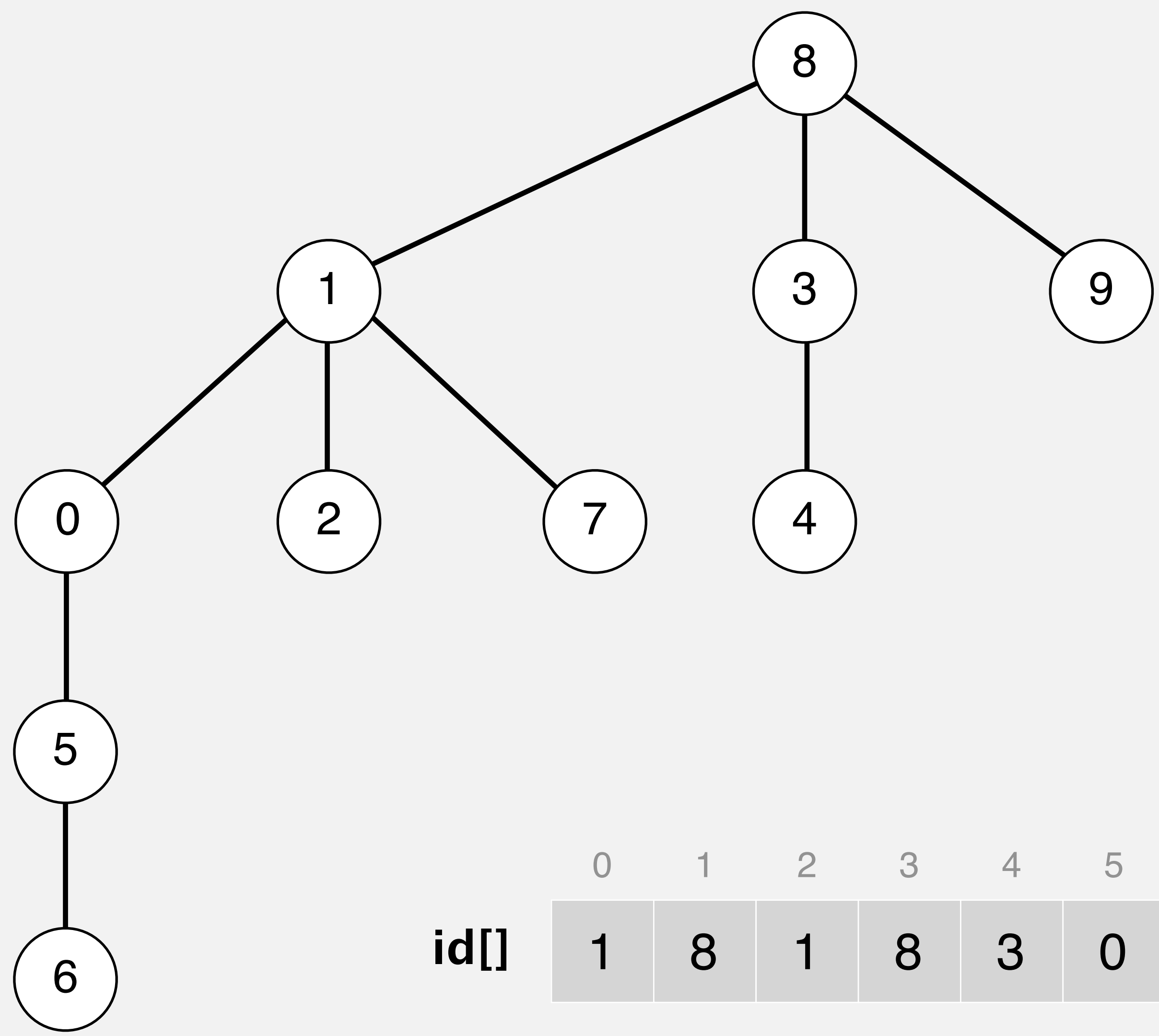
Quick-union demo

union(7, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union: Java implementation

```
public class QuickUnionUF {
   private int[] id;

   public QuickUnionUF(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }
```

set id of each object to itself
(N array accesses)

Quick-union: Java implementation

```
public class QuickUnionUF {
  private int[] id;

  public QuickUnionUF(int N) {
    id = new int[N];
    for (int i = 0; i < N; i++) id[i] = i;
  }

  public int find(int i) {
```

set id of each object to itself

(N array accesses)

Quick-union: Java implementation

```
public class QuickUnionUF {
  private int[] id;

  public QuickUnionUF(int N) {
    id = new int[N];
    for (int i = 0; i < N; i++) id[i] = i;
  }

  public int find(int i) {
    while (i != id[i]) i = id[i];
    return i;
  }

  public void union(int p, int q)  {
    int i = find(p);
    int j = find(q);
    id[i] = j;
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

Quick-union: Java implementation

```
public class QuickUnionUF {
   private int[] id;

   public QuickUnionUF(int N) {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   public int find(int i) {
      while (i != id[i]) i = id[i];
      return i;
   }

   public void union(int p, int q)  {
      int i = find(p);
      int j = find(q);
      id[i] = j;
   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find/connected too expensive (could be N array accesses).

http://algs4.cs.princeton.edu

1.5 Union-Find

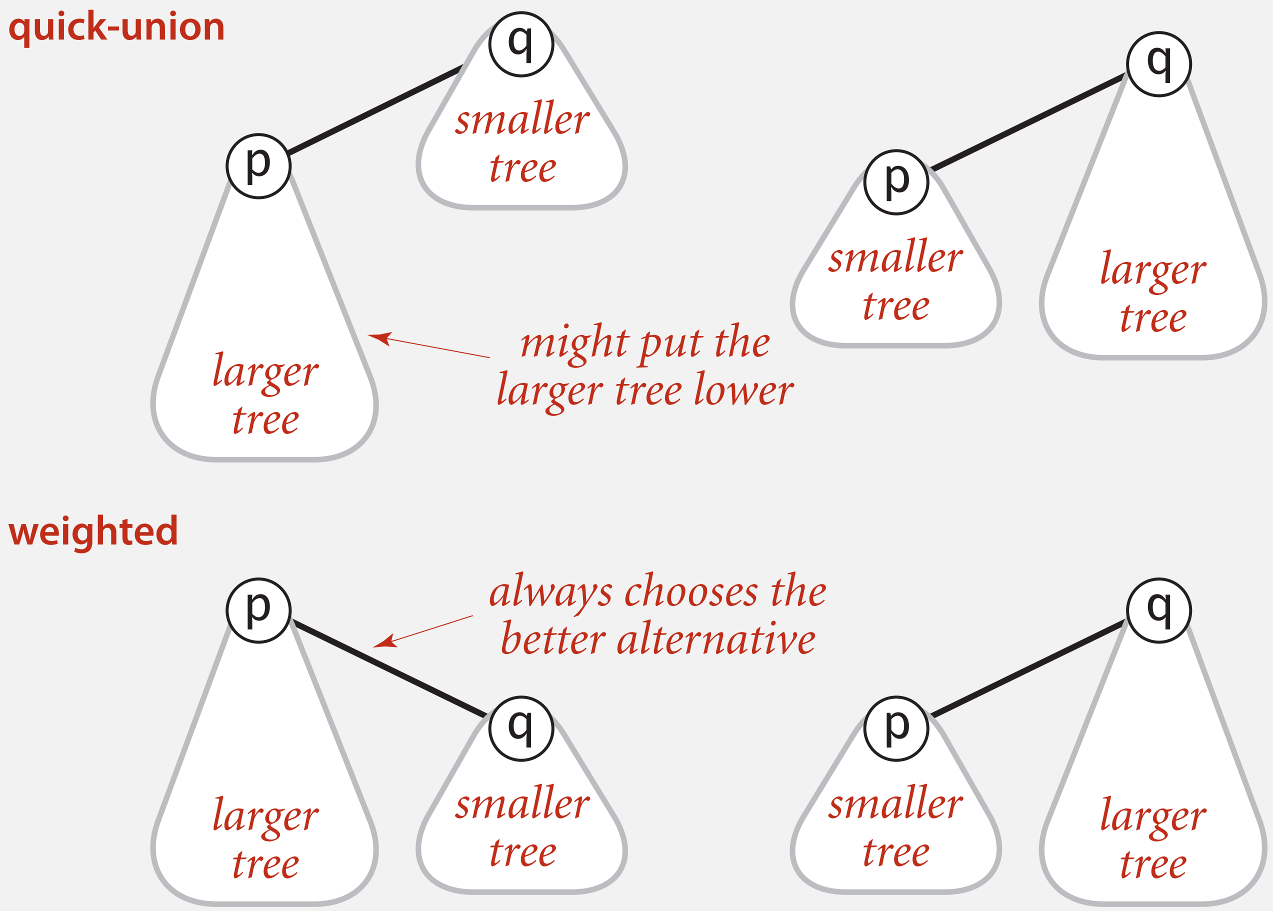
- dynamic connectivity
- quick find
- quick union
- improvements
- applications

Improvement 1: weighting

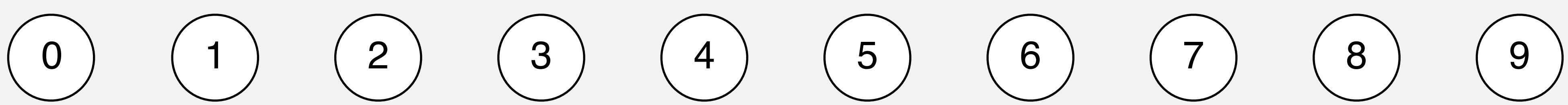
Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives:
union by height or "rank"



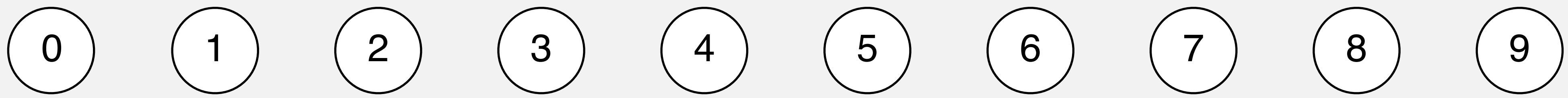
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Weighted quick-union demo

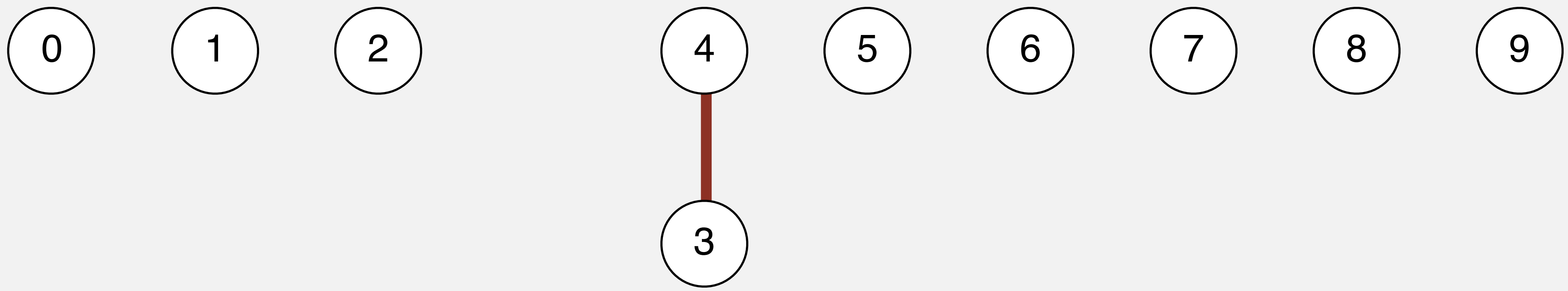
union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

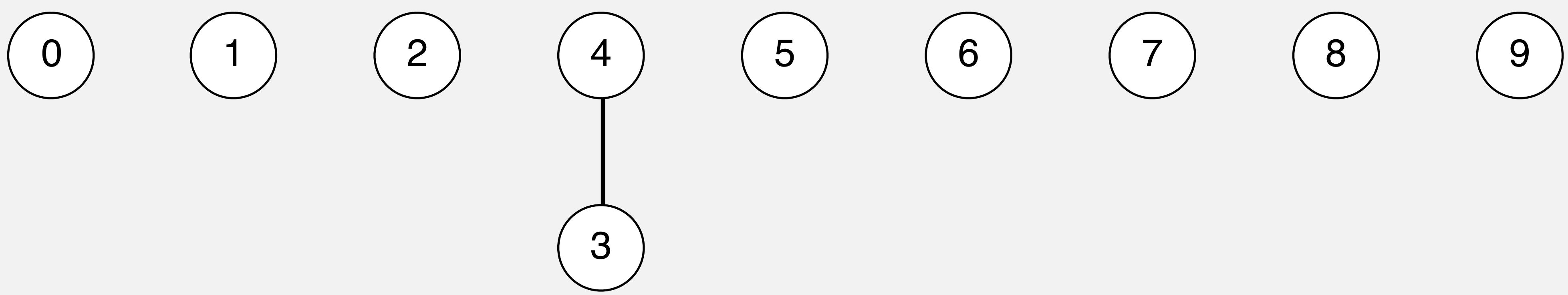
Weighted quick-union demo

union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

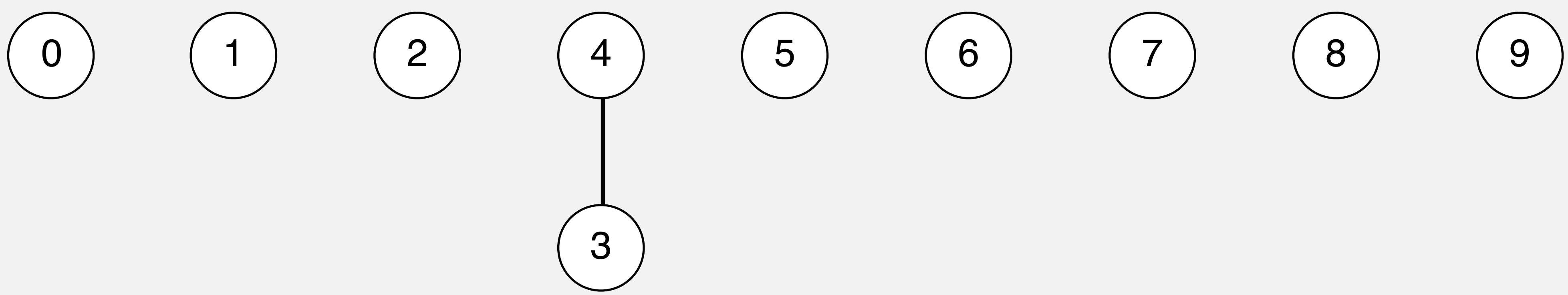
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

Weighted quick-union demo

union(3, 8)

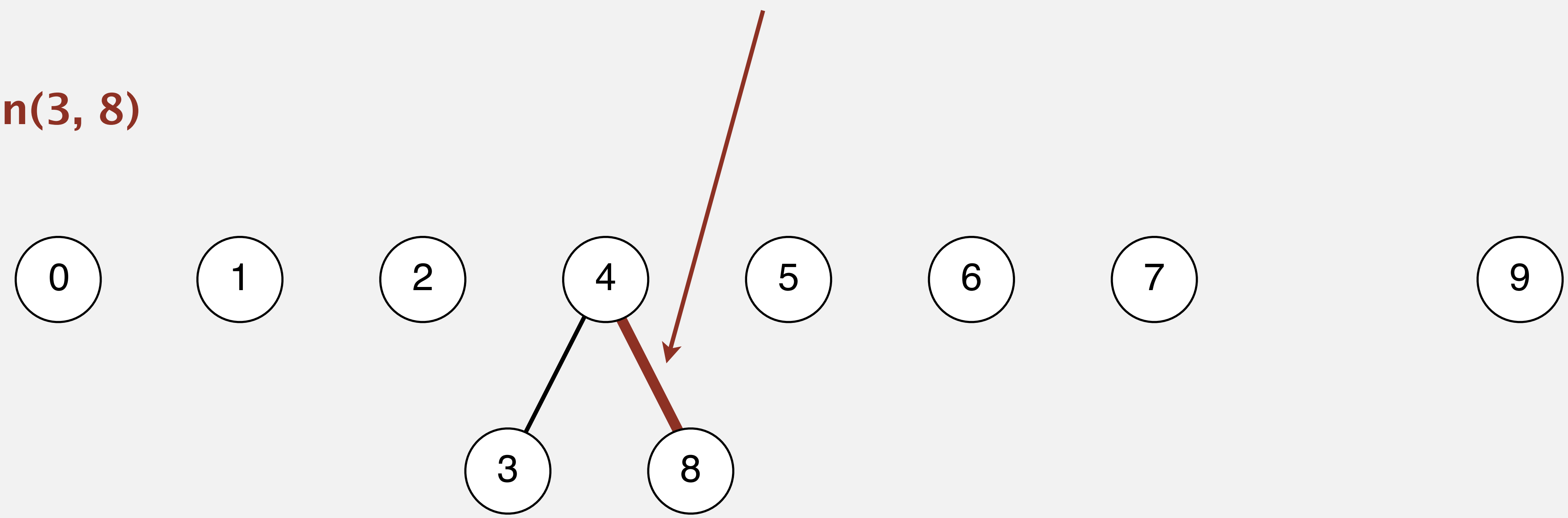


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

Weighted quick-union demo

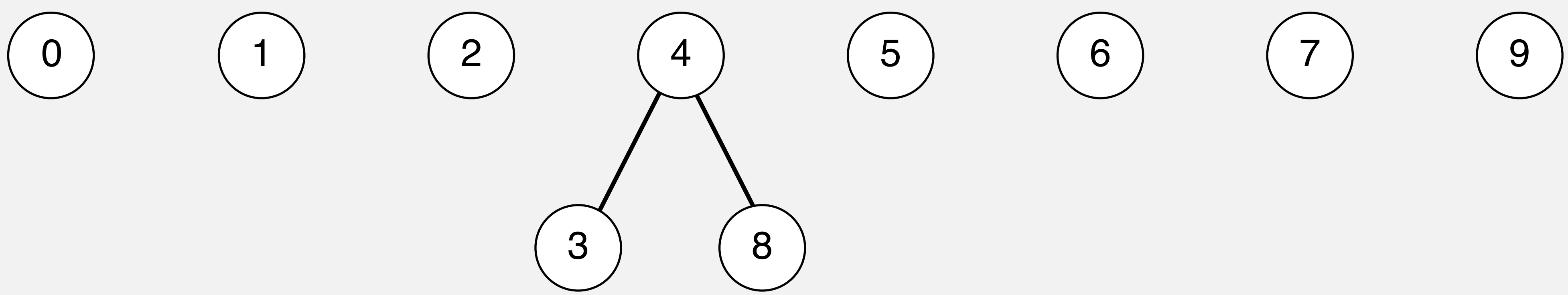
weighting: make 8 point to 4 (instead of 4 to 8)

union(3, 8)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

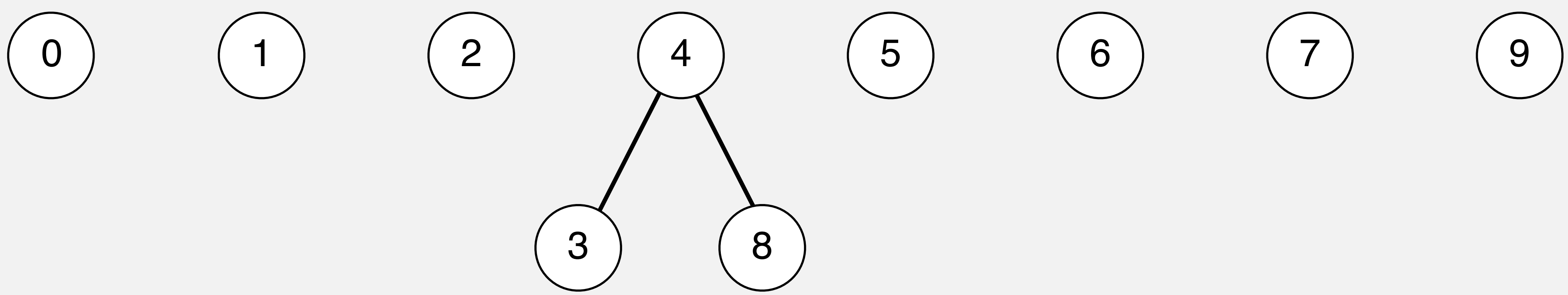
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

Weighted quick-union demo

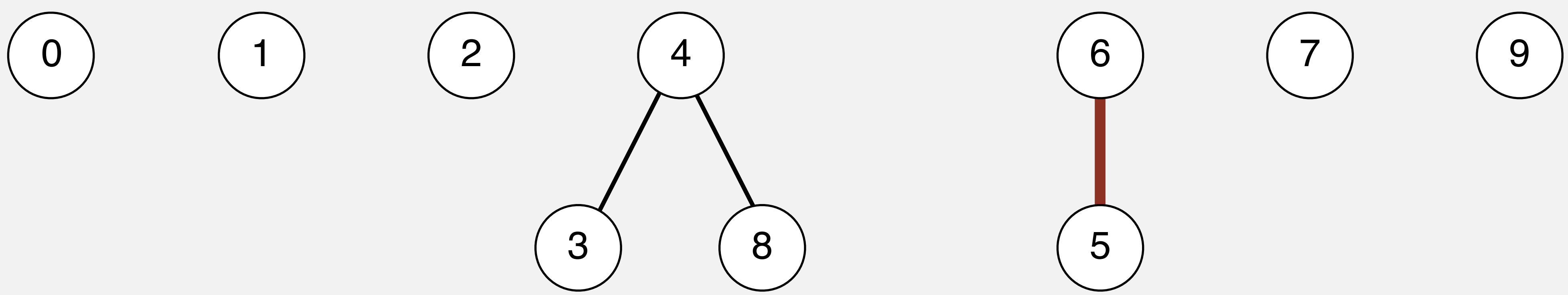
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

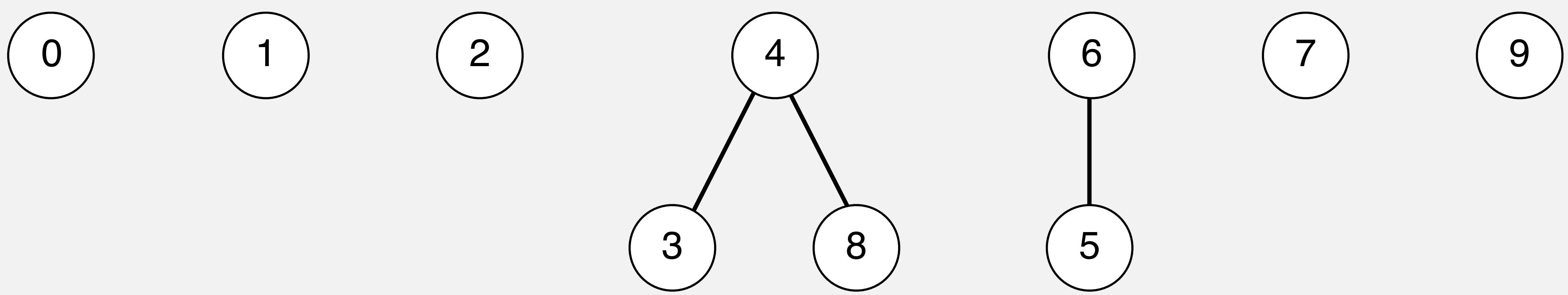
Weighted quick-union demo

union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

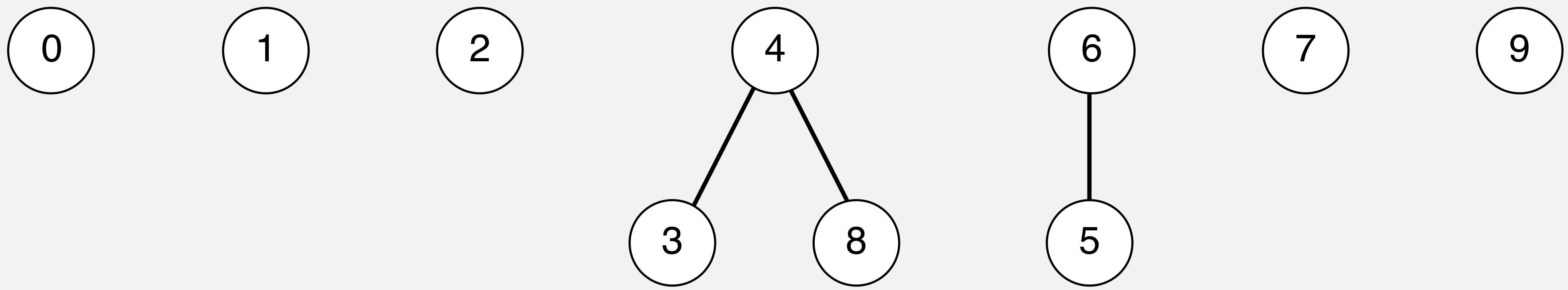
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

Weighted quick-union demo

union(9, 4)

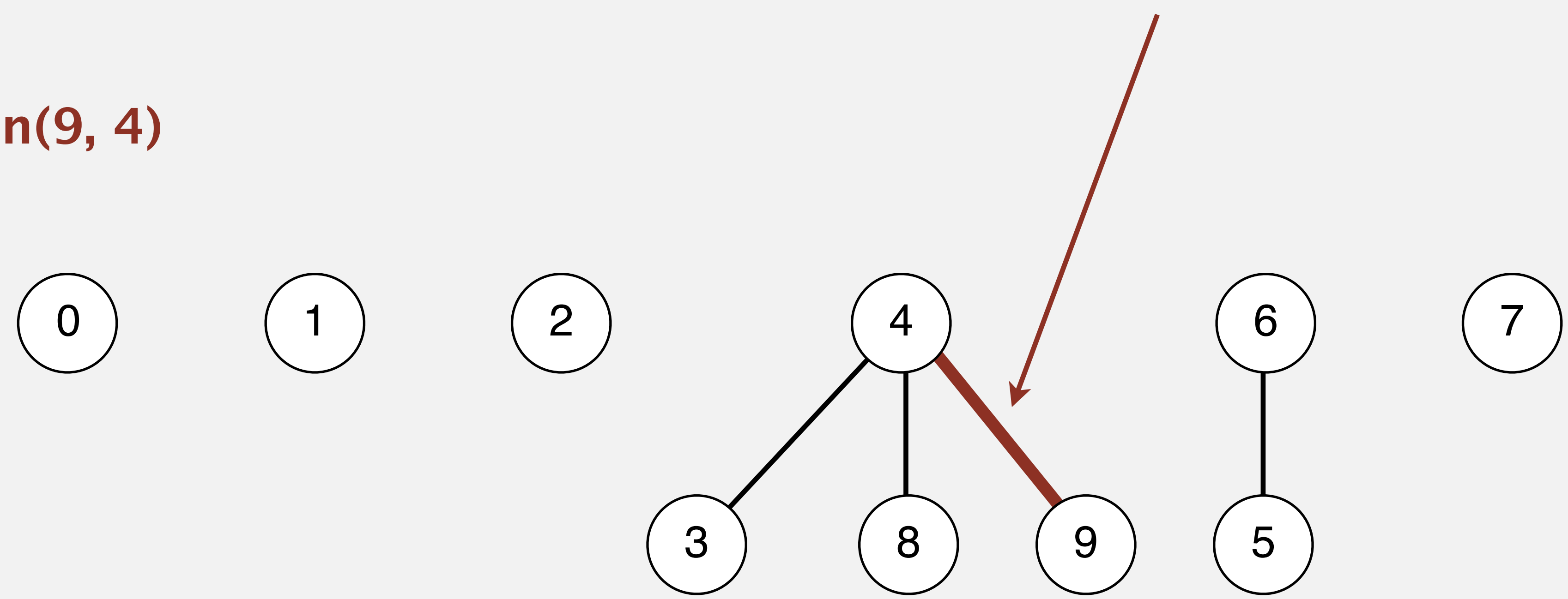


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

Weighted quick-union demo

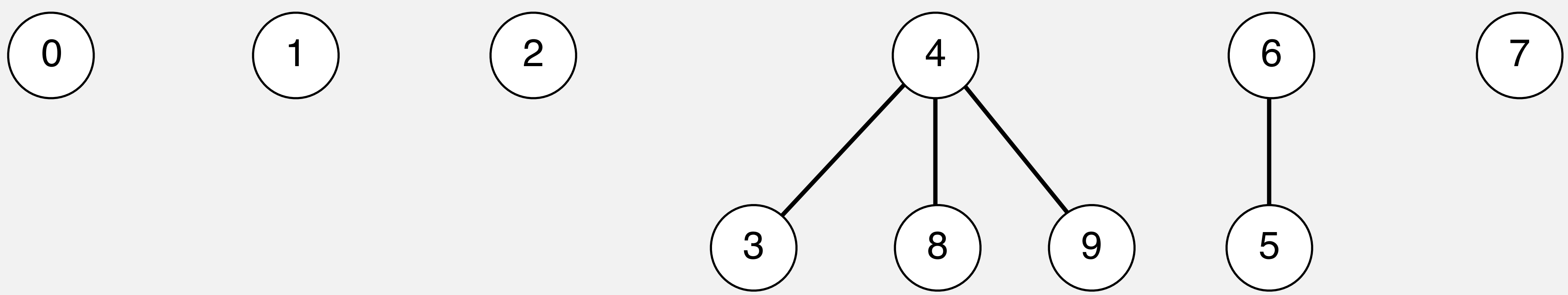
weighting: make 9 point to 4

union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

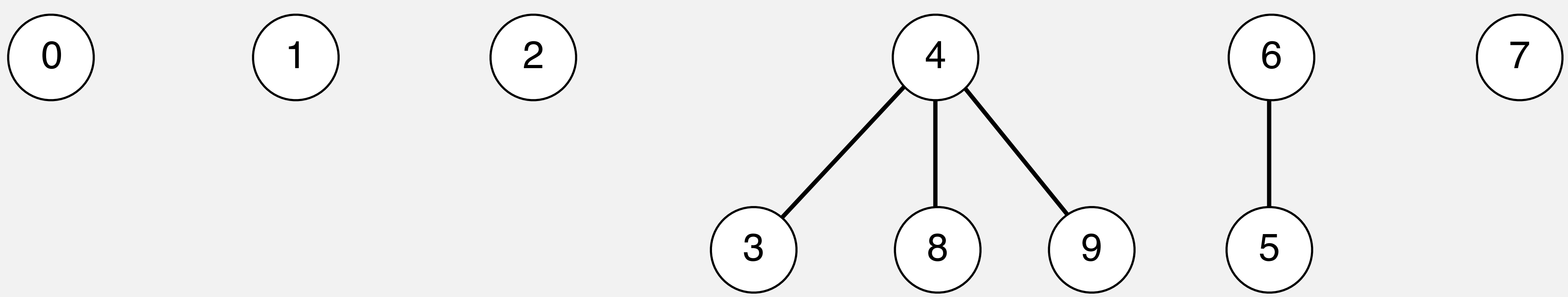
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

Weighted quick-union demo

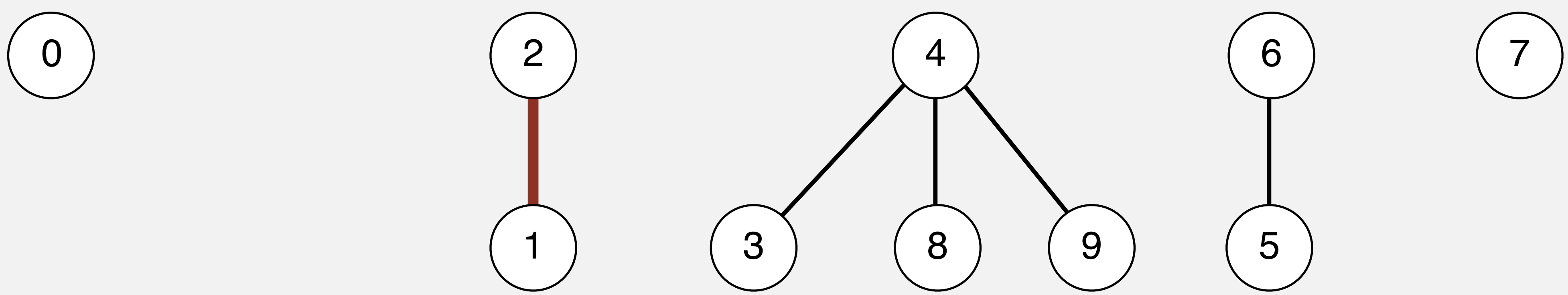
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

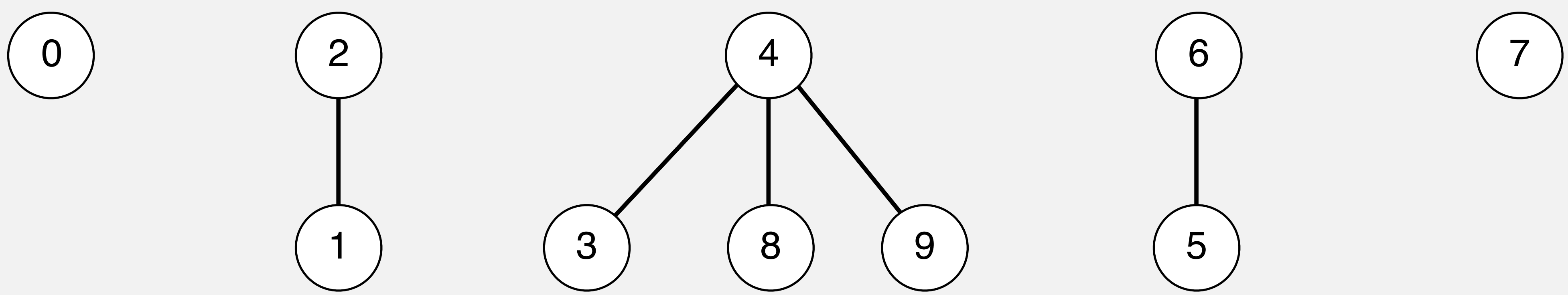
Weighted quick-union demo

union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

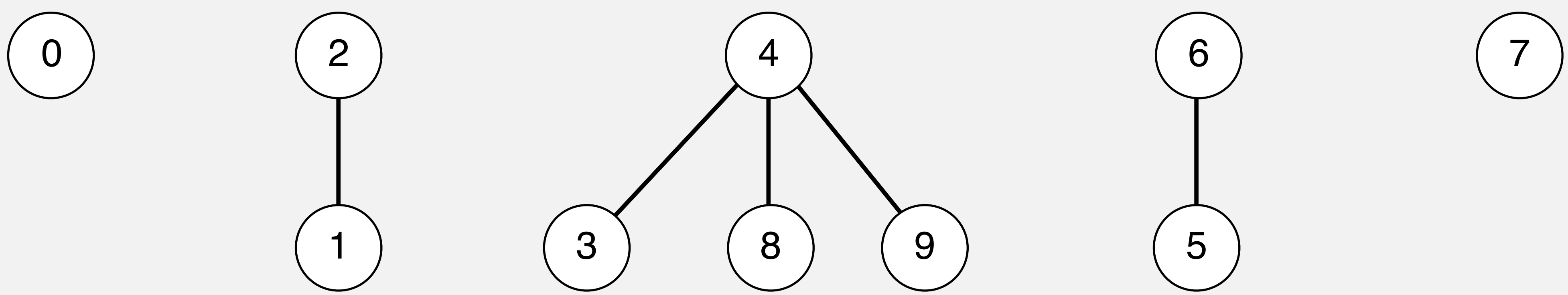
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

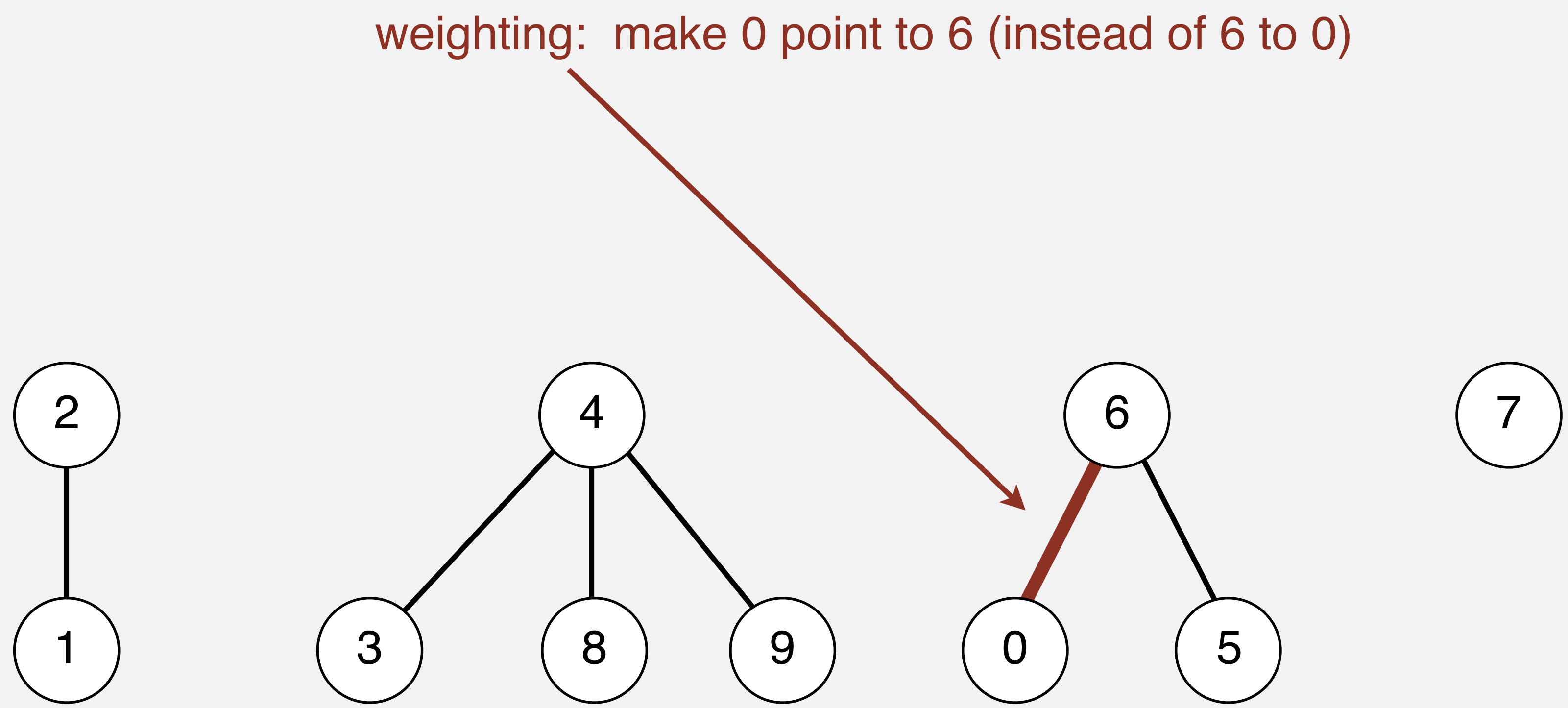
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

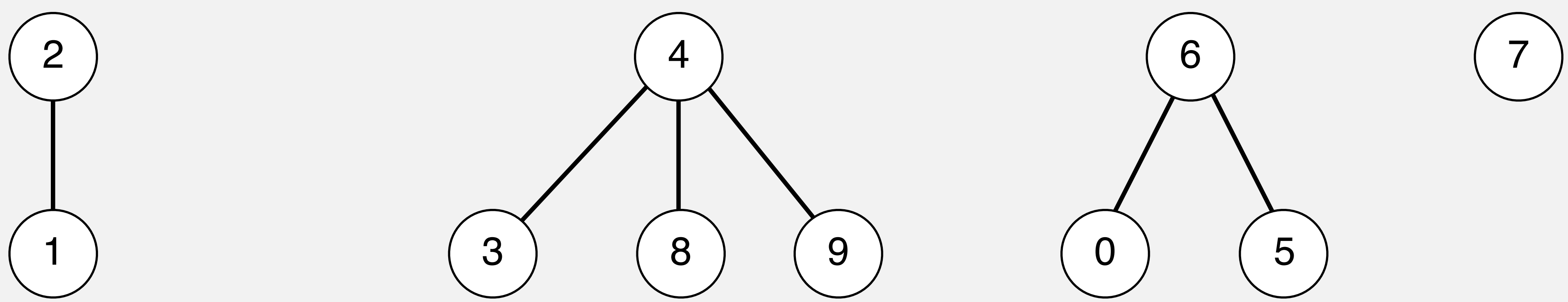
Weighted quick-union demo

union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

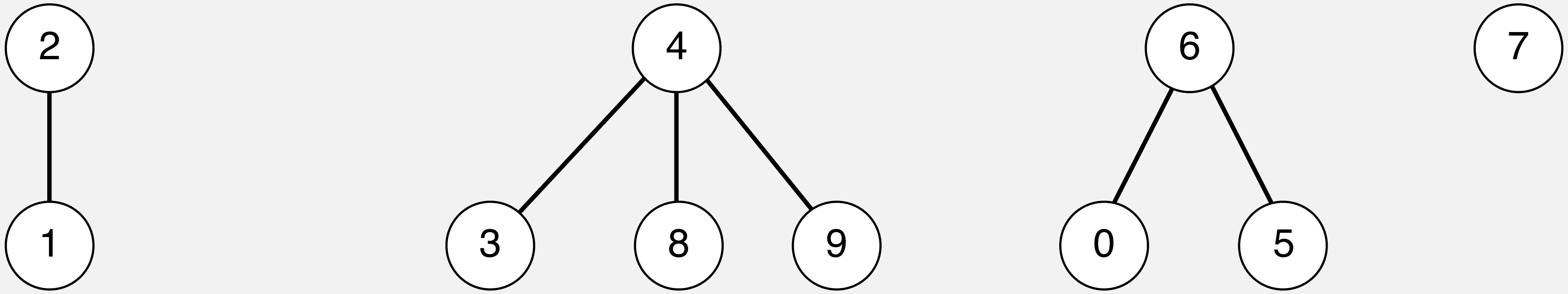
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

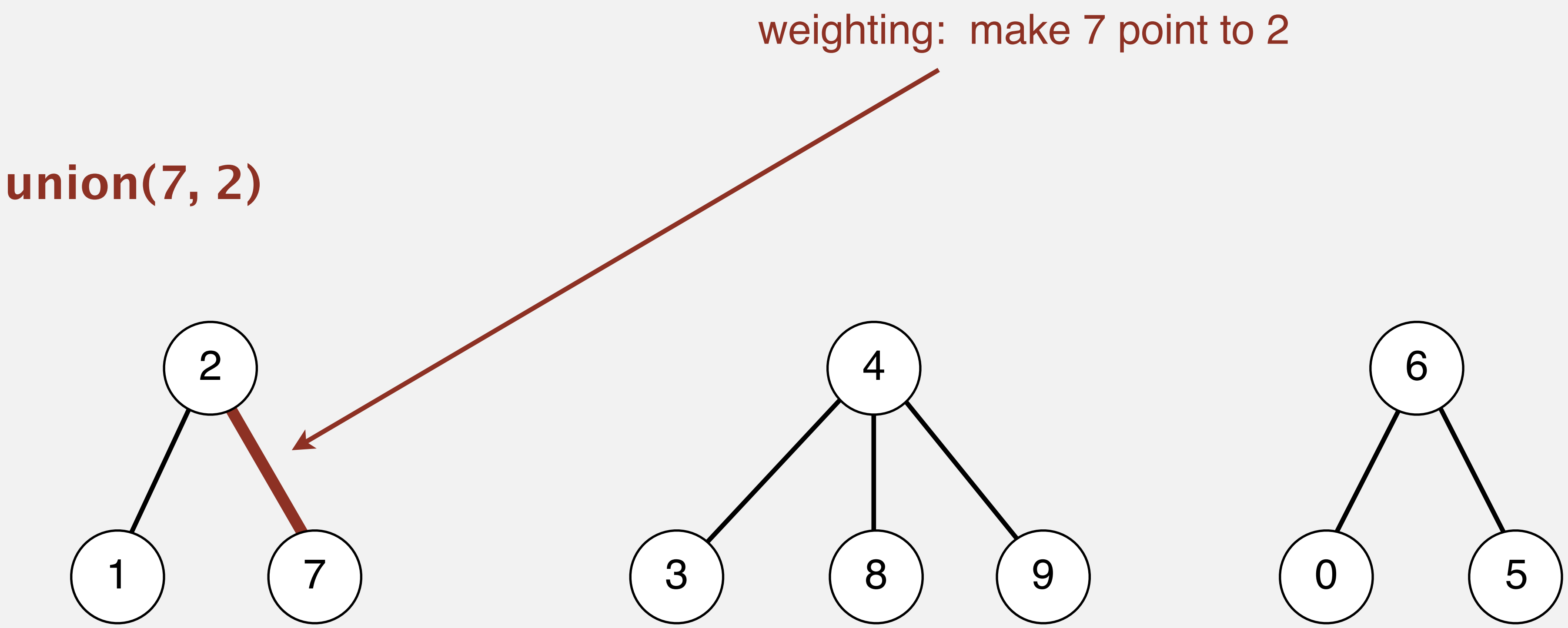
Weighted quick-union demo

union(7, 2)



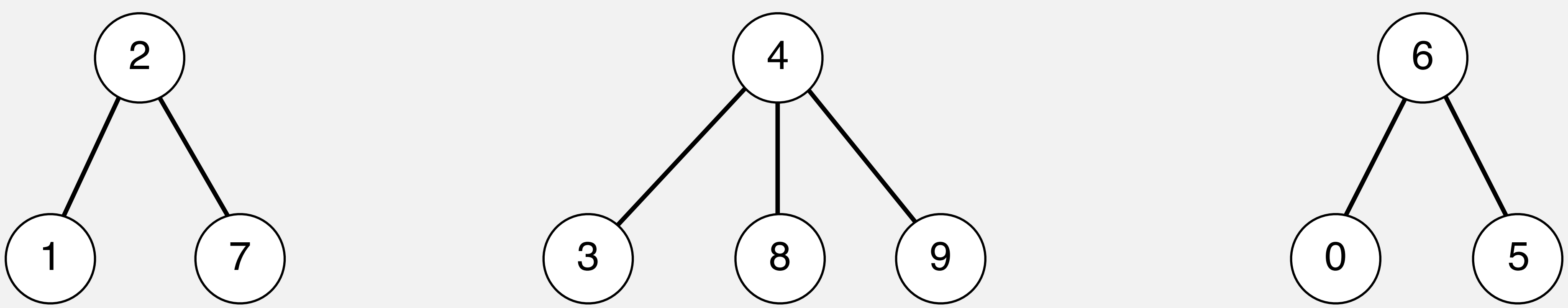
	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

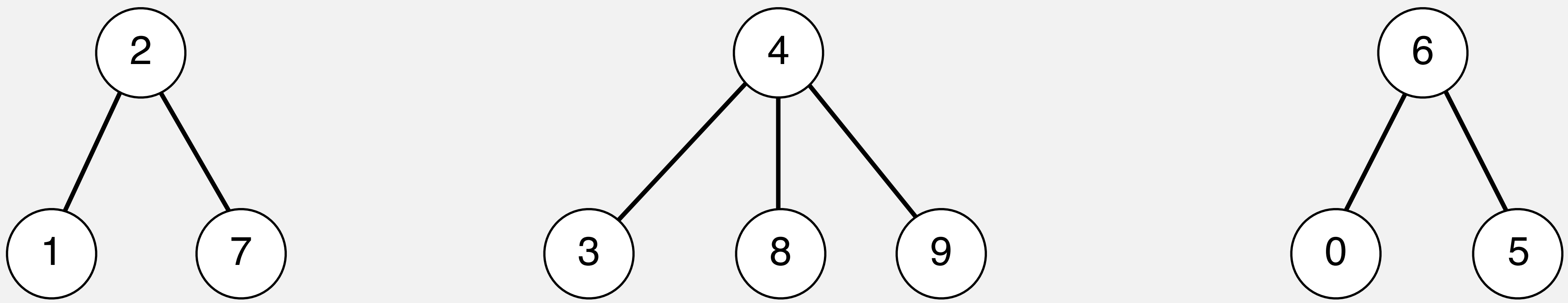
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

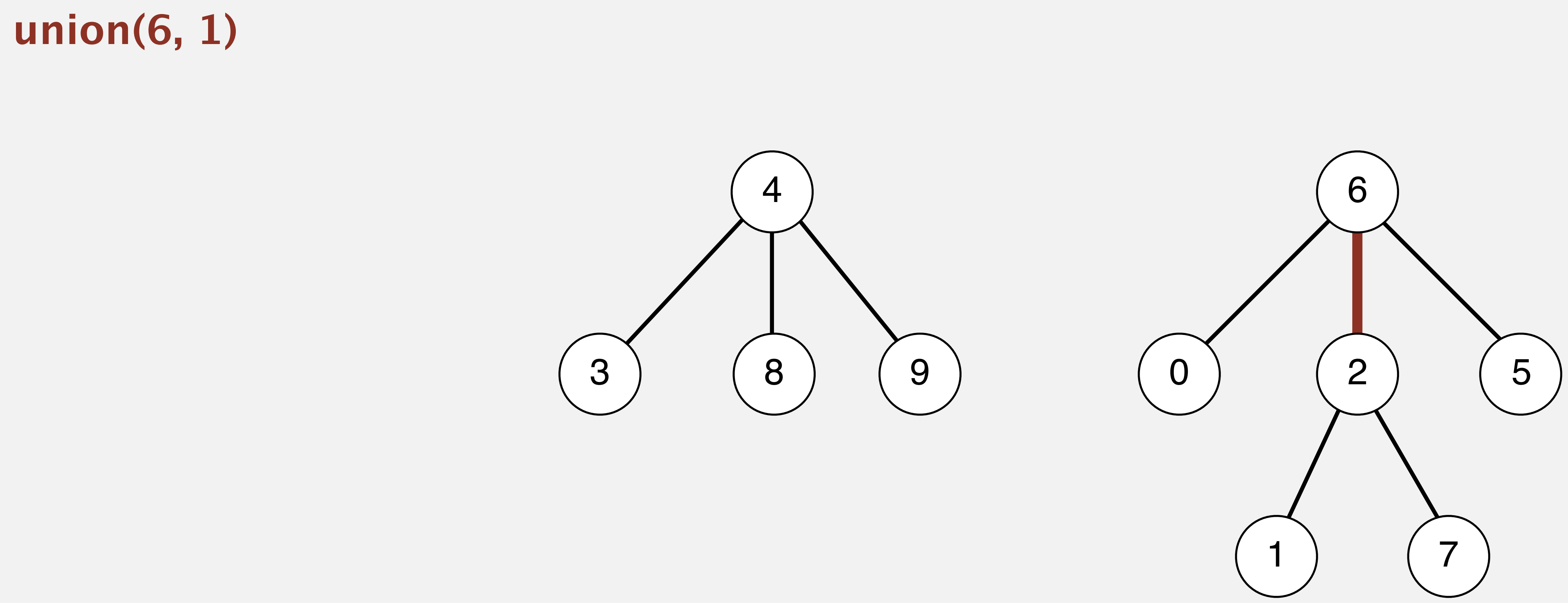
Weighted quick-union demo

union(6, 1)



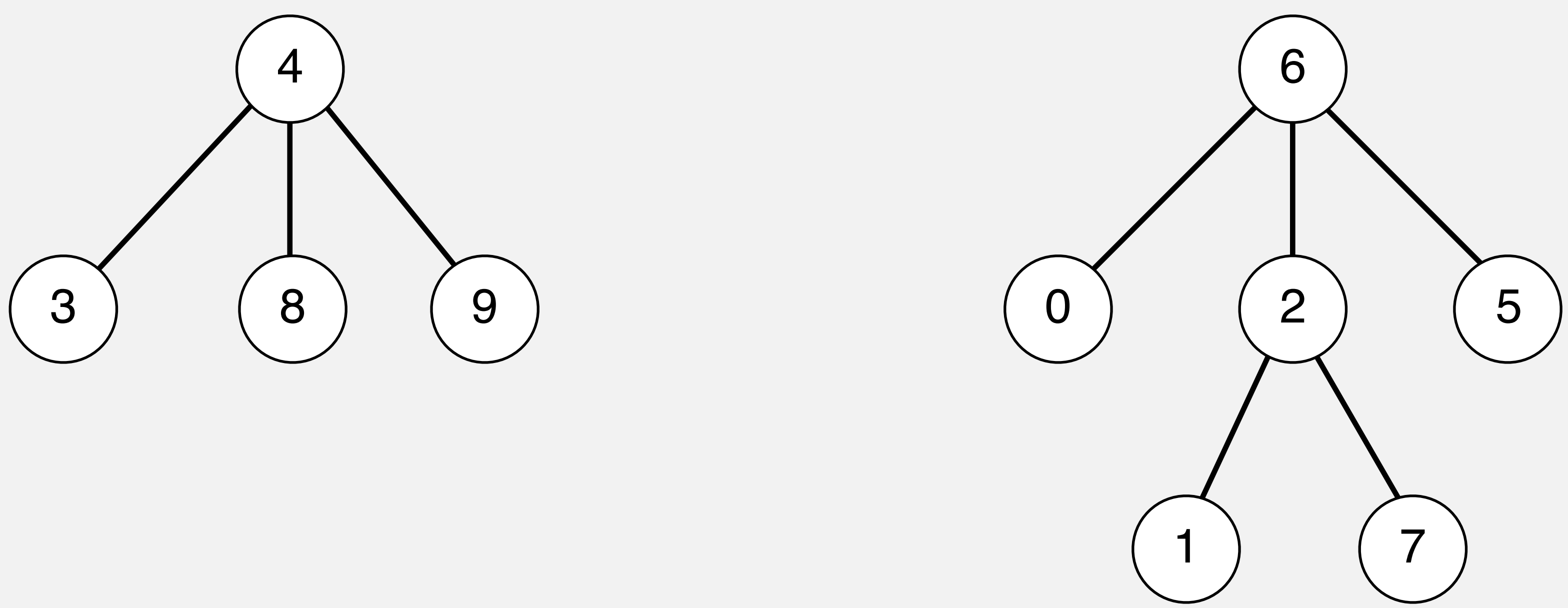
	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

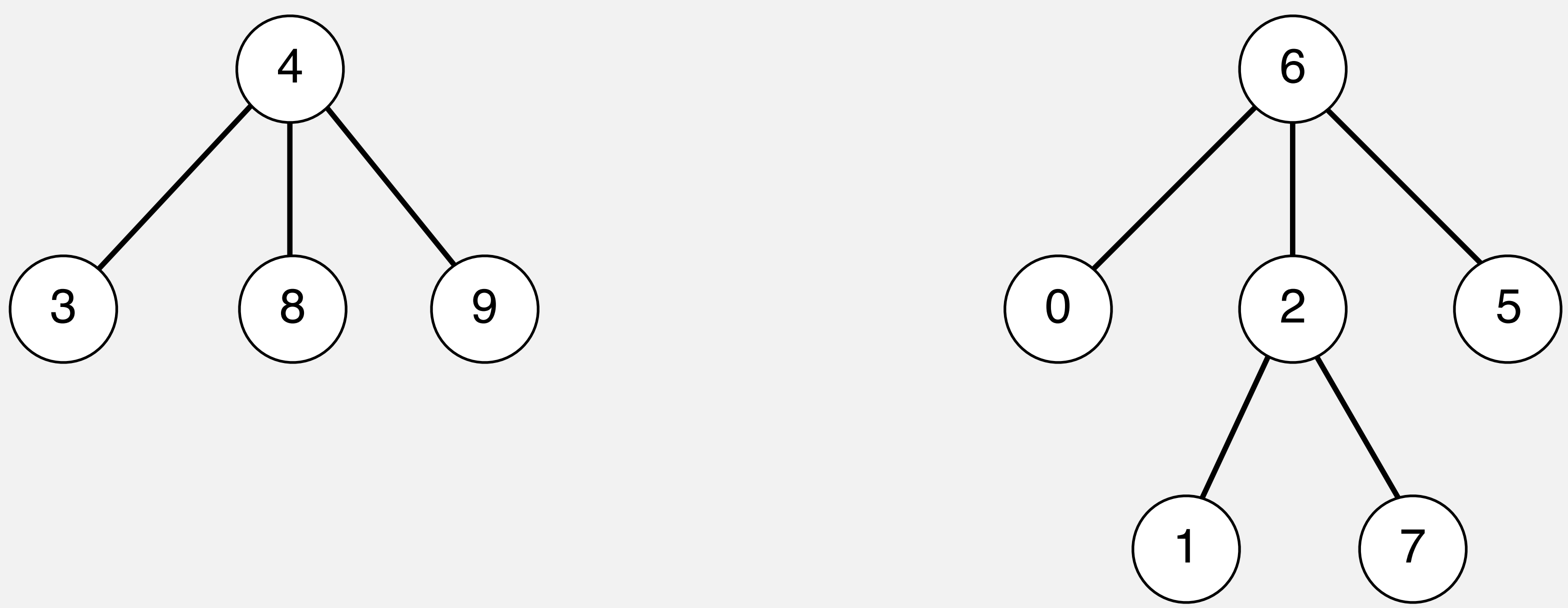
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

Weighted quick-union demo

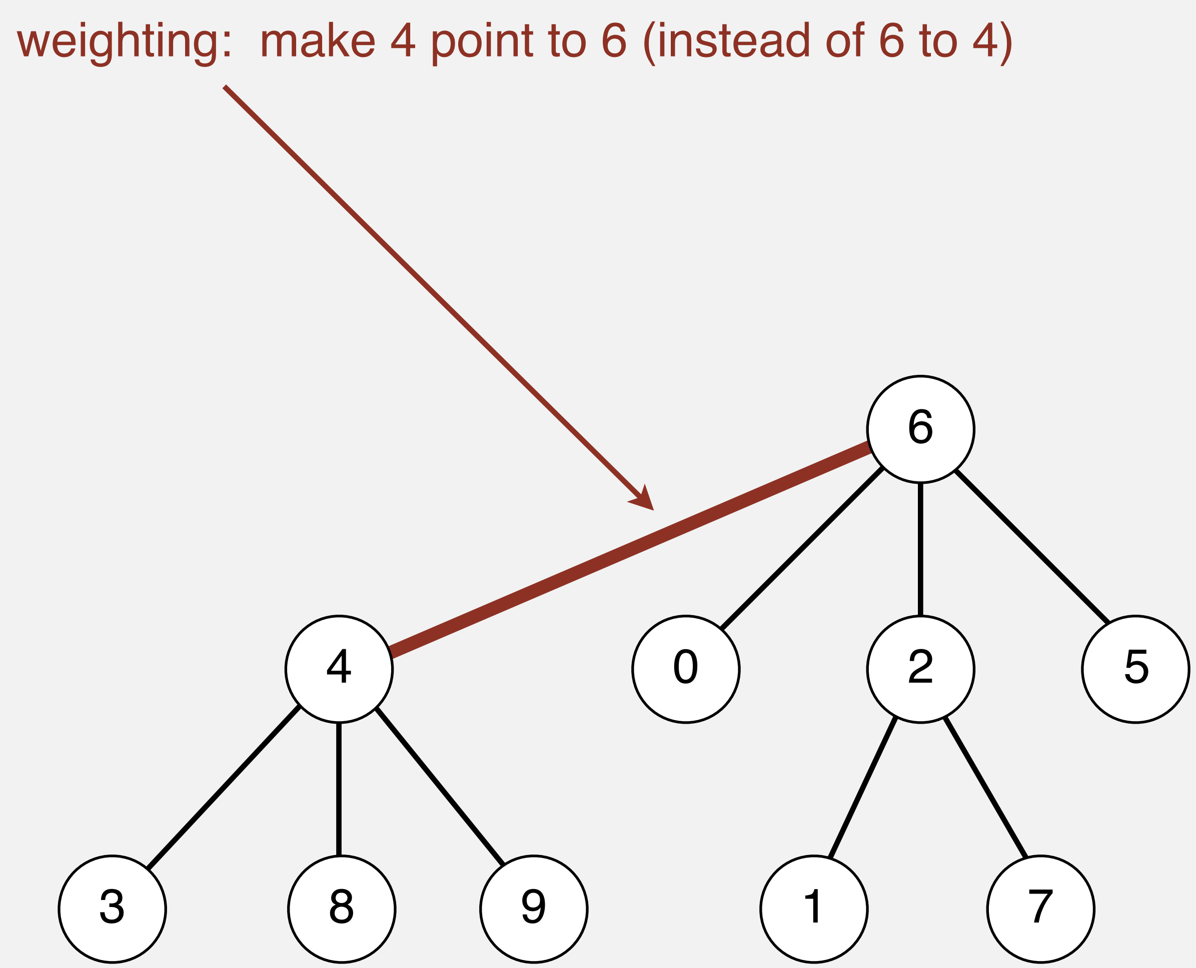
union(7, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

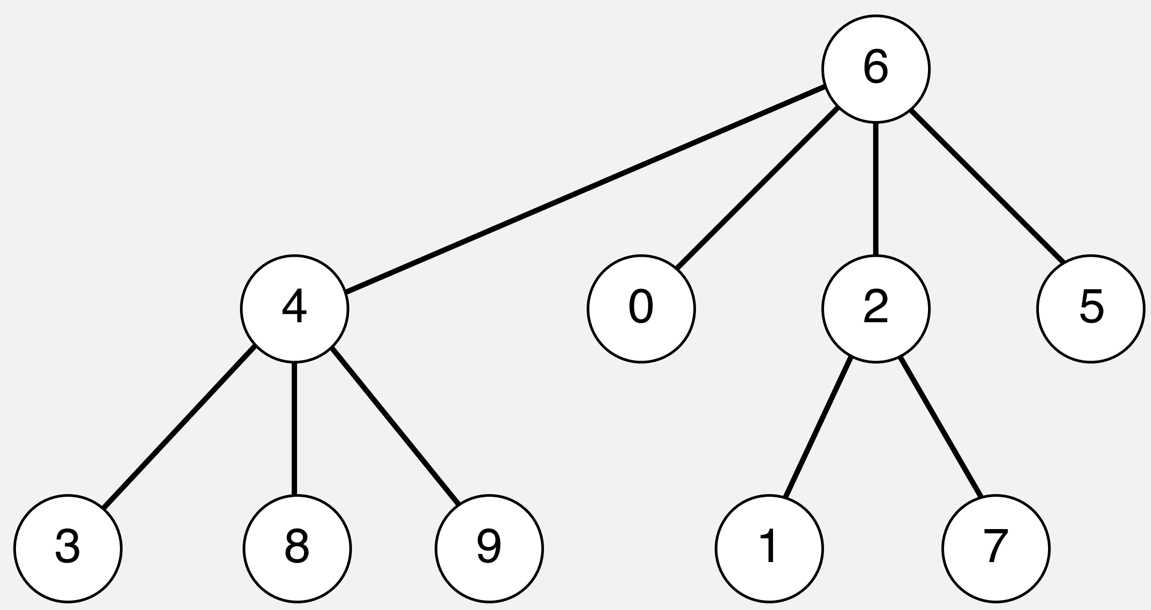
Weighted quick-union demo

union(7, 3)



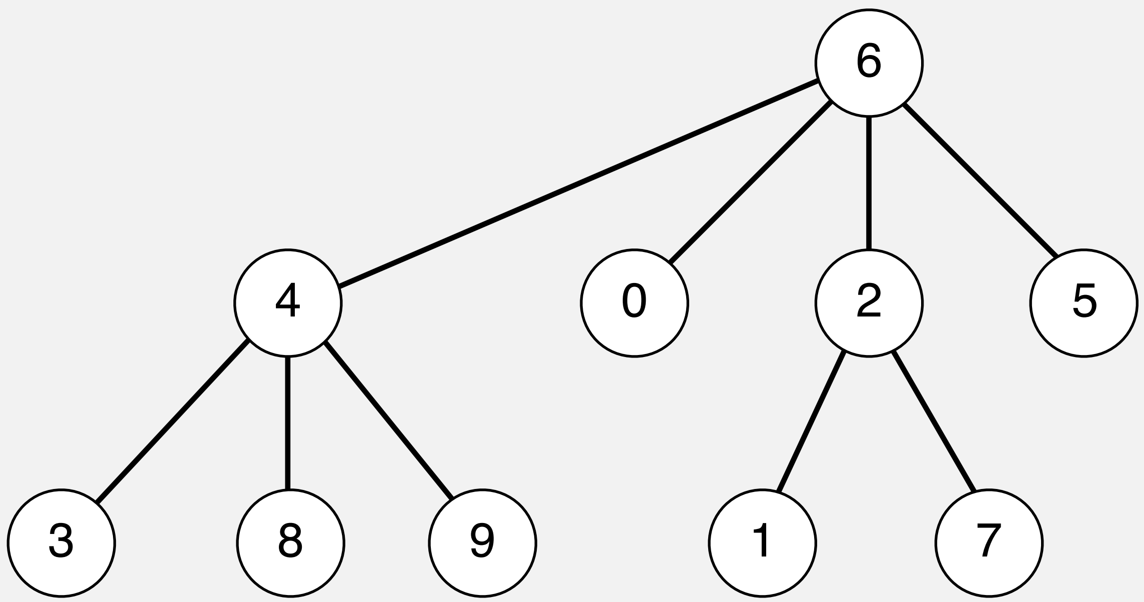
	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

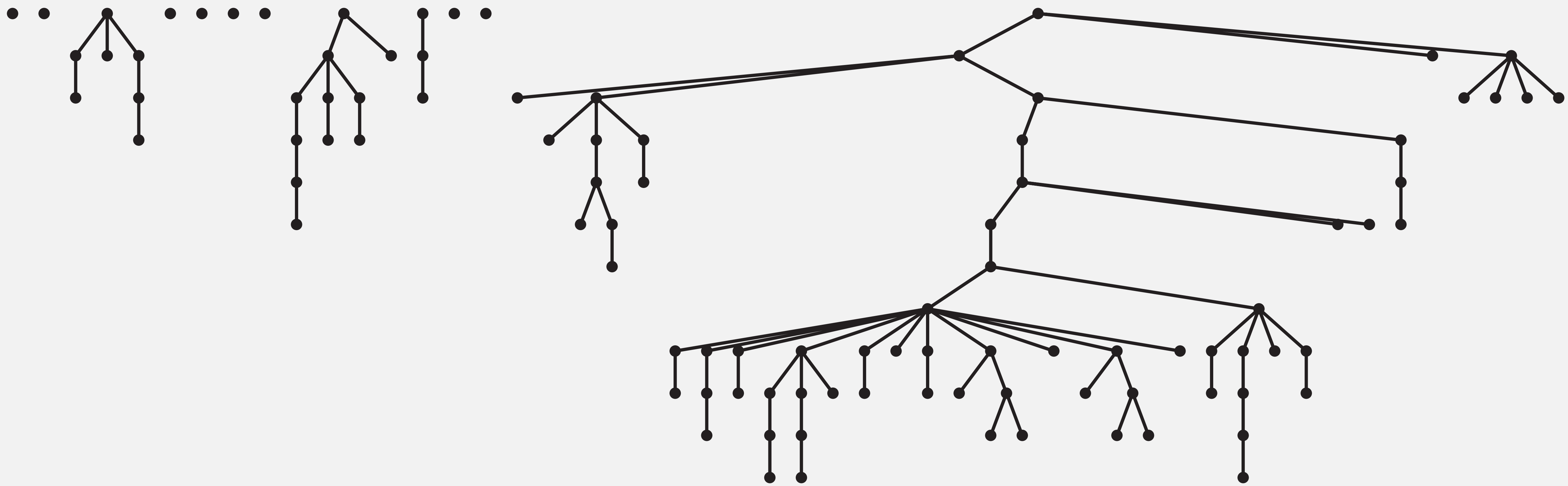
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Quick-union and weighted quick-union example

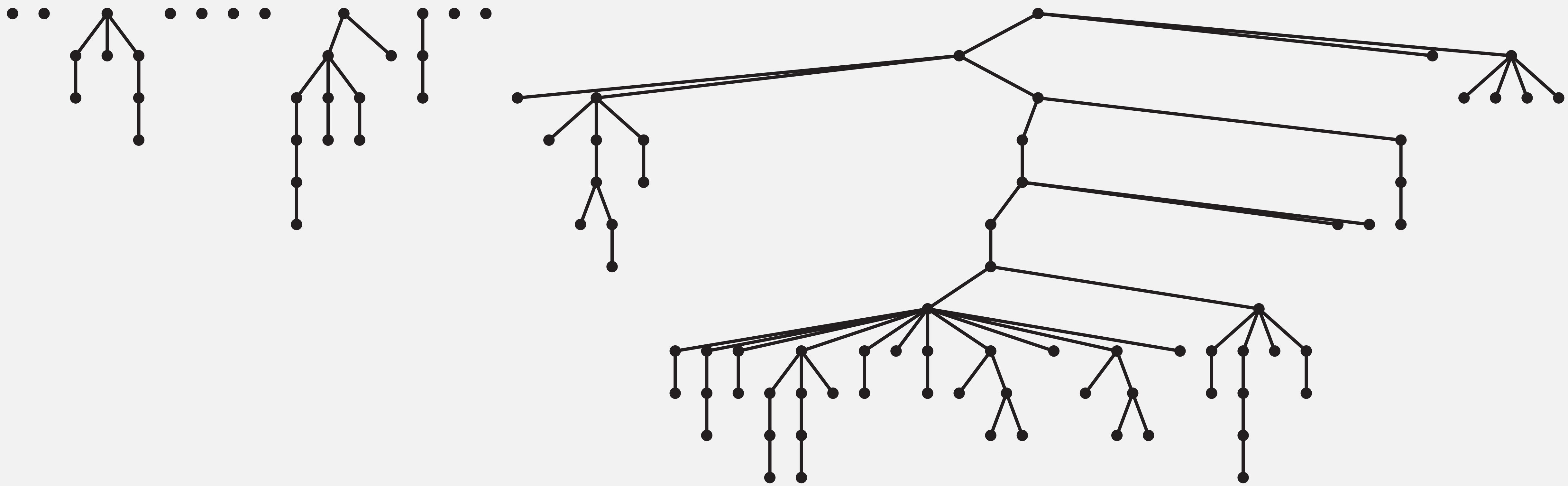
quick-union



average distance to root: 5.11

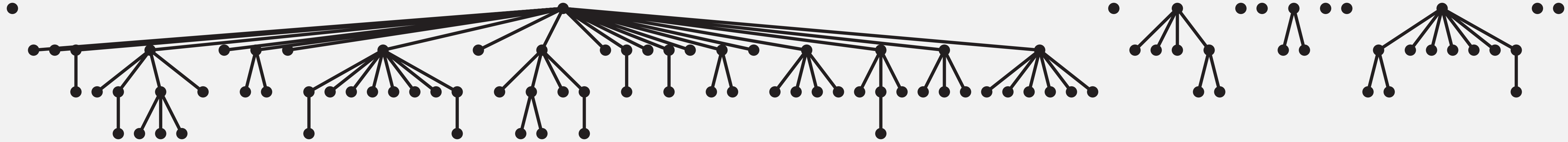
Quick-union and weighted quick-union example

quick-union



average distance to root: 5.11

weighted



average distance to root: 1.52

Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array sz[i] to count number of objects in the tree rooted at i.

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Find/connected. Identical to quick-union.

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array sz[i] to count number of objects in the tree rooted at i.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the sz[] array.

```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

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Running time.

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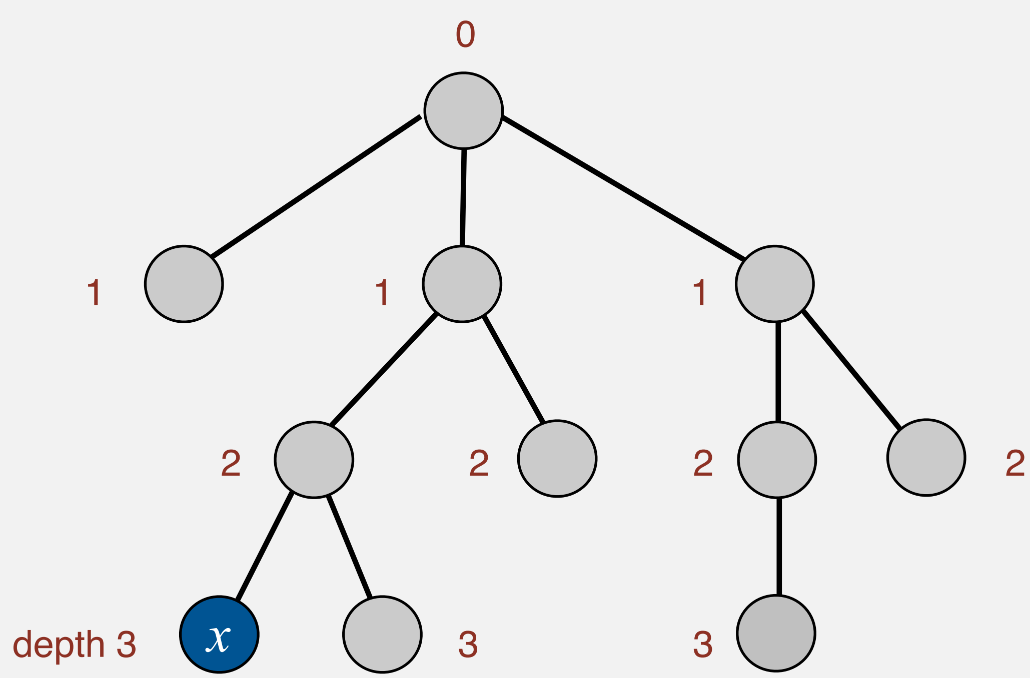
lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.



N = 11
depth(x) = 3 ≤ lg N

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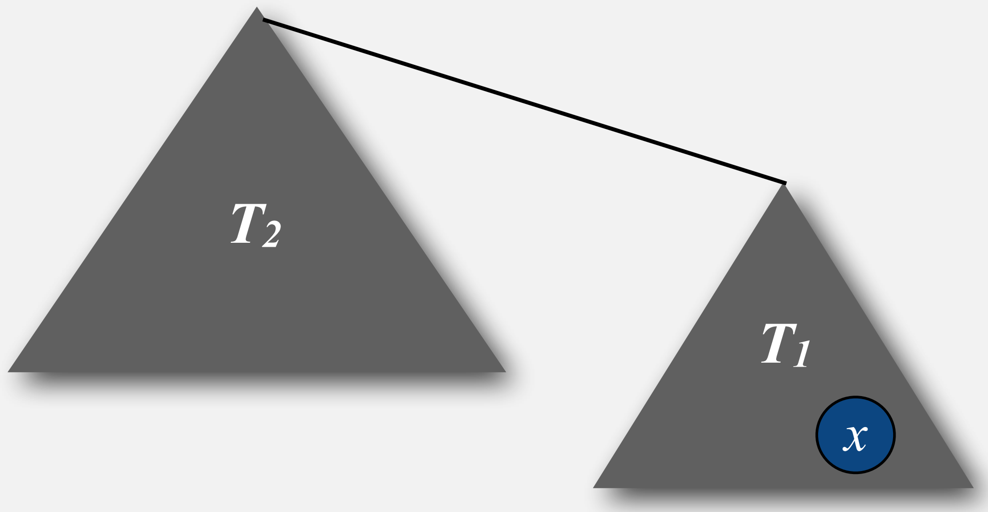

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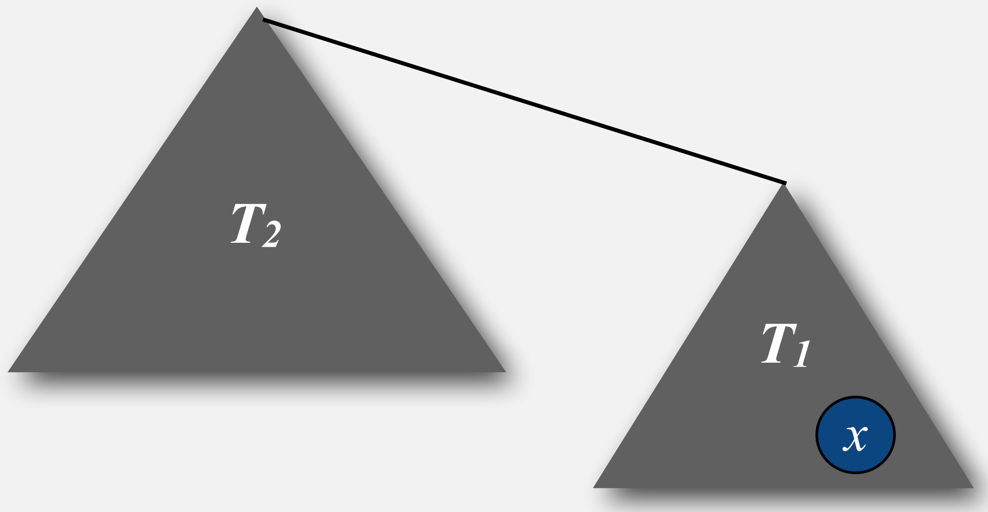
- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.



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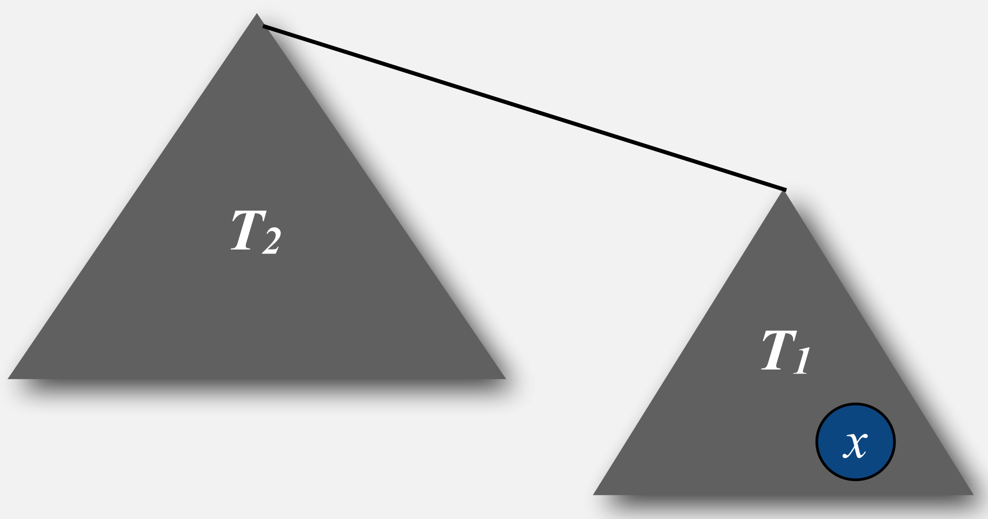

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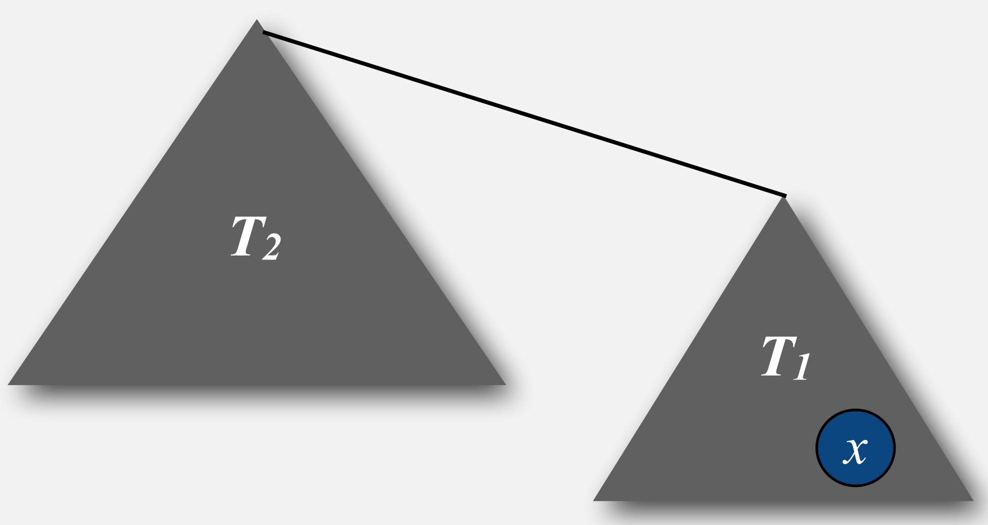
1



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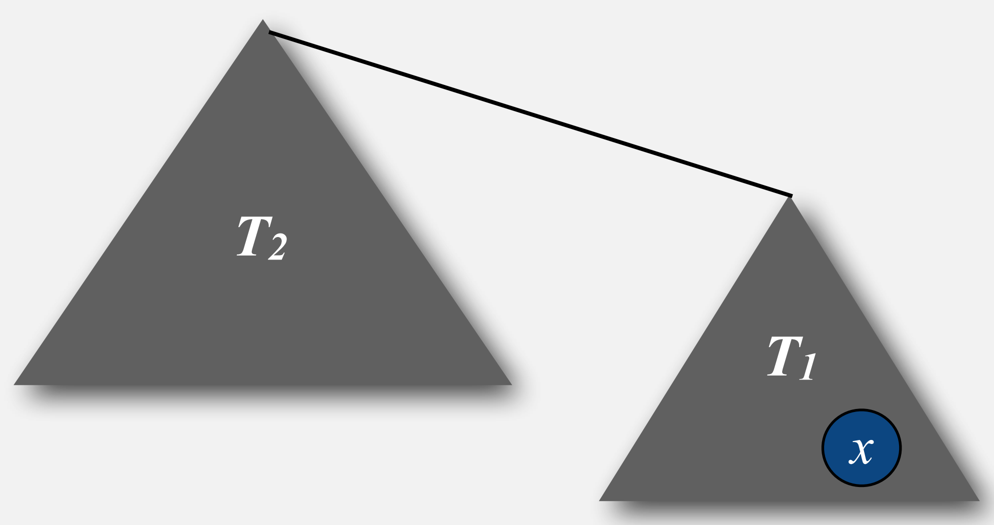
1

2



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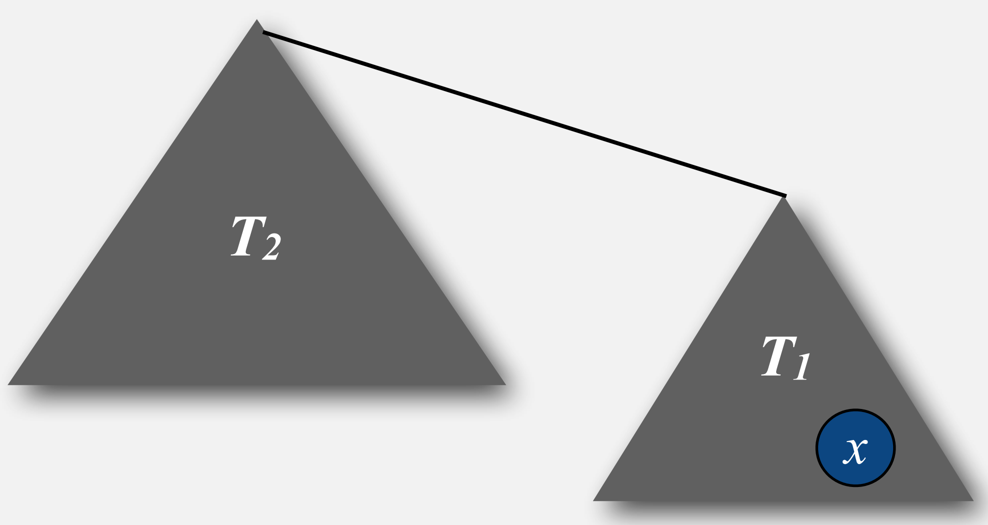
1

2

4



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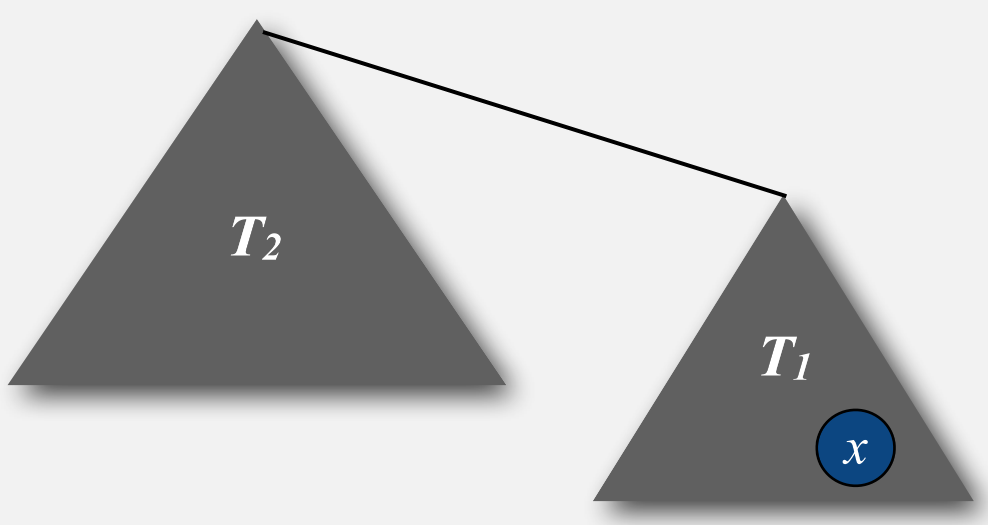
1

2

4

8



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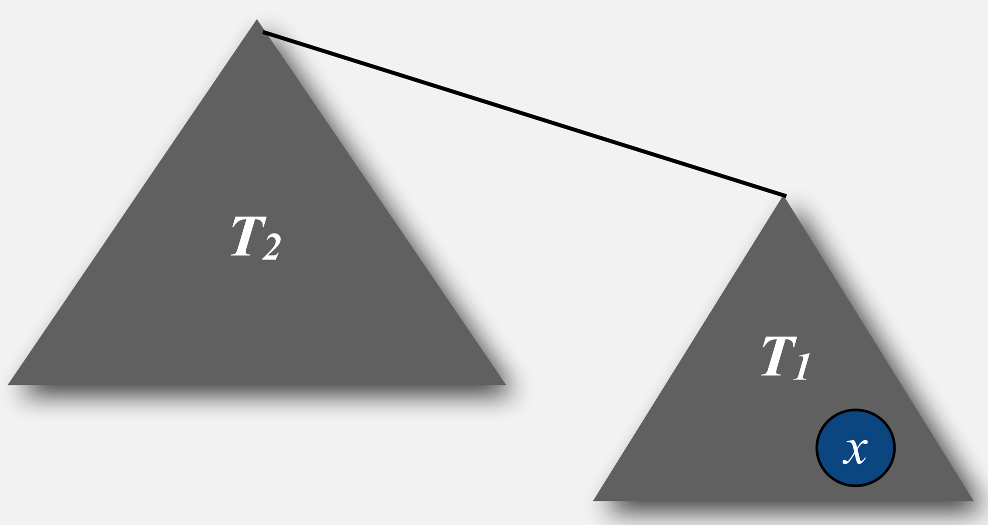

1

2

4

8

16

Weighted quick-union analysis

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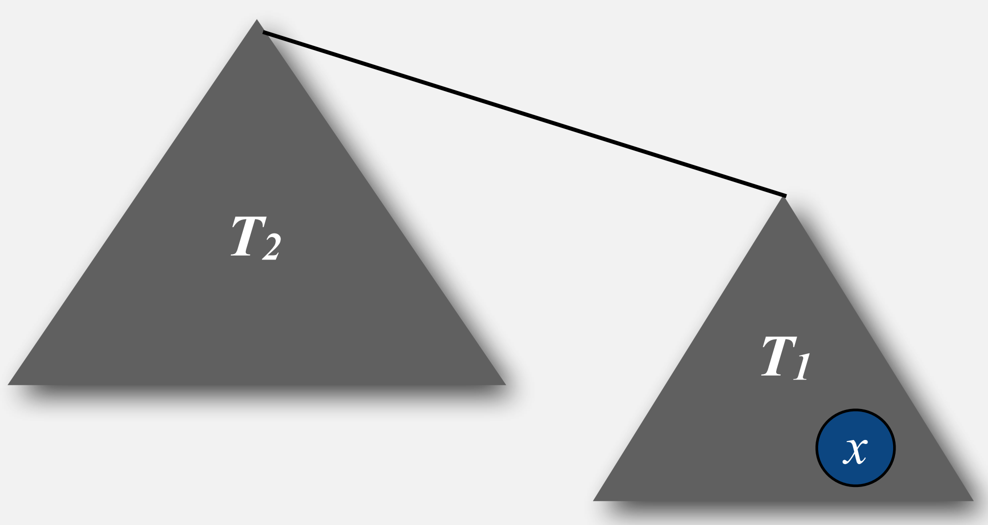

1
2
4
8
16
⋮

Weighted quick-union analysis

Running time.

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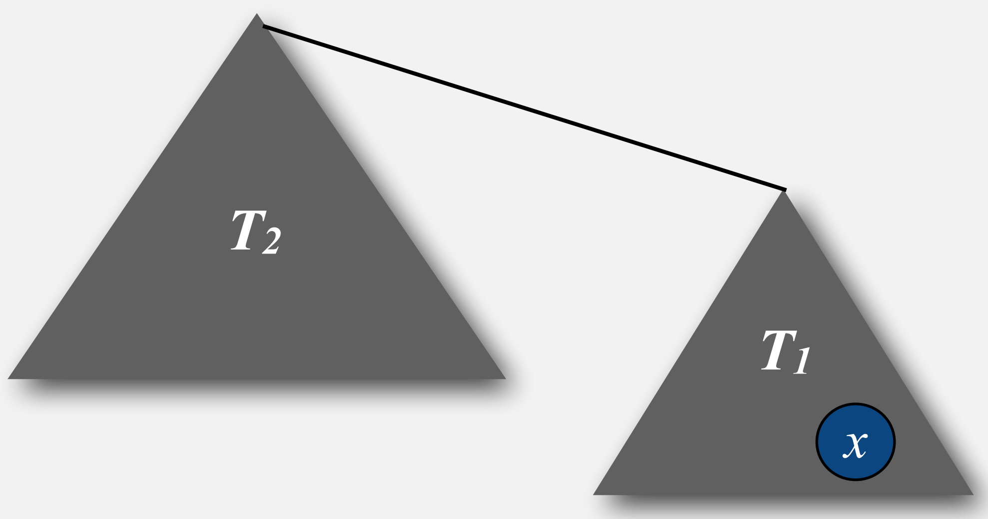

1

2

4

8

16

⋮

N

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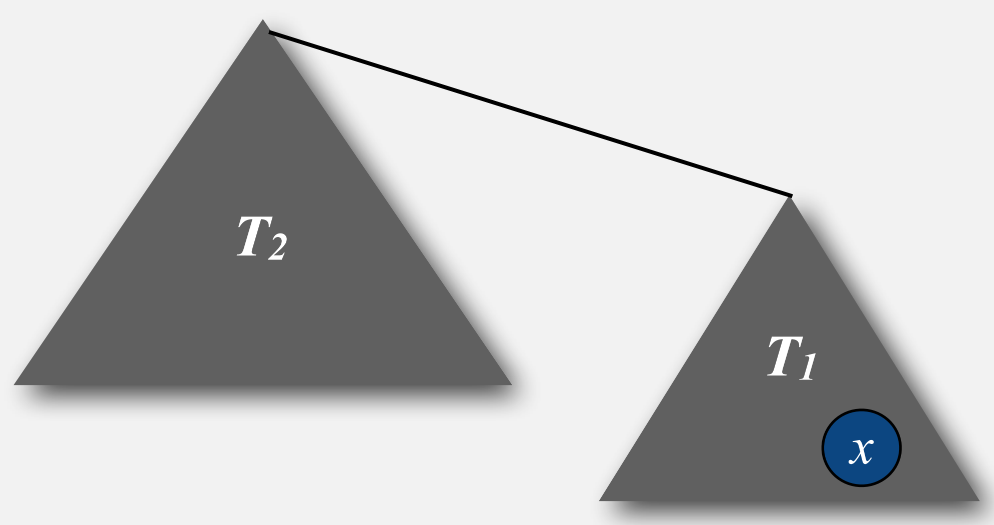

$$\left.\begin{array}{c} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ \vdots \\ N \end{array}\right\} \lg N$$

Weighted quick-union analysis

Running time.

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algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

† includes cost of finding roots

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

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quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

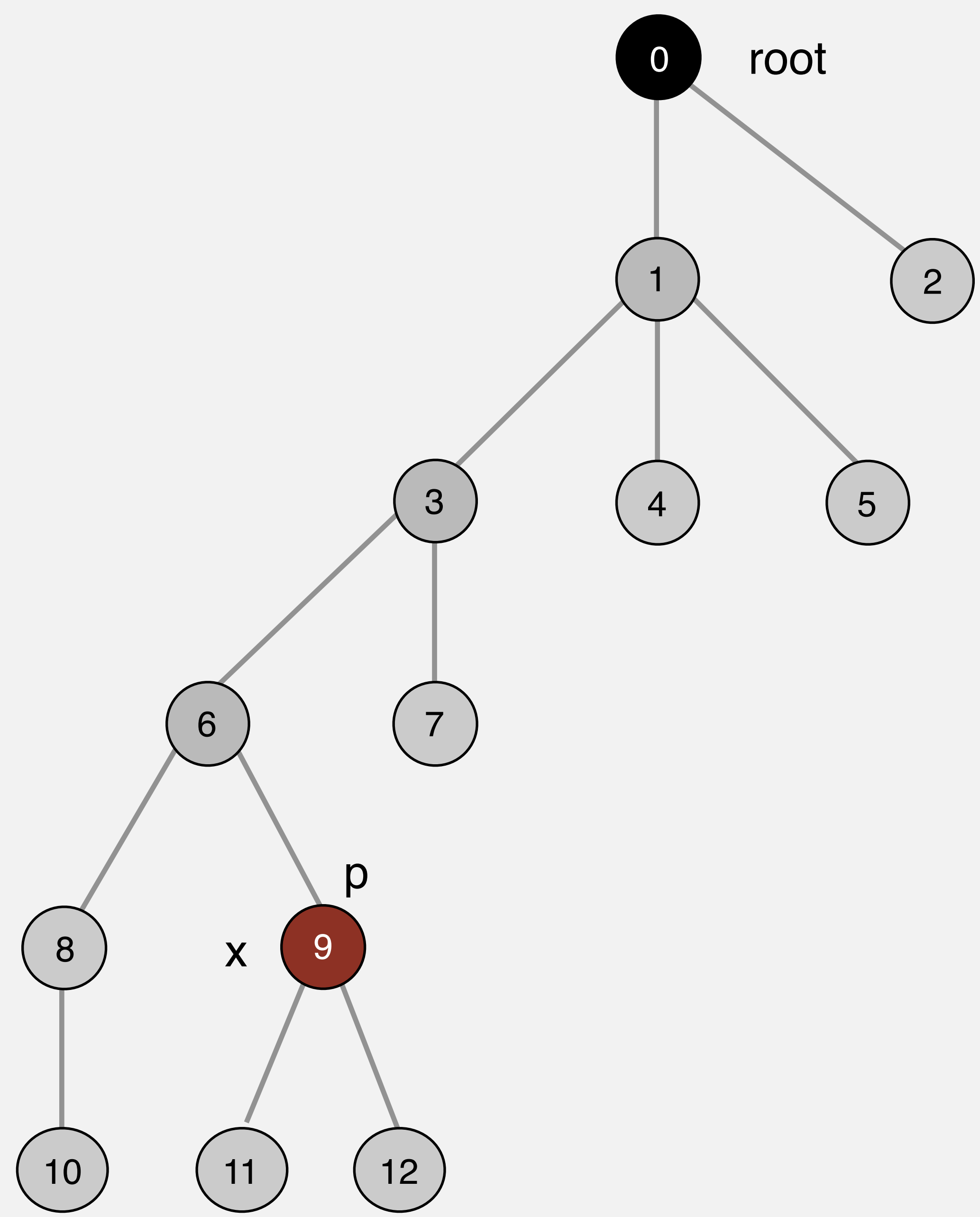
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

Improvement 2: path compression

Quick union with path compression. Just after computing the root of p,
set the id[] of each examined node to point to that root.



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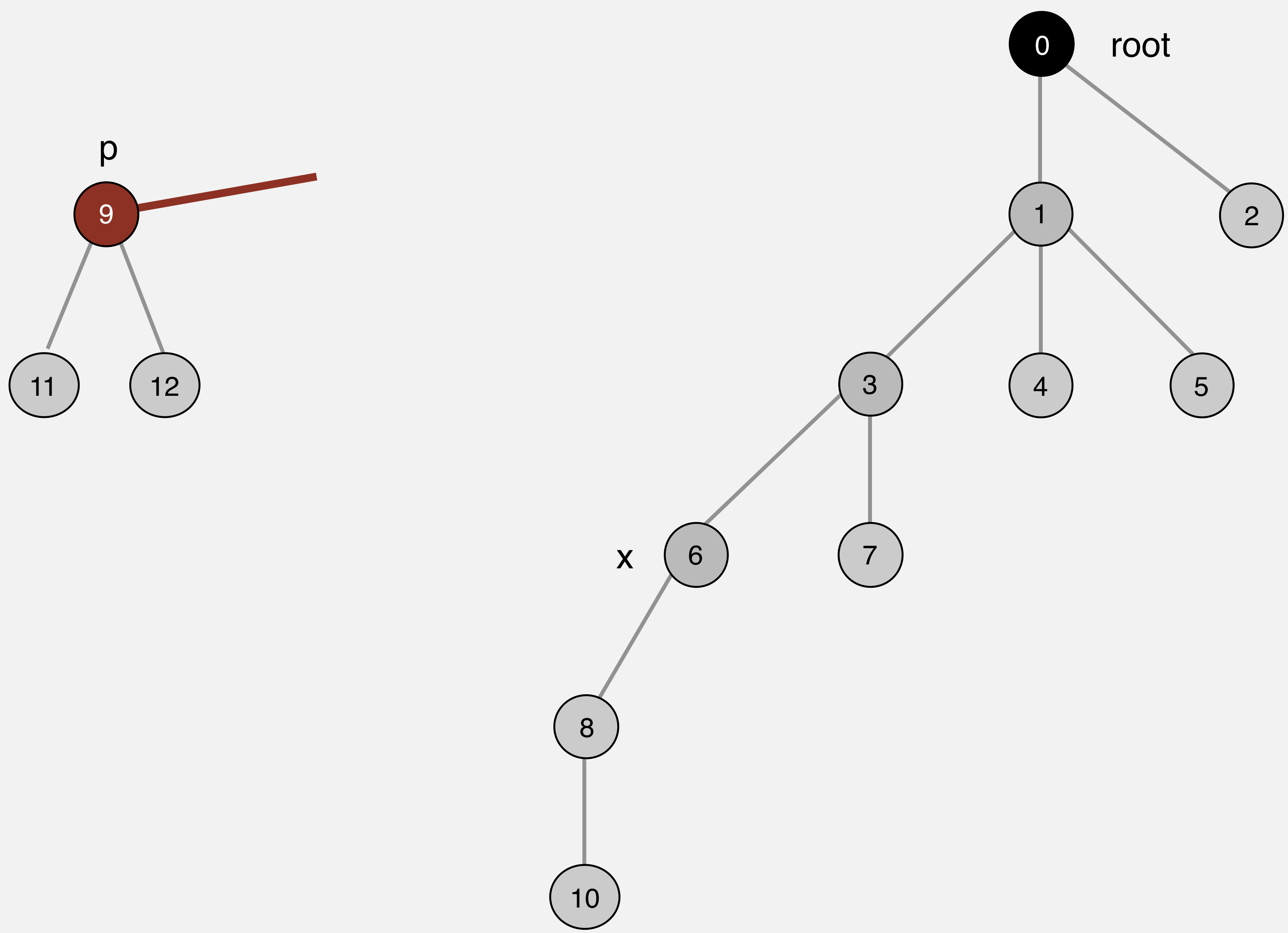

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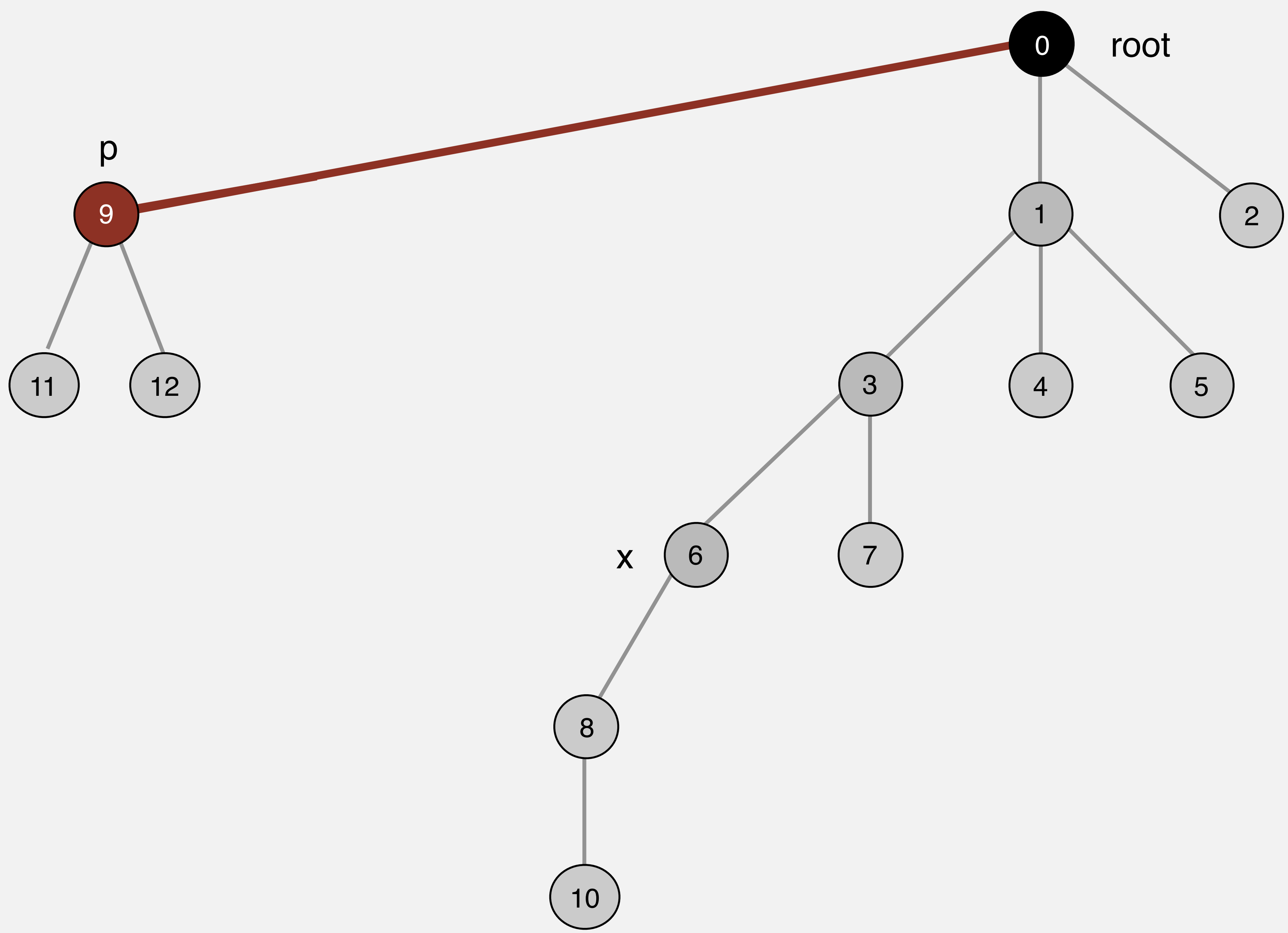

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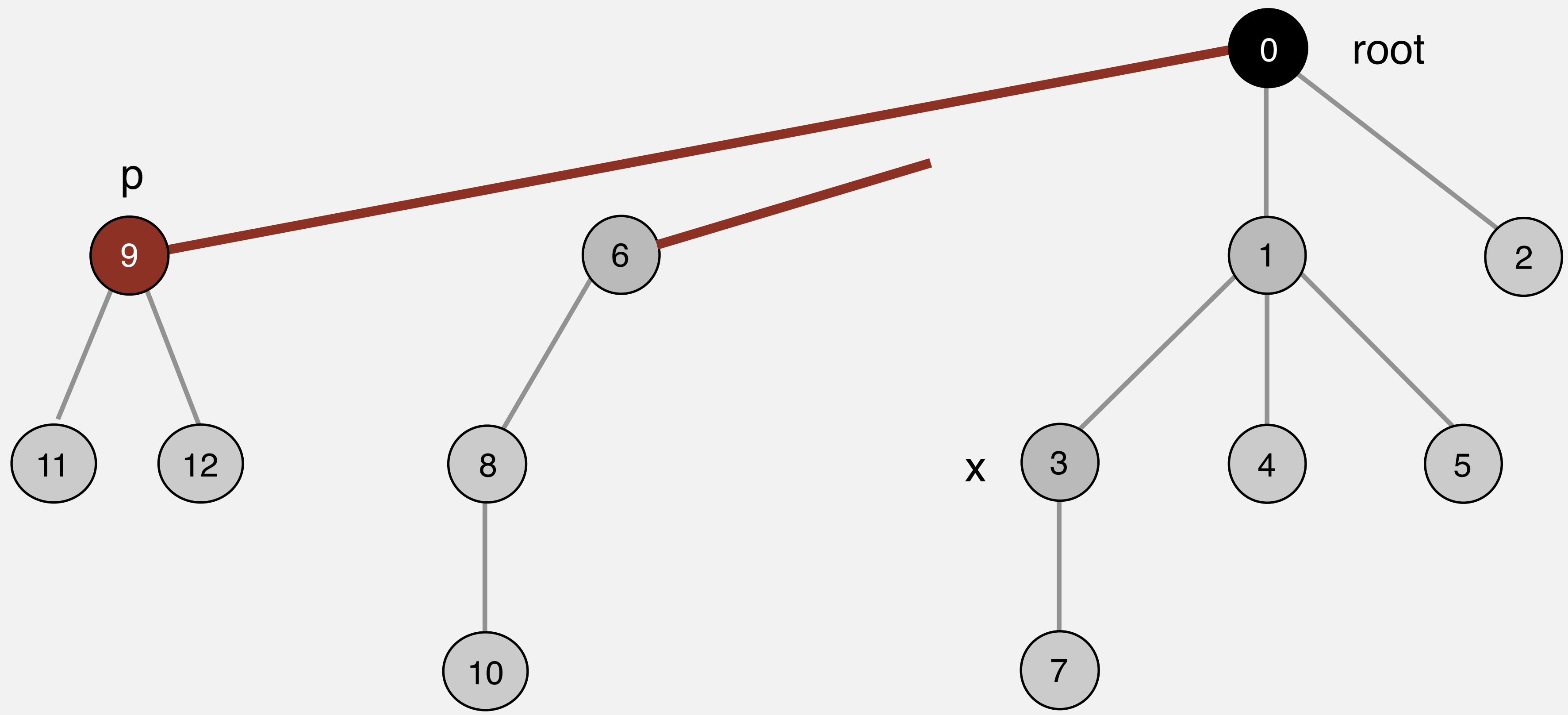

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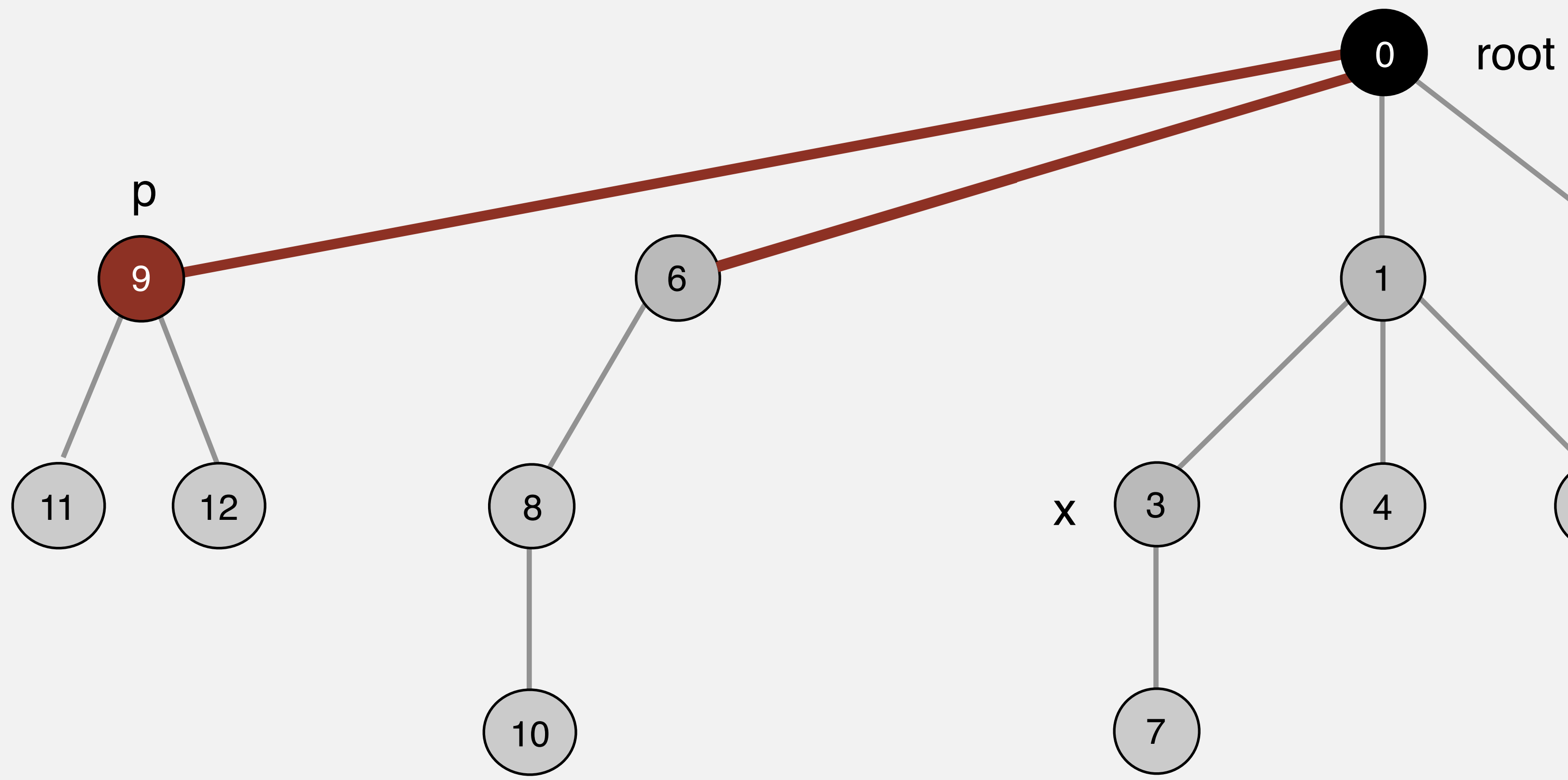

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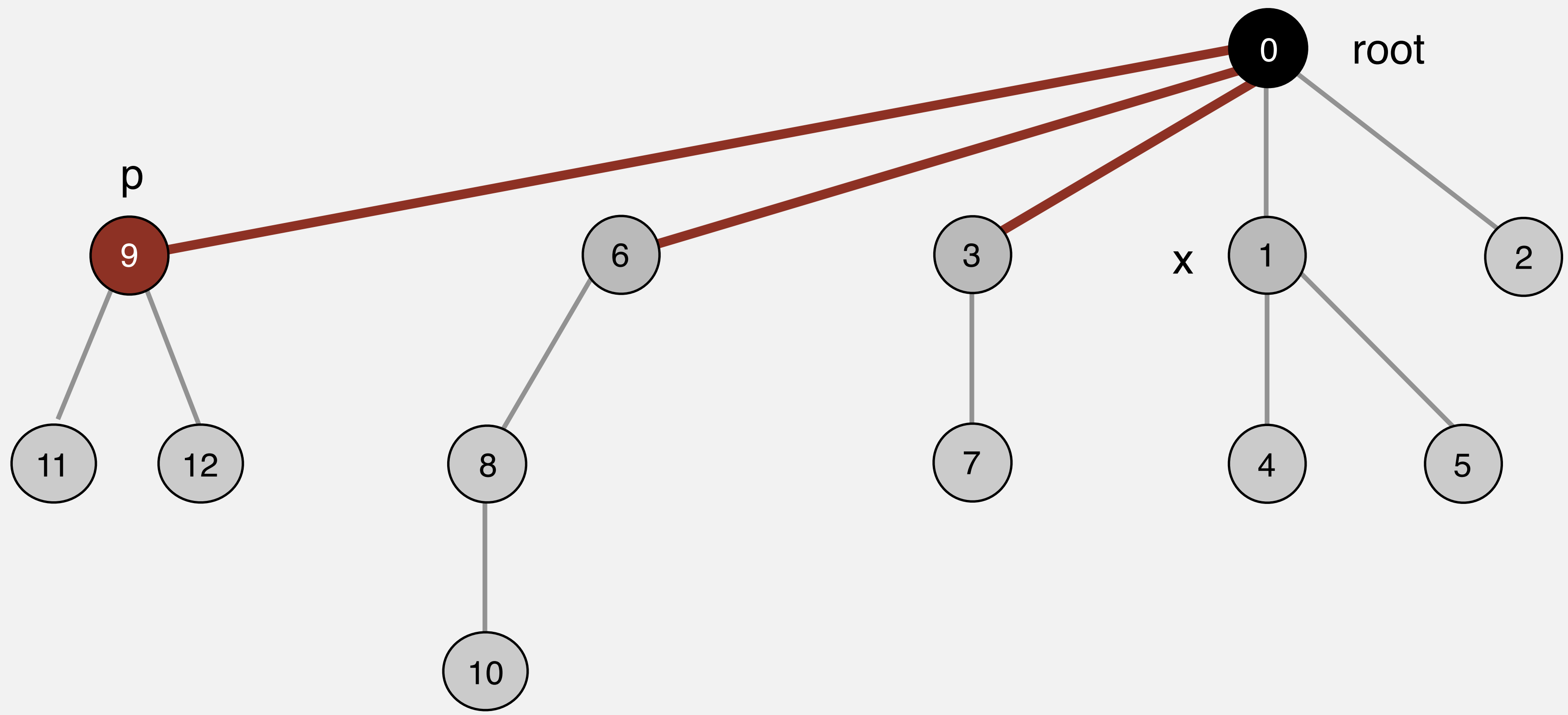

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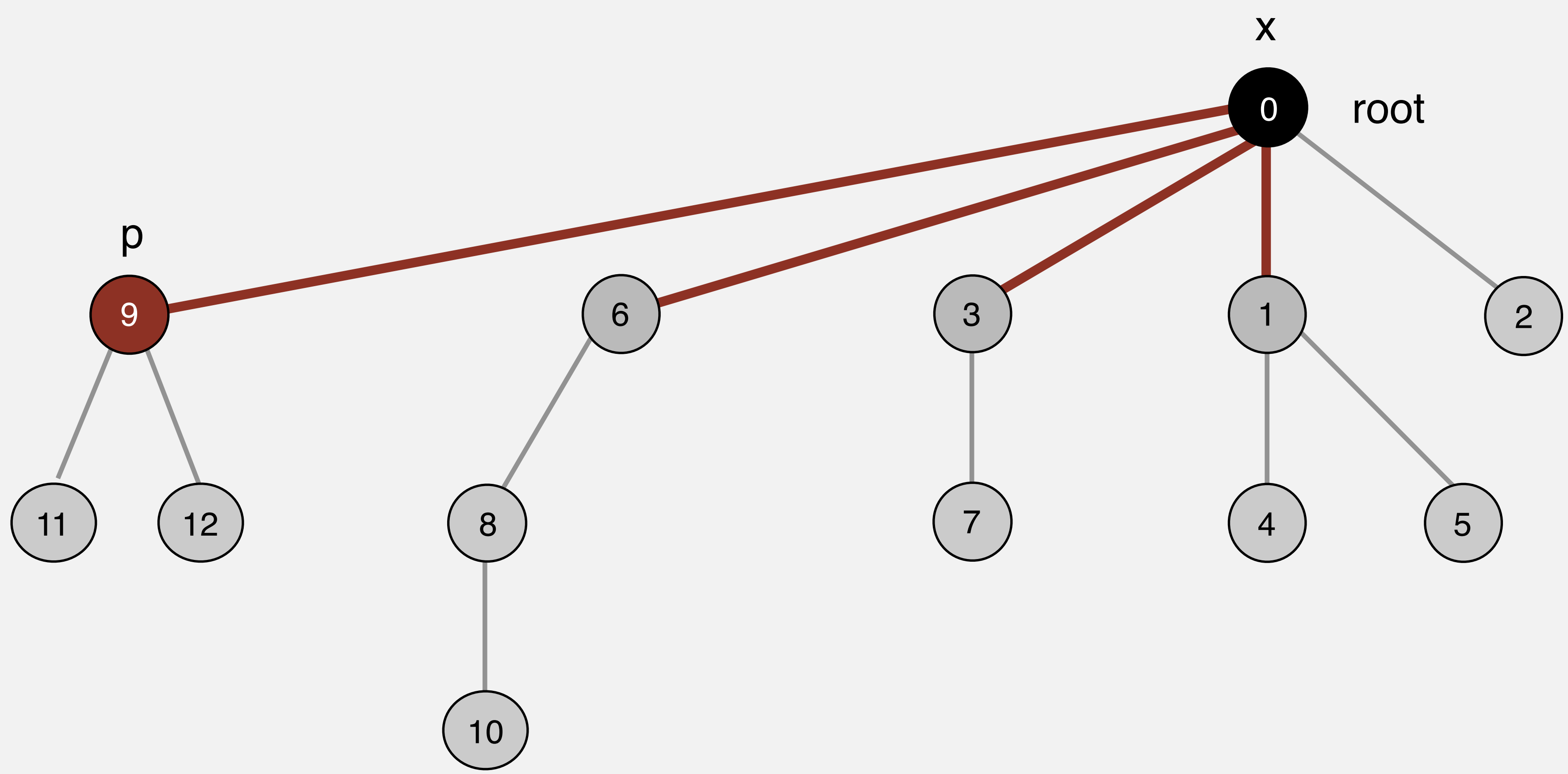

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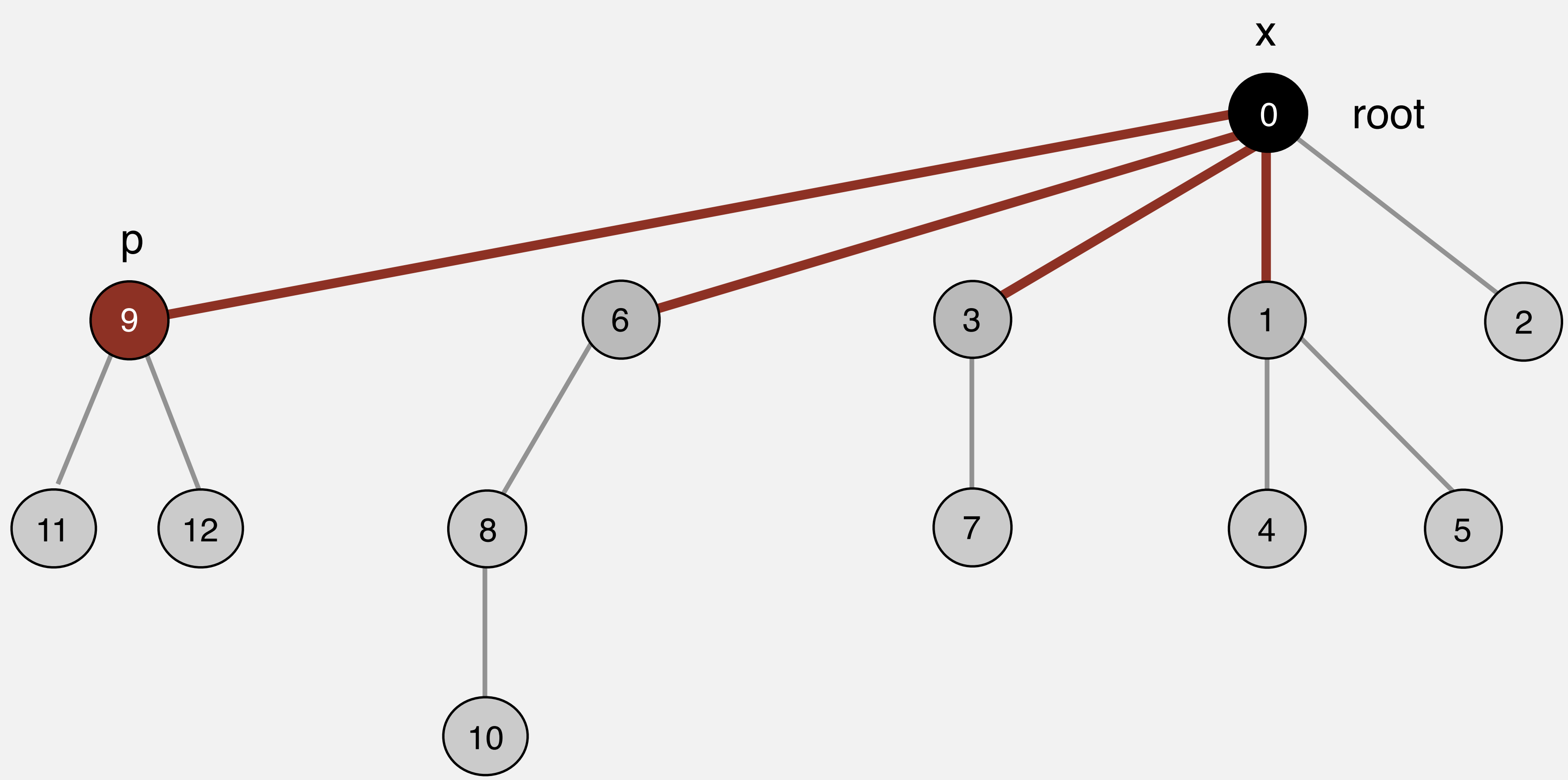

Bottom line. Now, find() has the side effect of compressing the tree.

Path compression: Java implementation

Two-pass implementation: add second loop to find() to set the id[] of each examined node to the root.

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Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

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public int find(int i) {
   while (i != id[i]) {
      id[i] = id[id[i]];
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only one extra line of code !

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In practice. No reason not to! Keeps tree almost completely flat.

Weighted quick-union with path compression: amortized analysis

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

iterated lg function

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Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of M union–find ops on N objects makes ≤ $c\,(N + M \lg^* N)$ array accesses.

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- Analysis can be improved to $N + M\,\alpha(M, N)$.
- Simple algorithm with fascinating mathematics.

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Linear-time algorithm for M union-find ops on N objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

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Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

Summary

Key point. Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
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quick-union	M N
weighted QU	N + M log N
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order of growth for M union-find operations on a set of N objects

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order of growth for M union-find operations on a set of N objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

Union-find applications

- Percolation.
- Games (Go, Hex).
- ✓ Dynamic connectivity.
- Least common ancestor.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.
- Morphological attribute openings and closings.
- Matlab's bwlabel() function in image processing.