Introduction to Category Theory and Homological Algebra

Lectures by Alexander Pavlov
Notes by Aleksandr Krasilnikov
kras1lnikoff.av@gmail.com

HSE University
Faculty of Mathematics
Spring 2025

Contents

Contents by Lecture

Last updated September 5, 2025.

LECTURE 1. JANUARY 21, 2025

COURSE OUTLINE

(1) Categories, functors, natural transformations

(2) Adjoint functors

(3) Limits and colimits

(4) Abelian categories

(5) Resolutions and derived functors

COURSE GRADING

(1) 3 quizzes: 10% each, 30% total

(2) Midterm exam: 30%

(3) Final exam: 30%

(4) Seminars: 10%

(5) Homework: ∼10%

1 Categories and Functors

1.1 Definition and Examples of Categories

Definition. A **category** $\underline{C}$ consists of the following data:

(i) a class of **objects** $\mathrm{Ob}(\underline{C})$;

(ii) for every $X, Y \in \mathrm{Ob}(\underline{C})$ a set (or a class) of **morphisms** $\mathrm{Hom}_{\underline{C}}(X, Y)$;

(iii) for every $X \in \mathrm{Ob}(\underline{C})$ an **identity morphism** $\mathrm{id}_X \equiv 1_X \in \mathrm{Hom}_{\underline{C}}(X, X)$;

(iv) for every $X, Y, Z \in \mathrm{Ob}(\underline{C})$ a **composition rule**

$$\circ : \mathrm{Hom}_{\underline{C}}(X, Y) \times \mathrm{Hom}_{\underline{C}}(Y, Z) \to \mathrm{Hom}_{\underline{C}}(X, Z)$$

satisfying the usual associativity and unitality relations.

A category $\underline{C}$ is called **locally small** if each hom-set $\mathrm{Hom}_{\underline{C}}(X, Y)$ is a set.

Remark. There are plenty of ways to denote the set of morphisms. For example,

$$\mathrm{Hom}_{\underline{C}}(X, Y) \equiv \underline{C}(X, Y) \equiv \mathrm{Maps}_{\underline{C}}(X, Y) \equiv \mathrm{Arr}_{\underline{C}}(X, Y).$$

Definition. Given a category $\underline{C}$, there is an **opposite category** $\underline{C}^{\mathrm{op}}$, whose

(i) objects are the same as in $\underline{C}$, i.e. $\mathrm{Ob}(\underline{C}^{\mathrm{op}}) := \mathrm{Ob}(\underline{C})$;

(ii) morphisms are «reversed», i.e. for all $X, Y \in \mathrm{Ob}(\underline{C}^{\mathrm{op}})$

$$\mathrm{Hom}_{\underline{C}^{\mathrm{op}}}(X,Y) := \mathrm{Hom}_{\underline{C}}(Y,X)$$

with the natural composition rule $f \circ_{\mathrm{op}} g := g \circ f$.

Examples:

(1) $\underline{\text{Sets}}$ is the category of sets and functions;

(2) $\underline{\text{Mon}}$ is the category of monoids and monoid homomorphisms;

(3) $\underline{\text{Grps}}$ is the category of group and group homomorphisms;

(4) $\underline{\text{Rings}}$ is the category of (all) rings and ring homomorphisms;

(5) $\underline{\text{CRings}}$ is the category of commutative rings;

(6) $\underline{\text{Fields}}$ is the category of fields and field homomorphisms;

(7) For a monoid M, there is a category $\underline{M\text{-Sets}}$, where

 (i) objects are sets X with an M-**action** $M \times X \to X$;

 (ii) morphisms are **equivariant maps**, i.e. functions $\varphi : X \to Y$ such that

$$\varphi(mx) = m\varphi(x)$$

for every $m \in M$ and $x \in X$. This can be expressed as a **commutative diagram**

$$\begin{tikzcd}
M \times X \arrow[r, "1_M \times \varphi"] \arrow[d] & M \times Y \arrow[d] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

meaning that tracing elements through all possible paths yields the same result:

$$\begin{tikzcd}
(m,x) \arrow[r, mapsto] \arrow[d, mapsto] & (m, \varphi(x)) \arrow[d, mapsto] \\
mx \arrow[r] & \varphi(mx) = m\varphi(x)
\end{tikzcd}$$

(8) $\underline{R\text{-Mod}}$ and $\underline{\text{Mod-}R}$ are the categories of left and right modules over a ring R.
If R is commutative, these notions coincide and we write

$$\underline{R\text{-Mod}} \equiv \underline{\text{Mod-}R} \equiv \underline{\text{Mod}}(R).$$

Moreover, if R is a field, this is the category of vector spaces

$$\underline{\text{Mod}}(R) \equiv \underline{\text{Vect}}(R).$$

(9) Top is the category of topological spaces and continuous maps.

(10) The category of metric spaces Met:

(i) Objects are pairs (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ a metric.

(ii) Morphisms $(X, d_1) \to (Y, d_2)$ are functions $f : X \to Y$ which are

a) isometries (the rigid version of Met), i.e.

$$d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X;$$

b) Lipschitz maps (categorically richer version of Met), i.e.

$$\exists K > 0 : \quad d(f(x), f(y)) \leq Kd(x, y) \quad \text{for all } x, y \in X.$$

(11) C^∞-manifolds, analytic manifolds, schemes, ...

(12) hTop is the homotopy category of topological spaces, which

(i) objects are the same as in Top;

(ii) morphisms are homotopy classes of continuous maps.

Definition (intuitive). A category is called **concrete** if objects are sets with additional structure and hom-sets are maps of sets respecting the additional structure.

Theorem (w/o proof). hTop *is not a concrete category.*

(13) The category of **binary relations** Rel:

(i) Objects of Ob(Rel) are sets.

(ii) Morphisms $X \to Y$ are binary relations $R \subset X \times Y$ with the usual notion of composition.

(14) Every monoid M can be considered a category $\underline{M}$, where

(i) there is a single object $*$;

(ii) morphisms $* \to *$ are labeled by elements of M, where composition is plain multiplication.

Definition. A **groupoid** $\underline{C}$ is a category where every arrow is inverible, which formally means that for every $f \in \underline{C}(X, Y)$ there is $g \in \underline{C}(Y, X)$ such that $fg = 1_Y$ and $gf = 1_X$.

Example. If G is a group, then $\underline{G}$ is a groupoid.

(15) If X is a topological space, then $\Pi_1(X)$, the **fundamental groupoid** of X:

(i) Objects of $\Pi_1(X)$ are points of X.

(ii) Morphisms $x \to y$ are homotopy classes of all paths from x to y.

This is a generalization of the fundamental group of X, since

$$\mathrm{Hom}_{\Pi_1(X)}(x, x) = \pi_1(X, x).$$

1.2 Special Morphisms

Let $f \in \underline{C}(X, Y)$ be a morphism in $\underline{C}$. We say that

Definitions:

(1) f is an **isomorphism** (iso) if it has an inverse $f^{-1} \in \underline{C}(Y, X)$.

(2) f is a **monomorphism** (mono) if $fg_1 = fg_2$ implies $g_1 = g_2$ for any $g_1, g_2 \in \underline{C}(Z, X)$.

(3) $f \in \underline{C}(X, Y)$ is an **epimorphism** (epi) if $f \in \underline{C}^{\mathrm{op}}(Y, X)$ is a monomorphism.

(4) Suppose $Y \xrightarrow{g} X \xrightarrow{f} Y$ with $f \circ g = 1_Y$. Then g is called a **section** (or a **coretraction**) of f and g is called a **retraction** (or a **cosection**) of g.

Lemma (Basic properties of monomorphisms and epimorphisms)**.**

(1) Composition of monomorphisms (resp. epimorphisms) is monic (resp. epic);

(2) If fg is monic, then g is monic;

(3) If fg is epic, then f is epic;

(4) Any section is a monomorphism;

(5) Any retraction is an epimorphism.

Proof. EXERCISE. □

Note that there are non-invertible maps that are both monomorphisms and epimorphisms.

Lemma. *For $f : X \to Y$, the following are equivalent:*

(1) f is an isomorphism;

(2) f is a retraction and a monomorphism;

(3) f is a section and an epimorphism.

Proof. EXERCISE. □

Remarks:

(1) Sections are also called **split monomorphisms**.

(2) Similarly, retractions are called **split epimorphisms**.

Definition. A category $\underline{C}'$ is a **subcategory** of a category $\underline{C}$ if

(i) its objects form a subclass of objects of $\underline{C}$, i.e. $\mathrm{Ob}(\underline{C}') \subset \mathrm{Ob}(\underline{C})$;

(ii) its morphisms are $\underline{C}'(X, Y) \subset \underline{C}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\underline{C}')$.

A subcategory is called **full** if $\underline{C}'(X, Y) = \underline{C}(X, Y)$ for each $X, Y \in \mathrm{Ob}(\underline{C}')$.

Lecture 2. January 28, 2025

Textbooks

(1) Leinster T., *Basic Category Theory*;

(2) Riehl E., *Category Theory in Context*;

(3) Herrlich H., Strecker G., *Category Theory (3rd Edition)*;

(4) Adámek J. et al., *Abstract and Concrete Categories. The Joy of Cats*;

(5) Mac Lane S., *Categories for a Working Mathematician*;

(6) Kashiwara M., Schapira P., *Categories and Sheaves*.

1.3 Functors

Definition. Let $\underline{C}$ and $\underline{C}'$ be categories. A (covariant) **functor** $F : \underline{C} \to \underline{C}'$ consists of

(i) a (class) function $F : \mathrm{Ob}(\underline{C}) \to \mathrm{Ob}(\underline{C}')$;

(ii) for all $X, Y \in \mathrm{Ob}(\underline{C})$, a function $F : \mathrm{Hom}_{\underline{C}}(X, Y) \to \mathrm{Hom}_{\underline{C}'}(F(X), F(Y))$,

such that

(1) $F(1_X) = 1_{F(X)}$ for all $X \in \mathrm{Ob}(\underline{C})$;

(2) $F(g \circ f) = F(g) \circ F(f)$ for all $f \in \mathrm{Hom}_{\underline{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\underline{C}}(Y, Z)$.

Examples:

(1) The **identity functor** $\mathrm{Id}_{\underline{C}} : \underline{C} \to \underline{C}$, defined by $F(X) := X$ and $F(f) := f$.

(2) For $Y \in \mathrm{Ob}(\underline{C}')$, the **constant functor** $\Delta_Y : \underline{C} \to \underline{C}'$, defined by $\Delta_Y(X) = Y$ and $\Delta_Y(f) = 1_Y$.

(3) For $\underline{C}$ locally small and $A \in \mathrm{Ob}(\underline{C})$, the **representable functor** $h_A : \underline{C} \to \underline{\mathrm{Sets}}$, also written as

$$h_A(-) := \mathrm{Hom}_{\underline{C}}(A, -),$$

which maps X to $\mathrm{Hom}_{\underline{C}}(A, X)$ and $f : B \to B'$ to $h_A(f) : \mathrm{Hom}_{\underline{C}}(A, B) \to \mathrm{Hom}_{\underline{C}}(A, B')$, defined on $g : A \to B$ as $h_A(f)(g) := gf$. There is also the contravariant version

$$h^A : \underline{C}^{\mathrm{op}} \to \underline{\mathrm{Sets}}, \qquad h^A(-) := \mathrm{Hom}_{\underline{C}}(-, A).$$

(4) For $n \in \mathbb{Z}_{>0}$, the general linear group defines a functor

$$GL_n : \underline{\mathrm{CRings}} \to \underline{\mathrm{Grps}}, \quad R \mapsto GL_n(R).$$

(5) The abelianization functor

$$(-)_{ab} : \underline{\mathrm{Grps}} \to \underline{\mathrm{Ab}}, \quad G \mapsto G/[G, G].$$

(6) Algebra of continuous functions to $\mathbb{R}$ is a functor

$$C : \underline{\mathrm{Top}}^{\mathrm{op}} \to \underline{\mathrm{Alg}}(\mathbb{R}), \qquad X \mapsto C(X) := \underline{\mathrm{Cont}}(X, \mathbb{R}).$$

(7) For k a field, the dual vector space functor

$$D : \underline{\mathrm{Vect}}(k)^{\mathrm{op}} \to \underline{\mathrm{Vect}}(k), \qquad V \mapsto V^{\vee} := \mathrm{Hom}_k(V, k).$$

Definition. A **contravariant** functor $F : \underline{C} \to \underline{C}'$ is simply a covariant functor $F : \underline{C}^{\mathrm{op}} \to \underline{C}'$.

Definition. A functor $F : \underline{C} \to \underline{C}'$ is called

(1) **faithful** if $F : \mathrm{Hom}_{\underline{C}}(X, Y) \to \mathrm{Hom}_{\underline{C}'}(F(X), F(Y))$ is injective for all $X, Y \in \mathrm{Ob}(\underline{C})$;

(2) **full** if $F : \mathrm{Hom}_{\underline{C}}(X, Y) \to \mathrm{Hom}_{\underline{C}'}(F(X), F(Y))$ is surjective for all $X, Y \in \mathrm{Ob}(\underline{C})$;

(3) **fully faithful** (f.f.) if F is full and faithful;

(4) **essentially surjective** (e.s.) if every $Y \in \mathrm{Ob}(\underline{C}')$ is isomorphic to $F(X)$ for some $X \in \mathrm{Ob}(\underline{C})$.

Definition. Given a family of categories $\{\underline{C}_i\}_{i\in I}$, we define the **product** $\prod\limits_{i\in I} \underline{C}_i$ by saying that

(i) $\mathrm{Ob}(\prod\limits_{i\in I} \underline{C}_i) := \prod\limits_{i\in I} \mathrm{Ob}(\underline{C}_i)$;

(ii) $\prod\limits_{i\in I} \underline{C}_i(\{X_i\}, \{Y_i\}) := \prod\limits_{i\in I} \mathrm{Hom}_{\underline{C}_i}(X_i, Y_i)$.

Definition. A functor $F : \underline{C}_1 \times \underline{C}_2 \to \underline{D}$ is also called a **bifunctor**.

Examples:

(1) Hom-set is a bifunctor

$$\mathrm{Hom}_{\underline{C}}(-, -) : \underline{C}^{\mathrm{op}} \times \underline{C} \to \underline{\mathrm{Sets}}.$$

(2) For a ring R, tensor product is a bifunctor

$$- \otimes_R - : \underline{\text{Mod-}R} \times \underline{R\text{-Mod}} \to \underline{\mathrm{Ab}}.$$

Definition. Categories $\underline{C}$ and $\underline{C}'$ are called **isomorphic** (written $\underline{C} \cong \underline{C}'$) if there exist functors $F : \underline{C} \to \underline{C}'$ and $G : \underline{C}' \to \underline{C}$ such that $F \circ G = \mathrm{id}_{\underline{C}'}$ and $G \circ F = \mathrm{id}_{\underline{C}}$.

Examples:

(1) $\underline{\mathbb{Z}\text{-Mod}} \cong \underline{\mathrm{Ab}}$.

(2) For a finite group G,

$$\underline{\mathrm{Rep}_k}(G) \cong \underline{k[G]\text{-Mod}}.$$

1.4 Natural Transformations

Definition. Let $F, G : \underline{C} \to \underline{D}$ be functors. A **natural transformation**

$$\theta : F \Rightarrow G$$

is a morphism $\theta_X : F(X) \to G(X)$ (a **component** of the transformation θ) for each $X \in \mathrm{Ob}(\underline{C})$ such that (**naturality condition**) given any $f : X \to Y$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\theta_X"'] & F(Y) \arrow[d, "\theta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

is commutative.

Remarks:

(1) If F is a functor, there is the **identity transformation** $1_F : F \Rightarrow F$.

(2) For $\theta : F \Rightarrow G$ and $\lambda : G \Rightarrow H$, the **composition** $\lambda \circ \theta : F \Rightarrow H$

$$\begin{tikzcd} \underline{C} \arrow[r, bend left=50, "F"] \arrow[r, "G"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3, "\theta\Downarrow"] \arrow[r, phantom, shift right=3, "\lambda\Downarrow"] & \underline{D} \end{tikzcd}$$

is defined by $(\lambda \circ \theta)_X = \lambda_X \circ \theta_X$.

(3) $\theta : F \Rightarrow G$ is a **natural isomorphism** if θ_X is an isomorphism for each $X \in \mathrm{Ob}(\underline{C})$.

If we fix $\underline{C}$ and $\underline{D}$, there is the **category of functors**

$$\underline{\mathrm{Func}}(\underline{C}, \underline{D}) = [\underline{C}, \underline{D}] = \underline{D}^{\underline{C}}.$$

And for functors $F, G : \underline{C} \to \underline{D}$, there is a **category of natural transformations**

$$\underline{\mathrm{Nat}}(F, G) := [\underline{C}, \underline{D}](F, G).$$

Examples:

(1) A transformation from Id to $(-)_{ab}$:

$$\begin{tikzcd} \underline{\mathrm{Grps}} \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "(-)_{ab}"'] \arrow[r, phantom, "\Downarrow\theta"] & \underline{\mathrm{Grps}} \end{tikzcd} \qquad G \xrightarrow[\theta_G]{} G_{ab}$$

(2) A transformation from Id to D^2:

$$\begin{tikzcd} \underline{\mathrm{Vect}}(k) \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "D^2"'] \arrow[r, phantom, "\Downarrow\theta"] & \underline{\mathrm{Vect}}k \end{tikzcd} \qquad V \xrightarrow[\theta_V]{} V^{\vee\vee}$$

(3) Determinant of a matrix as a natural transformation:

$$\begin{tikzcd} \underline{\mathrm{CRings}} \arrow[r, bend left, "\underline{\mathrm{GL}}_n"] \arrow[r, bend right, "\underline{\mathrm{GL}}_1=\underline{\mathrm{G}}_m"'] \arrow[r, phantom, "\Downarrow\det"] & \underline{\mathrm{Grps}} \end{tikzcd}$$

(4) For I a set (considered as a discrete category):

$$[I, \underline{C}] = \prod_{i \in I} \underline{C}.$$

(5) Given $I = (\bullet \rightrightarrows \bullet)$:

$$[I, \underline{\mathrm{Sets}}] = \underline{\mathrm{Graphs}}.$$

(6) $[\underline{G}, \underline{\mathrm{Sets}}] = \underline{G\text{-}\mathrm{Sets}}$ and $[\underline{G}, \underline{\mathrm{Vect}}(k)] = \underline{\mathrm{Rep}_{\mathrm{k}}}(G)$.

1.5 Equivalence of Categories

Definition. Categories $\underline{C}$ and $\underline{D}$ are called **equivalent** if there are functors $F : \underline{C} \to \underline{D}$, $G : \underline{D} \to \underline{C}$ and natural isomorphisms $\alpha : G \circ F \Rightarrow \mathrm{Id}_{\underline{C}}$, $\beta : F \circ G \Rightarrow \mathrm{Id}_{\underline{D}}$. Notation: $\underline{C} \simeq \underline{D}$.

Lemma. *Let $F : \underline{C} \to \underline{D}$ and $i : \underline{D_0} \hookrightarrow \underline{D}$ such that $\underline{D_0}$ is full and for all $X \in \mathrm{Ob}(\underline{C})$ there is $Y \in \mathrm{Ob}(\underline{D_0})$ such that $F(X) = Y$. Then there exists a functor $F_0 : \underline{C} \to \underline{D_0}$ and a natural isomorphism $\theta_0 : F \xrightarrow{\sim} iF_0$.*

Proof. Using the axiom of choice, we choose for each $X \in \mathrm{Ob}(\underline{C})$ an object $Y \in \underline{D}_0$ and an isomorphism $\varphi_X : Y \to F(X)$. Now set $F_0(X) := Y$. Given $f : X \to X'$, let

$$F_0(X) \xrightarrow[\varphi_X]{} F(X) \xrightarrow[F(f)]{} F(X') \xrightarrow[\varphi_{X'}^{-1}]{} F_0(X')$$

be the value of $F_0(f)$. This defines a functor since for any $f : X \to X'$ and $g : X' \to X''$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"'] & F(X') \arrow[r, "F(g)"'] & F(X'') \\ Y = F_0(X) \arrow[u, "\varphi_X"] \arrow[r, "F_0(f)"'] & Y' = F_0(X') \arrow[u, "\varphi_{X'}"] \arrow[r, "F_0(g)"'] & Y'' = F_0(X'') \arrow[u, "\varphi_{X''}"] \end{tikzcd}$$

It is clear that $\varphi : F \Rightarrow iF_0$ is the required natural transformation. □

Corollary. *For any category $\underline{C}$ there is a subcategory $\mathrm{sk}(\underline{C})$, the **skeleton** of $\underline{C}$, such that $i : \mathrm{sk}(\underline{C}) \hookrightarrow \underline{C}$ is an equivalence of categories, and in $\mathrm{sk}(\underline{C})$ any two issomorphic objects are equal.*

Proof. EXERCISE. □

Lecture 3. February 4, 2025

Theorem. *A functor $F : \underline{C} \to \underline{D}$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Proof. We introduce the following notation:

$$\begin{tikzcd} \mathrm{sk}\,\underline{C} \arrow[rr, "j_D F i_C"] \arrow[d, hook, shift left, "i_C"] & & \mathrm{sk}\,\underline{D} \arrow[d, hook, shift left, "i_D"] \\ \underline{C} \arrow[rr, "F"] \arrow[u, shift left, "j_C"] & & \underline{D} \arrow[u, shift left, "j_D"] \end{tikzcd}$$

Note that $j_D F i_C$ is surjective on objects. We will show that it is also injective. Indeed,

$$\varphi : \underline{C}(A, B) \xrightarrow{\sim} \underline{D}(F(A), F(B))$$

is a bijection for any $A, B \in \mathrm{Ob}(\underline{C})$. Suppose $h : F(A) \xrightarrow{\sim} F(B)$ is an isomorphism. Then $\varphi^{-1}(h)$ is also an isomorphism, hence $A = B$ in $\mathrm{sk}(\underline{C})$. This shows that $j_D F i_C$ is an isomorphism of categories, so there is an inverse functor $K : \mathrm{sk}\,\underline{D} \to \mathrm{sk}\,\underline{C}$. It follows that F is an equivalence of categories and $i_D K j_C$ is a quasi-inverse for F. □

1.6 Yoneda's Lemma

Let $\underline{C}^\vee = [\underline{C}, \underline{\mathrm{Sets}}]$. Recall that $h_A = \underline{C}(A, -) : \underline{C} \to \underline{\mathrm{Sets}}$ is an object of $\mathrm{Ob}(\underline{C}^\vee)$.

Lemma (Yoneda). *For any $F \in \mathrm{Ob}(\underline{C}^\vee)$ and $A \in \mathrm{Ob}(\underline{C})$,*

$$\underline{C}^\vee(h_A, F) \overset{\varphi}{\cong} F(A).$$

Moreover, φ is natural in A and F.

Proof. Let $\theta \in \underline{C}^\vee(h_A, F)$ and $f : A \to B$. Then

$$\begin{tikzcd} 1_A \in h_A(A) \arrow[r, "\theta_A"] \arrow[d, "h_A(f)"'] & F(A) \arrow[d, "F(f)"] \\ h_A(B) \arrow[r, "\theta_B"'] & F(B) \end{tikzcd}$$

and hence

$$F(f) \circ \theta_A = \theta_B \circ h_A(f),$$

which implies

$$F(f) \circ \theta_A(1_A) = \theta_B\big(h_A(f)(1_A)\big) = \theta_B(f).$$

NATURALITY IN A:

$$\begin{tikzcd}
\underline{C}^\vee(h_A, F) \arrow[r, "\theta\mapsto\theta\circ h_f"] \arrow[d, "\theta\mapsto\theta_A(1_A)"'] & \underline{C}^\vee(h_B, F) \arrow[d, "\theta'\mapsto\theta'_B(1_B)"] \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

We note that

$$(\theta \circ h_f)(1_B) = \theta \circ (h_f)_B(1_B) = \theta_B(f \circ 1_B) = \theta_B(f) = F(f)\theta_A(1_A).$$

NATURALITY IN F:

Let $\alpha : F \Rightarrow F'$. It follows that $(\alpha \circ \theta)(1_A) = \alpha_A \circ \theta_A(1_A)$.

$$\begin{tikzcd}
\theta \in \underline{C}^\vee(h_A, F) \arrow[r, "\alpha\circ(-)"] \arrow[d, "\cong"'] & \underline{C}^\vee(h_A, F') \arrow[d, "\cong"] \\
F(A) \arrow[r, "\alpha_A"'] & F(B)
\end{tikzcd}$$

□

Corollary. *Functor $h_{(-)} : \underline{C}^{\text{op}} \to \underline{C}^\vee$ (defined by $A \mapsto h_A$) is fully faithful.*

Proof. By Yoneda's lemma, $\underline{C}^\vee(h_A, h_B) = h_B(A) = \underline{C}(B, A)$. □

Let $\underline{\hat{C}} = [\underline{C}^{\text{op}}, \underline{\text{Sets}}]$. The dual statement of Yoneda's lemma:

$$\underline{\hat{C}}(h^A, F) = F(A),$$

where F and $h^A = \underline{C}(-, A) \in \text{Ob}(\underline{\hat{C}})$. It follows that $h^{(-)} : \underline{C} \to \underline{\hat{C}}$ is fully faithful.

Definition. A functor $F : \underline{C} \to \underline{\text{Sets}}$ is called **representable** if there is a natural isomorphism $\theta : F \xrightarrow{\sim} h_A$ for some object $A \in \text{Ob}(\underline{C})$.

1.7 The Godement's Product (a.k.a. the Horizontal Composition)

Recall that for natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ its **vertical composition**

$$\begin{tikzcd}
\underline{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=50, "H"'{name=H}] & \underline{D}
\arrow[from=F, to=G, Rightarrow, "\alpha"']
\arrow[from=G, to=H, Rightarrow, "\beta"']
\end{tikzcd}$$

is defined by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ for all $X \in \mathrm{Ob}(\underline{C})$.

Now consider the following diagram:

$$\begin{tikzcd} \underline{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \underline{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \underline{C} \end{tikzcd}$$

We will define the **horizontal composition** $\beta * \alpha$,

$$\begin{tikzcd} \underline{A} \arrow[r, bend left, "HF"] \arrow[r, bend right, "KG"'] \arrow[r, phantom, "\beta*\alpha\Downarrow"] & \underline{C} \end{tikzcd}$$

by saying that given $X \in \mathrm{Ob}(\underline{C})$, one has

$$(\beta * \alpha)_X := \beta_{G(X)} \circ H(\alpha_X) = K(\alpha_X) \circ \beta_{F(X)},$$

taking advantage of the fact that the following diagram commutes:

$$\begin{tikzcd} HF(X) \arrow[r, "\beta_{F(X)}"] \arrow[d, "H(\alpha_X)"'] \arrow[dr, dashed, "(\beta*\alpha)_X" description] & KF(X) \arrow[d, "K(\alpha_X)"] \\ HG(X) \arrow[r, "\beta_{G(X)}"'] & KG(X) \end{tikzcd}$$

Examples:

(1) If $F = G$ and $\alpha = 1_F$,

$$\beta_F := \beta * 1_F : HF \to KF, \qquad (\beta F)_X = \beta_{F(X)}.$$

(2) If $H = K$ and $\beta = 1_H$,

$$H\alpha := 1_H * \alpha : HF \to HG, \qquad (H\alpha)_X = H(\alpha_X).$$

Redrawing the diargam using new notation,

$$\begin{tikzcd} HF \arrow[r, "\beta_F"] \arrow[d, "H\alpha"'] \arrow[dr, dashed, "\beta*\alpha" description] & KF \arrow[d, "K\alpha"] \\ HG \arrow[r, "\beta_G"'] & KG \end{tikzcd}$$

we note that

$$(1_K * \alpha) \circ (\beta * 1_F) = (\beta * 1_G) \circ (1_H * \alpha) = \beta * \alpha.$$

Proposition. *Properties of Godement's product:*

(1) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

(2) ***2-associativity*** *or* ***interchange law****:*

$$\begin{tikzcd}[column sep=huge]
\underline{A} \arrow[r, bend left=50, "F"] \arrow[r, "G"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & \underline{B} \arrow[r, bend left=50, "F'"] \arrow[r, "G'"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, shift left=3, "\alpha'\Downarrow"] \arrow[r, phantom, shift right=3, "\beta'\Downarrow"] & \underline{C}
\end{tikzcd}$$

$$(\beta' \circ \alpha') * (\beta \circ \alpha) = (\beta' * \beta) \circ (\alpha' * \alpha).$$

Proof. EXERCISE. □

Remark. Let $\underline{A}$ and $\underline{B}$ be categories. Define a functor

$$\mathrm{Comp} : [A, B] \times [B, C] \to [A, C]$$

on objects and morphisms by letting

(1) $\mathrm{Comp}(F, H) = H \circ F$;

(2) $\mathrm{Comp}(\alpha, \beta) = \beta * \alpha$.

Remark. Godement's product is useful when defining 2-categories.

LECTURE 4. FEBRUARY 11, 2025

1.8 Adjoint Functors

Definition. Functors $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$ are called **adjoint** (F is **left adjoint** of G and G is **right adjoint** of F; we write $F \vdash G$) if there is a binatural bijection

$$\underline{D}(F(X), Y) \cong \underline{C}(X, G(Y)).$$

MOTIVATION

Let k be a field and S be a set. We write

$$k^{(S)} = k^{\oplus S} := \bigoplus_{s \in S} k.$$

If $U : \underline{\text{Vect}} \to \underline{\text{Sets}}$ is the forgetful functor, then

$$\text{Hom}_{\underline{\text{Vect}}}(k^{(S)}, V) \cong \text{Hom}_{\underline{\text{Sets}}}(S, U(V)).$$

It turns out that the isomorphism is also binatural, implying that $k^{(-)}$ is left adjoint to U.

BINATURALITY

(1) Naturality in the first argument

$$\begin{tikzcd} X' \arrow[d, "h"'] & \underline{D}(F(X), Y) \arrow[r, "\varphi^{-1}_{X,Y}"] \arrow[d, "\underline{D}(F(h), Y)"'] & \underline{C}(X, G(Y)) \arrow[d, "\underline{C}(h, G(Y))"] \\ X & \underline{D}(F(X'), Y) \arrow[r, "\varphi^{-1}_{X',Y}"'] & \underline{C}(X', G(Y)) \end{tikzcd}$$

(2) Naturality in the second argument

$$\begin{tikzcd} Y \arrow[d, "h"'] & \underline{D}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\underline{D}(F(X), h)"'] & \underline{C}(X, G(Y)) \arrow[d, "\underline{C}(X, G(h))"] \\ Y' & \underline{D}(F(X), Y') \arrow[r, "\varphi_{X,Y'}"'] & \underline{C}(X, G(Y')) \end{tikzcd}$$

Given the bijection

$$\varphi_{X,Y} : \underline{D}(F(X), Y) \cong \underline{C}(X, G(Y)),$$

it is convenient to speak of transposes: for a morphism $g : F(X) \to Y$, its **transpose** is

$$\bar{g} := \varphi_{X,Y}(g) : X \to G(Y);$$

conversely, for $f : X \to G(Y)$, its **transpose** is

$$\bar{f} := \varphi^{-1}_{X,Y}(f) : F(X) \to Y.$$

These assignments are mutually inverse, so $\bar{\bar{g}} = g$ and $\bar{\bar{f}} = f$.

Then the two naturalities can be phrased succinctly in this language:

(1) For every $h : X' \to X$, transposition commutes with precomposition:

$$\overline{fh} \;=\; \bar{f}\,F(h).$$

(2) For every $h : Y \to Y'$, transposition commutes with postcomposition:

$$\overline{hg} \;=\; G(h)\,\bar{g}.$$

Let now $X \in \mathrm{Ob}(\underline{C})$ and set $Y = F(X)$. Define

$$\eta_X := \bar{1}_{F(X)} : X \to GF(X).$$

This map is called the **unit** of the adjunction: it is the image, under the adjunction bijection, of the identity on $F(X)$, and it is natural in X by binaturality of φ. In particular, for any $h : X' \to X$ one has the naturality relation

$$GF(h)\,\eta_{X'} \;=\; \eta_X\,h.$$

Dually, define

$$\varepsilon_Y := \bar{1}_{G(Y)} : FG(Y) \to Y.$$

This map is called the **counit** of the adjunction: it is the image, under φ^{-1}, of the identity on $G(Y)$, and it is natural in Y by binaturality of φ. The unit and counit satisfy the **triangle identities**, which express that transposition and transposition back act as the identity on both sides of the adjunction:

$$\varepsilon_{F(X)} \circ F(\eta_X) \;=\; 1_{F(X)}, \qquad G(\varepsilon_Y) \circ \eta_{G(Y)} \;=\; 1_{G(Y)}.$$

Using η and ε, the transposes admit explicit formulas. For $g : F(X) \to Y$, one has

$$\bar{g} : X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(g)} G(Y).$$

Conversely, for $f : X \to G(Y)$, its transpose is the composite

$$\bar{f} : F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\varepsilon_Y} Y.$$

By the **triangle identities**, these constructions are inverse to one another: $\bar{\bar{g}} = g$ and $\bar{\bar{f}} = f$.

Lemma (Triangular identities)**.** *If $F \vdash G$, then the following diagrams are commutative:*

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\
 & F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\
 & G
\end{tikzcd}$$

Proof. The following establishes the first identity and finishes the proof (by duality):

$$1_{F(X)} = \overline{\eta_X} = \overline{1_{GF(X)}\eta_X} \overset{(2)}{=} \overline{1_{GF(X)}}F(\eta_X) = \varepsilon_{F(X)}F(\eta_X).$$

□

Theorem. *Let $F : \underline{C} \to \underline{D}$, $G : \underline{D} \to \underline{C}$. There is a one-to-one correspondence between*

(a) adjunctions between F and G;

(b) natural transformations $\eta : 1_{\underline{C}} \to GF$, $\varepsilon : FG \to 1_{\underline{D}}$ satisfying triangular identities.

Proof. It is left to define a function from (b) to (a). Given $g : F(X) \to Y$ and $f : X \to G(Y)$, we set

$$\bar{g} := (X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(g)} G(Y)), \qquad \bar{f} := (F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\varepsilon_Y} Y).$$

We claim that maps $g \to \bar{g}$ and $f \to \bar{f}$ are inverse to each other. Indeed,

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "1"'] & FGF(X) \arrow[r, "FG(g)"] \arrow[d, "\varepsilon_{F(X)}"] & FG(Y) \arrow[d, "\varepsilon_Y"] \\
 & F(X) \arrow[r, "g"'] & Y
\end{tikzcd}$$

This follows from the following computation:

$$g = \varepsilon_Y FG(g)F(\eta_X) = \varepsilon_Y F(G(g)\eta_X) = \eta_Y F(\bar{g}) = \bar{\bar{g}}.$$

Finally, the chain of equalities

$$\overline{hg} = G(hg)\eta_X = G(h)G(g)\eta_X = G(h)\bar{g}$$

verifies the naturality conditions and finishes the proof. □

Proposition. *If a functor $F : \underline{C} \to \underline{D}$ admits a right adjoint, then this adjoint is unique up to isomorphism. Moreover, F admits a right adjoint if and only if the functor*

$$\underline{D}(F(-), Y) : \underline{C}^{\mathrm{op}} \to \underline{\mathrm{Sets}}$$

is representable for every $Y \in \mathrm{Ob}(\underline{D})$.

Proof. Consider the functor

$$F_* : \underline{D} \longrightarrow [\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}], \quad F_*(Y) := \underline{D}(F(-), Y),$$

and recall the Yoneda embedding $h^{(-)} : \underline{C} \to [\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}]$, where $h^X = \underline{C}(-, X)$.

UNIQUENESS

If $G, G' : \underline{D} \to \underline{C}$ are both right adjoint to F, then for each Y there are natural bijections

$$\underline{C}(-, GY) \cong \underline{D}(F(-), Y) \cong \underline{C}(-, G'Y),$$

natural in the $\underline{C}$-variable. By Yoneda, this yields natural isomorphisms $GY \cong G'Y$, hence $G \cong G'$.

EXISTENCE

The existence of a right adjoint to F is equivalent to a factorization of F_* through $h^{(-)}$:

$$\begin{tikzcd} \underline{D} \arrow[rr, "F_*"] \arrow[dr, dashed, "G_0"'] & & {[\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}]} \\ & \underline{C} \arrow[ur, "h^{(-)}"'] & \end{tikzcd}$$

that is, to the existence of isomorphisms

$$F_*(Y) \cong h^{G_0(Y)} \quad \text{or equivalently} \quad \underline{D}(F(-), Y) \cong \underline{C}(-, G_0(Y)),$$

natural in both variables. If $F \vdash G$, then for every Y the functor $\underline{D}(F(-), Y)$ is represented by $G(Y)$, so $F_* \cong h^{(-)} \circ G$. Conversely, assume each $F_*(Y)$ is representable. Choose for each Y an object $G_0(Y) \in \mathrm{Ob}(\underline{C})$ and a natural isomorphism

$$\theta_Y : h^{G_0(Y)} \xrightarrow{\cong} F_*(Y).$$

For a morphism $f : Y \to Y'$ in $\underline{D}$, define $G_0(f) : G_0(Y) \to G_0(Y')$ to be the unique arrow corresponding under Yoneda to the natural transformation

$$h^{G_0(Y)} \xrightarrow{\theta_Y} F_*(Y) \xrightarrow{F_*(f)} F_*(Y') \xrightarrow{\theta_{Y'}^{-1}} h^{G_0(Y')}.$$

This makes G_0 a functor and the θ_Y natural in Y. Consequently, for all $X \in \mathrm{Ob}(\underline{C})$, $Y \in \mathrm{Ob}(\underline{D})$ there are natural bijections

$$\underline{C}(X, G_0Y) \cong [\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}]\big(h^X, h^{G_0Y}\big) \cong [\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}]\big(h^X, F_*(Y)\big) \cong \underline{D}(FX, Y),$$

natural in both X and Y. Thus G_0 is a right adjoint to F. $\square$

1.9 Terminal and Initial Objects

Definitions:

(1) An object $T \in \mathrm{Ob}(\underline{C})$ is **terminal** if for all $X \in \mathrm{Ob}(\underline{C})$ there exists a unique map $f : X \to T$.

(2) An object $S \in \mathrm{Ob}(\underline{C})$ is **initial** if for all $X \in \mathrm{Ob}(\underline{C})$ there exists a unique map $f : S \to X$.

(3) An object $Z \in \mathrm{Ob}(\underline{C})$ is **zero** if Z is both initial and terminal.

(4) A map $f : X \to Y$ is **zero** if f factors through the zero object.

Examples:

(1) In $\underline{\mathrm{Sets}}$, $S = \varnothing$ is initial, and $T = \{\bullet\}$ is terminal.

(2) In $\underline{\mathrm{R - Mod}}$, $Z = 0$ is a zero object.

(3) In $\underline{\mathrm{Rings}}$, $S = \mathbb{Z}$ is initial, and $T = 0$ is terminal.

1.10 Comma Category

Definition. Given a diagram of categories and functors

$$\underline{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \underline{D},$$

the **comma category** $(F \downarrow G)$ is defined by the following data:

i) Its objects are triples (C, h, D), where $C \in \mathrm{Ob}(\underline{C}), D \in \mathrm{Ob}(\underline{D})$, and $h \in \mathcal{E}(F(C), G(D))$.

ii) Its morphisms $(C, h, D) \to (C', h', D')$ are pairs (f, g), where $f \in \underline{C}(C, C')$, $g \in \underline{D}(D, D')$ are such the following diagram is commutative:

$$\begin{tikzcd} F(C) \arrow[r, "h"] \arrow[d, "F(f)"'] & G(D) \arrow[d, "G(g)"] \\ F(C') \arrow[r, "h'"'] & G(D') \end{tikzcd}$$

Examples:

(1) Given an object $c \in \mathrm{Ob}(\underline{C})$, the **slice category** is the comma category for the diagram

$$\underline{C} \xrightarrow{1_{\underline{C}}} \underline{C} \xleftarrow{c} 1,$$

where $1 \xrightarrow{c} \underline{C}$ is the constant functor with value c. It is commonly denoted by

$$(1_{\underline{C}} \downarrow c) \equiv (c \downarrow \underline{C}) \equiv \underline{C} \,/\, c.$$

(2) Analogously, the **coslice category** is defined by the diagram

$$1 \xrightarrow{c} \underline{C} \xleftarrow{1_{\underline{C}}} \underline{C},$$

and is usually denoted by

$$(c \downarrow 1_{\underline{C}}) \equiv (\underline{C} \downarrow c) \equiv c \,/\, \underline{C}.$$

(3) The diagram

$$\underline{C} \xrightarrow{1_{\underline{C}}} \underline{C} \xleftarrow{1_{\underline{C}}} \underline{C}$$

defines the **arrow category** denoted as follows:

$$(1_{\underline{C}} \downarrow 1_{\underline{C}}) \equiv \underline{\mathrm{Arr}}(\underline{C}) \equiv (\underline{C} \downarrow \underline{C}).$$

LECTURE 5. FEBRUARY 18, 2025

IN THE LAST EPISODE...

Definition. A **universal arrow** is an initial object in $(C \downarrow G)$, where $C \in \mathrm{Ob}(\underline{C})$ and $G : \underline{C} \to \underline{D}$.

Lemma. *If functors $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$ are adjoint ($F \vdash G$), then $(F(C), \eta_C)$ is a universal arrow for any given object $C \in \mathrm{Ob}(\underline{C})$.*

Proof. Let $(D, h) \in \mathrm{Ob}(C \downarrow G)$; in other words, let $D \in \mathrm{Ob}(\underline{D})$ and $h \in \underline{C}(C, G(D))$. By adjunction there is a unique map $\bar{h} \in \underline{D}(F(C), D)$ closing the diagram below:

$$\begin{tikzcd} & C \arrow[dl, "\eta_C"'] \arrow[dr, "h"] & \\ GF(C) \arrow[rr, "G(\bar{h})"'] & & G(D) \end{tikzcd}$$

This proves the claim. □

Theorem. *Let $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$. There is a one-to-one correspondence between*

a) adjunctions $F \vdash G$;

b) natural transformations $\eta : 1_{\underline{C}} \to GF$ such that $(F(C), \eta_C)$ is a universal arrow for any $C \in \mathrm{Ob}(\underline{C})$.

Proof. Note that the previous lemma implies a) ⇒ b). To show that b) ⇒ a), let $D \in \mathrm{Ob}(\underline{D})$ and define $\varepsilon_D : FG(D) \to D$ to be the unique map

$$(FG(D), \eta_{G(D)}) \to (G(D), 1_{G(D)})$$

in $(G(D) \downarrow G)$. We get the following commutative diargram:

$$\begin{tikzcd} & G(D) \arrow[dl, "\eta_{G(D)}"'] \arrow[dr, "1_{G(D)}"] & \\ GFG(D) \arrow[rr, "G(\varepsilon_D)"] & & G(D) \end{tikzcd}$$

Our goal is to show that ε_D defines a natural transformation, and that η and ε satisfy triangle identities. This would suffice to imply that $F \vdash G$ by one of the previous theorems.

We argue that one of the Δ-identities follows from the definition of ε_D. For the rest, we provide brief proof sketches, which constist of commutative diagrams and encode the proof.

NATURALITY OF ε_D:

Given a morphism $g : D \to D'$, one has

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "1"] \arrow[ddr, "G(g)"'] & GFG(D) \arrow[d, "G(\varepsilon_D)"] \\
 & G(D) \arrow[d, "G(g)"] \\
 & G(D')
\end{tikzcd}$$

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[d, "G(g)"'] & GFG(D) \arrow[d, "GFG(g)"] \\
G(D') \arrow[r, "\eta_{G(D')}"'] \arrow[dr, "1"'] & GFG(D') \arrow[d, "G(\varepsilon_D)"] \\
 & G(D')
\end{tikzcd}$$

$$g \circ \varepsilon_D = \varepsilon_D \circ FG(g)$$

SECOND Δ-IDENTITY:

$$\begin{tikzcd}
 & C \arrow[dl, "\eta_C"'] \arrow[dr, "\eta_C"] & \\
GF(C) \arrow[rr, "G(1_{F(C)})"'] & & GF(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & GF(C) \arrow[d, "GF(\eta_C)"] \\
GF(C) \arrow[r, "\eta_{GF(C)}"'] \arrow[dr, "1"'] & GFGF(C) \arrow[d, "G(\varepsilon_{F(C)})"] \\
 & GF(C)
\end{tikzcd}$$

$$1_{F(C)} = \varepsilon_{F(C)} \circ F(\eta_C)$$

$\square$

Corollary. *$G : \underline{D} \to \underline{C}$ has a left adjoint iff for all $C \in \mathrm{Ob}(\underline{C})$ the category $(C \downarrow G)$ has an initial object.*

Proof. It suffices to show only the backward implication. Set $(F(C), \eta_C)$ to be the initial oobject in $(C \downarrow G)$. If $f : C \to C'$ set $F(f)$ to be the unique map such that

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}"'] & GF(C') \end{tikzcd}$$

From the diagram it follows that $\eta : 1 \to GF$ is a natural transformation, hence the result. □

2 Limits

2.1 Products

Definition. Let X and Y be objects of $\underline{C}$. A **product** of X and Y is a triple $(X \times Y, \pi_1, \pi_2)$ with an object $X \times Y \in \mathrm{Ob}(\underline{C})$ and **projection morphisms**

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y,$$

having the property that for any other triple (T, f_1, f_2) there exists a unique map

$$f = \langle f_1, f_2 \rangle : T \to X \times Y$$

such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dl, "f_1"'] \arrow[d, "f=\langle f_1,f_2\rangle"] \arrow[dr, "f_2"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

Remarks:

(1) Product is the terminal object in the category of triples (T, f_1, f_2).

(2) Product is unique, if it exists at all.

(3) $\underline{C}(T, X \times Y) \cong \underline{C}(T, X) \times \underline{C}(T, Y)$.

(4) Let t be a terminal object in $\underline{C}$. Then $t \times X \cong X$ for all $X \in \mathrm{Ob}(\underline{C})$.

Examples:

(1) In Sets, G-Sets, Mon, Rings the product of X and Y the product is the «usual» $X \times Y$.

(2) In Fields there are no products (if L, K are fields, $L, L \times K, K$ would have the same characteristic).

Definition. Given a family $\{X_i\}_{i \in I} \subset \mathrm{Ob}(\underline{C})$, its **product** is defined by the following data:

i) An object $P = \prod_{i \in I} X_i$;

ii) A family of **projections** $\pi_i : P \to X_i$,

with with the property that for any other such data $(T, f_i : T \to X_i)$ there exists a unique map

$$f = \langle f_i \rangle_{i \in I} : T \to \prod_{i \in I} X_i$$

such that $f_i = \pi_i f$ for all $i \in I$.

Examples:

(1) In the category of finitely generated R-modules there are only finite products.

(2) $\underline{\text{Sets}}, \underline{\text{Mon}}, \underline{\text{Rings}}, \underline{\text{CRings}}$ all have arbitrary large products.

Definition. The **diagonal map** $\Delta_X : X \to X \times X$ is the unique map $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$.

Definition. The **product of maps** $f : X \to X', g : Y \to Y'$:

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, dashed, "\exists! f \times g"'] & Y \arrow[d, "g"] \\ X' & X' \times Y' \arrow[l, "\pi_{X'}"] \arrow[r, "\pi_{Y'}"'] & Y' \end{tikzcd}$$

Proposition. *For $f : X \to Y$, $g : X \to Y'$, one has $\langle f, g \rangle = (f \times g) \circ \Delta_X$.*

Proof. EXERCISE. □

2.2 Equalizers

Definition. An **equalizer** of two maps $f, g : \underline{C}(X, Y)$ is a pair (E, e) with an object $E \in \mathrm{Ob}(\underline{C})$ and a map $e : E \to X$ such that $fe = ge$, with the property that for any other such pair (T, h) with $fh = gh$, there exists a unique map $\bar{h}$ satifying $e\bar{h} = h$. In other words, we have the following diagram:

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ T \arrow[u, dashed, "\exists! \bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Remarks:

(1) An equalizer is the terminal object in the category of such pairs.

(2) If $\underline{C}$ has a zero object and 0 denotes the zero map, the **kernel** of $f \in \underline{C}(X, Y)$ is

$$\ker f := \mathrm{eq}(f, 0).$$

(3) The definition of an equalizer can be generalized to families of maps.

Examples:

(1) In Sets, given $f, g : X \to Y$ one has
$$\mathrm{eq}(f,g) = \{x \in X \mid f(x) = g(x)\}.$$

(2) In Mon, Grps, we can put algebraic structure on (1).

(3) In Top, we can put topology on (1).

Proposition. *e is a monomorphism.*

Proof. EXERCISE. □

Proposition. *The following are equivalent:*

(1) $f = g$;

(2) $e = 1_X$;

(3) *e is an isomorphism;*

(4) *e is an epimorphism.*

Proof. CLEAR. □

LECTURE 6. FEBRUARY 25, 2025

2.3 Pullbacks

Definition. A **pullback** of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ is a triple $(X \times_Z Y, \pi_1, \pi_2)$,

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "t_2"] \arrow[d, "t_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

with the property that given any other such triple (T, t_1, t_2), there is a unique map $u : T \to X \times_Z Y$ such that $t_1 = \pi_1 u$ and $t_2 = \pi_2 u$.

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "t_2"] \arrow[ddr, bend right, "t_1"'] & & \\ & X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

Pullbacks are also called **cartesian squares** or **fibered products**. Notation:

$$X \times_Z Y = X_f \times_g Y = X \prod_Z Y.$$

Remarks:

(1) A pullback is the terminal object in the category of all triples (P, π_1, π_2) such that $f\pi_1 = g\pi_2$ (with morphisms as in the diagram above).

(2) If $t \in \mathrm{Ob}(\underline{C})$ is terminal, then
$$X \times_t Y \cong X \times Y.$$

(3) There is a cancellation property (EXERCISE):
$$X \times_Y (Y \times_Z W) \cong X \times_Z W.$$

(4) Pullbacks are binary products in $\underline{C}/Z$.

Examples:

(1) In Sets, one has
$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

(2) Further examples in Sets:

$$\begin{tikzcd} f^{-1}(x) \arrow[r, hook] \arrow[d] & Y \arrow[d, "f"] \\ \{x\} \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & Y \arrow[d, hook] \\ X \arrow[r, hook] & Z \end{tikzcd} \qquad \begin{tikzcd} R_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(3) In Mon, R_f is the **congruence** on M induced by f.

2.4 Limits of Functors

Definition. A **limit** of a functor $D : I \to \underline{C}$ is the terminal object in $(\Delta \downarrow D)$:

$$\underline{C} \xrightarrow{\Delta} [I, \underline{C}] \xleftarrow{D_*} \underline{1}.$$

Notation: $\lim_I D \equiv \lim D$.

In more detail, elements of this category are morphisms

$$\Delta_C \xrightarrow{p} D \in \mathrm{Ob}(\Delta \downarrow D),$$

meaning that for all $u : i \to j$ in I the diagram

$$\begin{tikzcd} & D(i) \arrow[dd, "D(u)"] \\ C \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & \\ & D(j) \end{tikzcd}$$

is commutative. Such object is called a **cone** (over F), and p_i are called **projections**.

Definition (Alternative)**.** A **limit** of F is the terminal object in the category of cones over F.

Note that terminal objects, products, equalizers and pullback are all examples of limits.

Let $\alpha : D \to E$ be a natural transformation of $D, E : I \to \underline{C}$.

$$\begin{tikzcd} \Delta_{\lim D} \arrow[r, "p"] \arrow[d, dashed, "\exists!\lim\alpha"'] & D \arrow[d, "\alpha"] \\ \Delta_{\lim E} \arrow[r, "\tilde{p}"] & E \end{tikzcd}$$

Since limit is a terminal object, there is a unique arrow $\lim \alpha : \lim D \to \lim E$.

It follows that lim is a functor $\lim_I : [I, \underline{C}] \to \underline{C}$. Note that

$$[I, \underline{C}](\Delta_C, D) \cong \underline{C}(C, \lim_I D),$$

hence lim is a right adjoint for the constant functor Δ.

Theorem. *If a category $\underline{C}$ has binary equalizers and all products indexed by* $\mathrm{Ob}(I)$ *and* $\mathrm{Arr}(I)$*, then for any* $D : I \to \underline{C}$ *there is a limit* $\lim_I D$*.*

Proof. We argue that the category of cones over F is isomorphic to the following category of cones:

$$T \xrightarrow{\ t\ } \prod_{i \in \mathrm{Ob}(I)} D(i) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \prod_{u : i \to j} D(j)$$

Where φ and ψ are uniquely defined by the universal property of product and the relations

$$\pi_u \varphi = \pi_j, \qquad \pi_u \psi = D(u)\pi_i.$$

□

Corollary. *There is a monomorphism* $\lim_I D \hookrightarrow \prod_{i \in \mathrm{Ob}(I)} D(i)$*.*

Example. In a concrete category $\underline{C}$,

$$\lim_I D \;=\; \left\{ (x_i) \in \prod D(i) \;\middle|\; D(u)(x_i) = x_j \quad \text{for all } u : i \to j \right\}.$$

2.5 Inverse (Projective) Limits

Definition. An **inverse (projective) system** in $\underline{C}$ is $D : I^{op} \to \underline{C}$, where I is an ordered set. In this case the limit of D is called **inverse** (**projective**).

Examples:

(1) Let R be a commutative ring, $I \subset R$ an ideal, and M an R module. Consider the diagram

$$\cdots \longrightarrow M/I^3M \longrightarrow M/I^2M \longrightarrow M/IM$$

Its limit

$$\varprojlim M/I^nM =: \hat{M}^I$$

is called the **completion** of M with respect to I.

(2) For a prime p, the **p-adic integers**

$$\varprojlim \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p.$$

Proposition. *If $\underline{C}$ has finile limits and direct limits, then $\underline{C}$ has all limits.*

Proof. EXERCISE. □

Definition. A category $\underline{C}$ is called **complete** if any $D : I \to \underline{C}$ has a limit (where I is small).

Theorem. *The following are equivalent:*

(1) $\underline{C}$ is complete;

(2) $\underline{C}$ has small products and binary equalizers;

(3) *$\underline{C}$ has small products and binary pullbacks;*

(4) *$\underline{C}$ has terminal object and small pullbacks;*

(5) *$\underline{C}$ has finite limits and inverse limits;*

3 Colimits

Definition (Formal)**.** A **colimit** of a functor $D : I \to \underline{C}$ is

$$\operatorname{colim}_I D := \left(\lim_{I^{op}} D^{op}\right)^{op}.$$

It is the dual notion to the limit. There are **cocones** and **coprojections**:

$$\begin{tikzcd}
 & D(i) \arrow[dl, "q_i"'] \arrow[dd, "D(u)"] \\
\operatorname{colim}_I D & \\
 & D(j) \arrow[ul, "q_j"]
\end{tikzcd}$$

Remarks:

(1) $\operatorname{colim}_I : [I, \underline{C}] \to \underline{C}$ is a functor.

(2) colim_I is the left adjoint to the constant functor $\Delta : \underline{C} \to [I, \underline{C}]$.

3.1 Coproducts

Definition. Let I be a set, $D : I \to \underline{C}$ a family of objects $D(i) = X_i$. A coproduct of X_i is

$$\coprod_{i \in I} X_i := \operatorname{colim}_I D.$$

Considering the binary case, there is the following universal property:

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[r, "i_1"'] & X \coprod Y \arrow[u, dashed, "\exists! {[f,g]}"] & Y \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Examples:

(1) In $\underline{\text{Sets}}$, the coproduct is the disjoint union $X \coprod Y$.

(2) In $\underline{\text{Top}}$, the same set can be endowed with topology.

(3) In $\underline{R\text{-Mod}}$, the coproduct is the disjoint union $X \oplus Y$.

Note that for an infinite set I,
$$\bigoplus_{i \in I} M_i \neq \prod_{i \in I} M_i.$$

(4) In $\underline{\text{Grps}}$, the coproduct is the **free product** $X * Y$.

Note that if $X = \langle S_X \mid R_X \rangle$ and $Y = \langle S_Y \mid R_Y \rangle$, then
$$X * Y = \langle S_X \coprod S_Y \mid R_X \coprod R_Y \rangle.$$

(5) In $\underline{\text{Rings}}$, the coproduct is the similarly defined free product $X * Y$.

It can also be defined in the following way:
$$R * S = \mathrm{T}_{\mathbb{Z}}(R \times S) \,/\, \big(r \otimes r' - rr', s \otimes s' - ss', 1_R - 1_S\big).$$

Lecture 7. March 11, 2025

3.2 Coequalizers

Let I be an indexing category consisting of two parallel arrows. A functor

$$D : I \to \underline{C}$$

thereby defines a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

The colimit of this diagram is the **coequalizer** of f and g:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "t"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! u"] \\ & & T \end{tikzcd}$$

This diagram is called a **cofork**.

Examples:

(1) A special case in <u>Sets</u>. Let $R \subset X \times X$ be an equivalence relation on X.

$$\begin{tikzcd} R \arrow[r, hook, "r"] & X \times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] \arrow[dr, "h"'] & X/R \arrow[d, dashed, "\exists! u"] \\ & & Z \end{tikzcd}$$

(2) In general, the coequalizer of f and g is

$$\mathrm{coeq}(f,g) = Y/R^E,$$

where R^E is the equivalence relation generated by

$$T = \{(f(x), g(x)) \in Y \times Y \mid x \in X\}.$$

3.3 Pushouts

Let I be the category $(\bullet \leftarrow \bullet \rightarrow \bullet)$ and $D : I \to \underline{C}$. The colimit of D

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "q_2"'] \arrow[ddr, "t_2"] & \\
X \arrow[r, "q_1"] \arrow[drr, "t_1"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! u"] & \\
 & & T
\end{tikzcd}$$

is called the **pushout** of f and g.

Examples:

(1) In $\underline{\text{Sets}}$, the pushout of f and g is

$$X \sqcup_Z Y := X \sqcup Y / R,$$

where R is the equivalence relation on $X \sqcup Y$ generated by

$$f(z) \sim g(z) \quad \text{for all } z \in Z.$$

In particular, if X, Y are subobjects of A,

$$\begin{tikzcd}
X \cap Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & X \cup Y
\end{tikzcd}$$

(2) In $\underline{\text{Top}}$, the pushout is

$$X \sqcup_Z Y := X \sqcup Y \;/\; R$$

with the quotient topology.

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d, hook] & D^2 \arrow[d] \\
D^2 \arrow[r] & S^2
\end{tikzcd}$$

(3) In $\underline{\text{Grps}}$, given the diagram

$$G \xleftarrow{f} K \xrightarrow{g} H$$

the pushout of f and g is the **amalgamated product**

$$G *_K H = G * H \;/\; N,$$

where N is the normailzation of $\{f(k)g(k)^{-1}\}_{k\in K}$.

Let $\mathbb{Z}/4\mathbb{Z} = \langle S\rangle$ and $\mathbb{Z}/6\mathbb{Z} = \langle T\rangle$,

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} = \langle S, T \mid S^4, T^6, S^2 = T^3\rangle \cong SL_2(\mathbb{Z}),$$

where the last isomorphism is defined by

$$S \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T \mapsto \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Remark. $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(t\mathrm{Id}) =\cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

(4) Seifert-Van Kampan Theorem:

If a topological space $X = U_1 \cup U_2$, $U_1, U_2, U_1 \cap U_2$ are path-connected, then

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] & U_2 \arrow[d] \\ U_1 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \pi_1(U_1 \cap U_2, x) \arrow[r] \arrow[d] & \pi_1(U_2, x) \arrow[d] \\ \pi_1(U_1, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

the last diagram is a pushout square for any $x \in U_1 \cap U_2$.

So the fundamental group «maps pushouts to pushouts».

(5) In Rings, the pushout is the **amalgamated product of rings**.

(6) In CRings, the pushout is the tensor product of algebras.

3.4 Direct limits

Definition. A **direct limit** is a colimit over a direct set.

Examples:

(1) In Sets, the direct limit of

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots$$

is the union

$$\varinjlim X_n = \bigcup_{n=0}^{\infty} X_n.$$

(2) In Grps, the direct limit of

$$S_1 \hookrightarrow S_2 \hookrightarrow S_2 \hookrightarrow \dots$$

is the group

$$\varinjlim S_n = \{\sigma \in S_{\mathbb{N}} \mid \sigma(n) = n \text{ for almost all } n\}.$$

Given a ring R, the direct limit of

$$GL_1(R) \hookrightarrow GL_2(R) \hookrightarrow GL_3(R) \hookrightarrow \dots$$

is the **quite general group**

$$\varinjlim GL_n(R) = GL(R).$$

(3) In $\underline{\mathrm{Ab}}$, the direct limit of

$$\mathbb{Z}/p\mathbb{Z} \xhookrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xhookrightarrow{p} \mathbb{Z}/p^3\mathbb{Z} \xhookrightarrow{p} \dots$$

is the p-**Prufer group**

$$\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty) = \mathbb{Z}[1/p] \,/\, \mathbb{Z}.$$

3.5 Functors and Limits

Definition. A functor $F : \underline{C} \to \underline{D}$ **preserves limits** if for any $D : I \to \underline{C}$ and its limiting cone

$$(L \xrightarrow{p_i} D(i))_{i \in I},$$

the cone

$$(F(L) \xrightarrow{F(p_i)} FD(i))_{i \in I}$$

is limiting for $FD : I \to \underline{D}$.

Theorem. *The functor h_X preserves limits:*

$$\underline{C}(X, \lim_I D) = \lim_I \underline{C}(X, D).$$

The isomorphism is natural in X and D.

Proof. Let $L \xrightarrow{p_i} D(i)$ and Z be a set, then

$$\begin{tikzcd} L \arrow[r, "p_1"] & D(i) \\ X \arrow[u, dashed, "u(z)"] \arrow[ur, "q_i(z)"'] & \end{tikzcd} \qquad \begin{tikzcd} \underline{C}(X, L) \arrow[r, "\underline{C}(X, p_i)"'] & \underline{C}(X, D(i)) \\ z \in Z \arrow[u, dashed] \arrow[ur, "q_i"'] & \end{tikzcd}$$

□

Corollary. $h^X(\operatorname{colim} D) = \lim h^X(D)$.

Corollary. *Let $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$ such that $F \vdash G$, then*

(1) F preserves colimits (LAPC);

(2) G preserves limits (RAPL).

Proof. We show (2):

$$\underline{C}(C, G(\lim_I D)) \cong \underline{D}(F(C), \lim_I D) \cong \lim_I \underline{D}(F(C), D(i)) \cong$$
$$\cong \lim_I \underline{C}(C, GD(i)) \cong \underline{C}(C, \lim_I GD(i)).$$

By Yoneda's lemma, it follows that

$$G(\lim_I D) \cong \lim_I GD.$$

□

Examples:

(1) Let R and S be unital associative rings, and ${}_R B_S$ be an R-S-bimodule.

$$\begin{tikzcd} \underline{S\text{-Mod}} \arrow[r, shift left, "F"] \arrow[r, phantom, "\top"] & \underline{R\text{-Mod}} \arrow[l, shift left, "G"] \end{tikzcd}$$

And functors F and G are defined by

$$F(M) = B \otimes_S M, \quad G(N) = \mathrm{Hom}_R(B, N).$$

The required isomorphism is

$$\mathrm{Hom}_R(B \otimes_S M, N) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(B, N)),$$

where

$$f \ \mapsto\ (m \mapsto f((-) \otimes m))$$

and

$$(b \otimes m \mapsto g(m)b) \ \leftarrow\!\shortmid\ g.$$

Since $B \otimes -$ preserves colimits:

a) $B \otimes (\bigoplus M_i) \cong \bigoplus (B \otimes M_i)$

b) it is right exact (=preserves cokernels), i.e. exactness of

$$M' \xrightarrow{f} M \to M'' \to 0$$

implies the exactness of

$$B \otimes M' \to B \otimes M \to B \otimes M'' \to 0.$$

Remark. Let $f : R \to S$ be a morhisms of rings, the **restriction of scalars** functor

$$f^* \equiv \mathrm{res}_f : \underline{S\text{-Mod}} \to \underline{R\text{-Mod}}$$

on an S-module N is simultaneously

$$f^*(N) \cong \mathrm{Hom}_S({}_S S_R, N) =_R S_S \otimes_S N.$$

This gives us adjunctions

$$f_! \equiv \mathrm{ind} \vdash f^* \vdash f_* \equiv \mathrm{coind},$$

where

$$f_!(M) =_S S_R \otimes_R M, \quad f_*(M) = \mathrm{Hom}_R({}_R S_S, M).$$

LECTURE 8. MARCH 18, 2025

IN THE LAST EPISODE...

Let $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$ be a pair of adjoint functors ($F \vdash G$).

Definition. A functor is called **continuous** (resp. **cocontinuous**) if it preserves all limits (resp. colimits).

We note that F is cocontinuous (LAPC) and G is continuous (RAPL).

Examples:

(1) Let R, S be rings, and ${}_RB_S$ an R-S-bimodule. Then

$$B_S \otimes_S (-) \vdash \operatorname{Hom}_R({}_RB, -),$$

hence in particular

$$B_S \otimes_S \operatorname{colim} N_i \cong \operatorname{colim} B_S \otimes_S N_i$$

and

$$\operatorname{Hom}_R({}_RB, \lim M_i) \cong \lim \operatorname{Hom}_R({}_RB, M_i).$$

(2) Since $\operatorname{colim}_I \vdash \Delta \vdash \lim_I$, we get (completely for free)

$$\lim_I \lim_J(-) \cong \lim_J \lim_I(-).$$

4 Adjoint Functor Theorems

4.1 Generators and Cogenerators of Categories

Definitions:

(1) A set of objects $\{G_i\}_{i\in I}$ in a category $\underline{C}$ is a set of **generators** (**separators**) of $\underline{C}$ if for all $f \neq g \in \underline{C}(X, Y)$ there exist $i \in I$ and $h : G_i \to X$ such that $fh \neq gh$.

(2) Object G is called a generator if $\{G\}$ is a generating set.

(3) A **cogenerator** is a generator in the opposite category.

Remarks:

(1) G is a generator if and only if $h_G = \underline{C}(G, -)$ is faithful.

(2) $\{G_i\}_{i\in I}$ is a generating set if and only if $\prod_{i\in I} h_{G_i}$ is faithful.

Examples:

(1) In Sets, any non-empty set is a generator. Moreover, any S with $|S| \geq 2$ is a cogenerator.

(2) In Top we can endow the corresponding sets with discrete (resp. indiscrete) topology.

(3) In Ab, a group $\mathbb{Z} \oplus A$ is a generator and $\mathbb{Q}/\mathbb{Z}$ is a cogenerator.

(4) In $\underline{R\text{-Mod}}$, R is a generator and $\mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R, \mathbb{Q}/\mathbb{Z})$ is a cogenerator.

(5) In $\underline{\mathrm{Grps}}$, $\mathbb{Z}$ is a generator.

Statement. *There are no cogenerators in* $\underline{\mathrm{Grps}}$.

Proof. Suppose Q is cogenerating. Given a non-trivial simple group G,

$$\mathrm{id}_G \neq 1 : G \to G$$

hence there is $\alpha : G \to Q$ such that $\alpha \neq 1$. Since $\ker \alpha$ is a proper normal subgroup of a simple group, α is injective. But there are simple groups of arbitrary large cardinalities (for example, $\mathrm{PSL}_2(K), K = \mathbb{C}(x_i)_{i \in I}$), a contradiction. $\square$

(6) In $\underline{\mathrm{Rings}}$, $\mathbb{Z}[x]$ is a generator. There are no cogenerators by the similar agrument: there are fields of arbitrary large cardinalities, and all non-trivial homomorphisms from them are injective.

(7) In $\underline{\mathrm{CHaus}}$, $\{*\}$ is a generator, and $[0, 1]$ is a cogenerator.

Definitions:

(1) Let $R \overset{f}{\hookrightarrow} C$ and $S \overset{g}{\hookrightarrow} C$ be monomorphisms in $\underline{C}$. We say that $(R, f) \sim (S, g)$ if there is an isomorphism $\tau : R \to S$ such that $g\tau = f$. Equivalence classes under this relation are called **subobjects** of C. We denote by $\mathrm{Sub}(C)$ the class of all subobjects of C.

(2) We say that $\underline{C}$ is **well-powered** if $\mathrm{Sub}(C)$ is a set for all $C \in \mathrm{Ob}(\underline{C})$.

Examples:

(1) All «everyday» categories are well-powered.

(2) An example of a not well-powered category would be any partially ordered class with a maximal element (which every element would be a distinct suboject of). In particular, $\underline{\mathrm{Ord}}^{\mathrm{op}}$.

Theorem (Electrification)**.** *Let* $\underline{C}$ *be a balanced category with finite intersections and a set of generators* $\{G_i\}$*. Then* $\underline{C}$ *is well-powered.*

Proof. Let $B \ncong B'$. If both j and j' are epic, then $B \cong B \cap B' \cong B'$, a contradiction.

$$\begin{tikzcd}
G_i \arrow[r, "\nexists \tilde{h}"] \arrow[dr, "b"'] & B \cap B' \arrow[r, hook, "j"] \arrow[d, hook, "j'"'] & B' \arrow[d, hook, "i'"] \\
& B \arrow[r, hook, "i"'] & C
\end{tikzcd}$$

Suppose (WLOG) j is not epic, i.e. there are $f \neq g : B \to D$ such that $fj = gj$.

There is $h : G_i \to B$ such that $fh \neq gh$. It is clear that h cannot be factored through $B \cap B'$. Hence, there is no $h' : G_i \to B'$ such that $i'h' = ih$. It implies that

$$\Phi : \mathrm{Sub}(\underline{C}) \to 2^{\underline{C}(G_i, C)}, \quad B \mapsto \underline{C}(G_i, B)$$

is injective. $\square$

Lemma. *Let $\underline{C}$ and $\underline{D}$ be complete categories and $F : \underline{C} \to \underline{D}$, $G : \underline{D} \to \underline{E}$ continuous functors. Then $F \downarrow G$ is complete, and the forgetful functors $P_{\underline{C}} : F \downarrow G \to \underline{C}$ and $P_{\underline{D}} : F \downarrow G \to \underline{D}$ are continuous.*

Proof. We provide a sketch of the proof:

$$\lim_I \big(F(C_i) \xrightarrow{f} G(C_i)\big) = \lim_I F(C_i) \xrightarrow{\lim f_i} G(C_i).$$

□

Theorem (Primeval AFT)**.** *Suppose $\underline{D}$ has all (not necessary small) limits. A functor $G : \underline{D} \to \underline{C}$ has a left adjoint if and only if G is continuous.*

Remark. For such $\underline{D}$ we have $|\underline{D}(X, Y)| \leq 1$ (i.e. $\underline{D}$ is **thin**).

Proof. One direction follows from RAPL. Now, by the previous lemma, $\underline{C} \downarrow G$ has all limits, hence $\lim \mathrm{Id}_{\underline{C}\downarrow G}$ is the initial object. □

Lemma. *If $\underline{D}$ is complete, well-powered with a cogenerator Q, then $\underline{D}$ has an initial object.*

Proof. Let $i = \bigcap\limits_{X \hookrightarrow Q} X$ and $D \in \mathrm{Ob}(\underline{D})$. For $f, g \in \underline{D}(i, D)$, their equalizer is a suboject of i, hence $f = g$. Set $S := \underline{D}(D, Q)$. The map $D \to Q^S$ is monic, therefore

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] & Q \arrow[d, "\Delta"] \\ D \arrow[r, hook] & Q^S \end{tikzcd}$$

is a pullback square. Now $i \to T \to D$ is a required map. □

Proposition. *Let $\underline{C}$ be a category with (small) coproducts and $\{G_I\}_{i \in I}$ be a set of objects. Then the following are equivalent:*

(1) $\{G_i\}_{i \in I}$ is a generating set.

(2) $G = \coprod G_i$ is a generator.

(3) For each $X \in \mathrm{Ob}(\underline{C})$ there is a set S and an epimorphism $G^{(S)} \to X$.

Theorem (SAFT)**.** *Let $\underline{D}$ be a complete well-powered category with a cogenerator Q. A functor $G : \underline{D} \to \underline{C}$ has a left adjoint if and only if G is continuous.*

Proof. Only one direction requires proof. We want to show that $C \downarrow G$ has an initial object for each $C \in \mathrm{Ob}(\underline{C})$. To do that we apply the previous lemma:

(1) $C \downarrow G$ is complete (DONE).

(2) $C \downarrow G$ is well-powered:

Since $C \downarrow G \xrightarrow{P_{\underline{D}}} \underline{D}$ is continuous, it preserves monomorphisms (since f is mono iff the corresponding square is a pullback).

We show that $P_{\underline{D}}$ reflects monomorphisms. Indeed,

$$\begin{tikzcd}
 & G(D'') \arrow[d, shift right, "G(k)"'] \arrow[d, shift left, "G(k')"] \\
C \arrow[ur, "f''"] \arrow[r, "f"] \arrow[dr, "f'"'] & G(D) \arrow[d, "G(h)"] \\
 & G(D')
\end{tikzcd}$$

If h is mono, $G(h)G(k) = G(h)G(k')$, we get $G(k) = G(k')$ since $G(h)$ is monic.

(3) $C \downarrow G$ has a cogenerator:

We show that $S = \underline{C}(C, G(D))$ is a generating set. Let $C \xrightarrow{f} G(D) \in \mathrm{Ob}(C \downarrow G)$. We have $D \hookrightarrow Q^T$, so the diagram

$$\begin{tikzcd}
 & G(D) \arrow[d, "G(j)"] \\
C \arrow[ur, "f"] \arrow[r, "G(j)f"'] & G(Q)^T
\end{tikzcd}$$

finishes the proof.

$\square$

Lecture 9. April 1, 2025

4.2 Eilenberg-Watts Theorems

(1) Let $G : \underline{R\text{-Mod}} \to \underline{S\text{-Mod}}$ be a continuous functor. By SAFT G has a left adjoint F. Then for any R-module M
$$G(M) = \mathrm{Hom}_S(S, G(M)) \cong \mathrm{Hom}_R(F(S), M).$$
Note that $F(S)$ is a bimodule, since we have $S \to \mathrm{End}_S(S) \to \mathrm{End}_R(F(S))$.

(2) Let $F : \underline{S\text{-Mod}} \to \underline{R\text{-Mod}}$ be a cocontinuous functor. By co-SAFT it has a right adjoint G. Then $G = \mathrm{Hom}(B, -)$, hence $F = B \otimes_S -$.

Definition. A set of objects $\{D_i\}_{i\in I}$ is called **weakly initial** if for all $D \in \mathrm{Ob}(\underline{D})$ there is $i \in I$ and a map $D_i \to D$.

Lemma. *Let $\underline{D}$ be a complete category. If $\underline{D}$ has a weakly initial set, then it has an initial object.*

Proof. Let $W := \prod D_i$ and $V := \mathrm{eq}(\underline{D}(W, W)) \hookrightarrow W$. We know that $gh = h$ for each $g \in \underline{D}(W, W)$.

1) Let $D \in \mathrm{Ob}(\underline{D})$. There is $D_i \to D$, hence $V \xrightarrow{h} W \xrightarrow{p_i} D_i \to D$ and $\underline{D}(V, D) \neq \varnothing$.

2) Suppose $f, g : V \to D$. We want to show that $f = g$.

$$\begin{tikzcd}
E \arrow[r, hook, "e"] & V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "h"] & D \\
W \arrow[u, "s"] \arrow[r, "hes"'] & W
\end{tikzcd}$$

Since $hesh = h$ and h is monic, $esh = 1_V$. Analogously, $she = 1_E$ given that e is monic. But e was a regular monomorphism, hence e is an isomorphism. Therefore, $f = g$.

□

Theorem (GAFT or Freyd's AFT)**.** *Let $\underline{D}$ be a complete category and $G : \underline{D} \to \underline{C}$ a continuous functor. Then G has a left adjoint if and only if for each $C \in \mathrm{Ob}(\underline{C})$ there is $\{D_i\}_{i\in I} \subset \mathrm{Ob}(\underline{D})$ with the property that for all $D \in \mathrm{Ob}(\underline{D})$ and any $f : C \to G(D)$ there exist $i \in I$, $\varphi : G \to G(D_i)$, $\bar{f} : D_i \to D$ such that $f = G(\bar{f}) \circ \varphi$.*

Proof. Clear, since this is equivalent to having a weakly initial set. □

5 Abelian Categories

5.1 Additive Categories

Definitions:

(1) An Ab-category (a **preadditive** category) is a category $\underline{\mathcal{A}}$ such that

a) $\underline{\mathcal{A}}(X,Y)$ is an abelian group for all $X,Y \in \underline{\mathcal{A}}$.

b) The composition map $\underline{\mathcal{A}}(X,Y) \times \underline{\mathcal{A}}(Y,Z) \to \underline{\mathcal{A}}(X,Z)$ is bilinear:

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f.$$

(2) If $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ are Ab-categories, a functor $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ is called **additive** if

$$\underline{\mathcal{A}}(X,Y) \to \underline{\mathcal{B}}(F(X),F(Y))$$

is $\mathbb{Z}$-linear for any $X,Y \in \underline{\mathcal{A}}$.

Examples:

(1) $\underline{R\text{-Mod}}$. In particular, $\underline{\text{Ab}}$ and $\underline{\text{Vect}}(k)$.

(2) $\underline{\text{Sh}}(X,R)$.

(3) For X a ringed space, $\underline{\text{Mod}}(\mathcal{O}_X)$.

(4) $\underline{\text{QCoh}}(X)$ and $\underline{\text{Coh}}(X)$.

Proposition. *For $Z \in \underline{\mathcal{A}}$ the following are equivalent:*

(1) Z is initial;

(2) Z is final;

(3) $1_Z = 0_Z$;

(4) $\underline{\mathcal{A}}(Z,Z) = \{0_Z\}$.

Proof. Clear. □

Definition. Let $\underline{\mathcal{A}}$ be an Ab-category and $X,Y \in \underline{\mathcal{A}}$. A **biproduct** of X and Y is a diagram

$$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

such that:

(1) $p_1 i_1 = 1_X$;

(2) $p_2 i_2 = 1_Y$;

(3) $1_{X \oplus Y} = i_1 p_1 + i_2 p_2$.

Note that it implies $p_1 i_2 = 0 = p_2 i_1$.

Theorem. *Let $X,Y \in \underline{\mathcal{A}}$. The following are equivalent:*

(1) X and Y have a biproduct.

(2) There exists $X \times Y$ and $X \times Y \cong X \oplus Y$ with projections p_1 and p_2.

(3) There exists $X \coprod Y$ and $X \coprod Y \cong X \oplus Y$ with inclusions i_1 and i_2.

Proof. We first show $(1) \Rightarrow (2)$. Indeed,

$$p_1 i_2 = p_1(i_1 p_1 + i_2 p_2) i_2 = p_1 i_2 + p_1 i_2 \implies p_1 i_2 = 0.$$

Similarly, $p_2 i_1 = 0$.

$$\begin{tikzcd} & T \arrow[dl, "t_1"'] \arrow[d, dashed, "\exists! h"'] \arrow[dr, "t_2"] & \\ X & X \oplus Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

Let $h = i_1 t_1 + i_2 t_2$. Then $p_1 h = t_1$ and $p_2 h = t_2$. For any other $h' : T \to X \otimes Y$ such that $p_i h' = t_i$, we get

$$h' = (i_1 p_1 + i_2 p_2) h' = i_1 t_1 + i_2 t_2 = h.$$

We will only show $(2) \Rightarrow (1)$. Inclusions i_1 and i_2 can be obtained by the universal property in the following way:

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[d, dashed, "\exists! i_1"'] \arrow[dr, "0"] & \\ X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

We get that $p_1 i_1 = 1_X$ and $p_2 i_2 = 1_Y$. Lastly,

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[d, dashed, "\exists!"'] \arrow[dr, "p_2"] & \\ X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

and $i_1 p_1 + i_2 p_2 = 1_{X \times Y}$, since

$$p_i(i_1 p_1 + i_2 p_2) = p_i.$$

□

Definition. An **additive** category is an Ab-category with a zero object and binary biproducts.

Proposition. *If $\underline{\mathcal{A}}$ is additive and $f, f' \in \underline{\mathcal{A}}(X, Y)$*

$$f + f' = \nabla_Y (f \sqcup f') \Delta_X,$$

where $\Delta_X : X \to X \times X$ and $\nabla : Y \times Y \to Y$.

Proof. EXERCISE. □

Proposition. *If $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ are additive, then $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ is additive if and only if F preserves biproducts.*

Proof. EXERCISE. □

We write

$$\underline{\mathrm{Func}}^{add}(\underline{\mathcal{A}}, \underline{\mathcal{B}}) = (\underline{\mathcal{A}}, \underline{\mathcal{B}}).$$

5.2 Abelian Categories

Let $\underline{\mathcal{A}}$ be an additive category with kernels and cokernels.

$$\begin{tikzcd}
\ker f \arrow[r, "k"'] & X \arrow[rr, "f"] \arrow[d] \arrow[drr, dashed, "\tilde{f}"] & & Y \arrow[r, "c"'] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[rr, dotted, "\bar{f}"'] & & \operatorname{im} f \arrow[u] &
\end{tikzcd}$$

We define $\operatorname{im} f := \ker c$ and $\operatorname{coim} f := \operatorname{coker} k$.

Definition. An additive category is called **abelian** if

AB1 Any map has a kernel and a cokernel.

AB2 For any map f the induced map $\bar{f}$ is an isomorphism.

Remarks:

(1) $\underline{\mathcal{A}}$ is abelian if and only if $\underline{\mathcal{A}}^{\text{op}}$ is abelian.

(2) Any abelian category is finitely (co)complete.

For a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ with $gf = 0$, one has

$$\begin{tikzcd}
X' \arrow[rr, "f"'] \arrow[dr, two heads] & & X \arrow[rr, "g"'] & & X'' \\
& \operatorname{im} f \arrow[ur, hook] \arrow[rr, dashed, "\varphi"] & & \ker g \arrow[ul, hook'] &
\end{tikzcd}$$

Note that φ is monic. We let $H = \operatorname{coker} \varphi$.

Definition. A **cochain complex** in $\underline{\mathcal{A}}$ is

$$X^\bullet = \qquad \cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots$$

with the property that $d_i d_{i-1} = 0$ for each $i \in \mathbb{Z}$.

We define
$$H^i(X^\bullet) = H(X^{i-1} \to X^i \to X^{i+1}).$$

Definition. $X^\bullet$ is called an **acyclic complex** (or an **exact sequence**) if $H^i(X^\bullet) = 0$.

Examples:

(1) $0 \to A \xrightarrow{f} B$ is exact if and only if f is monic.

(2) $A \xrightarrow{f} B \to 0$ is exact if and only if f is epic.

(3) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ (which is called a **short exact sequence**, a SES) is exact if and only if f is monic and $C = \operatorname{coker} f$ if and only if g is epic and $A = \ker f$.

Definition. A short exact sequence **splits** if there is $\varphi : X \to X' \oplus X''$ such that

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, "="'] & X \arrow[r] \arrow[d, "\varphi"] & X'' \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & X' \arrow[r] & X' \oplus X'' \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Proposition. *For a short exact sequence, the following are equivalent:*

(1) it splits;

(2) there is $\sigma : X'' \to X$ such that $g\sigma = 1_{X''}$;

(3) there is $\rho : X \to X'$ such that $\rho f = 1_{X'}$.

Proof. EXERCISE. □

5.3 Projective and Injective Objects

Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ be abelian categories and $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be a functor.

Definitions:

(1) F is **left exact** (Lex) if for any exact sequence $0 \to X' \to X \to X''$ the sequence

$$0 \to F(X') \to F(X) \to F(X'')$$

is exact.

(2) F is **right exact** (Rex) if for any exact sequence $0 \to X' \to X \to X''$ the sequence

$$F(X') \to F(X) \to F(X'') \to 0$$

is exact.

(3) F is **exact** if F is left exact and right exact.

Examples:

(1) $B \otimes -$ is right exact.

(2) $\operatorname{Hom}(B, -)$ is left exact.

Definition. An object $X \in \underline{\mathcal{A}}$ is **projective** (corresp. **injective**) if the functor $h_X = \operatorname{Hom}(X, -)$ (corresp. $h^X = \operatorname{Hom}(-, X)$) is exact.

Theorem. *Let $P \in \underline{R\text{-Mod}}$. The following are equivalent:*

(1) P is projective.

(2) For any epimorphism $\beta : M \to N$ and any $f : P \to N$ there is $\sigma : P \to M$ such that $\beta\sigma = f$.

$$\begin{tikzcd} M \arrow[r, "\beta"] & N \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\sigma"] & \end{tikzcd}$$

(3) Any short exact sequence $0 \to K \to M \to P \to 0$ splits.

(4) There is $K \in \underline{R\text{-Mod}}$ such that $P \oplus K \cong R^{(S)}$.

Proof. FOR NEXT TIME. $\square$

LECTURE 10. APRIL 8, 2025

IN THE PREVIOUS EPISODE...

Theorem. *Let $P \in \underline{R\text{-Mod}}$. The following are equivalent:*

(1) P is projective.

(2) For any epimorphism $\beta : M \to N$ and any $f : P \to N$ there exists $\sigma : P \to M$ such that $\beta\sigma = f$.

$$\begin{tikzcd} M \arrow[r, "\beta"] & N \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\sigma"] & \end{tikzcd}$$

(3) Any short exact sequence $0 \to K \to M \to P \to 0$ splits.

(4) There exists $K \in \underline{R\text{-Mod}}$ such that $P \oplus K \cong R^{(I)}$ for some set I.

Proof. (1) $\Rightarrow$ (2) is obvious, since $\mathrm{Hom}(P, \beta)$ is surjective. (2) $\Rightarrow$ (3) follows from the general criterion on splitting of exact sequences. We obtain (3) $\Rightarrow$ (4) by considering

$$0 \to K \to R^{(I)} \to P \to 0,$$

where I is the set of generators of P. It splits, hence the result.

Finally, we verify (4) $\Rightarrow$ (1). Let $0 \to M' \to M \to M'' \to 0$ and consider

$$0 \to \prod_{i \in I} M' \to \prod_{i \in I} M \to \prod_{i \in I} M'' \to 0.$$

Rewrite it as the sequence

$$0 \to \mathrm{Hom}(R^{(I)}, M') \to \mathrm{Hom}(R^{(I)}, M) \to \mathrm{Hom}(R^{(I)}, M'') \to 0,$$

where I is the set in (4). But since $R^{(I)} = P \oplus K$ we get

$$0 \to \mathrm{Hom}(K, M') \to \mathrm{Hom}(K, M) \to \mathrm{Hom}(K, M'') \to 0.$$

□

Proposition. *$\bigoplus_{i \in I} P_i$ is projective if and only if each P_i is projective.*

Proof. Note that $\mathrm{Hom}(\oplus P_i, -) = \prod \mathrm{Hom}(P_i, -)$. □

Theorem. *Let $E \in \underline{R\text{-Mod}}$. The following are equivalent:*

(1) E is injective (h^E is exact).

(2) For any monomorphism $\beta : M \to N$ and any $f : M \to P$ there exists $g : N \to P$ such that $g\beta = f$.

$$\begin{tikzcd} M \arrow[r, "\beta"] & N \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\sigma"] & \end{tikzcd}$$

(3) Any short exact sequence $0 \to E \to M \to N \to 0$ splits.

Proof. The proof is similar. □

Proposition. *$\prod_{i \in I} E_i$ is injective if and only if each E_i is injective.*

Theorem (Baer's criterion)**.** *An R-module E is injective if and only if for any left ideal $I \subset R$ has the property*

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "\varphi"'] & R \arrow[dl, dashed, "\exists\, \tilde{\varphi}"] \\ E & \end{tikzcd}$$

Proof. This condition is clearly necessary. Now let $A \subset B$ and $f : A \to E$. Consider a set X of all ordered pairs (A', g'), where $A \subset A' \subset B$ and $g' : A' \to E$ such that $g'\big|_A = f$.
There is a natural order on X, and by Zorn's lemma there is a maximal element (A_0, g_0). We want to show that $A_0 = B$. Assume the contrary, i.e. there is $b \in B \setminus A_0$. Let

$$I = \{r \in R \mid rb \in A_0\} \subset R.$$

It is clear that I is a left ideal of R. Define $\varphi : I \to E$ by

$$\varphi(r) := g_0(rb).$$

Let $A_1 = A_0 + Rb$ and $g_1 : A_1 \to E$, where

$$g_1(a_1 + rb) = g_1(a_0) + r\tilde{\varphi}(1).$$

It is routine to check that g_1 is well-defined. Hence (A_0, g_0) is not maximal, a contradiction. □

Definition. Let R be a (commutative) domain. An R-module is a **division module** if for every $m \in M \setminus \{0\}$ and $r \in R \setminus \{0\}$ there is a «quotient» $m' \in M$ such that $m = rm'$.

Proposition. *If D is a division module, then D/C is a division module for each submodule $C \subset D$.*

Proof. EXERCISE. □

Proposition (Relations between injective and division modules)**.**

(1) If R is a domain, then injective modules are division modules, i.e.$\mathrm{Inj}(R) \subset \mathrm{Div}(R)$.

(2) If R is a PID, then every division module is injective, i.e. $\mathrm{Inj}(R) = \mathrm{Div}(R)$.

(3) If R is a domain, then $Q(R)$, i.e. the field of fractions of R, is injective.

Proof. EXERCISE. □

Note that when $R = \mathbb{Z}$, abelian group $\mathbb{Q}$ is divisible, hence $\mathbb{Q}/\mathbb{Z}$ is injective.

Proposition. *$\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\underline{\mathrm{Ab}}$.*

Proof. Let $f : A \to B$ such that $f \neq 0$, i.e. there is $b \in B$ such that $b \in \operatorname{Im} f$ and $b \neq 0$. Define $\tilde{h} : \mathbb{Z}b \to \mathbb{Q}/\mathbb{Z}$ by

$$\tilde{h}(b) = \begin{cases} \frac{1}{n}, & n = \operatorname{ord} b \\ \frac{1}{2}, & \operatorname{ord} = \infty \end{cases}.$$

By injectivity, there is $h : B \to \mathbb{Q}/\mathbb{Z}$ such that $h(b) = \tilde{h}(b) \neq 0$. Hence $hf \neq 0$. □

Proposition. *If Q is an injective cogenerator in an abelian category $\underline{\mathcal{A}}$, then any object of $\underline{\mathcal{A}}$ is a subobject of an injective object.*

Proof. There is a monomorphism $X \hookrightarrow Q^S$ for each $X \in \mathrm{Ob}(\underline{\mathcal{A}})$ and some set S. □

Definition. An abelian category $\underline{\mathcal{A}}$ **has enough injectives** if every object can be embedded into an injective object.

Proposition. *Let R be a ring. Then $\mathrm{Hom}_{\mathbb{Z}}(R_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator in $\underline{R\text{-Mod}}$.*

Proof. It follows from the following chain of isomorphisms:

$$\mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(R_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R (-), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}).$$

□

Corollary. *In $\underline{R\text{-Mod}}$ there are enough injectives.*

Definition. An abelian category $\underline{\mathcal{A}}$ **has enough projectives** if for any $X \in \mathrm{Ob}(\underline{\mathcal{A}})$ there is a projective object P and an epimorphism $P \to X$.

Proposition. *In $\underline{R\text{-Mod}}$ there are enough projectives.*

Proof. Note that $R^{(S)} \to M \to 0$ and free module $R^{(S)}$ is clearly projective. □

Proposition. *If R is left Noetherian and $\{E_i\}_{i \in I}$ is a set of injective R-modules, then $\bigoplus E_i$ is injective.*

Proof. We check the Baer's criterion:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "f"'] \arrow[dr] & R \arrow[d, dashed, "g"] \\ \bigoplus\limits_{i\in I} E_i & \bigoplus\limits_{i\in I_0} E_i \arrow[l, hook'] \end{tikzcd}$$

Since I is finitely generated, $\operatorname{Im} f$ is contained in some $\bigoplus\limits_{i\in I_0} E_i$ where I_0 is finite, and the result follows from injectivity of $\bigoplus\limits_{i\in I_0} E_i$. $\square$

Theorem (Bass-Papp)**.** *Let R be a ring. If countable direct sums of injective objects in $\underline{R}$-$\underline{\text{Mod}}$ are injective, then R is Noetherian.*

Proof. Assume the contrary. Let $I_1 \subsetneq I_2 \subsetneq \dots$ and $I = \bigcup\limits_{n\geq 1} I_n$ be ideals of R. Consider injective module E_n such that $I/I_n \hookrightarrow E_n$, and let $\pi_n : I \to I/I_n$.

Let $\pi : I \to \prod\limits_{n\geq 1} I/I_n$. Note that the image of π is in $\bigoplus\limits_{n\geq 1} I/I_n$, so we can compose it with the map $\bigoplus\limits_{n\geq 1} I/I_n \to \bigoplus\limits_{n\geq 1} E_n$ and get $f : I \to \bigoplus\limits_{n\geq 1} E_n$. By assumtion $\bigoplus\limits_{n\geq 1} E_n$ is injective, hence there is $g : R \to \bigoplus\limits_{n\geq 1} E_n$ extending f. Let $g(1) = (e_n)$, $e_n \in E_n$. For each $m \geq 1$ there is some $a_m \in I \setminus I_m$, and it follows that $g(a_m) = f(a_m) = \pi(a_m) \neq 0$. But $g(a_m) = (a_m e_n)$, so $e_m \neq 0$ for any $m \geq 0$. This is a contradiction, since $g(1) = (e_m) \in \bigoplus\limits_{n\geq 1} E_n$. $\square$

Definition. A monomorphism $A \hookrightarrow B$ is called **essential** if for all $S \in B$ we have $A \cap S \neq 0$.

Example. In $\mathbb{Z}$-modules, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is essential.

Definition. An **injective hull** (or **envelope**) of a module M is an injective module E such that $M \hookrightarrow E$ is essential. We denote E by $E(M)$, which is unique up to isomorphism.

Let M and E be R-modules such that E is injective. Then (without proof)

$$\begin{tikzcd} & E(M) \arrow[dd, hook, dashed] \\ M \arrow[ur, hook] \arrow[dr, hook] & \\ & E \end{tikzcd}$$

and the following exact sequence splits:

$$0 \to E(M) \to E \to E/E(M) \to 0.$$

Theorem (w/o proof)**.** *For any R-module there is an injective hull.*

LECTURE 11. APRIL 15, 2025

6 Resolutions and Derived Functors

REFERENCES

- J. Rotman, *An Introduction to Homological Algebra*.
- C. Weibel, *An Introduction to Homological Algebra*.

6.1 Complexes

Definitions:

(1) If $\underline{\mathcal{A}}$ is an abelian category, the category $\underline{\mathrm{Ch}}(\underline{\mathcal{A}})$ of chain complexes has as objects diagrams

$$C = \left(\cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots\right),$$

with differentials $d_i : C_i \to C_{i-1}$ such that $d_i\, d_{i+1} = 0$ for all $i \in \mathbb{Z}$ (equivalently, $d^2 = 0$).

(2) The category $\underline{\mathrm{CoCh}}(\underline{\mathcal{A}})$ of cochain complexes has as objects diagrams

$$C = \left(\cdots \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots\right),$$

with differentials $d^i : C^i \to C^{i+1}$ such that $d^{i+1}\, d^i = 0$ for all $i \in \mathbb{Z}$.

(3) For a chain complex C, define the **cycles**, **boundaries**, and **homology** by

$$Z_i(C) := \ker(d_i), \qquad B_i(C) := \mathrm{im}(d_{i+1}), \qquad H_i(C) := Z_i(C) \Big/ B_i(C).$$

(4) Given chain complexes C, C', put for each $d \in \mathbb{Z}$

$$\underline{\mathrm{Hom}}^d(C, C') := \prod_{i \in \mathbb{Z}} \mathrm{Hom}(C_i, C'_{i+d}).$$

Elements of $\underline{\mathrm{Hom}}^d(C, C')$ are **homogeneous maps of degree** d: these are families $u = (u_i)$ with $u_i : C_i \to C'_{i+d}$. Define a differential $D : \underline{\mathrm{Hom}}^d(C, C') \to \underline{\mathrm{Hom}}^{d-1}(C, C')$ by

$$D(u) := d' \circ u - (-1)^d u \circ d.$$

Then $D^2 = 0$, so $\underline{\mathrm{Hom}}(C, C') := (\underline{\mathrm{Hom}}^\bullet(C, C'), D)$ is a complex. A **chain map** is a degree 0 cycle in $\underline{\mathrm{Hom}}(C, C')$, i.e. a family $f = (f_i)$ with $f_i : C_i \to C'_i$ and $d' f = f d$.

(5) Two chain maps $f, g : C \to C'$ are **homotopic** if $f - g = D(h)$ for some $h \in \underline{\mathrm{Hom}}^1(C, C')$, equivalently,

$$f_i - g_i = d'_{i+1} h_i + h_{i-1} d_i \quad \text{for all } i \in \mathbb{Z}.$$

The **homotopy category** $\underline{\mathrm{K}}(\underline{\mathcal{A}})$ has the same objects as $\underline{\mathrm{Ch}}(\underline{\mathcal{A}})$ and morphisms

$$\mathrm{Hom}_{\underline{\mathrm{K}}(\underline{\mathcal{A}})}(C, C') := H_0\big(\underline{\mathrm{Hom}}(C, C')\big),$$

i.e. chain-homotopy classes of chain maps $C \to C'$.

Definitions:

(1) Chain complexes C and D are **homotopy equivalent** if there are chain maps $f : C \to D$ and $g : D \to C$ with $fg \sim 1_D$ and $gf \sim 1_C$.

(2) If $u : C \to C'$ is a chain map, then for all $i \in \mathbb{Z}$

$$u\big(Z_i(C)\big) \subseteq Z_i(C'), \qquad u\big(B_i(C)\big) \subseteq B_i(C'),$$

so there are induced maps $Z_i(u) : Z_i(C) \to Z_i(C')$, $B_i(u) : B_i(C) \to B_i(C')$, and hence

$$H_i(u) : H_i(C) \to H_i(C').$$

A chain map u is a **quasi-isomorphism** if $H_i(u)$ is an isomorphism for all $i \in \mathbb{Z}$.

6.2 Snake lemma

Lemma (Snake Lemma)**.**

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

If the rows are exact, then there is a natural long exact sequence

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'') \to 0.$$

Proposition. *Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a short exact sequence of chain complexes. Then there are connecting morphisms $\delta_n : H_n(C) \to H_{n-1}(A)$ forming a long exact sequence*

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots .$$

Proof. Apply the Snake Lemma to the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Z_n(A) \arrow[r] \arrow[d] & Z_n(B) \arrow[r] \arrow[d] & Z_n(C) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_{n-1}(A) \arrow[r] & B_{n-1}(B) \arrow[r] & B_{n-1}(C) \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are induced by the differentials. The connecting morphism δ_n lands in H_{n-1} after passing to the quotients by boundaries, giving the long exact sequence. □

Proposition. *Given a morphism of short exact sequences of chain complexes*

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r] & B'_\bullet \arrow[r] & C'_\bullet \arrow[r] & 0 \end{tikzcd}$$

the induced long exact sequences in homology form a commutative diagram. Moreover, the connecting morphisms are natural, i.e.

$$H_{n-1}(f) \circ \delta_n \;=\; \delta'_n \circ H_n(h) \quad \textit{for all } n \in \mathbb{Z}.$$

6.3 Resolutions

Definitions:

(1) A **left resolution** of an object $M \in \underline{\mathcal{A}}$ is a chain complex

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

with augmentation $\varepsilon : P_0 \to M$, such that $P_i = 0$ for $i < 0$ and the sequence is exact; equivalently,

$$H_0(P) \cong M, \qquad H_i(P) = 0 \text{ for } i > 0.$$

A left resolution is **projective** if $P_i \in \mathrm{Proj}(\underline{\mathcal{A}})$ for all $i \geq 0$.

(2) A **right resolution** of M is a cochain complex

$$0 \to M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

with $I^j = 0$ for $j < 0$, which is exact except at degree 0; equivalently,

$$H^0(I) \cong M, \qquad H^j(I) = 0 \text{ for } j > 0.$$

It is **injective** if $I^j \in \mathrm{Inj}(\underline{\mathcal{A}})$ for all $j \geq 0$.

Examples:

(1) If $\underline{\mathcal{A}} = \underline{R\text{-Mod}}$, then every module has a projective (indeed, free) resolution: choose an epimorphism $R^{(I_1)} \twoheadrightarrow M$ with kernel K_1, then an epimorphism $R^{(I_2)} \twoheadrightarrow K_1$ with kernel K_2, etc., obtaining an exact sequence

$$\cdots \to R^{(I_2)} \to R^{(I_1)} \to M \to 0,$$

whose left part $\cdots \to R^{(I_2)} \to R^{(I_1)} \to 0$ is a projective resolution of M.

(2) The modules K_i appearing here are called **syzygies** of M.

(3) If $\underline{\mathcal{A}} = \underline{R\text{-Mod}}$, then every module has an injective resolution: embed M into an injective module E^0 (e.g. an injective hull), let $C^1 := \mathrm{coker}(M \to E^0)$, embed C^1 into an injective E^1, and continue to obtain

$$0 \to M \to E^0 \to E^1 \to E^2 \to \cdots,$$

which is an injective resolution of M.

Proposition (Comparison for projective resolutions)**.** *Let $f : M \to M'$ be a morphism in $\underline{\mathcal{A}}$. Let $P_\bullet \xrightarrow{\varepsilon} M$ and $Q_\bullet \xrightarrow{\varepsilon'} M'$ be left resolutions with P_i projective for all $i \geq 0$. Then there exists a chain map*

$$\tilde{f} : P_\bullet \to Q_\bullet$$

lifting f, i.e. $\varepsilon' \circ \tilde{f}_0 = f \circ \varepsilon$; moreover, $\tilde{f}$ is unique up to chain homotopy.

Proof. Construct $\tilde{f}_0 : P_0 \to Q_0$ by lifting $f\varepsilon$ along the epimorphism ε' using the projectivity of P_0. Inductively, having $\tilde{f}_{i-1}$ with $\varepsilon' d'_1 \cdots d'_i \tilde{f}_i = f\varepsilon d_1 \cdots d_i$, use the exactness of $Q_\bullet$ at Q_{i-1} and the projectivity of P_i to obtain $\tilde{f}_i : P_i \to Q_i$ such that $d'_i \tilde{f}_i = \tilde{f}_{i-1} d_i$.

To show uniqueness, let $\tilde{f}, \tilde{g} : P_\bullet \to Q_\bullet$ be two lifts of f. Put $h := \tilde{f} - \tilde{g}$ and define $s_n := 0$ for $n < 0$. Since $\varepsilon' h_0 = 0$ and $Q_\bullet$ is exact at Q_0, there exists $s_0 : P_0 \to Q_1$ with $d'_1 s_0 = h_0$. Assume $s_0, \ldots, s_{n-1}$ are chosen so that $h_k = d'_{k+1} s_k + s_{k-1} d_k$ for all $k < n$. Set

$$\alpha_n := h_n - s_{n-1} d_n : P_n \to Q_n.$$

Then $d'_n \alpha_n = d'_n h_n - (h_{n-1} - s_{n-2} d_{n-1}) d_n = 0$, so by exactness at Q_n there exists $s_n : P_n \to Q_{n+1}$ with $d'_{n+1} s_n = \alpha_n$. Hence for all n

$$h_n = d'_{n+1} s_n + s_{n-1} d_n,$$

which means $h = D(s)$. Therefore $\tilde{f}$ and $\tilde{g}$ are chain-homotopic. □

Corollary. *Let $M \in \underline{\mathcal{A}}$. Any two projective resolutions of M are chain-homotopy equivalent. Equivalently, if $P_\bullet \xrightarrow{\varepsilon} M$ and $P'_\bullet \xrightarrow{\varepsilon'} M$ are projective resolutions, then there exist chain maps*

$$f : P_\bullet \to P'_\bullet, \qquad g : P'_\bullet \to P_\bullet$$

with $gf \sim 1_{P_\bullet}$ and $fg \sim 1_{P'_\bullet}$.

LECTURE 12. APRIL 22, 2025

Definition. Let $f : A \to B$ be a chain map. The **cone** of f is the complex $C(f)$ defined degreewise by

$$C(f)_n := B_n \oplus A_{n-1} = B_n \oplus A[1]_n,$$

with differential given, in the decomposition $B_n \oplus A_{n-1} \to B_{n-1} \oplus A_{n-2}$, by the block matrix

$$d_n^{C(f)} = \begin{pmatrix} d_n^B & -f_{n-1} \\ 0 & -d_{n-1}^A \end{pmatrix}, \quad \text{i.e.} \quad d^{C(f)}(b,a) = \big(d_n^B b - f_{n-1}(a),\, -d_{n-1}^A a\big).$$

One checks that $d^{C(f)} \circ d^{C(f)} = 0$.

Remark. The **shift functor** $[1] : \underline{\mathrm{Ch}}(\underline{A}) \to \underline{\mathrm{Ch}}(\underline{A})$ is defined by

$$(A[1])_n := A_{n-1}, \qquad d_n^{A[1]} := -\, d_{n-1}^A,$$

and for a chain map f set $\big(f[1]\big)_n := f_{n-1}$.

Proposition (Properties of cones)**.**

(1) There is a short exact sequence of chain complexes

$$0 \longrightarrow B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \longrightarrow 0,$$

where $i(b) := (b, 0)$ and $p(b,a) := -a$.

(2) If the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is a commutative square of chain complexes, then the map

$$h \oplus g[1] : C(f) \longrightarrow C(f'), \qquad (b,a) \longmapsto \big(h(b),\, g(a)\big),$$

is a chain map.

(3) In the situation of (2), there is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "h"'] & C(f) \arrow[r, "p"] \arrow[d, "h\oplus g[1]"] & A[1] \arrow[r] \arrow[d, "g[1]"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "i'"] & C(f') \arrow[r, "p'"] & A'[1] \arrow[r] & 0 \end{tikzcd}$$

commuting with the indicated arrows.

Let $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ be a short exact sequence in an abelian category $\underline{A}$. Take projective resolutions $\varepsilon^A : Q_\bullet \twoheadrightarrow A$ and $\varepsilon^C : P_\bullet \twoheadrightarrow C$. By projectivity choose a lift $\psi_0 : P_0 \to B$ with $q\,\psi_0 = \varepsilon_0^C$. Using the induced map on cycles, pick $\varphi_0 : P_1 \to Q_0$ so that

$$i\,\varepsilon_0^A\,\varphi_0 \;=\; \psi_0\,d_1^P.$$

Inductively, choose maps $\varphi_n : P_{n+1} \to Q_n$ making $i\,\varepsilon_n^A\,\varphi_n = \psi_n\,d_{n+1}^P$ with suitable lifts $\psi_n : P_n \to B$ extending ψ_{n-1}. This yields a chain map

$$\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet[1].$$

Lemma (Horseshoe Lemma)**.**

(1) Put $S_\bullet := C\big(\varphi_\bullet[-1]\big)$. Then there is a short exact sequence of complexes

$$0 \longrightarrow Q_\bullet \longrightarrow S_\bullet \longrightarrow P_\bullet \longrightarrow 0,$$

and $S_\bullet$ is a projective resolution of B fitting into a commutative diagram with $\varepsilon^A, \varepsilon^B, \varepsilon^C$.

(2) The construction in (1) is natural with respect to morphisms of short exact sequences.

We omit the proof of this lemma but reformulate the statement (2):

Lemma. *Given a morphism of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "i"] \arrow[d, "a"'] & B \arrow[r, two heads, "q"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "i'"] & B' \arrow[r, two heads, "q'"] & C' \arrow[r] & 0
\end{tikzcd}$$

choose projective resolutions $P_\bullet(A), P_\bullet(C)$ and $P_\bullet(A'), P_\bullet(C')$, together with chain maps lifting a and c. Then the above construction yields a chain map

$$P_\bullet(B) \longrightarrow P_\bullet(B'),$$

which makes the diagram of short exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & P_\bullet(A) \arrow[r] \arrow[d] & P_\bullet(B) \arrow[r] \arrow[d] & P_\bullet(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_\bullet(A') \arrow[r] & P_\bullet(B') \arrow[r] & P_\bullet(C') \arrow[r] & 0
\end{tikzcd}$$

commute.

6.4 Derived Functors

Definition. Let $F : \underline{A} \to \underline{B}$ be a right exact additive functor between abelian categories and assume $\underline{A}$ has enough projectives. For $A \in \mathrm{Ob}(\underline{A})$, choose a projective resolution $P_\bullet \twoheadrightarrow A$ and define

$$(\mathrm{L}_i F)(A) := H_i\big(F(P_\bullet)\big) \qquad (i \geq 0).$$

Independent of the choice of $P_\bullet$ up to canonical isomorphism, this yields **left derived functors**

$$\mathrm{L}_i F : \underline{A} \to \underline{B}.$$

Definition. Similarly, if $F : \underline{A} \to \underline{B}$ is left exact additive and $\underline{A}$ has enough injectives, then for $A \in \mathrm{Ob}(\underline{A})$ and an injective resolution $A \hookrightarrow E^\bullet$ one defines **right derived functors**

$$(\mathrm{R}^i F)(A) := H^i\big(F(E^\bullet)\big) \qquad (i \geq 0).$$

Proposition (Properties of derived functors)**.**

(1) $\mathrm{L}_0 F \cong F$.

(2) The functors $\mathrm{L}_i F$ *are independent of the chosen projective resolution.*

(3) Each $\mathrm{L}_i F$ *is additive.*

Proof.

(1) If $0 \to P_1 \to P_0 \twoheadrightarrow A \to 0$ is exact with P_j projective, then applying F gives an exact sequence $F(P_1) \to F(P_0) \to F(A) \to 0$, and hence $H_0(F(P_\bullet)) \cong F(A)$.

(2) Any two resolutions of A are connected by a chain homotopy equivalence, and applying F preserves homotopies, so the induced maps on homology are isomorphisms.

(3) For a morphism $f : A \to A'$ and chosen projective resolutions, any lift $f_\bullet$ of f induces a chain map whose effect on homology defines $\mathrm{L}_i F(f)$. In particular, $f_\bullet + g_\bullet$ lifts $f + g$.

□

Remark. For any projective object P and any $i \geq 1$ one has $L_i F(P) = 0$.

In the category of R-modules there are two canonical examples.

Examples:

(1) Given $M \in \mathrm{Ob}(\underline{R\text{-Mod}})$, the functor $h_M := \mathrm{Hom}(M, -)$ is left exact. Then

$$\mathrm{Ext}^i_R(M, N) := R^i\, h_M(N).$$

(2) Given $M \in \mathrm{Ob}(\underline{R\text{-Mod}})$, the functor $t_M := M \otimes -$ is left exact. Then

$$\mathrm{Tor}^R_i(M, N) := L^i\, t_M(N).$$

Theorem (Derived long exact sequence)**.** *Let* $0 \to A' \xrightarrow{i} A \xrightarrow{q} A'' \to 0$ *be a short exact sequence in an abelian category* $\underline{A}$, *and let* $F : \underline{A} \to \underline{B}$ *be a right exact additive functor. Assume* $\underline{A}$ *has enough projectives. Then there is a natural long exact sequence*

$$\cdots \longrightarrow \mathrm{L}_n F(A') \xrightarrow{u_n} \mathrm{L}_n F(A) \xrightarrow{v_n} \mathrm{L}_n F(A'') \xrightarrow{\partial_n} \mathrm{L}_{n-1} F(A') \longrightarrow$$

$$\cdots \longrightarrow \mathrm{L}_1 F(A'') \xrightarrow{\partial_1} \mathrm{L}_0 F(A') \longrightarrow \mathrm{L}_0 F(A) \longrightarrow \mathrm{L}_0 F(A'') \to 0.$$

The maps u_n *and* v_n *are induced by* i *and* q, *and* ∂_n *is the connecting morphism. The construction is functorial in morphisms of short exact sequences.*

Proof. We outline a proof sketch. Choose projective resolutions $P'_\bullet \twoheadrightarrow A'$ and $P''_\bullet \twoheadrightarrow A''$. By the Horseshoe Lemma there exists a short exact sequence of complexes

$$0 \longrightarrow P'_\bullet \longrightarrow P_\bullet \longrightarrow P''_\bullet \longrightarrow 0,$$

with degreewise splittings $P_n \cong P'_n \oplus P''_n$. Applying F yields a short exact sequence of complexes

$$0 \longrightarrow F(P'_\bullet) \longrightarrow F(P_\bullet) \longrightarrow F(P''_\bullet) \longrightarrow 0,$$

so passing to homology gives the desired long exact sequence in which $H_n(F(P_\bullet)) = \mathrm{L}_n F(-)$. Naturality follows from the functoriality of the horseshoe and the connecting morphism in homology. $\square$

Example. Let $0 \to A' \xrightarrow{i} A \xrightarrow{q} A'' \to 0$ be a short exact sequence of left R-modules and M a right R-module. Then there is a long exact sequence

$$\ldots \to \mathrm{Tor}_1^R(M, A') \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, A'') \to M \otimes A' \to M \otimes A \to M \otimes M'' \to 0.$$

LECTURE 13. APRIL 29, 2025

Proposition (Dimension shifting). *Let F be right exact. Let*

$$0 \longrightarrow K_{m+1} \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be an exact sequence with P_i projective for $i = 0, \ldots, m$. Denote syzygies $K_0 := A$ and, for $i \geq 0$, $K_{i+1} := \ker(d_i : P_i \to K_i)$. Then:

(1) For $i \geq m+2$, there are natural isomorphisms

$$L_i F(A) \;\simeq\; L_{i-1} F(K_1) \;\simeq\; \cdots \;\simeq\; L_{i-m-1} F(K_{m+1}).$$

(2) There is a short exact sequence

$$0 \longrightarrow L_{m+1} F(A) \longrightarrow F(K_{m+1}) \longrightarrow F(P_m) \longrightarrow F(K_m) \longrightarrow 0.$$

Proof. For the short exact sequence $0 \to K_{i+1} \to P_i \to K_i \to 0$ with P_i projective, the long exact sequence of left derived functors gives $L_j F(P_i) = 0$ for $j \geq 1$ and isomorphisms $L_j F(K_i) \simeq L_{j-1} F(K_{i+1})$ for $j \geq 2$. Iterating, starting from $0 \to K_1 \to P_0 \to A \to 0$ yields the chain in (1). For (2), apply the long exact sequence to $0 \to K_{m+1} \to P_m \to K_m \to 0$ to obtain

$$0 \to L_1 F(K_m) \to F(K_{m+1}) \to F(P_m) \to F(K_m) \to 0,$$

then identify $L_1 F(K_m) \simeq L_{m+1} F(A)$ by (1). Consider the short exact sequences

$$0 \to K_{m+2} \to P_{m+1} \xrightarrow{d_{m+1}} K_{m+1} \to 0$$

and

$$0 \to K_{m+1} \xrightarrow{i} P_m \xrightarrow{d_m} K_m \to 0.$$

Applying F and using $L_j F(P) = 0$ for projective P, get the commutative diagram with exact rows

$$\begin{tikzcd}
F(P_{m+2}) \arrow[r] \arrow[d, equal] & \ker F(d_{m+1}) \arrow[r] \arrow[d] & \ker F(i) \arrow[r] \arrow[d] & 0 \\
F(P_{m+2}) \arrow[r] \arrow[d] & F(P_{m+1}) \arrow[r] \arrow[d, "F(d_{m+1})"] & F(K_{m+1}) \arrow[r] \arrow[d, "F(i)"] & 0 \\
0 \arrow[r] & F(P_m) \arrow[r, "\sim"] \arrow[d] & F(P_m) \arrow[r] \arrow[d] & 0 \\
& F(K_m) \arrow[r, "\sim"] & \operatorname{coker} F(i) &
\end{tikzcd}$$

Snake Lemma yields an isomorphism $\ker F(i) \cong L_1 F(K_m)$, and by iterating the isomorphisms $L_j F(K_r) \cong L_{j-1} F(K_{r+1})$ one gets $\ker F(i) \cong L_{m+1} F(A)$, which finishes the proof. $\square$

Definition. An object Q is called F-**acyclic** if $L_i F(Q) = 0$ for all $i \geq 1$.

Definition. An F-**acyclic resolution** of an object A is a left resolution

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow A \longrightarrow 0,$$

with each Q_i F-acyclic.

Example. Let $F(-) := M \otimes_R (-)$. A left R-module N is F-acyclic iff $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. The following are equivalent:

(1) N is flat.

(2) $\mathrm{Tor}_1^R(M,N) = 0$ for all $M \in \underline{\text{Mod-}R}$.

(3) $\mathrm{Tor}_i^R(M,N) = 0$ for all $M \in \underline{\text{Mod-}R}$ and all $i \geq 1$.

Proposition. *If $Q_\bullet \to A$ is an F-acyclic resolution, then for every $n \geq 0$*

$$L_n F(A) \;\simeq\; H_n\big(F(Q_\bullet)\big).$$

Proof. Set $K_i := \ker(d_i : Q_i \to Q_{i-1})$ for $i \geq 1$ and $K_0 := A$. For each short exact sequence

$$0 \to K_{i+1} \to Q_i \to K_i \to 0$$

with Q_i being F-acyclic, the long exact sequence of left derived functors yields isomorphisms

$$L_n F(K_i) \cong L_{n-1} F(K_{i+1})$$

for all $n \geq 2$ and an exact segment

$$0 \longrightarrow L_1 F(K_i) \longrightarrow F(K_{i+1}) \longrightarrow F(Q_i) \longrightarrow F(K_i) \longrightarrow 0.$$

Iterating gives $L_n F(A) \cong L_1 F(K_{n-1})$. For $i = 0$ the segment reads

$$0 \longrightarrow L_1 F(A) \longrightarrow F(K_1) \xrightarrow{F(j)} F(Q_0) \longrightarrow F(A) \longrightarrow 0,$$

which identifies

$$L_1 F(A) \cong \ker F(j) \cong H_1(F(Q_\bullet)),$$

and the result follows after shifting the indices

$$L_n F(A) \cong L_1 F(K_{n-1}) \cong H_1\Big(F\big(\ldots \to Q_{n-1} \to K_{n-1} \to 0\big)\Big) \cong H_n(F(Q_\bullet)).$$

□

6.5 Ext and Tor

Let R be a ring and A, B left R-modules. If $P_\bullet \to A$ is a projective resolution, define

$$\mathrm{Tor}_i^R(A,B) \;:=\; H_i\big(P_\bullet \otimes_R B\big) \qquad (i \geq 0).$$

Also if $Q_\bullet \to B$ is a projective resolution, let

$$\mathrm{tor}_i^R(A,B) \;\cong\; H_i\big(A \otimes_R Q_\bullet\big) \qquad (i \geq 0).$$

Theorem (Balancing Tor). *For all left R-modules A, B and all $n \geq 0$ there is a natural isomorphism*

$$\mathrm{Tor}_n^R(A,B) \;\cong\; \mathrm{tor}_n^R(A,B).$$

Proof. Let $P_\bullet \to A$ and $Q_\bullet \to B$ be projective resolutions and set $K_i := \ker(P_i \to P_{i-1})$ with $K_0 = A$, and $V_j := \ker(Q_j \to Q_{j-1})$ with $V_0 = B$. By dimension shifting,

$$\mathrm{Tor}_n^R(A,B) \;\cong\; \mathrm{Tor}_1^R(K_i,B) \quad \text{and} \quad \mathrm{tor}_n^R(A,B) \;\cong\; \mathrm{tor}_1^R(A,V_j) \qquad (i+j=n,\ i,j \geq 1).$$

Consider the short exact sequences $0 \to K_i \to P_{i-1} \to K_{i-1} \to 0$ and $0 \to V_j \to Q_{j-1} \to V_{j-1} \to 0$. Tensoring element-wise gives a commutative 3×3 diagram with exact rows and columns

$$\begin{array}{ccccccc}
K_i \otimes V_j & \longrightarrow & P_{i-1} \otimes V_j & \longrightarrow & K_{i-1} \otimes V_j & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_i \otimes Q_{j-1} & \longrightarrow & P_{i-1} \otimes Q_{j-1} & \longrightarrow & K_{i-1} \otimes Q_{j-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_i \otimes V_{j-1} & \longrightarrow & P_{i-1} \otimes V_{j-1} & \longrightarrow & K_{i-1} \otimes V_{j-1} & \longrightarrow & 0
\end{array}$$

The Snake Lemma identifies $\mathrm{Tor}_1^R(K_{i-1},V_j)$ with the kernel of the top horizontal map and also identifies $\mathrm{tor}_1^R(K_i,V_{j-1})$ with the kernel of the left vertical map; exactness in the middle square shows these kernels coincide, hence

$$\mathrm{Tor}_1^R(K_i,V_{j-1}) \;\cong\; \mathrm{tor}_1^R(K_i,V_{j-1}).$$

Applying dimension shifting back yields $\mathrm{Tor}_n^R(A,B) \cong \mathrm{tor}_n^R(A,B)$ for all $n \geq 0$. □

Theorem (Ext via resolutions). *For left R-modules A, B and $n \geq 0$, if $B \to E^\bullet$ is an injective resolution and $P_\bullet \to A$ is a projective resolution, then*

$$\mathrm{Ext}_R^n(A,B) \;=\; H^n\big(\mathrm{Hom}(A,E^\bullet)\big) \;\cong\; H^n\big(\mathrm{Hom}(P_\bullet,B)\big).$$

6.6 Dimensions

Definition. For $M \in \underline{R\text{-Mod}}$, the **projective dimension** $\mathrm{pd}_R(M)$ is the minimal integer n such that there exists a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

The **injective dimension** $\mathrm{id}_R(M)$ is defined dually via an injective resolution

$$0 \to M \to I^0 \to \cdots \to I^n \to 0.$$

Example. Over $R = \mathbb{Z}$ and for $M = \mathbb{Z}/n$, one has $\mathrm{pd}_R(M) = 1$, since there is a projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow M \longrightarrow 0.$$

Proposition. *If R is a PID, then $\mathrm{pd}_R(M) \leq 1$ for all R-modules M, and dually $\mathrm{id}_R(M) \leq 1$ for all M.*

Proof. EXERCISE. □

Theorem. *Let M be a left R-module and $d \geq 0$. The following are equivalent:*

(1) $\mathrm{pd}_R(M) \leq d$.

(2) $\mathrm{Ext}^n_R(M, N) = 0$ *for all* $n > d$ *and all* N.

(3) $\mathrm{Ext}^{d+1}_R(M, N) = 0$ *for all* N.

(4) If $0 \to K_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$ *is exact with each* P_i *projective, then* K_d *is projective.*

Proof. Implications $4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$ are clear. We show $3 \Rightarrow 4$. By dimension shifting,

$$\mathrm{Ext}^{d+1}_R(M, N) \cong \mathrm{Ext}^1_R(K_d, N)$$

for all N. Hence $\mathrm{Ext}^1_R(K_d, -) = 0$, and the lemma below shows that K_d is projective. $\square$

Lemma. *A left R-module P is projective if and only if* $\mathrm{Ext}^1_R(P, N) = 0$ *for all* N.

Proof. The forward implication is clear. Now, given a short exact sequence

$$0 \to N' \to N \xrightarrow{g} N'' \to 0,$$

apply $\mathrm{Hom}_R(P, -)$ to get an exact row

$$0 \to \mathrm{Hom}(P, N') \to \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N'') \to \mathrm{Ext}^1_R(P, N') = 0.$$

Thus $\mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N'')$ is an epimorphism, so P is projective. $\square$

A dual statement of the theorem also holds for injective dimensions.

Theorem. *For a left module N and integer $d \geq 0$, the following are equivalent:*

(1) $\mathrm{id}_R(N) \leq d$.

(2) $\mathrm{Ext}^n_R(M, N) = 0$ *for all* $n > d$ *and all* M.

(3) $\mathrm{Ext}^{d+1}_R(M, N) = 0$ *for all* M.

(4) If $0 \to N \to E^0 \to \cdots \to E^{d-1} \to K^d \to 0$ *is exact with each* E_i *injective, then* K_d *is injective.*

Lemma. *A left R-module N is injective if and only if* $\mathrm{Ext}^1_R(R/I, N) = 0$ *for all left ideals I.*

Proof. Follows from the long exact sequence for $I \to R \to R/I$ and $F = \mathrm{Hom}(-, N)$:

$$\mathrm{Hom}(R, N) \to \mathrm{Hom}(I, N) \to \mathrm{Ext}^1_R(R/I, N) \to 0.$$

$\square$

Theorem. *The following numbers are equal:*

(1) $\sup\{\mathrm{id}_R N \mid N \in \underline{R\text{-Mod}}\}$.

(2) $\sup\{\mathrm{pd}_R M \mid M \in \underline{R\text{-Mod}}\}$.

(3) $\sup\{\mathrm{pd}_R M \mid M$ *is finitely generated*$\}$.

(4) $\sup\{\mathrm{pd}_R I \mid I \leq R$ *a left ideal*$\}$.

(5) $\sup\{d \mid \exists M, N \in \underline{R\text{-Mod}}$ *with* $\mathrm{Ext}^d_R(M, N) \neq 0\}$.

Proof. From the theorems on projective and injective dimensions it follows that $(1) = (5) = (2)$. Moreover, $(2) \geq (3) \geq (4)$ is immediate. If $d := (4) < \infty$ and $N \in \underline{R\text{-Mod}}$, take

$$0 \to N \to E^0 \to \cdots \to E^d \to K \to 0.$$

to be an injective resolution of N. For every left ideal $I \leq R$, dimension shifting gives

$$\mathrm{Ext}^{d+1}_R(R/I, N) \cong \mathrm{Ext}^1_R(R/I, K).$$

Since $\mathrm{pd}_R(R/I) \leq d$, the left side vanishes, hence $\mathrm{Ext}^1_R(R/I, K) = 0$ for all I. As we showed, this forces K injective, so $\mathrm{id}_R N \leq d$. Thus $(1) \leq (4)$, and all five numbers are equal. □

Definition. This common value is called the **left global dimension** of R, and is denoted $\mathrm{lgl}\, R$.

LECTURE 14. MAY 18, 2025

Definition. For a right R-module M, the **flat dimension** $\mathrm{fd}_R(M)$ is the least $d \in \mathbb{Z}_{\geq 0}$ such that there exists an exact sequence

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with each F_i flat; if no such d exists, set $\mathrm{fd}_R(M) = \infty$.

Lemma. *Fix $d \geq 0$. For a right R-module M, the following are equivalent:*

(1) $\mathrm{fd}_R(M) \leq d$.

(2) $\mathrm{Tor}_i^R(M, N) = 0$ *for all $i > d$ and all $N \in \underline{\text{Mod-}R}$.*

(3) $\mathrm{Tor}_{d+1}^R(M, N) = 0$ *for all $N \in \underline{\text{Mod-}R}$.*

(4) If $0 \to K_d \to F_{d-1} \to \cdots \to F_0 \to M \to 0$ is exact with each F_i flat, then K_d is flat.

Theorem. *The following numbers are equal:*

(1) $\sup\{\mathrm{fd}_R(M) \mid M \in \underline{R\text{-Mod}}\}$.

(2) $\sup\{\mathrm{fd}_R(R/I) \mid I \leq R$ *a right ideal*$\}$.

(3) $\sup\{\mathrm{fd}_R(N) \mid N \in \underline{\text{Mod-}R}\}$.

(4) $\sup\{\mathrm{fd}_R(R/I) \mid I \leq R$ *a left ideal*$\}$.

(5) $\sup\{\, d \mid \exists\, M \in \underline{R\text{-Mod}},\ N \in \underline{\text{Mod-}R}$ *with* $\mathrm{Tor}_d^R(M, N) \neq 0 \,\}$.

Definition. The **weak global dimension** of R is the common value above, and is denoted $\mathrm{wgl}(R)$.

7 Delta Functors

7.1 Definition and Basic Properties

Fix abelian categories $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$.

Definition (Delta Functor)**.** A **(cohomological) delta functor** $F = \{F^n, \delta_C^n\} : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ is:

(1) a collection of additive functors $F^n : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ for $n \geq 0$;

(2) for any short exact sequence $0 \to A \to B \to C \to 0$, a collection $\delta_C^n : F^n(C) \to F^{n+1}(A)$ for $n \geq 0$ of morphisms in $\underline{\mathcal{B}}$.

Satisfying the following axioms:

(i) For any short exact sequence $0 \to A \to B \to C \to 0$, the following is exact:

$$0 \to F^0(A) \to F^0(B) \to F^0(C) \xrightarrow{\delta_C^0} F^1(A) \to F^1(B) \to F^1(C) \xrightarrow{\delta_C^1} \cdots$$

(ii) For any commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

the diagram

$$\begin{tikzcd} F^n(C) \arrow[r, "\delta^n_C"] \arrow[d] & F^{n+1}(A) \arrow[d] \\ F^n(C') \arrow[r, "\delta^n_{C'}"] & F^{n+1}(A') \end{tikzcd}$$

commutes, i.e. δ^n is natural.

Remark. $F^\bullet \in \mathrm{Lex}(\underline{\mathcal{A}}, \underline{\mathcal{B}})$.

Example. $\{R^i F\}$ is a δ-functor.

Definition (Morphism of Delta Functors)**.** A morphism $t : (F^n, \delta^n_F) \to (G^n, \delta^n_G)$ of δ-functors is a collection of natural transformations $t^n : F^n \to G^n$ for $n \geq 0$ such that for any short exact sequence $0 \to A \to B \to C \to 0$ in $\underline{\mathcal{A}}$, the diagram

$$\begin{tikzcd} F^n(C) \arrow[r, "\delta^n_F"] \arrow[d, "t^n"] & F^{n+1}(A) \arrow[d, "t^{n+1}"] \\ G^n(C) \arrow[r, "\delta^n_G"] & G^{n+1}(A) \end{tikzcd}$$

commutes.

Definition (Homological Delta Functor)**.** A **homological δ-functor** is a functor

$$K^\bullet : \mathrm{Lex}(\underline{\mathcal{A}}, \underline{\mathcal{B}}) \to \delta\text{-}\mathrm{Func}(\underline{\mathcal{A}}, \underline{\mathcal{B}}).$$

Definition (Effaceability; Grothendieck)**.** Let $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be an additive fuctor.

(1) F is **effaceable** if for every $A \in \underline{\mathcal{A}}$ there exists a monomorphism $u : A \hookrightarrow E$ such that $F(u) = 0$.

(2) F is **coeffaceable** if for every $A \in \underline{\mathcal{A}}$ there exists an epimorphism $v : P \twoheadrightarrow A$ such that $F(v) = 0$.

Lemma. *Assume that $\underline{\mathcal{A}}$ has enough injectives. Then F is effaceable if and only if $F(I) = 0$ for all $I \in \mathrm{Inj}(\underline{\mathcal{A}})$.*

Proof. ($\Rightarrow$) Given injective I, choose a monomorphism $u : I \hookrightarrow E$ with $F(u) = 0$. Since I is injective, u splits, hence $E \simeq I \oplus I'$. Applying F, the map $F(u)$ identifies with the inclusion $F(I) \to F(I) \oplus F(I')$, so $0 = F(u)$ forces $\mathrm{id}_{F(I)} = 0$ and thus $F(I) = 0$. ($\Leftarrow$) For any A pick a monomorphism $i : A \hookrightarrow I$ with I injective. Then $F(I) = 0$ by assumption, hence $F(i) = 0$, proving effaceability. $\square$

7.2 Universal δ-functors

Definition. A δ-functor $F = (F^n, \delta_F^n) : \underline{A} \to \underline{B}$ is **universal** if for every δ-functor $G = (G^n, \delta_G^n)$ and natural transformation $t : F^0 \to G^0$ there exists a unique morphism of δ-functors $\{t^n\}_{n \geq 0} : F \to G$ with $t^0 = t$.

Lemma. *If a universal δ-functor exists, it is unique up to a unique isomorphism of δ-functors.*

Theorem. *If a cohomological δ-functor (F^n, δ_F^n) is effaceable in all degrees $n \geq 1$, then F is universal.*

Proof. Given $t^0 : F^0 \to G^0$, construct t^n by induction on n. Suppose t^k are defined for $k \leq n$. For each A choose a short exact sequence $0 \to A \xrightarrow{i} B \to C \to 0$ with $F^{n+1}(i) = 0$ (effaceability). Exactness gives a factorization

$$F^n(C) \longrightarrow \operatorname{coker} F^n(i) \xrightarrow{\bar{\delta}_F^n} F^{n+1}(A).$$

By functoriality of cokernels, t_B^n and t_C^n induce $z_A^n : \operatorname{coker} F^n(i) \to \operatorname{coker} G^n(i)$. Define t_A^{n+1} as the unique map making the following diagram commute:

$$\begin{tikzcd}
F^n(B) \arrow[r] \arrow[d, "t_B^n"] & F^n(C) \arrow[r] \arrow[d, "t_C^n"] & \operatorname{coker} F^n(i) \arrow[r, "\bar{\delta}_F^n"] \arrow[d, "z_A^n"] & F^{n+1}(A) \arrow[d, "t_A^{n+1}"] \\
G^n(B) \arrow[r] & G^n(C) \arrow[r] & \operatorname{coker} G^n(i) \arrow[r, "\bar{\delta}_G^n"] & G^{n+1}(A)
\end{tikzcd}$$

Naturality of connecting morphisms shows this t_A^{n+1} is well defined and natural in A; uniqueness follows by exactness and the same diagram chase. $\square$

Corollary. *If $F : \underline{A} \to \underline{B}$ is left exact additive, then its right derived functors R^nF with the standard connecting morphisms form a universal δ-functor.*

7.3 Universal Coefficients Theorem

Theorem (Künneth formula). *Let $P = (\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots)$ be a chain complex of flat right R-modules such that the images $d(P_n) \subseteq P_{n-1}$ are flat for all $n \in \mathbb{Z}$. For every $n \in \mathbb{Z}$ and every $M \in$ $\underline{R\text{-Mod}}$ there is a natural short exact sequence*

$$0 \longrightarrow H_n(P) \otimes_R M \longrightarrow H_n(P \otimes_R M) \longrightarrow \operatorname{Tor}_1^R\big(H_{n-1}(P), M\big) \longrightarrow 0. \tag{$\star$}$$

Proof. By the long exact sequence of Tor for $0 \to Z_n \to P_n \to d(P_n) \to 0$ and the flatness of P_n and $d(P_n)$, each $Z_n := \ker(d_n)$ is flat. Hence there is a short exact sequence of complexes

$$0 \longrightarrow Z_\bullet \longrightarrow P_\bullet \longrightarrow dP_\bullet \longrightarrow 0,$$

where the differentials on $Z_\bullet$ and $dP_\bullet$ are zero. Tensoring with M remains exact and yields a long exact sequence in homology

$$H_{n+1}(dP_\bullet \otimes_R M) \to H_n(Z_\bullet \otimes_R M) \to H_n(P_\bullet \otimes_R M) \to H_n(dP_\bullet \otimes_R M).$$

Since the outer complexes have zero differentials, one identifies

$$H_n(Z_\bullet \otimes_R M) \simeq Z_n \otimes_R M, \quad H_n(dP_\bullet \otimes_R M) \simeq d(P_n) \otimes_R M, \quad H_{n+1}(dP_\bullet \otimes_R M) \simeq d(P_{n+1}) \otimes_R M.$$

The short exact sequence

$$0 \to d(P_{n+1}) \to Z_n \to H_n(P) \to 0$$

identifies the left term with $\mathrm{Tor}_1^R(H_n(P), M)$ and gives an exact segment

$$0 \to \mathrm{Tor}_1^R\big(H_n(P), M\big) \to Z_n \otimes_R M \to H_n(P \otimes_R M) \to d(P_n) \otimes_R M \to 0.$$

Finally, the short exact sequence $0 \to d(P_n) \to Z_{n-1} \to H_{n-1}(P) \to 0$ identifies

$$\mathrm{coker}\big(Z_n \otimes_R M \to H_n(P \otimes_R M)\big) \simeq \mathrm{Tor}_1^R(H_{n-1}(P), M).$$

□

Corollary. *If $R = \mathbb{Z}$, the sequence $(\star)$ splits (non-naturally); hence*

$$H_n(P \otimes_{\mathbb{Z}} M) \simeq H_n(P) \otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(P), M\big).$$

Proof. Over $\mathbb{Z}$, subgroups of free abelian groups are free; thus each $d(P_n) \subseteq P_{n-1}$ is free, and there exists a decomposition $P_{n-1} \cong \mathbb{Z}^{I_{n-1}} \oplus d(P_n)$. Passing to tensors, $d(P_n) \otimes M$ is a direct summand of $P_{n-1} \otimes M$. Modding out $Z_n \otimes M$ and $\ker(d_n \otimes 1_M)$ by the common image $d_{n+1} \otimes 1_M$ shows that $H_n(P) \otimes M$ is a direct summand of $H_n(P \otimes M)$; the complement is $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M)$ by $(\star)$. □

Definition. Let $P \in \underline{\mathrm{Ch}}(\underline{\text{Mod-}R})$ be a chain complex of right R-modules and $Q \in \underline{\mathrm{Ch}}(\underline{R\text{-Mod}})$ a chain complex of left R-modules. Define the tensor product complex $(P \otimes_R Q, d)$ by

$$(P \otimes_R Q)_n \;=\; \bigoplus_{p+q=n} P_p \otimes_R Q_q,$$

with differential on homogeneous tensors given by

$$d(a \otimes b) \;=\; d_P(a) \otimes b \;+\; (-1)^p\, a \otimes d_Q(b), \qquad a \in P_p,\ b \in Q_q.$$

Theorem (Künneth formula for complexes)**.** *If each P_n and $d(P_n) \subseteq P_{n-1}$ is flat (as right R-modules) for all n, then for every n there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \longrightarrow H_n(P \otimes_R Q) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R\big(H_p(P), H_q(Q)\big) \longrightarrow 0.$$

If R is a PID, this sequence (noncanonically) splits.

7.4 Topological Application of the Künneth Formula

Let X be a topological space. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$ from the standard simplex $\Delta^n = \langle e_0, \ldots, e_n \rangle$ (the convex hull of the vertices $e_0, \ldots, e_n$).

Definition (Singular chains)**.** The group of **singular n-chains** is the free abelian group on singular n-simplices,

$$S_n(X) := \mathbb{Z}\langle \sigma \mid \sigma : \Delta^n \to X \rangle \;\simeq\; \bigoplus_{\sigma} \mathbb{Z}\,[\sigma].$$

For a generator corresponding to σ, write $[p_0, \ldots, p_n] := [\sigma(e_0), \ldots, \sigma(e_n)]$. The boundary map $\partial : S_n(X) \to S_{n-1}(X)$ is

$$\partial[p_0, \ldots, p_n] = \sum_{k=0}^{n} (-1)^k [p_0, \ldots, \widehat{p_k}, \ldots, p_n] \;=\; \sum_{k=0}^{n} (-1)^k\, [\sigma \circ d^k(e_0, \ldots, \widehat{e_k}, \ldots, e_n)],$$

where $d^k : \Delta^{n-1} \hookrightarrow \Delta^n$ is the k-th face inclusion.

For an abelian group M, **homology with coefficients** in M is defined by

$$H_n(X,M) := H_n\big(S_\bullet(X)\otimes_{\mathbb{Z}} M\big), \qquad M\in \underline{\mathrm{Ab}}.$$

By the universal coefficients theorem,

$$H_n(X,M) \;\simeq\; H_n(X,\mathbb{Z})\otimes_{\mathbb{Z}} M \,\oplus\, \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(X,\mathbb{Z}),\, M\big) \quad (\text{e.g. } M=\mathbb{Z}/n).$$

Theorem (Eilenberg–Zilber)**.** *There is a natural chain homotopy equivalence*

$$S_\bullet(X\times Y)\simeq S_\bullet(X)\otimes_{\mathbb{Z}} S_\bullet(Y),$$

whence an isomorphism $H_n(X\times Y,\mathbb{Z})\simeq H_n(S_\bullet(X)\otimes S_\bullet(Y))$.

Combining with Künneth, for all spaces X, Y one obtains the (integral) Künneth decomposition

$$H_n(X\times Y) \;\simeq\; \bigoplus_{p+q=n} H_p(X)\otimes_{\mathbb{Z}} H_q(Y) \,\oplus\, \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}\big(H_p(X), H_q(Y)\big).$$

Theorem (Universal Coefficient Theorem)**.** *Let P be a chain complex of projective right R-modules with P_n and $d(P_n)$ projective for all n. For any left R-module M there is a (noncanonically) split short exact sequence*

$$0\longrightarrow \mathrm{Ext}^1_R\big(H_{n-1}(P),\, M\big)\longrightarrow H^n\big(\mathrm{Hom}_R(P,M)\big)\longrightarrow \mathrm{Hom}_R\big(H_n(P),\, M\big)\longrightarrow 0.$$

Application

For a topological space X and an abelian group M, define singular cohomology with coefficients by

$$H^n(X,M) := H^n\big(\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), M)\big).$$

Then the universal coefficient theorem over $\mathbb{Z}$ gives a (noncanonical) decomposition

$$H^n(X,M) \;\simeq\; \mathrm{Hom}\big(H_n(X,\mathbb{Z}), M\big)\oplus \mathrm{Ext}^1_{\mathbb{Z}}\big(H_{n-1}(X,\mathbb{Z}), M\big).$$

7.5 Ext^1 and Extensions

Definitions:

(1) For objects A, C in an abelian category $\underline{\mathcal{A}}$, an **extension** of A by C is a short exact sequence

$$0\longrightarrow A\xrightarrow{i} B\xrightarrow{p} C\longrightarrow 0.$$

Write $E=(A\xrightarrow{i} B\xrightarrow{p} C)$.

(2) A morphism $r=(\alpha,\beta,\gamma): E\to E'$ between $E=(A\to B\to C)$ and $E'=(A'\to B'\to C')$ is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0 \end{tikzcd}$$

(in particular, $p'\beta=\gamma p$ and $\beta i = i'\alpha$).

(3) Two extensions of the same A, C are **equivalent**, $E\sim E'$, if there exists an isomorphism of extensions $(\mathrm{id}_A,\beta,\mathrm{id}_C): E\to E'$ (equivalently, $\beta: B\xrightarrow{\sim} B'$). Denote by $\mathrm{Ext}(C,A)$ the set of equivalence classes of extensions of A by C.

FUNCTORIALITY

Given $E \in \mathrm{Ext}(C, A)$ and a morphism $\gamma : C' \to C$, there exists a unique class $\gamma^* E \in \mathrm{Ext}(C', A)$ together with a morphism of extensions $(\mathrm{id}_A, \beta, \gamma) : \gamma^* E \to E$ fitting into a pullback square

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \times_C C' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

Dually, for $\alpha : A \to A'$ there exists a unique class $\alpha_* E \in \mathrm{Ext}(C, A')$ with a morphism of extensions $(\alpha, \beta, \mathrm{id}_C) : E \to \alpha_* E$ obtained by the pushout

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] & A' \sqcup_A B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

SUM OF EXTENSIONS

For $E_1, E_2 \in \mathrm{Ext}(C, A)$, define their **sum** by the following formula:

$$E_1 + E_2 \ := \ \nabla_{A*}\big(\Delta_C^*(E_1 \oplus E_2)\big) \in \mathrm{Ext}(C, A),$$

where $\Delta_C : C \to C \oplus C$ is the diagonal and $\nabla_A : A \oplus A \to A$ the codiagonal.

The zero element is the split extension and $-E = (-\mathrm{id}_A)_* E$.

ADDITIVITY

For all $\alpha : A \to A'$ and $\gamma : C' \to C$:

$$\alpha_*(E_1 + E_2) = \alpha_* E_1 + \alpha_* E_2, \qquad (E_1 + E_2) \circ \gamma = E_1 \circ \gamma + E_2 \circ \gamma.$$

Write $E \circ \gamma := \gamma^* E$. The operations are additive in the morphisms:

$$(\alpha_1 + \alpha_2)_* E = (\alpha_1)_* E + (\alpha_2)_* E, \qquad E \circ (\gamma_1 + \gamma_2) = E \circ \gamma_1 + E \circ \gamma_2.$$

Let E be an extension $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$. Choose a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \to 0.$$

Pick a lift $\alpha : P_0 \to B$ with $p \circ \alpha = \epsilon$. Since $p\,\alpha\,d_1 = 0$, there is a unique map $\delta_1 : P_1 \to A$ with $i\,\delta_1 = \alpha d_1$. Because $d_1 d_2 = 0$, one has $\delta_1 d_2 = 0$, so δ_1 is a 1–cocycle in $\mathrm{Hom}_R(P_\bullet, A)$. Put

$$\Psi([E]) := [\delta_1] \in H^1\big(\mathrm{Hom}_R(P_\bullet, A)\big) = \mathrm{Ext}^1_R(C, A).$$

Lemma. *The map Ψ is well-defined.*

Proof. If α' is another lift with $p\alpha' = \epsilon$, then $\alpha' - \alpha = i\,s$ for a unique $s : P_0 \to A$. The corresponding cocycles satisfy $\delta_1' - \delta_1 = s\,d_1$, a coboundary. Hence $[\delta_1'] = [\delta_1]$ in Ext^1, so Ψ is well defined. $\square$

Let $[d_1] \in \operatorname{Ext}^1(C, A)$ be represented by a cocycle $d_1 : P_1 \to A$ with $d_1 d_2 = 0$, where

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_0} P_0 \longrightarrow C \to 0$$

is a projective resolution of C. Put $B_2 := \operatorname{im}(d_2)$ and let $\bar{d}_1 : P_1/B_2 \to A$ be induced by d_1. Define

$$\Theta([d_1]) := \Big[\bar{d}_1 : 0 \longrightarrow P_1/B_2 \longrightarrow P_0 \longrightarrow C \longrightarrow 0\Big],$$

that is, the pushout of $0 \to P_1/B_2 \to P_0 \to C \to 0$ along $\bar{d}_1$. We argue that Θ is well-defined.

Lemma. *If $d_1' = d_1 + s\, d_2$ for some $s : P_0 \to A$, then $\Theta([d_1'])$ is equivalent to $\Theta([d_1])$.*

Proof. Let $\bar{d}_1' : P_1/B_2 \to A$ be induced by d_1'. Consider the cocartesian squares

$$\begin{tikzcd} P_1/B_2 \arrow[r] \arrow[d, "\bar{d}_1"'] & P_0 \arrow[d] \\ A \arrow[r] & B_{d_1} \end{tikzcd} \qquad \begin{tikzcd} P_1/B_2 \arrow[r] \arrow[d, "\bar{d}_1'"'] & P_0 \arrow[d] \\ A \arrow[r] & B_{d_1'} \end{tikzcd}$$

The universal property of pushouts applied to the pair of maps $(\mathrm{id}_A, \mathrm{id}_{P_0})$ and $(\mathrm{id}_A - s\,p, \mathrm{id}_{P_0})$ (where $p : P_0 \twoheadrightarrow C$) yields mutually inverse isomorphisms $B_{d_1} \xrightarrow{\sim} B_{d_1'}$ commuting with the structure maps to A and C. Hence the bottom short exact rows

$$0 \longrightarrow A \longrightarrow B_{d_1} \longrightarrow C \longrightarrow 0, \qquad 0 \longrightarrow A \longrightarrow B_{d_1'} \longrightarrow C \longrightarrow 0$$

represent the same class in $\operatorname{Ext}(C, A)$. $\square$

Proposition (w/o proof)**.** *The maps Ψ and Θ are inverse bijections.*