Introduction to Category Theory and Homological Algebra

Chapter 2

1. Are projections for products epimorphisms?
2. Let $f\colon X \to Y$ and $g\colon X \to Y'$. Show that $\langle f, g\rangle = (f \times g) \circ \Delta_X$.
3. Show that there are no products in the category of fields.
4. Show that a map of an equalizer is a mono.
5. Let T be a terminal object. Show that $X \times_T Y \cong X \times Y$.
6. Show that $X \times_Y (Y \times_Z W) \cong X \times_Z W$.
7. Show that pullback of a mono is mono.
8. Show that if a category has binary products and equalizers than it has pullbacks.
9. Show that a category has finite products and equalizers if and only if it has pullbacks and a terminal object.
10. Show that $f\colon X \to Y$ is a monomorphism if and only if

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pullback square.

11. Show that $\varprojlim R[x_1, \ldots, x_n]/I^n \cong R[[x_1, \ldots, x_n]]$, where $I = (x_1, \ldots, x_n)$.
12. Show that if a category has finite limits and inverse limits that it has all (small) limits.
13. Show that if a category I has an initial object i_0 then $\lim_I D \cong D(i_0)$.
14. Consider a group G as a category with one objects. Does this category has pullbacks? Products? Equalizers?
15. Show that if a category has equalizers then any idempotent splits.
16. Let $X\colon \underline{G} \to \underline{Sets}$ be a functor, find $\lim_{\underline{G}} X$.
17. Find when a limit of an identity functor exists and what it is?
18. Find $\varinjlim S_n$.
19. Find $\varinjlim GL_n(R)$.
20. Find $\varinjlim \mathbb{Z}/p^n\mathbb{Z}$.
21. Show that a functor is representable if and only if the category of elements of the functor has an initial object.
22. Let I be a non-empty connected category. Show that if there are limits of shape I in the category $\underline{C}$ then there are limits of shape I in the category $\underline{C}/C$.
23. Let $X\colon \underline{G} \to \underline{Sets}$ be a functor, find $\operatorname{colim}_{\underline{G}} X$.
24. Let $h_X\colon \underline{C} \to \underline{Sets}$. Find $\operatorname{colim} h_X$.
25. Show that the categories of groups and rings don't have cogenerators.