Introduction to Category Theory and Homological Algebra

Chapter 1

1. Show that in a concrete category surjections (injections) are epimorpisms (monomorphisms respectively).
2. Show that in a concrete category retractions (sections) are surjections (injections respectively).
3. Show that for a map f in a category the following are equivalent
   (a) f is an isomorphism.
   (b) f is an epimorphism and section.
   (c) f is a monomorphism and retraction.
4. Analyze the following examples of an epimorphism (monomorphism) in a concrete category which is not surjective (injective respectively).
   (a) Map $\mathbb{N} \to \mathbb{Z}$ in the category of monoids.
   (b) Map $\mathbb{Z} \to \mathbb{Q}$ in the category of (commutative) rings.
   (c) Covering map in the category of pointed, connected, locally path connected topological spaces.
   (d) Map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ in the category of divisible groups.
5. Show that in the category of sets the following conditions for a map f are equivalent
   (a) f is a surjection.
   (b) f is an epimorphism.
   (c) f is a retraction.
6. Show that in the category of sets the following conditions for a map f are equivalent
   (a) f is a injection.
   (b) f is an monomorphism.
   (c) f is a section.
7. Show that in the category of topological spaces epimorphisms are the same as surjections and monomorphisms are the same as injections.
8. Show that in the category of Hausdorff topological spaces epimorphisms are the same as dominant maps and monomorphisms are the same as injections.
9. Show that in the category of compact Hausdorff topological spaces epimorphisms are the same as surjections and monomorphisms are the same as injections.
10. Let R be a ring. Show that in the category of R-modules epimorphisms are the same as surjections and monomorphisms are the same as injections.
11. Show that in the category of groups epimorphisms are the same as surjections and monomorphisms are the same as injections.
12. Is image of a functor a subcategory?
13. Let $F\colon \underline{C} \to \underline{D}$ be a faithful functor. Show that if $F(f)$ is an epimorphism (monomorphism), then f is epimorphism (monomorphism respectively).

14. Is there a functor $F\colon \underline{Grps} \to \underline{Grps}$ such that $F(G) = Z(G)$ for any group G?

15. Show that map f is epimorphism (monomorphism) if and only if $h^A(f)$ ($h_A(f)$ respectively) is injective for any object A.

16. Are the following categories equivalent to their opposite categories

    (a) the category of sets $\underline{\text{Sets}}$,

    (b) the category of binary relations $\underline{\text{Rel}}$,

    (c) the category of finite abelian groups $\underline{FinAb}$?

17. Show that the center of a category is a commutative monoid.

18. Let G and H be groups. Find all functors from $\underline{G}$ to $\underline{H}$ and natural transformations between those functors.

19. For the following forgetful functors find left adjoint

    (a) $U\colon \underline{Ab} \to \underline{Groups}$,

    (b) $U\colon \underline{Sets_*} \to \underline{Sets}$,

    (c) $U\colon \underline{Rings_*} \to \underline{Rings}$,

    (d) $U\colon \underline{Rings} \to \underline{Mon}$ is taking multiplicative (additive) monoid of the ring,

    (e) $U\colon \underline{CHaus} \to \underline{Top}$,

    (f) $U\colon \underline{Haus} \to \underline{Top}$.

20. For the following forgetful functors find left and right adjoint

    (a) $U\colon \underline{G - Sets} \to \underline{Sets}$,

    (b) $U\colon \underline{Top} \to \underline{Sets}$,

    (c) $U\colon \underline{Groups} \to \underline{Mon}$.