# Introduction to Category Theory and Homological Algebra

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# LECTURE 1. JANUARY 21, 2025

### **COURSE OUTLINE**

- (1) Categories, functors, natural transformations
- (2) Adjoint functors
- (3) Limits and colimits
- (4) Abelian categories
- (5) Resolutions and derived functors

#### **COURSE GRADING**

(1) 3 quizzes: 10% each, 30% total

(2) Midterm exam: 30%

(3) Final exam: 30%

(4) Seminars: 10%

(5) Homework:  $\sim$ 10%

# 1 Categories and Functors

# 1.1 Definition and Examples of Categories

**Definition.** A **category**  $\underline{C}$  consists of the following data:

- (i) a class of **objects**  $Ob(\underline{C})$ ;
- (ii) for every  $X, Y \in Ob(\underline{C})$  a set (or a class) of **morphisms**  $Hom_{\underline{C}}(X, Y)$ ;
- (iii) for every  $X \in Ob(\underline{C})$  an **identity morphism**  $id_X \equiv 1_X \in Hom_{\underline{C}}(X, X)$ ;
- (iv) for every  $X, Y, Z \in \mathsf{Ob}(\underline{C})$  a **composition rule**

$$\circ: \operatorname{Hom}_{\underline{C}}(X,Y) \times \operatorname{Hom}_{\underline{C}}(Y,Z) \to \operatorname{Hom}_{\underline{C}}(X,Z)$$

satisfying the usual associativity and unitality relations.

A category  $\underline{C}$  is called **locally small** if each hom-set  $\operatorname{Hom}_{\underline{C}}(X,Y)$  is a set.

Remark. There are plenty of ways to denote the set of morphisms. For example,

$$\operatorname{Hom}_{\underline{C}}(X,Y) \equiv \underline{C}(X,Y) \equiv \operatorname{Maps}_{\underline{C}}(X,Y) \equiv \operatorname{Arr}_{\underline{C}}(X,Y).$$

**Definition.** Given a category  $\underline{C}$ , there is an **opposite category**  $\underline{C}^{op}$ , whose

(i) objects are the same as in  $\underline{C}$ , i.e.  $Ob(\underline{C}^{op}) := Ob(\underline{C})$ ;

(ii) morphisms are «reversed», i.e. for all  $X, Y \in Ob(C^{op})$ 

$$\operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(X,Y) := \operatorname{Hom}_{\mathcal{C}}(Y,X)$$

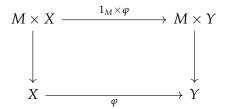
with the natural composition rule  $f \circ_{op} g := g \circ f$ .

### **Examples:**

- (1) Sets is the category of sets and functions;
- (2) Mon is the category of monoids and monoid homomorphisms;
- (3) Grps is the category of group and group homomorphisms;
- (4) Rings is the category of (all) rings and ring homomorphisms;
- (5) CRings is the category of commutative rings;
- (6) Fields is the category of fields and field homomorphisms;
- (7) For a monoid M, there is a category M-Sets, where
  - (i) objects are sets X with an M-action  $M \times X \to X$ ;
  - (ii) morphisms are **equivariant maps**, i.e. functions  $\varphi : X \to Y$  such that

$$\varphi(mx) = m\varphi(x)$$

for every  $m \in M$  and  $x \in X$ . This can be expressed as a **commutative diagram** 



meaning that tracing elements through all possible paths yields the same result:

$$(m,x) \longmapsto (m,\varphi(x))$$

$$\downarrow \qquad \qquad \downarrow$$

$$mx \longrightarrow \varphi(mx) = m\varphi(x)$$

(8)  $\underline{R}$ -Mod and  $\underline{Mod}$ - $\underline{R}$  are the categories of left and right modules over a ring R.

If *R* is commutative, these notions coincide and we write

$$R$$
-Mod  $\equiv$  Mod- $R$   $\equiv$  Mod( $R$ ).

Moreover, if *R* is a field, this is the category of vector spaces

$$\underline{\text{Mod}}(R) \equiv \underline{\text{Vect}}(R).$$

- (9) Top is the category of topological spaces and continuous maps.
- (10) The category of metric spaces Met:
  - (i) Objects are pairs (X, d), where X is a set and  $d: X \times X \to \mathbb{R}$  a metric.
  - (ii) Morphisms  $(X, d_1) \rightarrow (Y, d_2)$  are functions  $f: X \rightarrow Y$  which are
    - a) isometries (the rigid version of Met), i.e.

$$d(f(x), f(y)) = d(x, y)$$
 for all  $x, y \in X$ ;

b) Lipschitz maps (categorically richer version of Met), i.e.

$$\exists K > 0: \quad d(f(x), f(y)) \le Kd(x, y) \quad \text{ for all } x, y \in X.$$

- (11)  $C^{\infty}$ -manifolds, analytic manifolds, schemes, ...
- (12) hTop is the homotopy category of topological spaces, which
  - (i) objects are the same as in Top;
  - (ii) morphisms are homotopy classes of continuous maps.

**Definition** (intuitive). A category is called **concrete** if objects are sets with additional structure and hom-sets are maps of sets respecting the additional structure.

**Theorem** (w/o proof). hTop is not a concrete category.

- (13) The category of **binary relations** Rel:
  - (i) Objects of Ob(Rel) are sets.
  - (ii) Morphisms  $X \to Y$  are binary relations  $R \subset X \times Y$  with the usual notion of composition.
- (14) Every monoid M can be considered a category  $\underline{M}$ , where
  - (i) there is a single object \*;
  - (ii) morphisms  $* \rightarrow *$  are labeled by elements of M, where composition is plain multiplication.

**Definition.** A **groupoid**  $\underline{C}$  is a category where every arrow is inverible, which formally means that for every  $f \in \underline{C}(X,Y)$  there is  $g \in \underline{C}(Y,X)$  such that  $fg = 1_Y$  and  $gf = 1_X$ .

**Example.** If G is a group, then G is a groupoid.

- (15) If *X* is a topological space, then  $\Pi_1(X)$ , the **fundamental groupoid** of *X*:
  - (i) Objects of  $\Pi_1(X)$  are points of X.
  - (ii) Morphisms  $x \rightarrow y$  are homotopy classes of all paths from x to y.

This is a generalization of the fundamental group of *X*, since

$$\text{Hom}_{\Pi_1(X)}(x,x) = \pi_1(X,x).$$

# 1.2 Special Morphisms

Let  $f \in \underline{C}(X, Y)$  be a morphism in  $\underline{C}$ . We say that

### **Definitions:**

- (1) f is an **isomorphism** (iso) if it has an inverse  $f^{-1} \in \underline{C}(Y, X)$ .
- (2) f is a **monomorphism** (mono) if  $fg_1 = fg_2$  implies  $g_1 = g_2$  for any  $g_1, g_2 \in \underline{C}(Z, X)$ .
- (3)  $f \in \underline{C}(X,Y)$  is an **epimorphism** (epi) if  $f \in \underline{C}^{op}(Y,X)$  is a monomorphism.
- (4) Suppose  $Y \xrightarrow{g} X \xrightarrow{f} Y$  with  $f \circ g = 1_Y$ . Then g is called a **section** (or a **coretraction**) of f and g is called a **retraction** (or a **cosection**) of g.

Lemma (Basic properties of monomorphisms and epimorphisms).

- (1) Composition of monomorphisms (resp. epimorphisms) is monic (resp. epic);
- (2) *If f g is monic, then g is monic;*
- (3) If fg is epic, then f is epic;
- (4) Any section is a monomorphism;
- (5) Any retraction is an epimorphism.

*Proof.* Exercise.

Note that there are non-invertible maps that are both monomorphisms and epimorphisms.

**Lemma.** For  $f: X \to Y$ , the following are equivalent:

- (1) f is an isomorphism;
- (2) f is a retraction and a monomorphism;
- (3) f is a section and an epimorphism.

Proof. Exercise.

#### **Remarks:**

- (1) Sections are also called **split monomorphisms**.
- (2) Similarly, retractions are called **split epimorphisms**.

**Definition.** A category  $\underline{C}'$  is a **subcategory** of a category  $\underline{C}$  if

- (i) its objects form a subclass of objects of  $\underline{C}$ , i.e.  $\mathsf{Ob}(\underline{C}') \subset \mathsf{Ob}(\underline{C})$ ;
- (ii) its morphisms are  $\underline{C'}(X,Y) \subset \underline{C}(X,Y)$  for all  $X,Y \in \mathrm{Ob}(\underline{C'})$ .

A subcategory is called **full** if  $\underline{C'}(X,Y) = \underline{C}(X,Y)$  for each  $X,Y \in \text{Ob}(\underline{C'})$ .

1.3 FUNCTORS 5

# LECTURE 2. JANUARY 28, 2025

### **TEXTBOOKS**

- (1) Leinster T., Basic Category Theory;
- (2) Riehl E., Category Theory in Context;
- (3) Herrlich H., Strecker G., Category Theory (3rd Edition);
- (4) Adámek J. et al., Abstract and Concrete Categories. The Joy of Cats;
- (5) Mac Lane S., Categories for a Working Mathematician;
- (6) Kashiwara M., Schapira P., Categories and Sheaves.

### 1.3 Functors

**Definition.** Let  $\underline{C}$  and  $\underline{C}'$  be categories. A (covariant) **functor**  $F : \underline{C} \to \underline{C}'$  consists of

- (i) a (class) function  $F : Ob(\underline{C}) \to Ob(\underline{C}')$ ;
- (ii) for all  $X, Y \in Ob(\underline{C})$ , a function  $F : Hom_{\underline{C}}(X, Y) \to Hom_{\underline{C}'}(F(X), F(Y))$ ,

such that

- (1)  $F(1_X) = 1_{F(X)}$  for all  $X \in Ob(\underline{C})$ ;
- (2)  $F(g \circ f) = F(g) \circ F(f)$  for all  $f \in \text{Hom}_C(X, Y)$  and  $g \in \text{Hom}_C(Y, Z)$ .

### **Examples:**

- (1) The **identity functor**  $\mathrm{Id}_{\mathcal{C}}: \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ , defined by F(X) := X and F(f) := f.
- (2) For  $Y \in Ob(\underline{C}')$ , the **constant functor**  $\Delta_Y : \underline{C} \to \underline{C}'$ , defined by  $\Delta_Y(X) = Y$  and  $\Delta_Y(f) = 1_Y$ .
- (3) For  $\underline{C}$  locally small and  $A \in \text{Ob}(\underline{C})$ , the **representable functor**  $h_A : \underline{C} \to \underline{\text{Sets}}$ , also written as

$$h_A(-) := \operatorname{Hom}_{\underline{C}}(A, -),$$

which maps X to  $\operatorname{Hom}_{\underline{C}}(A,X)$  and  $f:B\to B'$  to  $h_A(f):\operatorname{Hom}_{\underline{C}}(A,B)\to\operatorname{Hom}_{\underline{C}}(A,B')$ , defined on  $g:A\to B$  as  $h_A(f)(g):=gf$ . There is also the contravariant version

$$h^A : \underline{C}^{op} \to \underline{Sets}, \qquad h^A(-) := \operatorname{Hom}_{\mathcal{C}}(-, A).$$

(4) For  $n \in \mathbb{Z}_{>0}$ , the general linear group defines a functor

$$GL_n : \underline{\mathsf{CRings}} \to \underline{\mathsf{Grps}}, \quad R \mapsto GL_n(R).$$

(5) The abelianization functor

$$(-)_{ab}: \underline{\mathsf{Grps}} \to \underline{\mathsf{Ab}}, \quad G \mapsto G/[G,G].$$

(6) Algebra of continuous functions to  $\mathbb{R}$  is a functor

$$C: \operatorname{Top}^{\operatorname{op}} \to \operatorname{Alg}(\mathbb{R}), \qquad X \mapsto C(X) := \underline{\operatorname{Cont}}(X, \mathbb{R}).$$

(7) For *k* a field, the dual vector space functor

$$D: \operatorname{Vect}(k)^{\operatorname{op}} \to \operatorname{Vect}(k), \qquad V \mapsto V^{\vee} := \operatorname{Hom}_k(V, k).$$

**Definition.** A **contravariant** functor  $F : \underline{C} \to \underline{C}'$  is simply a covariant functor  $F : \underline{C}^{op} \to \underline{C}'$ .

**Definition.** A functor  $F : \underline{C} \to \underline{C}'$  is called

- (1) **faithful** if  $F : \text{Hom}_{C}(X, Y) \to \text{Hom}_{C'}(F(X), F(Y))$  is injective for all  $X, Y \in \text{Ob}(\underline{C})$ ;
- (2) **full** if  $F : \text{Hom}_{\underline{C}}(X, Y) \to \text{Hom}_{\underline{C}'}(F(X), F(Y))$  is surjective for all  $X, Y \in \text{Ob}(\underline{C})$ ;
- (3) **fully faithful** (f.f.) if *F* is full and faithful;
- (4) **essentially surjective** (e.s.) if every  $Y \in Ob(\underline{C}')$  is isomorphic to F(X) for some  $X \in Ob(\underline{C})$ .

**Definition.** Given a family of categories  $\{\underline{C}_i\}_{i\in I}$ , we define the **product**  $\prod_{i\in I}\underline{C}_i$  by saying that

(i) 
$$Ob(\prod_{i \in I} \underline{C}_i) := \prod_{i \in I} Ob(\underline{C}_i);$$

(ii) 
$$\prod_{i \in I} \underline{C}_i(\{X_i\}, \{Y_i\}) := \prod_{i \in I} \operatorname{Hom}_{\underline{C}_i}(X_i, Y_i).$$

**Definition.** A functor  $F : \underline{C}_1 \times \underline{C}_2 \to \underline{D}$  is also called a **bifunctor**.

### **Examples:**

(1) Hom-set is a bifunctor

$$\operatorname{Hom}_C(-,-):C^{\operatorname{op}}\times C\to\operatorname{Sets}.$$

(2) For a ring *R*, tensor product is a bifunctor

$$-\otimes_R -: \underline{\mathsf{Mod}} - R \times \underline{\mathsf{R-Mod}} \to \underline{\mathsf{Ab}}.$$

**Definition.** Categories  $\underline{C}$  and  $\underline{C}'$  are called **isomorphic** (written  $\underline{C} \cong \underline{C}'$ ) if there exist functors  $F : \underline{C} \to \underline{C}'$  and  $G : \underline{C}' \to \underline{C}$  such that  $F \circ G = \mathrm{id}_{\underline{C}'}$  and  $G \circ F = \mathrm{id}_{\underline{C}}$ .

### **Examples:**

- (1)  $\mathbb{Z}$ -Mod  $\cong$  Ab.
- (2) For a finite group *G*,

$$\underline{\operatorname{Rep}}_k(G) \cong \underline{k[G]\operatorname{-Mod}}.$$

# 1.4 Natural Transformations

**Definition.** Let  $F, G : \underline{C} \to \underline{D}$  be functors. A **natural transformation** 

$$\theta: F \Rightarrow G$$

is a morphism  $\theta_X : F(X) \to G(X)$  (a **component** of the transformation  $\theta$ ) for each  $X \in Ob(\underline{C})$  such that (**naturality condition**) given any  $f : X \to Y$ , the diagram

$$F(X) \xrightarrow{F(f)} F(Y)$$

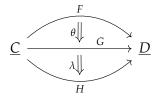
$$\theta_X \downarrow \qquad \qquad \qquad \downarrow \theta_Y$$

$$G(X) \xrightarrow{G(f)} G(Y)$$

is commutative.

#### **Remarks:**

- (1) If *F* is a functor, there is the **identity transformation**  $1_F : F \Rightarrow F$ .
- (2) For  $\theta : F \Rightarrow G$  and  $\lambda : G \Rightarrow H$ , the **composition**  $\lambda \circ \theta : F \Rightarrow H$



is defined by  $(\lambda \circ \theta)_X = \lambda_X \circ \theta_X$ .

(3)  $\theta: F \Rightarrow G$  is a **natural isomorphism** if  $\theta_X$  is an isomorphism for each  $X \in Ob(\underline{C})$ .

If we fix  $\underline{C}$  and  $\underline{D}$ , there is the **category of functors** 

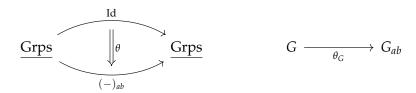
$$\underline{\operatorname{Func}}(\underline{C},\underline{D}) = [\underline{C},\underline{D}] = \underline{D}^{\underline{C}}.$$

And for functors F, G :  $\underline{C} \rightarrow \underline{D}$ , there is a **category of natural transformations** 

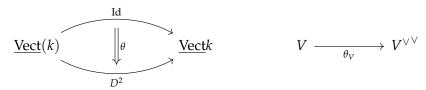
$$\underline{\mathrm{Nat}}(F,G) := [\underline{C},\underline{D}](F,G).$$

#### **Examples:**

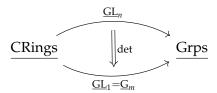
(1) A transformation from Id to  $(-)_{ab}$ :



(2) A transformation from Id to  $D^2$ :



(3) Determinant of a matrix as a natural transformation:



(4) For *I* a set (considered as a discrete category):

$$[I,\underline{C}] = \prod_{i \in I} \underline{C}.$$

(5) Given  $I = (\bullet \Rightarrow \bullet)$ :

$$[I, \underline{\mathsf{Sets}}] = \mathsf{Graphs}.$$

(6)  $[\underline{G}, \underline{\operatorname{Sets}}] = \underline{G} \cdot \underline{\operatorname{Sets}}$  and  $[\underline{G}, \underline{\operatorname{Vect}}(k)] = \operatorname{Rep}_k(G)$ .

# 1.5 Equivalence of Categories

**Definition.** Categories  $\underline{C}$  and  $\underline{D}$  are called **equivalent** if there are functors  $F : \underline{C} \to \underline{D}$ ,  $G : \underline{D} \to \underline{C}$  and natural isomorphisms  $\alpha : G \circ F \Rightarrow \mathrm{Id}_{\underline{C}}$ ,  $\beta : F \circ G \Rightarrow \mathrm{Id}_{\underline{D}}$ . Notation:  $\underline{C} \simeq \underline{D}$ .

**Lemma.** Let  $F: \underline{C} \to \underline{D}$  and  $i: \underline{D_0} \hookrightarrow \underline{D}$  such that  $\underline{D_0}$  is full and for all  $X \in Ob(\underline{C})$  there is  $Y \in Ob(\underline{D_0})$  such that F(X) = Y. Then there exists a functor  $F_0: \underline{C} \to D_0$  and a natural isomorphism  $\theta_0: F \xrightarrow{\sim} iF_0$ .

*Proof.* Using the axiom of choice, we choose for each  $X \in \text{Ob}(\underline{C})$  an object  $Y \in \underline{D}_0$  and an isomorphism  $\varphi_X : Y \to F(X)$ . Now set  $F_0(X) := Y$ . Given  $f : X \to X'$ , let

$$F_0(X) \xrightarrow{\varphi_X} F(X) \xrightarrow{F(f)} F(X') \xrightarrow{\varphi_{X'}^{-1}} F_0(X')$$

be the value of  $F_0(f)$ . This defines a functor since for any  $f: X \to X'$  and  $g: X' \to X''$ 

$$F(X) \xrightarrow{F(f)} F(X') \xrightarrow{F(g)} F(X'')$$

$$\varphi_{X} \uparrow \qquad \qquad \varphi_{X'} \uparrow \qquad \qquad \varphi_{X''} \uparrow$$

$$Y = F_{0}(X) \xrightarrow{F_{0}(f)} Y' = F_{0}(X') \xrightarrow{F_{0}(g)} Y'' = F_{0}(X'')$$

It is clear that  $\varphi : F \Rightarrow iF_0$  is the required natural transformation.

**Corollary.** For any category  $\underline{C}$  there is a subcategory  $\operatorname{sk}(\underline{C})$ , the **skeleton** of  $\underline{C}$ , such that  $i : \operatorname{sk}(\underline{C}) \hookrightarrow \underline{C}$  is an equivalence of categories, and in  $\operatorname{sk}(C)$  any two issomorphic objects are equal.

# LECTURE 3. FEBRUARY 4, 2025

**Theorem.** A functor  $F : \underline{C} \to \underline{D}$  is an equivalence of categories if and only if F is fully faithful and essentially surjective.

*Proof.* We introduce the following notation:

$$\begin{array}{ccc}
\operatorname{sk}\underline{C} & \xrightarrow{j_D F i_C} & \operatorname{sk}\underline{D} \\
j_C & & & & & \downarrow \\
f_C & & & & \downarrow \\
C & \xrightarrow{F} & & D
\end{array}$$

Note that  $j_D Fi_C$  is surjective on objects. We will show that it is also injective. Indeed,

$$\varphi : \underline{C}(A, B) \xrightarrow{\sim} \underline{D}(F(A), F(B))$$

is a bijection for any  $A, B \in \mathrm{Ob}(\underline{C})$ . Suppose  $h : F(A) \xrightarrow{\sim} F(B)$  is an isomorphism. Then  $\varphi^{-1}(h)$  is also an isomorphism, hence A = B in  $\mathrm{sk}(\underline{C})$ . This shows that  $j_D F i_C$  is an isomorphism of categories, so there is an inverse functor  $K : \mathrm{sk}\,\underline{D} \to \mathrm{sk}\,\underline{C}$ . It follows that F is an equivalence of categories and  $i_D K j_C$  is a quasi-inverse for F.

## 1.6 Yoneda's Lemma

Let  $\underline{C}^{\vee} = [\underline{C}, \underline{Sets}]$ . Recall that  $h_A = \underline{C}(A, -) : \underline{C} \to \underline{Sets}$  is an object of  $Ob(\underline{C}^{\vee})$ .

**Lemma** (Yoneda). *For any*  $F \in Ob(\underline{C}^{\vee})$  *and*  $A \in Ob(\underline{C})$ ,

$$\underline{C}^{\vee}(h_A,F) \stackrel{\varphi}{\cong} F(A).$$

*Moreover,*  $\varphi$  *is natural in A and F.* 

*Proof.* Let  $\theta \in \underline{C}^{\vee}(h_A, F)$  and  $f : A \to B$ . Then

$$\begin{array}{ccc}
1_A \in h_A(A) & \xrightarrow{\theta_A} & F(A) \\
\downarrow h_A(f) & & \downarrow F(f) \\
h_A(B) & \xrightarrow{\theta_B} & F(B)
\end{array}$$

and hence

$$F(f)\circ\theta_A=\theta_B\circ h_A(f),$$

which implies

$$F(f) \circ \theta_A(1_A) = \theta_B(h_A(f)(1_A)) = \theta_B(f).$$

NATURALITY IN A:

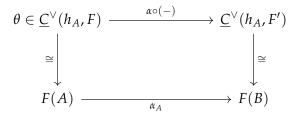
$$\underline{\underline{C}}^{\vee}(h_A, F) \xrightarrow{\theta \mapsto \theta \circ h_f} \underline{\underline{C}}^{\vee}(h_B, F) 
\theta \mapsto \theta_A(1_A) \downarrow \qquad \qquad \downarrow \theta' \mapsto \theta'_B(1_B) 
F(A) \xrightarrow{F(f)} F(B)$$

We note that

$$(\theta \circ h_f)(1_B) = \theta \circ (h_f)_B(1_B) = \theta_B(f \circ 1_B) = \theta_B(f) = F(f)\theta_A(1_A).$$

NATURALITY IN F:

Let  $\alpha : F \Rightarrow F'$ . It follows that  $(\alpha \circ \theta)(1_A) = \alpha_A \circ \theta_A(1_A)$ .



**Corollary.** Functor  $h_{(-)}:\underline{C}^{\mathrm{op}}\to\underline{C}^{\vee}$  (defined by  $A\mapsto h_A$ ) is fully faithful.

*Proof.* By Yoneda's lemma, 
$$\underline{C}^{\vee}(h_A, h_B) = h_B(A) = \underline{C}(B, A)$$
.

Let  $\hat{\underline{C}} = [\underline{C}^{op}, \underline{Sets}]$ . The dual statement of Yoneda's lemma:

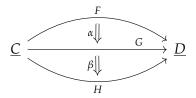
$$\hat{C}(h^A, F) = F(A),$$

where F and  $h^A = \underline{C}(-, A) \in \text{Ob}(\underline{\hat{C}})$ . It follows that  $h^{(-)} : \underline{C} \to \underline{\hat{C}}$  is fully faithful.

**Definition.** A functor  $F: \underline{C} \to \underline{Sets}$  is called **representable** if there is a natural isomorphism  $\theta: F \xrightarrow{\sim} h_A$  for some object  $A \in Ob(\underline{C})$ .

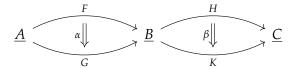
# 1.7 The Godement's Product (a.k.a. the Horizontal Composition)

Recall that for natural transformations  $\alpha : F \Rightarrow G$  and  $\beta : G \Rightarrow H$  its **vertical composition** 

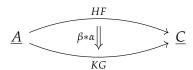


is defined by  $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$  for all  $X \in Ob(\underline{C})$ .

Now consider the following diagram:



We will define the **horizontal composition**  $\beta * \alpha$ ,



by saying that given  $X \in Ob(\underline{C})$ , one has

$$(\beta * \alpha)_X := \beta_{G(X)} \circ H(\alpha_X) = K(\alpha_X) \circ \beta_{F(X)}$$

taking advantage of the fact that the following diagram commutes:

$$HF(X) \xrightarrow{\beta_{F(X)}} KF(X)$$

$$H(\alpha_X) \downarrow \qquad \qquad \downarrow K(\alpha_X)$$

$$HG(X) \xrightarrow{\beta_{G(X)}} KG(X)$$

### **Examples:**

(1) If F = G and  $\alpha = 1_F$ ,

$$\beta_F := \beta * 1_F : HF \to KF, \qquad (\beta F)_X = \beta_{F(X)}.$$

(2) If H = K and  $\beta = 1_H$ ,

$$H\alpha := 1_H * \alpha : HF \rightarrow HG$$
,  $(H\alpha)_X = H(\alpha_X)$ .

Redrawing the diargam using new notation,

$$HF \xrightarrow{\beta_F} KF$$

$$H\alpha \downarrow \qquad \qquad \beta * \alpha \qquad \qquad \downarrow K\alpha$$

$$HG \xrightarrow{\beta_G} KG$$

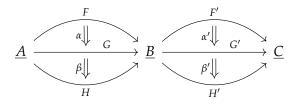
we note that

$$(1_K * \alpha) \circ (\beta * 1_F) = (\beta * 1_G) \circ (1_H * \alpha) = \beta * \alpha.$$

**Proposition.** *Properties of Godement's product:* 

(1) 
$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$$
.

(2) **2-associativity** or interchange law:



$$(\beta' \circ \alpha') * (\beta \circ \alpha) = (\beta' * \beta) \circ (\alpha' * \alpha).$$

Proof. EXERCISE.

**Remark.** Let  $\underline{A}$  and  $\underline{B}$  be categories. Define a functor

$$Comp: [A, B] \times [B, C] \rightarrow [A, C]$$

on objects and morphisms by letting

- (1)  $Comp(F, H) = H \circ F$ ;
- (2)  $Comp(\alpha, \beta) = \beta * \alpha$ .

**Remark.** Godement's product is useful when defining 2-categories.

## LECTURE 4. FEBRUARY 11, 2025

# 1.8 Adjoint Functors

**Definition.** Functors  $F : \underline{C} \to \underline{D}$  and  $G : \underline{D} \to \underline{C}$  are called **adjoint** (F is **left adjoint** of G and G is **right adjoint** of F; we write  $F \vdash G$ ) if there is a binatural bijection

$$\underline{D}(F(X), Y) \cong \underline{C}(X, G(Y)).$$

#### **MOTIVATION**

Let *k* be a field and *S* be a set. We write

$$k^{(S)} = k^{\oplus S} := \bigoplus_{s \in S} k.$$

If  $U : \underline{\text{Vect}} \to \underline{\text{Sets}}$  is the forgetful functor, then

$$\operatorname{Hom}_{\operatorname{Vect}}(k^{(S)}, V) \cong \operatorname{Hom}_{\operatorname{Sets}}(S, U(V)).$$

It turns out that the isomorphism is also binatural, implying that  $k^{(-)}$  is left adjoint to U.

#### **BINATURALITY**

(1) Naturality in the first argument

(2) Naturality in the second argument

Given the bijection

$$\varphi_{X,Y}: D(F(X),Y) \cong C(X,G(Y)),$$

it is convenient to speak of transposes: for a morphism  $g : F(X) \to Y$ , its **transpose** is

$$\bar{g} := \varphi_{X,Y}(g) : X \to G(Y);$$

conversely, for  $f: X \to G(Y)$ , its **transpose** is

$$\bar{f} := \varphi_{X,Y}^{-1}(f) : F(X) \to Y.$$

These assignments are mutually inverse, so  $\bar{g}=g$  and  $\bar{f}=f$ .

Then the two naturalities can be phrased succinctly in this language:

(1) For every  $h: X' \to X$ , transposition commutes with precomposition:

$$\overline{fh} = \overline{f} F(h).$$

(2) For every  $h: Y \to Y'$ , transposition commutes with postcomposition:

$$\overline{hg} = G(h) \bar{g}.$$

Let now  $X \in \text{Ob}(\underline{C})$  and set Y = F(X). Define

$$\eta_X := \overline{1}_{F(X)} : X \to GF(X).$$

This map is called the **unit** of the adjunction: it is the image, under the adjunction bijection, of the identity on F(X), and it is natural in X by binaturality of  $\varphi$ . In particular, for any  $h: X' \to X$  one has the naturality relation

$$GF(h) \eta_{X'} = \eta_X h.$$

Dually, define

$$\varepsilon_Y := \overline{1}_{G(Y)} : FG(Y) \to Y.$$

This map is called the **counit** of the adjunction: it is the image, under  $\varphi^{-1}$ , of the identity on G(Y), and it is natural in Y by binaturality of  $\varphi$ . The unit and counit satisfy the **triangle identities**, which express that transposition and transposition back act as the identity on both sides of the adjunction:

$$\varepsilon_{F(X)} \circ F(\eta_X) \; = \; 1_{F(X)}, \qquad G(\varepsilon_Y) \circ \eta_{G(Y)} \; = \; 1_{G(Y)}.$$

Using  $\eta$  and  $\varepsilon$ , the transposes admit explicit formulas. For  $g : F(X) \to Y$ , one has

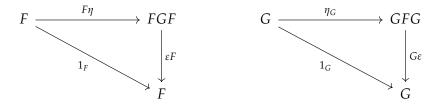
$$\bar{g}: X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(g)} G(Y).$$

Conversely, for  $f: X \to G(Y)$ , its transpose is the composite

$$\bar{f}: F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\varepsilon_Y} Y.$$

By the **triangle identities**, these constructions are inverse to one another:  $\bar{g} = g$  and  $\bar{f} = f$ .

**Lemma** (Triangular identities). *If*  $F \vdash G$ , *then the following diagrams are commutative:* 



*Proof.* The following establishes the first identity and finishes the proof (by duality):

$$1_{F(X)} = \overline{\eta_X} = \overline{1_{GF(X)}\eta_X} \stackrel{(2)}{=} \overline{1_{GF(X)}}F(\eta_X) = \varepsilon_{F(X)}F(\eta_X).$$

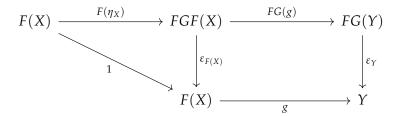
**Theorem.** Let  $F : \underline{C} \to \underline{D}$ ,  $G : \underline{D} \to \underline{C}$ . There is a one-to-one correspondence between

- (a) adjunctions between F and G;
- (b) natural transformations  $\eta:1_{\underline{C}}\to GF$ ,  $\epsilon:FG\to 1_{\underline{D}}$  satisfying triangular identities.

*Proof.* It is left to define a function from (b) to (a). Given  $g : F(X) \to Y$  and  $f : X \to G(Y)$ , we set

$$\bar{g} := (X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(g)} G(Y)), \qquad \bar{f} := (F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\varepsilon_Y} Y).$$

We claim that maps  $g \to \bar{g}$  and  $f \to \bar{f}$  are inverse to each other. Indeed,



This follows from the following computation:

$$g = \varepsilon_Y FG(g)F(\eta_X) = \varepsilon_Y F(G(g)\eta_X) = \eta_Y F(\bar{g}) = \bar{g}.$$

Finally, the chain of equalities

$$\overline{hg} = G(hg)\eta_X = G(h)G(g)\eta_X = G(h)\bar{g}$$

verifies the naturality conditions and finishes the proof.

**Proposition.** *If a functor*  $F : \underline{C} \to \underline{D}$  *admits a right adjoint, then this adjoint is unique up to isomorphism. Moreover,* F *admits a right adjoint if and only if the functor* 

$$D(F(-),Y): C^{\mathrm{op}} \to \mathrm{Sets}$$

is representable for every  $Y \in Ob(D)$ .

Proof. Consider the functor

$$F_*: \underline{D} \longrightarrow [\underline{C}^{op}, \underline{Sets}], \quad F_*(Y) := \underline{D}(F(-), Y),$$

and recall the Yoneda embedding  $h^{(-)}: \underline{C} \to [\underline{C}^{op}, \underline{Sets}]$ , where  $h^X = \underline{C}(-, X)$ .

#### UNIQUENESS

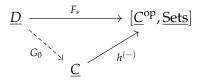
If  $G, G' : \underline{D} \to \underline{C}$  are both right adjoint to F, then for each Y there are natural bijections

$$\underline{C}(-,GY) \cong \underline{D}(F(-),Y) \cong \underline{C}(-,G'Y),$$

natural in the  $\underline{C}$ -variable. By Yoneda, this yields natural isomorphisms  $GY \cong G'Y$ , hence  $G \cong G'$ .

### **EXISTENCE**

The existence of a right adjoint to F is equivalent to a factorization of  $F_*$  through  $h^{(-)}$ :



that is, to the existence of isomorphisms

$$F_*(Y) \cong h^{G_0(Y)}$$
 or equivalently  $\underline{D}(F(-),Y) \cong \underline{C}(-,G_0(Y))$ ,

natural in both variables. If  $F \vdash G$ , then for every Y the functor  $\underline{D}(F(-), Y)$  is represented by G(Y), so  $F_* \cong h^{(-)} \circ G$ . Conversely, assume each  $F_*(Y)$  is representable. Choose for each Y an object  $G_0(Y) \in \mathrm{Ob}(\underline{C})$  and a natural isomorphism

$$\theta_Y: h^{G_0(Y)} \xrightarrow{\cong} F_*(Y).$$

For a morphism  $f: Y \to Y'$  in  $\underline{D}$ , define  $G_0(f): G_0(Y) \to G_0(Y')$  to be the unique arrow corresponding under Yoneda to the natural transformation

$$h^{G_0(Y)} \xrightarrow{\theta_Y} F_*(Y) \xrightarrow{F_*(f)} F_*(Y') \xrightarrow{\theta_{Y'}^{-1}} h^{G_0(Y')}.$$

This makes  $G_0$  a functor and the  $\theta_Y$  natural in Y. Consequently, for all  $X \in Ob(\underline{C})$ ,  $Y \in Ob(\underline{D})$  there are natural bijections

$$\underline{C}(X, G_0Y) \cong [\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}](h^X, h^{G_0Y}) \cong [\underline{C}^{\mathrm{op}}, \underline{\mathrm{Sets}}](h^X, F_*(Y)) \cong \underline{D}(FX, Y),$$

natural in both X and Y. Thus  $G_0$  is a right adjoint to F.

# 1.9 Terminal and Initial Objects

#### **Definitions:**

- (1) An object  $T \in Ob(\underline{C})$  is **terminal** if for all  $X \in Ob(\underline{C})$  there exists a unique map  $f : X \to T$ .
- (2) An object  $S \in \text{Ob}(\underline{C})$  is **initial** if for all  $X \in \text{Ob}(\underline{C})$  there exists a unique map  $f : S \to X$ .
- (3) An object  $Z \in Ob(\underline{C})$  is **zero** if Z is both initial and terminal.
- (4) A map  $f: X \to Y$  is **zero** if f factors through the zero object.

### **Examples:**

- (1) In Sets,  $S = \emptyset$  is initial, and  $T = \{\bullet\}$  is terminal.
- (2) In R Mod, Z = 0 is a zero object.
- (3) In Rings,  $S = \mathbb{Z}$  is initial, and T = 0 is terminal.

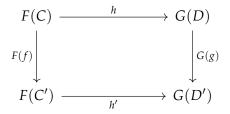
# 1.10 Comma Category

**Definition.** Given a diagram of categories and functors

$$C \xrightarrow{F} \mathcal{E} \xleftarrow{G} D$$

the **comma category** ( $F \downarrow G$ ) is defined by the following data:

- i) Its objects are triples (C, h, D), where  $C \in Ob(\underline{C})$ ,  $D \in Ob(\underline{D})$ , and  $h \in \mathcal{E}(F(C), G(D))$ .
- ii) Its morphisms  $(C, h, D) \to (C', h', D')$  are pairs (f, g), where  $f \in \underline{C}(C, C')$ ,  $g \in \underline{D}(D, D')$  are such the following diagram is commutative:



### **Examples:**

(1) Given an object  $c \in Ob(\underline{C})$ , the **slice category** is the comma category for the diagram

$$\underline{C} \xrightarrow{1_{\underline{C}}} \underline{C} \xleftarrow{c} 1$$
,

where  $1 \xrightarrow{c} \underline{C}$  is the constant functor with value c. It is commonly denoted by

$$(1_{\underline{C}} \downarrow c) \equiv (c \downarrow \underline{C}) \equiv \underline{C} / c.$$

(2) Analogously, the **coslice category** is defined by the diagram

$$1 \xrightarrow{c} C \xleftarrow{1_{\underline{C}}} C,$$

and is usually denoted by

$$(c \downarrow 1_{\underline{C}}) \equiv (\underline{C} \downarrow c) \equiv c / \underline{C}.$$

(3) The diagram

$$C \xrightarrow{1_{\underline{C}}} C \xleftarrow{1_{\underline{C}}} C$$

defines the arrow category denoted as follows:

$$(1_{\underline{C}} \downarrow 1_{\underline{C}}) \equiv \underline{Arr}(\underline{C}) \equiv (\underline{C} \downarrow \underline{C}).$$

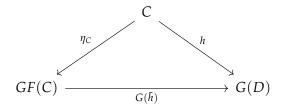
# LECTURE 5. FEBRUARY 18, 2025

IN THE LAST EPISODE...

**Definition.** A **universal arrow** is an initial object in  $(C \downarrow G)$ , where  $C \in Ob(\underline{C})$  and  $G : \underline{C} \to \underline{D}$ .

**Lemma.** If functors  $F : \underline{C} \to \underline{D}$  and  $G : \underline{D} \to \underline{C}$  are adjoint  $(F \vdash G)$ , then  $(F(C), \eta_C)$  is a universal arrow for any given object  $C \in Ob(\underline{C})$ .

*Proof.* Let  $(D,h) \in Ob(C \downarrow G)$ ; in other words, let  $D \in Ob(\underline{D})$  and  $h \in \underline{C}(C,G(D))$ . By adjunction there is a unique map  $\bar{h} \in \underline{D}(F(C),D)$  closing the diagram below:



This proves the claim.

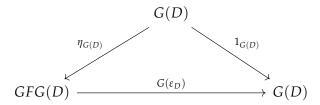
**Theorem.** Let  $F : \underline{C} \to \underline{D}$  and  $G : \underline{D} \to \underline{C}$ . There is a one-to-one correspondence between

- *a)* adjunctions  $F \vdash G$ ;
- *b)* natural transformations  $\eta: 1_C \to GF$  such that  $(F(C), \eta_C)$  is a universal arrow for any  $C \in Ob(\underline{C})$ .

*Proof.* Note that the previous lemma implies a)  $\Rightarrow$  b). To show that b)  $\Rightarrow$  a), let  $D \in Ob(\underline{D})$  and define  $\varepsilon_D : FG(D) \to D$  to be the unique map

$$(FG(D),\eta_{G(D)})\to (G(D),1_{G(D)})$$

in  $(G(D) \downarrow G)$ . We get the following commutative diargram:

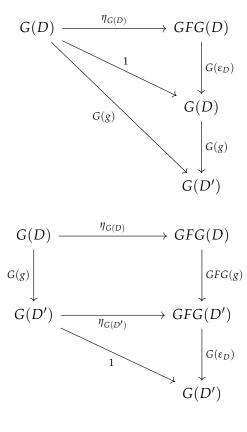


Our goal is to show that  $\varepsilon_D$  defines a natural transformation, and that  $\eta$  and  $\varepsilon$  satisfy triangle identities. This would suffice to imply that  $F \vdash G$  by one of the previous theorems.

We argue that one of the  $\Delta$ -identities follows from the definition of  $\varepsilon_D$ . For the rest, we provide brief proof sketches, which constist of commutative diagrams and encode the proof.

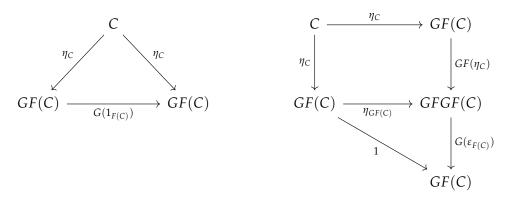
## NATURALITY OF $\varepsilon_D$ :

Given a morphism  $g: D \rightarrow D'$ , one has



$$g \circ \varepsilon_D = \varepsilon_D \circ FG(g)$$

### SECOND $\Delta$ -IDENTITY:



 $1_{F(C)} = \varepsilon_{F(C)} \circ F(\eta_C)$ 

**Corollary.**  $G: \underline{D} \to \underline{C}$  has a left adjoint iff for all  $C \in Ob(\underline{C})$  the category  $(C \downarrow G)$  has an initial object.

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*Proof.* It suffices to show only the backward implication. Set  $(F(C), \eta_C)$  to be the initial oobject in  $(C \downarrow G)$ . If  $f : C \to C'$  set F(f) to be the unique map such that

$$\begin{array}{ccc}
C & \xrightarrow{\eta_C} & GF(C) \\
\downarrow^f & & \downarrow^{GF(f)} \\
C' & \xrightarrow{\eta_{C'}} & GF(C')
\end{array}$$

From the diagram it follows that  $\eta: 1 \to GF$  is a natural transformation, hence the result.

### 2 Limits

### 2.1 Products

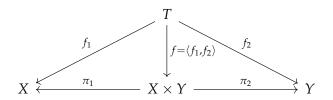
**Definition.** Let *X* and *Y* be objects of  $\underline{C}$ . A **product** of *X* and *Y* is a triple  $(X \times Y, \pi_1, \pi_2)$  with an object  $X \times Y \in \text{Ob}(\underline{C})$  and **projection morphisms** 

$$X \stackrel{\pi_1}{\longleftarrow} X \times Y \stackrel{\pi_2}{\longrightarrow} Y$$

having the property that for any other triple  $(T, f_1, f_2)$  there exists a unique map

$$f = \langle f_1, f_2 \rangle : T \to X \times Y$$

such that the following diagram commutes:



#### **Remarks:**

- (1) Product is the terminal object in the category of triples  $(T, f_1, f_2)$ .
- (2) Product is unique, if it exists at all.
- (3)  $C(T, X \times Y) \cong C(T, X) \times C(T, Y)$ .
- (4) Let *t* be a terminal object in  $\underline{C}$ . Then  $t \times X \cong X$  for all  $X \in Ob(\underline{C})$ .

### **Examples:**

- (1) In Sets, G-Sets, Mon, Rings the product of X and Y the product is the «usual»  $X \times Y$ .
- (2) In <u>Fields</u> there are no products (if L, K are fields, L,  $L \times K$ , K would have the same characteristic).

**Definition.** Given a family  $\{X_i\}_{i\in I}\subset \mathrm{Ob}(\underline{C})$ , its **product** is defined by the following data:

i) An object  $P = \prod_{i \in I} X_i$ ;

2.2 EQUALIZERS 21

ii) A family of **projections**  $\pi_i : P \to X_i$ ,

with with the property that for any other such data  $(T, f_i : T \to X_i)$  there exists a unique map

$$f = \langle f_i \rangle_{i \in I} : T \to \prod_{i \in I} X_i$$

such that  $f_i = \pi_i f$  for all  $i \in I$ .

### **Examples:**

- (1) In the category of finitely generated *R*-modules there are only finite products.
- (2) Sets, Mon, Rings, CRings all have arbitrary large products.

**Definition.** The **diagonal map**  $\Delta_X : X \to X \times X$  is the unique map  $\langle id_X, id_X \rangle$ .

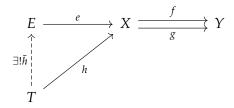
**Definition.** The **product of maps**  $f: X \to X', g: Y \to Y'$ :

**Proposition.** *For*  $f: X \to Y$ ,  $g: X \to Y'$ , *one has*  $\langle f, g \rangle = (f \times g) \circ \Delta_X$ .

Proof. Exercise. □

# 2.2 Equalizers

**Definition.** An **equalizer** of two maps  $f,g:\underline{C}(X,Y)$  is a pair (E,e) with an object  $E\in \mathrm{Ob}(\underline{C})$  and a map  $e:E\to X$  such that fe=ge, with the property that for any other such pair (T,h) with fh=gh, there exists a unique map  $\bar{h}$  satisfying  $e\bar{h}=h$ . In other words, we have the following diagram:



#### **Remarks:**

- (1) An equalizer is the terminal object in the category of such pairs.
- (2) If  $\underline{C}$  has a zero object and 0 denotes the zero map, the **kernel** of  $f \in \underline{C}(X,Y)$  is

$$\ker f := \operatorname{eq}(f, 0).$$

(3) The definition of an equalizer can be generalized to families of maps.

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# **Examples:**

(1) In Sets, given f, g :  $X \rightarrow Y$  one has

$$eq(f,g) = \{x \in X \mid f(x) = g(x)\}.$$

- (2) In  $\underline{Mon}$ , Grps, we can put algebraic structure on (1).
- (3) In Top, we can put topology on (1).

**Proposition.** *e is a monomorphism.* 

Proof. EXERCISE.

**Proposition.** *The following are equivalent:* 

- (1) f = g;
- (2)  $e = 1_X$ ;
- (3) e is an isomorphism;
- (4) e is an epimorphism.

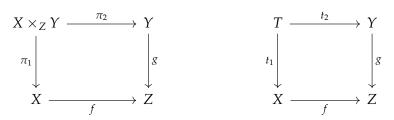
Proof. CLEAR.

2.3 PULLBACKS 23

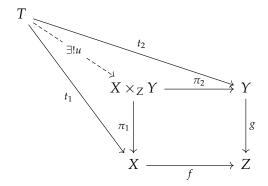
# LECTURE 6. FEBRUARY 25, 2025

# 2.3 Pullbacks

**Definition.** A **pullback** of the diagram  $X \xrightarrow{f} Z \xleftarrow{g} Y$  is a triple  $(X \times_Z Y, \pi_1, \pi_2)$ ,



with the property that given any other such triple  $(T, t_1, t_2)$ , there is a unique map  $u : T \to X \times_Z Y$  such that  $t_1 = \pi_1 u$  and  $t_2 = \pi_2 u$ .



Pullbacks are also called **cartesian squares** or **fibered products**. Notation:

$$X \times_Z Y = X_f \times_g Y = X \prod_Z Y.$$

#### **Remarks:**

- (1) A pullback is the terminal object in the category of all triples  $(P, \pi_1, \pi_2)$  such that  $f\pi_1 = g\pi_2$  (with morphisms as in the diagram above).
- (2) If  $t \in Ob(\underline{C})$  is terminal, then

$$X \times_t Y \cong X \times Y$$
.

(3) There is a cancellation property (EXERCISE):

$$X \times_Y (Y \times_Z W) \cong X \times_Z W$$
.

(4) Pullbacks are binary products in C/Z.

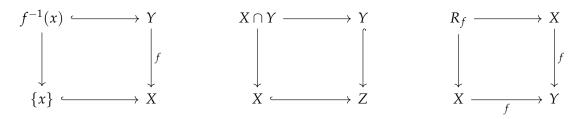
#### **Examples:**

(1) In <u>Sets</u>, one has

$$X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

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(2) Further examples in Sets:



(3) In Mon,  $R_f$  is the **congruence** on M induced by f.

# 2.4 Limits of Functors

**Definition.** A **limit** of a functor  $D: I \to \underline{C}$  is the terminal object in  $(\Delta \downarrow D)$ :

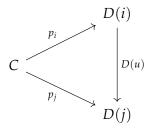
$$\underline{C} \xrightarrow{\Delta} [I,\underline{C}] \xleftarrow{D_*} \underline{1}.$$

Notation:  $\lim_{I} D \equiv \lim_{I} D$ .

In more detail, elements of this category are morphisms

$$\Delta_C \xrightarrow{p} D \in \mathrm{Ob}(\Delta \downarrow D),$$

meaning that for all  $u : i \rightarrow j$  in I the diagram

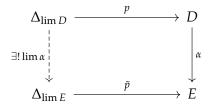


is commutative. Such object is called a **cone** (over F), and  $p_i$  are called **projections**.

**Definition** (Alternative). A **limit** of *F* is the terminal object in the category of cones over *F*.

Note that terminal objects, products, equalizers and pullback are all examples of limits.

Let  $\alpha: D \to E$  be a natural transformation of  $D, E: I \to \underline{C}$ .



Since limit is a terminal object, there is a unique arrow  $\lim \alpha : \lim D \to \lim E$ .

It follows that  $\lim$  is a functor  $\lim_{I} : [I, \underline{C}] \to \underline{C}$ . Note that

$$[I,\underline{C}](\Delta_{C},D)\cong\underline{C}(C,\lim_{I}D),$$

hence  $\lim$  is a right adjoint for the constant functor  $\Delta$ .

**Theorem.** *If a category*  $\subseteq$  *has binary equalizers and all products indexed by* Ob(I) *and* Arr(I)*, then for any*  $D: I \to C$  *there is a limit*  $\lim_{I} D$ .

*Proof.* We argue that the category of cones over *F* is isomorphic to the following category of cones:

$$T \xrightarrow{t} \prod_{i \in Ob(I)} D(i) \xrightarrow{\varphi} \prod_{u: i \to j} D(j)$$

Where  $\varphi$  and  $\psi$  are uniquely defined by the universal property of product and the relations

$$\pi_u \varphi = \pi_i, \qquad \pi_u \psi = D(u)\pi_i.$$

**Corollary.** There is a monomorphism  $\lim_I D \hookrightarrow \prod_{i \in Ob(I)} D(i)$ .

**Example.** In a concrete category  $\underline{C}$ ,

$$\lim_{I} D = \{(x_i) \in \prod D(i) \mid D(u)(x_i) = x_j \quad \text{ for all } u : i \to j\}.$$

# 2.5 Inverse (Projective) Limits

**Definition.** An **inverse (projective) system** in  $\underline{C}$  is  $D: I^{op} \to \underline{C}$ , where I is an ordered set. In this case the limit of D is called **inverse (projective)**.

### **Examples:**

(1) Let R be a commutative ring,  $I \subset R$  an ideal, and M an R module. Consider the diagram

$$\cdots \longrightarrow M/I^3M \longrightarrow M/I^2M \longrightarrow M/IM$$

Its limit

$$\lim M/I^nM=:\hat{M}^I$$

is called the **completion** of *M* with respect to *I*.

(2) For a prime p, the **p-adic integers** 

$$\lim_{\longleftarrow} \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p.$$

**Proposition.** *If C has finile limits and direct limits, then C has all limits.* 

**Definition.** A category  $\underline{C}$  is called **complete** if any  $D: I \to \underline{C}$  has a limit (where I is small).

**Theorem.** *The following are equivalent:* 

- (1) C is complete;
- (2)  $\underline{C}$  has small products and binary equalizers;

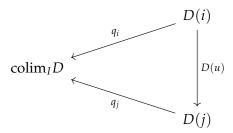
- (3)  $\underline{C}$  has small products and binary pullbacks;
- (4)  $\underline{C}$  has terminal object and small pullbacks;
- (5) C has finite limits and inverse limits;

### 3 Colimits

**Definition** (Formal). A **colimit** of a functor  $D: I \to \underline{C}$  is

$$\operatorname{colim}_{I} D := \left( \lim_{I^{op}} D^{op} \right)^{op}.$$

It is the dual notion to the limit. There are **cocones** and **coprojections**:



#### **Remarks:**

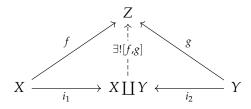
- (1)  $\operatorname{colim}_{I} : [I, \underline{C}] \to \underline{C}$  is a functor.
- (2) colim<sub>*I*</sub> is the left adjoint to the constant functor  $\Delta : \underline{C} \rightarrow [I,\underline{C}]$ .

# 3.1 Coproducts

**Definition.** Let *I* be a set,  $D: I \to \underline{C}$  a family of objects  $D(i) = X_i$ . A coproduct of  $X_i$  is

$$\coprod_{i\in I} X_i := \operatorname{colim}_I D.$$

Considering the binary case, there is the following universal property:



### **Examples:**

- (1) In Sets, the coproduct is the disjoint union  $X \coprod Y$ .
- (2) In Top, the same set can be endowed with topology.

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(3) In <u>R-Mod</u>, the coproduct is the disjoint union  $X \oplus Y$ .

Note that for an infinite set *I*,

$$\bigoplus_{i\in I} M_i \neq \prod_{i\in I} M_i.$$

(4) In Grps, the coproduct is the **free product** X \* Y.

Note that if  $X = \langle S_X \mid R_X \rangle$  and  $Y = \langle S_Y \mid R_Y \rangle$ , then

$$X * Y = \langle S_X \coprod S_Y \mid R_X \coprod R_Y \rangle.$$

(5) In  $\underline{\underline{\text{Rings}}}$ , the coproduct is the similarly defined free product X \* Y. It can also be defined in the following way:

$$R * S = T_{\mathbb{Z}}(R \times S) / (r \otimes r' - rr', s \otimes s' - ss', 1_R - 1_S).$$

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# 3.2 Coequalizers

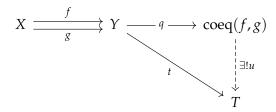
Let *I* be an indexing category consisting of two parallel arrows. A functor

$$D:I\to C$$

thereby defines a diagram

$$X \xrightarrow{f} Y$$

The colimit of this diagram is the **coequalizer** of *f* and *g*:



This diagram is called a cofork.

### **Examples:**

(1) A special case in <u>Sets</u>. Let  $R \subset X \times X$  be an equivalence relation on X.

(2) In general, the coequalizer of f and g is

$$\operatorname{coeq}(f,g) = Y/R^E,$$

where  $R^E$  is the equivalence relation generated by

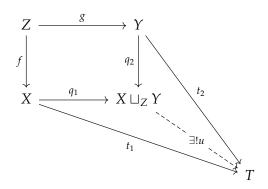
$$T = \{(f(x), g(x)) \in Y \times Y \mid x \in X\}.$$

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# 3.3 Pushouts

Let *I* be the category  $(\bullet \leftarrow \bullet \rightarrow \bullet)$  and  $D: I \rightarrow \underline{C}$ . The colimit of *D* 



is called the **pushout** of *f* and *g*.

### **Examples:**

(1) In Sets, the pushout of f and g is

$$X \sqcup_Z Y := X \sqcup Y/R$$
,

where *R* is the equivalence relation on  $X \sqcup Y$  generated by

$$f(z) \sim g(z)$$
 for all  $z \in Z$ .

In particular, if X, Y are subobjects of A,

$$\begin{array}{cccc} X \cap Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup Y \end{array}$$

(2) In Top, the pushout is

$$X \sqcup_Z Y := X \sqcup Y / R$$

with the quotient topology.

$$S^{1} \longleftrightarrow D^{2}$$

$$\downarrow \qquad \qquad \downarrow$$

$$D^{2} \longleftrightarrow S^{2}$$

(3) In Grps, given the diagram

$$G \stackrel{f}{\leftarrow} K \stackrel{g}{\rightarrow} H$$

the pushout of f and g is the **amalgamated product** 

$$G *_K H = G * H / N$$
,

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where *N* is the normalization of  $\{f(k)g(k)^{-1}\}_{k\in K}$ .

Let  $\mathbb{Z}/4\mathbb{Z} = \langle S \rangle$  and  $\mathbb{Z}/6\mathbb{Z} = \langle T \rangle$ ,

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} = \langle S, T \mid S^4, T^6, S^2 = T^3 \rangle \cong SL_2(\mathbb{Z}),$$

where the last isomorphism is defined by

$$S \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T \mapsto \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

**Remark.**  $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(t\text{Id}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$ 

(4) Seifert-Van Kampan Theorem:

If a topological space  $X = U_1 \cup U_2$ ,  $U_1$ ,  $U_2$ ,  $U_1 \cap U_2$  are path-connected, then

$$I_1 \cap U_2 \longrightarrow U_2 \qquad \qquad \pi_1(U_1 \cap U_2, x) \longrightarrow \pi_1(U_2, x)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$U_1 \longrightarrow X \qquad \qquad \pi_1(U_1, x) \longrightarrow \pi_1(X, x)$$

the last diagram is a pushout square for any  $x \in U_1 \cap U_2$ .

So the fundamental group «maps pushouts to pushouts».

- (5) In Rings, the pushout is the **amalgamated product of rings**.
- (6) In CRings, the pushout is the tensor product of algebras.

### 3.4 Direct limits

**Definition.** A **direct limit** is a colimit over a direct set.

### **Examples:**

(1) In <u>Sets</u>, the direct limit of

$$X_0 \subset \longrightarrow X_1 \subset \longrightarrow X_2 \subset \longrightarrow \dots$$

is the union

$$\underline{\lim} X_n = \bigcup_{n=0}^{\infty} X_n.$$

(2) In Grps, the direct limit of

$$S_1 \hookrightarrow S_2 \hookrightarrow S_2 \hookrightarrow \ldots$$

is the group

$$\varinjlim S_n = \{ \sigma \in S_{\mathbb{N}} \mid \sigma(n) = n \text{ for almost all } n \}.$$

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Given a ring *R*, the direct limit of

$$GL_1(R) \hookrightarrow GL_2(R) \hookrightarrow GL_3(R) \hookrightarrow \dots$$

is the quite general group

$$\underline{\lim} \ GL_n(R) = GL(R).$$

(3) In Ab, the direct limit of

$$\mathbb{Z}/p\mathbb{Z} \stackrel{p}{\longleftarrow} \mathbb{Z}/p^2\mathbb{Z} \stackrel{p}{\longleftarrow} \mathbb{Z}/p^3\mathbb{Z} \stackrel{p}{\longleftarrow} \dots$$

is the p-Prufer group

$$\underline{\lim} \ \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^{\infty}) = \mathbb{Z}[1/p] / \mathbb{Z}.$$

#### 3.5 **Functors and Limits**

**Definition.** A functor  $F : \underline{C} \to \underline{D}$  **preserves limits** if for any  $D : I \to \underline{C}$  and its limiting cone

$$(L \xrightarrow{p_i} D(i))_{i \in I},$$

the cone

$$(F(L) \xrightarrow{F(p_i)} FD(i))_{i \in I}$$

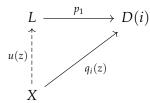
is limiting for  $FD: I \to \underline{D}$ .

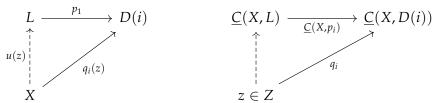
**Theorem.** *The functor*  $h_X$  *preserves limits:* 

$$\underline{C}(X, \lim_{I} D) = \lim_{I} \underline{C}(X, D).$$

*The isomorphism is natural in X and D.* 

*Proof.* Let  $L \xrightarrow{p_i} D(i)$  and Z be a set, then





**Corollary.**  $h^X(\operatorname{colim} D) = \lim h^X(D)$ .

**Corollary.** *Let*  $F : \underline{C} \to \underline{D}$  *and*  $G : \underline{D} \to \underline{C}$  *such that*  $F \vdash G$ *, then* 

- (1) F preserves colimits (LAPC);
- (2) G preserves limits (RAPL).

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*Proof.* We show (2):

$$\underline{C}(C, G(\lim_{I} D)) \cong \underline{D}(F(C), \lim_{I} D) \cong \lim_{I} \underline{D}(F(C), D(i)) \cong$$
$$\cong \lim_{I} \underline{C}(C, GD(i)) \cong \underline{C}(C, \lim_{I} GD(i)).$$

By Yoneda's lemma, it follows that

$$G(\lim_{I} D) \cong \lim_{I} GD.$$

**Examples:** 

(1) Let R and S be unital associative rings, and  $RB_S$  be an R-S-bimodule.

$$\underbrace{S\text{-Mod}}_{G} \xrightarrow{F} \underbrace{R\text{-Mod}}_{G}$$

And functors *F* and *G* are defined by

$$F(M) = B \otimes_S M$$
,  $G(N) = \operatorname{Hom}_R(B, N)$ .

The required isomorphism is

$$\operatorname{Hom}_R(B \otimes_S M, N) \cong \operatorname{Hom}_S(M, \operatorname{Hom}_R(B, N)),$$

where

$$f \mapsto (m \mapsto f((-) \otimes m))$$

and

$$(b \otimes m \mapsto g(m)b) \leftarrow g.$$

Since  $B \otimes -$  preserves colimits:

- a)  $B \otimes (\bigoplus M_i) \cong \bigoplus (B \otimes M_i)$
- b) it is right exact (=preserves cokernels), i.e. exactness of

$$M' \xrightarrow{f} M \to M'' \to 0$$

implies the exactness of

$$B \otimes M' \to B \otimes M \to B \otimes M'' \to 0.$$

**Remark.** Let  $f: R \to S$  be a morhisms of rings, the **restriction of scalars** functor

$$f^* \equiv \operatorname{res}_f : \underline{S\text{-Mod}} \to \underline{R\text{-Mod}}$$

on an S-module N is simultaneously

$$f^*(N) \cong \operatorname{Hom}_S({}_SS_R, N) =_R S_S \otimes_S N.$$

This gives us adjunctions

$$f_! \equiv \operatorname{ind} \vdash f^* \vdash f_* \equiv \operatorname{coind}$$

where

$$f_!(M) =_S S_R \otimes_R M$$
,  $f_*(M) = \operatorname{Hom}_R({}_RS_S, M)$ .

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IN THE LAST EPISODE...

Let  $F : \underline{C} \to \underline{D}$  and  $G : \underline{D} \to \underline{C}$  be a pair of adjoint functors  $(F \vdash G)$ .

**Definition.** A functor is called **continuous** (resp. **cocontinuous**) if it preserves all limits (resp. colimits).

We note that *F* is cocontinuous (LAPC) and *G* is continuous (RAPL).

### **Examples:**

(1) Let R, S be rings, and  $RB_S$  an R-S-bimodule. Then

$$B_S \otimes_S (-) \vdash \operatorname{Hom}_R({}_RB, -),$$

hence in particular

$$B_S \otimes_S \operatorname{colim} N_i \cong \operatorname{colim} B_S \otimes_S N_i$$

and

$$\operatorname{Hom}_R({}_RB, \operatorname{lim} M_i) \cong \operatorname{lim} \operatorname{Hom}_R({}_RB, M_i).$$

(2) Since  $\operatorname{colim}_I \vdash \Delta \vdash \lim_I$ , we get (completely for free)

$$\lim_{I} \lim_{I} (-) \cong \lim_{I} \lim_{I} (-).$$

# 4 Adjoint Functor Theorems

# 4.1 Generators and Cogenerators of Categories

#### **Definitions:**

- (1) A set of objects  $\{G_i\}_{i\in I}$  in a category  $\underline{C}$  is a set of **generators** (**separators**) of  $\underline{C}$  if for all  $f \neq g \in \underline{C}(X,Y)$  there exist  $i \in I$  and  $h : G_i \to X$  such that  $fh \neq gh$ .
- (2) Object G is called a generator if  $\{G\}$  is a generating set.
- (3) A **cogenerator** is a generator in the opposite category.

#### **Remarks:**

- (1) *G* is a generator if and only if  $h_G = \underline{C}(G, -)$  is faithful.
- (2)  ${G_i}_{i \in I}$  is a generating set if and only if  $\prod_{i \in I} h_{G_i}$  is faithful.

#### **Examples:**

- (1) In Sets, any non-empty set is a generator. Moreover, any S with  $|S| \ge 2$  is a cogenerator.
- (2) In Top we can endow the corresponding sets with discrete (resp. indiscrete) topology.
- (3) In  $\underline{Ab}$ , a group  $\mathbb{Z} \oplus A$  is a generator and  $\mathbb{Q}/\mathbb{Z}$  is a cogenerator.

- (4) In <u>R-Mod</u>, *R* is a generator and  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}R,\mathbb{Q}/\mathbb{Z})$  is a cogenerator.
- (5) In Grps,  $\mathbb{Z}$  is a generator.

**Statement.** There are no cogenerators in Grps.

*Proof.* Suppose *Q* is cogenerating. Given a non-trivial simple group *G*,

$$id_G \neq 1: G \rightarrow G$$

hence there is  $\alpha : G \to Q$  such that  $\alpha \neq 1$ . Since ker  $\alpha$  is a proper normal subgroup of a simple group,  $\alpha$  is injective. But there are simple groups of arbitrary large cardinalities (for example,  $PSL_2(K)$ ,  $K = \mathbb{C}(x_i)_{i \in I}$ ), a contradiction.

- (6) In Rings,  $\mathbb{Z}[x]$  is a generator. There are no cogenerators by the similar agrument: there are fields of arbitrary large cardinalities, and all non-trivial homomorphisms from them are injective.
- (7) In CHaus,  $\{*\}$  is a generator, and [0,1] is a cogenerator.

#### **Definitions:**

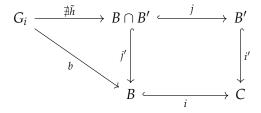
- (1) Let  $R \stackrel{f}{\hookrightarrow} C$  and  $S \stackrel{g}{\hookrightarrow} C$  be monomorphisms in  $\underline{C}$ . We say that  $(R, f) \sim (S, g)$  if there is an isomorphism  $\tau : R \to S$  such that  $g\tau = f$ . Equivalence classes under this relation are called **subobjects** of C. We denote by  $\operatorname{Sub}(C)$  the class of all subobjects of C.
- (2) We say that  $\underline{C}$  is **well-powered** if Sub(C) is a set for all  $C \in Ob(\underline{C})$ .

#### **Examples:**

- (1) All «everyday» categories are well-powered.
- (2) An example of a not well-powered category would be any partially ordered class with a maximal element (which every element would be a distinct suboject of). In particular, Ord<sup>op</sup>.

**Theorem** (Electrification). Let  $\underline{C}$  be a balanced category with finite intersections and a set of generators  $\{G_i\}$ . Then  $\underline{C}$  is well-powered.

*Proof.* Let  $B \ncong B'$ . If both j and j' are epic, then  $B \cong B \cap B' \cong B'$ , a contradiction.



Suppose (WLOG) j is not epic, i.e. there are  $f \neq g : B \rightarrow D$  such that fj = gj.

There is  $h: G_i \to B$  such that  $fh \neq gh$ . It is clear that h cannot be factored through  $B \cap B'$ . Hence, there is no  $h': G_i \to B'$  such that i'h' = ih. It implies that

$$\Phi: \operatorname{Sub}(C) \to 2^{\underline{C}(G_i,C)}, \quad B \mapsto C(G_i,B)$$

is injective.  $\Box$ 

**Lemma.** Let  $\underline{C}$  and  $\underline{D}$  be complete categories and  $F : \underline{C} \to \underline{D}$ ,  $G : \underline{D} \to \underline{E}$  continuous functors. Then  $F \downarrow G$  is complete, and the forgetful functors  $P_{\underline{C}} : F \downarrow G \to \underline{C}$  and  $P_{\underline{D}} : F \downarrow G \to \underline{D}$  are continuous.

*Proof.* We provide a sketch of the proof:

$$\lim_{I} \left( F(C_i) \xrightarrow{f} G(C_i) \right) = \lim_{I} F(C_i) \xrightarrow{\lim_{I} f_i} G(C_i).$$

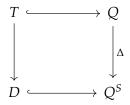
**Theorem** (Primeval AFT). *Suppose*  $\underline{D}$  *has all* (not necessary small) limits. A functor  $G : \underline{D} \to \underline{C}$  has a left adjoint if and only if G is continuous.

**Remark.** For such  $\underline{D}$  we have  $|\underline{D}(X,Y)| \leq 1$  (i.e.  $\underline{D}$  is **thin**).

*Proof.* One direction follows from RAPL. Now, by the previous lemma,  $\underline{C} \downarrow G$  has all limits, hence  $\lim \operatorname{Id}_{C \downarrow G}$  is the initial object.

**Lemma.** If  $\underline{D}$  is complete, well-powered with a cogenerator Q, then  $\underline{D}$  has an initial object.

*Proof.* Let  $i = \bigcap_{X \hookrightarrow Q} X$  and  $D \in Ob(\underline{D})$ . For  $f, g \in \underline{D}(i, D)$ , their equalizer is a suboject of i, hence f = g. Set  $S := \underline{D}(D, Q)$ . The map  $D \to Q^S$  is monic, therefore



is a pullback square. Now  $i \rightarrow T \rightarrow D$  is a required map.

**Proposition.** Let  $\underline{C}$  be a category with (small) coproducts and  $\{G_I\}_{i\in I}$  be a set of objects. Then the following are equivalent:

- (1)  $\{G_i\}_{i\in I}$  is a generating set.
- (2)  $G = \coprod G_i$  is a generator.
- (3) For each  $X \in \text{Ob}(\underline{C})$  there is a set S and an epimorphism  $G^{(S)} \to X$ .

**Theorem** (SAFT). Let  $\underline{D}$  be a complete well-powered category with a cogenerator Q. A functor  $G : \underline{D} \to \underline{C}$  has a left adjoint if and only if G is continuous.

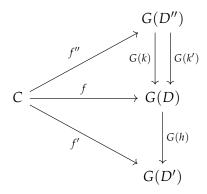
*Proof.* Only one direction requires proof. We want to show that  $C \downarrow G$  has an initial object for each  $C \in Ob(C)$ . To do that we apply the previous lemma:

(1)  $C \downarrow G$  is complete (DONE).

### (2) $C \downarrow G$ is well-powered:

Since  $C \downarrow G \xrightarrow{P_{\underline{D}}} \underline{D}$  is continuous, it preserves monomorphisms (since f is mono iff the corresponding square is a pullback).

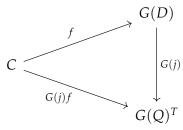
We show that  $P_D$  reflects monomorphisms. Indeed,



If *h* is mono, G(h)G(k) = G(h)G(k'), we get G(k) = G(k') since G(h) is monic.

#### (3) $C \downarrow G$ has a cogenerator:

We show that  $S = \underline{C}(C, G(D))$  is a generating set. Let  $C \xrightarrow{f} G(D) \in Ob(C \downarrow G)$ . We have  $D \hookrightarrow Q^T$ , so the diagram



finishes the proof.

### LECTURE 9. APRIL 1, 2025

### 4.2 Eilenberg-Watts Theorems

(1) Let  $G : \underline{R\text{-Mod}} \to \underline{S\text{-Mod}}$  be a continuous functor. By SAFT G has a left adjoint F. Then for any R-module M

$$G(M) = \operatorname{Hom}_S(S, G(M)) \cong \operatorname{Hom}_R(F(S), M).$$

Note that F(S) is a bimodule, since we have  $S \to \operatorname{End}_S(S) \to \operatorname{End}_R(F(S))$ .

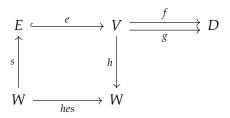
(2) Let  $F : \underline{S\text{-Mod}} \to \underline{R\text{-Mod}}$  be a cocontinuous functor. By co-SAFT it has a right adjoint G. Then  $G = \operatorname{Hom}(B, -)$ , hence  $F = B \otimes_S -$ .

**Definition.** A set of objects  $\{D_i\}_{i\in I}$  is called **weakly initial** if for all  $D \in Ob(\underline{D})$  there is  $i \in I$  and a map  $D_i \to D$ .

**Lemma.** Let  $\underline{D}$  be a complete category. If  $\underline{D}$  has a weakly initial set, then it has an initial object.

*Proof.* Let  $W := \prod D_i$  and  $V := eq(\underline{D}(W, W)) \hookrightarrow W$ . We know that gh = h for each  $g \in \underline{D}(W, W)$ .

- 1) Let  $D \in Ob(\underline{D})$ . There is  $D_i \to D$ , hence  $V \xrightarrow{h} W \xrightarrow{p_i} D_i \to D$  and  $\underline{D}(V, D) \neq \emptyset$ .
- 2) Suppose  $f, g : V \to D$ . We want to show that f = g.



Since hesh = h and h is monic,  $esh = 1_V$ . Analogously,  $she = 1_E$  given that e is monic. But e was a regular monomorphism, hence e is an isomorphism. Therefore, f = g.

**Theorem** (GAFT or Freyd's AFT). Let  $\underline{D}$  be a complete category and  $G : \underline{D} \to \underline{C}$  a continuous functor. Then G has a left adjoint if and only if for each  $C \in Ob(\underline{C})$  there is  $\{D_i\}_{i \in I} \subset Ob(\underline{D})$  with the property that for all  $D \in Ob(\underline{D})$  and any  $f : C \to G(D)$  there exist  $i \in I$ ,  $\varphi : G \to G(D_i)$ ,  $\overline{f} : D_i \to D$  such that  $f = G(\overline{f}) \circ \varphi$ .

*Proof.* Clear, since this is equivalent to having a weakly initial set.

# 5 Abelian Categories

# 5.1 Additive Categories

#### **Definitions:**

(1) An Ab-category (a **preadditive** category) is a category  $\underline{A}$  such that

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- a)  $\underline{A}(X,Y)$  is an abelian group for all  $X,Y \in \underline{A}$ .
- b) The composition map  $\underline{\mathcal{A}}(X,Y) \times \underline{\mathcal{A}}(Y,Z) \to \underline{\mathcal{A}}(X,Z)$  is bilinear:

$$(g_1+g_2)\circ f=g_1\circ f+g_2\circ f.$$

(2) If  $\underline{A}$  and  $\underline{B}$  are Ab-categories, a functor  $F : \underline{A} \to \underline{B}$  is called **additive** if

$$\underline{\mathcal{A}}(X,Y) \to \underline{\mathcal{B}}(F(X),F(Y))$$

is  $\mathbb{Z}$ -linear for any  $X, Y \in \underline{\mathcal{A}}$ .

### **Examples:**

- (1) <u>R-Mod</u>. In particular, <u>Ab</u> and  $\underline{\text{Vect}}(k)$ .
- (2)  $\underline{\operatorname{Sh}}(X,R)$ .
- (3) For *X* a ringed space,  $\underline{\text{Mod}}(\mathcal{O}_X)$ .
- (4) QCoh(X) and  $\underline{Coh}(X)$ .

**Proposition.** *For*  $Z \in \underline{A}$  *the following are equivalent:* 

- (1) Z is initial;
- (2) Z is final;
- (3)  $1_Z = 0_Z$ ;
- $(4) \ \underline{\mathcal{A}}(Z,Z) = \{0_Z\}.$

Proof. Clear.

**Definition.** Let  $\underline{A}$  be an Ab-category and  $X, Y \in \underline{A}$ . A **biproduct** of X and Y is a diagram

$$X \xleftarrow{p_1} X \oplus Y \xrightarrow{p_2} Y$$

such that:

- (1)  $p_1i_1 = 1_X$ ;
- (2)  $p_2i_2 = 1_Y$ ;
- (3)  $1_{X \oplus Y} = i_1 p_1 + i_2 p_2$ .

Note that it implies  $p_1i_2 = 0 = p_2i_1$ .

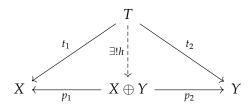
**Theorem.** *Let*  $X, Y \in \underline{A}$ *. The following are equivalent:* 

- (1) X and Y have a biproduct.
- (2) There exists  $X \times Y$  and  $X \times Y \cong X \oplus Y$  with projections  $p_1$  and  $p_2$ .
- (3) There exists  $X \coprod Y$  and  $X \coprod Y \cong X \oplus Y$  with inclusions  $i_1$  and  $i_2$ .

*Proof.* We first show  $(1) \Rightarrow (2)$ . Indeed,

$$p_1i_2 = p_1(i_1p_1 + i_2p_2)i_2 = p_1i_2 + p_1i_2 \implies p_1i_2 = 0.$$

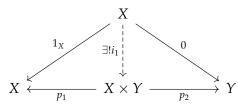
Similarly,  $p_2i_1 = 0$ .



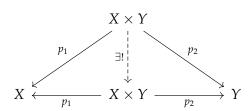
Let  $h = i_1t_1 + i_2t_2$ . Then  $p_1h = t_1$  and  $p_2h = t_2$ . For any other  $h' : T \to X \otimes Y$  such that  $p_ih' = t_i$ , we get

$$h' = (i_1p_1 + i_2p_2)h' = i_1t_1 + i_2t_2 = h.$$

We will only show  $(2) \Rightarrow (1)$ . Inclusions  $i_1$  and  $i_2$  can be obtained by the universal property in the following way:



We get that  $p_1i_1 = 1_X$  and  $p_2i_2 = 1_Y$ . Lastly,



and  $i_1p_1 + i_2p_2 = 1_{X \times Y}$ , since

$$p_i(i_1p_1 + i_2p_2) = p_i.$$

**Definition.** An **additive** category is an Ab-category with a zero object and binary biproducts.

**Proposition.** *If*  $\underline{A}$  *is additive and* f ,  $f' \in \underline{A}(X,Y)$ 

$$f + f' = \nabla_Y (f \sqcup f') \Delta_X,$$

where  $\Delta_X: X \to X \times X$  and  $\nabla: Y \times Y \to Y$ .

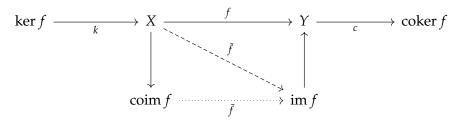
**Proposition.** *If*  $\underline{A}$  *and*  $\underline{B}$  *are additive, then*  $F : \underline{A} \to \underline{B}$  *is additive if and only if* F *preserves biproducts.* 

We write

$$\underline{\operatorname{Func}}^{add}(\underline{\mathcal{A}},\underline{\mathcal{B}}) = (\underline{\mathcal{A}},\underline{\mathcal{B}}).$$

## 5.2 Abelian Categories

Let  $\underline{A}$  be an additive category with kernels and cokernels.



We define im  $f := \ker c$  and coim  $f := \operatorname{coker} k$ .

**Definition.** An additive category is called **abelian** if

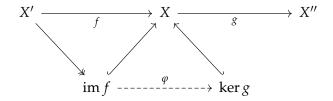
AB1 Any map has a kernel and a cokernel.

AB2 For any map f the induced map  $\bar{f}$  is an isomorphism.

#### **Remarks:**

- (1)  $\underline{A}$  is abelian if and only if  $\underline{A}^{op}$  is abelian.
- (2) Any abelian category is finitely (co)complete.

For a complex  $X' \xrightarrow{f} X \xrightarrow{g} X''$  with gf = 0, one has



Note that  $\varphi$  is monic. We let  $H = \operatorname{coker} \varphi$ .

**Definition.** A cochain complex in  $\underline{A}$  is

$$X^{\bullet} = \cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^{i} \xrightarrow{d^{i}} X^{i+1} \longrightarrow \cdots$$

with the property that  $d_i d_{i-1} = 0$  for each  $i \in \mathbb{Z}$ .

We define

$$H^i(X^{\bullet}) = H(X^{i-1} \to X^i \to X^{i+1}).$$

**Definition.**  $X^{\bullet}$  is called an **acyclic complex** (or an **exact sequence**) if  $H^{i}(X^{\bullet}) = 0$ .

#### **Examples:**

- (1)  $0 \to A \xrightarrow{f} B$  is exact if and only if f is monic.
- (2)  $A \xrightarrow{f} B \to 0$  is exact if and only if f is epic.

(3)  $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$  (which is called a **short exact sequence**, a SES) is exact if and only if f is monic and  $C = \operatorname{coker} f$  if and only if g is epic and  $A = \ker f$ .

**Definition.** A short exact sequence **splits** if there is  $\varphi : X \to X' \oplus X''$  such that

is commutative.

**Proposition.** *For a short exact sequence, the following are equivalent:* 

- (1) it splits;
- (2) there is  $\sigma: X'' \to X$  such that  $g\sigma = 1_{X''}$ ;
- (3) there is  $\rho: X \to X'$  such that  $\rho f = 1_{X'}$ .

Proof. EXERCISE.

# 5.3 Projective and Injective Objects

Let  $\underline{A}$  and  $\underline{B}$  be abelian categories and  $F : \underline{A} \to \underline{B}$  be a functor.

#### **Definitions:**

(1) *F* is **left exact** (Lex) if for any exact sequence  $0 \to X' \to X \to X''$  the sequence

$$0 \to F(X') \to F(X) \to F(X'')$$

is exact.

(2) *F* is **right exact** (Rex) if for any exact sequence  $0 \to X' \to X \to X''$  the sequence

$$F(X') \to F(X) \to F(X'') \to 0$$

is exact.

(3) *F* is **exact** if *F* is left exact and right exact.

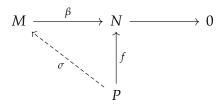
#### **Examples:**

- (1)  $B \otimes -$  is right exact.
- (2) Hom(B, -) is left exact.

**Definition.** An object  $X \in \underline{A}$  is **projective** (corresp. **injective**) if the functor  $h_X = \text{Hom}(X, -)$  (corresp.  $h^X = \text{Hom}(-, X)$ ) is exact.

**Theorem.** *Let*  $P \in \underline{R\text{-Mod}}$ . *The following are equivalent:* 

- (1) P is projective.
- (2) For any epimorphism  $\beta: M \to N$  and any  $f: P \to N$  there is  $\sigma: P \to M$  such that  $\beta \sigma = f$ .



- (3) Any short exact sequence  $0 \to K \to M \to P \to 0$  splits.
- (4) There is  $K \in \underline{R}\text{-Mod}$  such that  $P \oplus K \cong R^{(S)}$ .

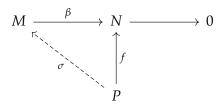
*Proof.* FOR NEXT TIME.

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IN THE PREVIOUS EPISODE...

**Theorem.** Let  $P \in \underline{R}\text{-Mod}$ . The following are equivalent:

- (1) P is projective.
- (2) For any epimorphism  $\beta: M \to N$  and any  $f: P \to N$  there exists  $\sigma: P \to M$  such that  $\beta \sigma = f$ .



- (3) Any short exact sequence  $0 \to K \to M \to P \to 0$  splits.
- (4) There exists  $K \in \underline{R}\text{-Mod}$  such that  $P \oplus K \cong R^{(I)}$  for some set I.

*Proof.* (1)  $\Rightarrow$  (2) is obvious, since Hom(P, $\beta$ ) is surjective. (2)  $\Rightarrow$  (3) follows from the general criterion on splitting of exact sequences. We obtain (3)  $\Rightarrow$  (4) by considering

$$0 \to K \to R^{(I)} \to P \to 0$$

where *I* is the set of generators of *P*. It splits, hence the result.

Finally, we verify (4)  $\Rightarrow$  (1). Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  and consider

$$0 o \prod_{i \in I} M' o \prod_{i \in I} M o \prod_{i \in I} M'' o 0.$$

Rewrite it as the sequence

$$0 \to \operatorname{Hom}(R^{(I)}, M') \to \operatorname{Hom}(R^{(I)}, M) \to \operatorname{Hom}(R^{(I)}, M'') \to 0,$$

where *I* is the set in (4). But since  $R^{(I)} = P \oplus K$  we get

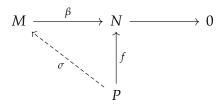
$$0 \to \operatorname{Hom}(K,M') \to \operatorname{Hom}(K,M) \to \operatorname{Hom}(K,M'') \to 0.$$

**Proposition.**  $\bigoplus_{i \in I} P_i$  is projective if and only if each  $P_i$  is projective.

*Proof.* Note that 
$$Hom(\oplus P_i, -) = \prod Hom(P_i, -)$$
.

**Theorem.** *Let*  $E \in \underline{R}\text{-Mod}$ . *The following are equivalent:* 

- (1) E is injective ( $h^E$  is exact).
- (2) For any monomorphism  $\beta: M \to N$  and any  $f: M \to P$  there exists  $g: N \to P$  such that  $g\beta = f$ .

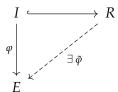


(3) Any short exact sequence  $0 \to E \to M \to N \to 0$  splits.

*Proof.* The proof is similar.

**Proposition.**  $\prod_{i \in I} E_i$  is injective if and only if each  $E_i$  is injective.

**Theorem** (Baer's criterion). An R-module E is injective if and only if for any left ideal  $I \subset R$  has the property



*Proof.* This condition is clearly necessary. Now let  $A \subset B$  and  $f : A \to E$ . Consider a set X of all ordered pairs (A',g'), where  $A \subset A' \subset B$  and  $g' : A' \to E$  such that  $g'|_A = f$ .

There is a natural order on X, and by Zorn's lemma there is a maximal element  $(A_0, g_0)$ . We want to show that  $A_0 = B$ . Assume the contrary, i.e. there is  $b \in B \setminus A_0$ . Let

$$I = \{r \in R \mid rb \in A_0\} \subset R.$$

It is clear that *I* is a left ideal of *R*. Define  $\varphi$  :  $I \to E$  by

$$\varphi(r) := g_0(rb).$$

Let  $A_1 = A_0 + Rb$  and  $g_1 : A_1 \rightarrow E$ , where

$$g_1(a_1 + rb) = g_1(a_0) + r\tilde{\varphi}(1).$$

It is routine to check that  $g_1$  is well-defined. Hence  $(A_0, g_0)$  is not maximal, a contradiction.

**Definition.** Let R be a (commutative) domain. An R-module is a **division module** if for every  $m \in M \setminus \{0\}$  and  $r \in R \setminus \{0\}$  there is a "quotient"  $m' \in M$  such that m = rm'.

**Proposition.** *If* D *is a division module, then* D/C *is a division module for each submodule*  $C \subset D$ .

Proof. Exercise.

**Proposition** (Relations between injective and division modules).

- (1) If R is a domain, then injective modules are division modules, i.e.  $Inj(R) \subset Div(R)$ .
- (2) If R is a PID, then every division module is injective, i.e. Inj(R) = Div(R).
- (3) If R is a domain, then Q(R), i.e. the field of fractions of R, is injective.

Note that when  $R = \mathbb{Z}$ , abelian group Q is divisible, hence  $\mathbb{Q}/\mathbb{Z}$  is injective.

**Proposition.**  $\mathbb{Q}/\mathbb{Z}$  is an injective cogenerator of Ab.

*Proof.* Let  $f:A\to B$  such that  $f\neq 0$ , i.e. there is  $b\in B$  such that  $b\in \operatorname{Im} f$  and  $b\neq 0$ . Define  $\tilde{h}:\mathbb{Z}b\to\mathbb{Q}/\mathbb{Z}$  by

$$\tilde{h}(b) = \begin{cases} \frac{1}{n}, & n = \text{ord } b \\ \frac{1}{2}, & \text{ord } = \infty \end{cases}.$$

By injectivity, there is  $h: B \to \mathbb{Q}/\mathbb{Z}$  such that  $h(b) = \tilde{h}(b) \neq 0$ . Hence  $hf \neq 0$ .

**Proposition.** If Q is an injective cogenerator in an abelian category  $\underline{A}$ , then any object of  $\underline{A}$  is a subobject of an injective object.

*Proof.* There is a monomorphism  $X \hookrightarrow Q^S$  for each  $X \in \text{Ob}(\underline{\mathcal{A}})$  and some set S.

**Definition.** An abelian category  $\underline{A}$  has enough injectives if every object can be embedded into an injective object.

**Proposition.** Let R be a ring. Then  $\operatorname{Hom}_{\mathbb{Z}}(R_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$  is an injective cogenerator in <u>R-Mod</u>.

*Proof.* It follows from the following chain of isomorphisms:

$$\operatorname{Hom}_{R}(-,\operatorname{Hom}_{\mathbb{Z}}(R_{\mathbb{Z}},\mathbb{Q}/\mathbb{Z}))\cong \operatorname{Hom}_{\mathbb{Z}}(R\otimes_{R}(-),\mathbb{Q}/\mathbb{Z})\cong \operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z}).$$

**Corollary.** *In R-Mod there are enough injectives.* 

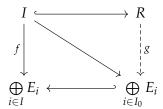
**Definition.** An abelian category  $\underline{A}$  has enough projectives if for any  $X \in Ob(\underline{A})$  there is a projective object P and an epimorphism  $P \to X$ .

**Proposition.** *In* <u>R-Mod</u> *there are enough projectives.* 

*Proof.* Note that  $R^{(S)} \to M \to 0$  and free module  $R^{(S)}$  is clearly projective.

**Proposition.** *If* R *is left Noetherian and*  $\{E_i\}_{i\in I}$  *is a set of injective* R*-modules, then*  $\bigoplus E_i$  *is injective.* 

*Proof.* We check the Baer's criterion:



Since I is finitely generated, Im f is contained in some  $\bigoplus_{i \in I_0} E_i$  where  $I_0$  is finite, and the result follows from injectivity of  $\bigoplus_{i \in I_0} E_i$ .

**Theorem** (Bass-Papp). Let R be a ring. If countable direct sums of injective objects in  $\underline{R\text{-Mod}}$  are injective, then R is Noetherian.

*Proof.* Assume the contrary. Let  $I_1 \subsetneq I_2 \subsetneq \ldots$  and  $I = \bigcup_{n \geq 1} I_n$  be ideals of R. Consider injective module  $E_n$  such that  $I/I_n \hookrightarrow E_n$ , and let  $\pi_n : I \to I/I_n$ .

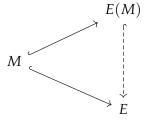
Let  $\pi:I\to\prod_{n\geq 1}I/I_n$ . Note that the image of  $\pi$  is in  $\bigoplus_{n\geq 1}I/I_n$ , so we can compose it with the map  $\bigoplus_{n\geq 1}I/I_n\to\bigoplus_{n\geq 1}E_n$  and get  $f:I\to\bigoplus_{n\geq 1}E_n$ . By assumtion  $\bigoplus_{n\geq 1}E_n$  is injective, hence there is  $g:R\to\bigoplus_{n\geq 1}E_n$  extending f. Let  $g(1)=(e_n),e_n\in E_n$ . For each  $m\geq 1$  there is some  $a_m\in I\setminus I_m$ , and it follows that  $g(a_m)=f(a_m)=\pi(a_m)\neq 0$ . But  $g(a_m)=(a_me_n)$ , so  $e_m\neq 0$  for any  $m\geq 0$ . This is a contradiction, since  $g(1)=(e_m)\in\bigoplus_{n\geq 1}E_n$ .

**Definition.** A monomorphism  $A \hookrightarrow B$  is called **essential** if for all  $S \in B$  we have  $A \cap S \neq 0$ .

**Example.** In  $\mathbb{Z}$ -modules,  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  is essential.

**Definition.** An **injective hull** (or **envelope**) of a module M is an injective module E such that  $M \hookrightarrow E$  is essential. We denote E by E(M), which is unique up to isomorphism.

Let *M* and *E* be *R*-modules such that *E* is injective. Then (without proof)



and the following exact sequence splits:

$$0 \to E(M) \to E \to E/E(M) \to 0$$
.

**Theorem** (w/o proof). *For any R-module there is an injective hull.* 

### LECTURE 11. APRIL 15, 2025

### 6 Resolutions and Derived Functors

#### REFERENCES

- J. Rotman, An Introduction to Homological Algebra.
- C. Weibel, An Introduction to Homological Algebra.

## 6.1 Complexes

#### **Definitions:**

(1) If  $\underline{A}$  is an abelian category, the category  $\underline{\mathrm{Ch}}(\underline{A})$  of chain complexes has as objects diagrams

$$C = \left(\cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots\right),$$

with differentials  $d_i: C_i \to C_{i-1}$  such that  $d_i d_{i+1} = 0$  for all  $i \in \mathbb{Z}$  (equivalently,  $d^2 = 0$ ).

(2) The category  $\underline{CoCh}(\underline{A})$  of cochain complexes has as objects diagrams

$$C = \left(\cdots \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots\right),$$

with differentials  $d^i: C^i \to C^{i+1}$  such that  $d^{i+1} d^i = 0$  for all  $i \in \mathbb{Z}$ .

(3) For a chain complex *C*, define the **cycles**, **boundaries**, and **homology** by

$$Z_i(C) := \ker(d_i), \qquad B_i(C) := \operatorname{im}(d_{i+1}), \qquad H_i(C) := Z_i(C) / B_i(C).$$

(4) Given chain complexes C, C', put for each  $d \in \mathbb{Z}$ 

$$\underline{\mathrm{Hom}}^d(C,C'):=\prod_{i\in\mathbb{Z}}\mathrm{Hom}(C_i,C'_{i+d}).$$

Elements of  $\underline{\mathrm{Hom}}^d(C,C')$  are **homogeneous maps of degree** d: these are families  $u=(u_i)$  with  $u_i:C_i\to C'_{i+d}$ . Define a differential  $D:\underline{\mathrm{Hom}}^d(C,C')\to\underline{\mathrm{Hom}}^{d-1}(C,C')$  by

$$D(u) := d' \circ u - (-1)^d u \circ d.$$

Then  $D^2 = 0$ , so  $\underline{\text{Hom}}(C, C') := (\underline{\text{Hom}}^{\bullet}(C, C'), D)$  is a complex. A **chain map** is a degree 0 cycle in  $\underline{\text{Hom}}(C, C')$ , i.e. a family  $f = (f_i)$  with  $f_i : C_i \to C'_i$  and d'f = fd.

(5) Two chain maps  $f,g:C\to C'$  are **homotopic** if f-g=D(h) for some  $h\in \underline{\mathrm{Hom}}^1(C,C')$ , equivalently,

$$f_i - g_i = d'_{i+1}h_i + h_{i-1}d_i$$
 for all  $i \in \mathbb{Z}$ .

The **homotopy category** K(A) has the same objects as Ch(A) and morphisms

$$\text{Hom}_{K(\mathcal{A})}(C,C') := H_0(\underline{\text{Hom}}(C,C')),$$

i.e. chain-homotopy classes of chain maps  $C \rightarrow C'$ .

#### **Definitions:**

- (1) Chain complexes C and D are **homotopy equivalent** if there are chain maps  $f: C \to D$  and  $g: D \to C$  with  $fg \sim 1_D$  and  $gf \sim 1_C$ .
- (2) If  $u: C \to C'$  is a chain map, then for all  $i \in \mathbb{Z}$

$$u(Z_i(C)) \subseteq Z_i(C'), \qquad u(B_i(C)) \subseteq B_i(C'),$$

so there are induced maps  $Z_i(u): Z_i(C) \to Z_i(C')$ ,  $B_i(u): B_i(C) \to B_i(C')$ , and hence

$$H_i(u): H_i(C) \to H_i(C').$$

A chain map u is a **quasi-isomorphism** if  $H_i(u)$  is an isomorphism for all  $i \in \mathbb{Z}$ .

### 6.2 Snake lemma

Lemma (Snake Lemma).

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

$$\downarrow f' \qquad \downarrow f \qquad \downarrow f''$$

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

If the rows are exact, then there is a natural long exact sequence

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'') \to 0.$$

**Proposition.** Let  $0 \to A_{\bullet} \to B_{\bullet} \to C_{\bullet} \to 0$  be a short exact sequence of chain complexes. Then there are connecting morphisms  $\delta_n : H_n(C) \to H_{n-1}(A)$  forming a long exact sequence

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$$

*Proof.* Apply the Snake Lemma to the diagram with exact rows

$$0 \longrightarrow Z_n(A) \longrightarrow Z_n(B) \longrightarrow Z_n(C) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow B_{n-1}(A) \longrightarrow B_{n-1}(B) \longrightarrow B_{n-1}(C) \longrightarrow 0$$

where the vertical arrows are induced by the differentials. The connecting morphism  $\delta_n$  lands in  $H_{n-1}$  after passing to the quotients by boundaries, giving the long exact sequence.

**Proposition.** Given a morphism of short exact sequences of chain complexes

$$0 \longrightarrow A_{\bullet} \longrightarrow B_{\bullet} \longrightarrow C_{\bullet} \longrightarrow 0$$

$$\downarrow f_{\bullet} \qquad \downarrow g_{\bullet} \qquad \downarrow h_{\bullet}$$

$$0 \longrightarrow A'_{\bullet} \longrightarrow B'_{\bullet} \longrightarrow C'_{\bullet} \longrightarrow 0$$

the induced long exact sequences in homology form a commutative diagram. Moreover, the connecting morphisms are natural, i.e.

$$H_{n-1}(f) \circ \delta_n = \delta'_n \circ H_n(h)$$
 for all  $n \in \mathbb{Z}$ .

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### 6.3 Resolutions

#### **Definitions:**

(1) A **left resolution** of an object  $M \in \underline{A}$  is a chain complex

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

with augmentation  $\varepsilon: P_0 \to M$ , such that  $P_i = 0$  for i < 0 and the sequence is exact; equivalently,

$$H_0(P) \cong M$$
,  $H_i(P) = 0$  for  $i > 0$ .

A left resolution is **projective** if  $P_i \in \text{Proj}(\underline{A})$  for all  $i \geq 0$ .

(2) A **right resolution** of *M* is a cochain complex

$$0 \to M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

with  $I^{j} = 0$  for j < 0, which is exact except at degree 0; equivalently,

$$H^0(I) \cong M$$
,  $H^j(I) = 0$  for  $j > 0$ .

It is **injective** if  $I^j \in \text{Inj}(\underline{A})$  for all  $j \geq 0$ .

### **Examples:**

(1) If  $\underline{A} = \underline{R}\text{-Mod}$ , then every module has a projective (indeed, free) resolution: choose an epimorphism  $R^{(I_1)} \to M$  with kernel  $K_1$ , then an epimorphism  $R^{(I_2)} \to K_1$  with kernel  $K_2$ , etc., obtaining an exact sequence

$$\cdots \rightarrow R^{(I_2)} \rightarrow R^{(I_1)} \rightarrow M \rightarrow 0,$$

whose left part  $\cdots \to R^{(I_2)} \to R^{(I_1)} \to 0$  is a projective resolution of M.

- (2) The modules  $K_i$  appearing here are called **syzygies** of M.
- (3) If  $\underline{A} = \underline{R\text{-Mod}}$ , then every module has an injective resolution: embed M into an injective module  $E^0$  (e.g. an injective hull), let  $C^1 := \operatorname{coker}(M \to E^0)$ , embed  $C^1$  into an injective  $E^1$ , and continue to obtain

$$0 \to M \to E^0 \to E^1 \to E^2 \to \cdots$$
 ,

which is an injective resolution of *M*.

**Proposition** (Comparison for projective resolutions). Let  $f: M \to M'$  be a morphism in  $\underline{\mathcal{A}}$ . Let  $P_{\bullet} \xrightarrow{\varepsilon} M$  and  $Q_{\bullet} \xrightarrow{\varepsilon'} M'$  be left resolutions with  $P_i$  projective for all  $i \geq 0$ . Then there exists a chain map

$$\tilde{f}:P_{ullet} o Q_{ullet}$$

*lifting* f , i.e.  $\varepsilon' \circ \tilde{f}_0 = f \circ \varepsilon$ ; moreover,  $\tilde{f}$  is unique up to chain homotopy.

*Proof.* Construct  $\tilde{f}_0: P_0 \to Q_0$  by lifting  $f\varepsilon$  along the epimorphism  $\varepsilon'$  using the projectivity of  $P_0$ . Inductively, having  $\tilde{f}_{i-1}$  with  $\varepsilon' d'_1 \cdots d'_i \tilde{f}_i = f\varepsilon d_1 \cdots d_i$ , use the exactness of  $Q_{\bullet}$  at  $Q_{i-1}$  and the projectivity of  $P_i$  to obtain  $\tilde{f}_i: P_i \to Q_i$  such that  $d'_i \tilde{f}_i = \tilde{f}_{i-1} d_i$ .

To show uniqueness, let  $\tilde{f}$ ,  $\tilde{g}$  :  $P_{\bullet} \to Q_{\bullet}$  be two lifts of f. Put  $h := \tilde{f} - \tilde{g}$  and define  $s_n := 0$  for n < 0. Since  $\varepsilon' h_0 = 0$  and  $Q_{\bullet}$  is exact at  $Q_0$ , there exists  $s_0 : P_0 \to Q_1$  with  $d'_1 s_0 = h_0$ . Assume  $s_0, \ldots, s_{n-1}$  are chosen so that  $h_k = d'_{k+1} s_k + s_{k-1} d_k$  for all k < n. Set

$$\alpha_n := h_n - s_{n-1}d_n : P_n \to Q_n.$$

Then  $d'_n \alpha_n = d'_n h_n - (h_{n-1} - s_{n-2} d_{n-1}) d_n = 0$ , so by exactness at  $Q_n$  there exists  $s_n : P_n \to Q_{n+1}$  with  $d'_{n+1} s_n = \alpha_n$ . Hence for all n

$$h_n = d'_{n+1}s_n + s_{n-1}d_n,$$

which means h = D(s). Therefore  $\tilde{f}$  and  $\tilde{g}$  are chain-homotopic.

**Corollary.** Let  $M \in \underline{A}$ . Any two projective resolutions of M are chain-homotopy equivalent. Equivalently, if  $P_{\bullet} \stackrel{\varepsilon}{\to} M$  and  $P'_{\bullet} \stackrel{\varepsilon'}{\to} M$  are projective resolutions, then there exist chain maps

$$f: P_{\bullet} \to P'_{\bullet}, \qquad g: P'_{\bullet} \to P_{\bullet}$$

with  $gf \sim 1_{P_{\bullet}}$  and  $fg \sim 1_{P'_{\bullet}}$ .

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**Definition.** Let  $f: A \to B$  be a chain map. The **cone** of f is the complex C(f) defined degreewise by

$$C(f)_n := B_n \oplus A_{n-1} = B_n \oplus A[1]_n,$$

with differential given, in the decomposition  $B_n \oplus A_{n-1} \to B_{n-1} \oplus A_{n-2}$ , by the block matrix

$$d_n^{C(f)} = \begin{pmatrix} d_n^B & -f_{n-1} \\ 0 & -d_{n-1}^A \end{pmatrix}$$
, i.e.  $d^{C(f)}(b,a) = (d_n^B b - f_{n-1}(a), -d_{n-1}^A a)$ .

One checks that  $d^{C(f)} \circ d^{C(f)} = 0$ .

**Remark.** The **shift functor** [1] :  $\underline{Ch}(\underline{A}) \rightarrow \underline{Ch}(\underline{A})$  is defined by

$$(A[1])_n := A_{n-1}, \qquad d_n^{A[1]} := -d_{n-1}^A,$$

and for a chain map f set  $(f[1])_n := f_{n-1}$ .

**Proposition** (Properties of cones).

(1) There is a short exact sequence of chain complexes

$$0 \longrightarrow B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \longrightarrow 0,$$

where i(b) := (b,0) and p(b,a) := -a.

(2) *If the square* 

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
g \downarrow & & \downarrow h \\
A' & \xrightarrow{f'} & B'
\end{array}$$

is a commutative square of chain complexes, then the map

$$h \oplus g[1] : C(f) \longrightarrow C(f'), \qquad (b,a) \longmapsto (h(b), g(a)),$$

is a chain map.

(3) In the situation of (2), there is a morphism of short exact sequences

$$0 \longrightarrow B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \longrightarrow 0$$

$$\downarrow h \downarrow \qquad \qquad \downarrow h \oplus g[1] \qquad \qquad \downarrow g[1]$$

$$0 \longrightarrow B' \xrightarrow{i'} C(f') \xrightarrow{p'} A'[1] \longrightarrow 0$$

commuting with the indicated arrows.

Let  $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$  be a short exact sequence in an abelian category  $\underline{A}$ . Take projective resolutions  $\varepsilon^A : Q_{\bullet} \to A$  and  $\varepsilon^C : P_{\bullet} \to C$ . By projectivity choose a lift  $\psi_0 : P_0 \to B$  with  $q \psi_0 = \varepsilon_0^C$ . Using the induced map on cycles, pick  $\varphi_0 : P_1 \to Q_0$  so that

$$i \, \varepsilon_0^A \, \varphi_0 = \psi_0 \, d_1^P$$
.

Inductively, choose maps  $\varphi_n: P_{n+1} \to Q_n$  making  $i \, \varepsilon_n^A \, \varphi_n = \psi_n \, d_{n+1}^P$  with suitable lifts  $\psi_n: P_n \to B$  extending  $\psi_{n-1}$ . This yields a chain map

$$\varphi_{\bullet}: P_{\bullet} \longrightarrow Q_{\bullet}[1].$$

Lemma (Horseshoe Lemma).

(1) Put  $S_{\bullet} := C(\varphi_{\bullet}[-1])$ . Then there is a short exact sequence of complexes

$$0 \longrightarrow Q_{\bullet} \longrightarrow S_{\bullet} \longrightarrow P_{\bullet} \longrightarrow 0$$

and  $S_{\bullet}$  is a projective resolution of B fitting into a commutative diagram with  $\varepsilon^A$ ,  $\varepsilon^B$ ,  $\varepsilon^C$ .

(2) The construction in (1) is natural with respect to morphisms of short exact sequences.

We omit the proof of this lemma but reformulate the statement (2):

**Lemma.** Given a morphism of short exact sequences

$$0 \longrightarrow A \stackrel{i}{\smile} B \stackrel{q}{\longrightarrow} C \longrightarrow 0$$

$$\downarrow b \qquad \downarrow c \qquad \downarrow c$$

$$0 \longrightarrow A' \stackrel{i'}{\smile} B' \stackrel{q'}{\longrightarrow} C' \longrightarrow 0$$

choose projective resolutions  $P_{\bullet}(A)$ ,  $P_{\bullet}(C)$  and  $P_{\bullet}(A')$ ,  $P_{\bullet}(C')$ , together with chain maps lifting a and c. Then the above construction yields a chain map

$$P_{\bullet}(B) \longrightarrow P_{\bullet}(B'),$$

which makes the diagram of short exact sequences of complexes

$$0 \longrightarrow P_{\bullet}(A) \longrightarrow P_{\bullet}(B) \longrightarrow P_{\bullet}(C) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow P_{\bullet}(A') \longrightarrow P_{\bullet}(B') \longrightarrow P_{\bullet}(C') \longrightarrow 0$$

commute.

### 6.4 Derived Functors

**Definition.** Let  $F : \underline{A} \to \underline{B}$  be a right exact additive functor between abelian categories and assume  $\underline{A}$  has enough projectives. For  $A \in Ob(\underline{A})$ , choose a projective resolution  $P_{\bullet} \twoheadrightarrow A$  and define

$$(L_iF)(A) := H_i(F(P_{\bullet})) \qquad (i \ge 0).$$

Independent of the choice of  $P_{\bullet}$  up to canonical isomorphism, this yields **left derived functors** 

$$L_iF:A\to B.$$

**Definition.** Similarly, if  $F : \underline{A} \to \underline{B}$  is left exact additive and  $\underline{A}$  has enough injectives, then for  $A \in \text{Ob}(\underline{A})$  and an injective resolution  $A \hookrightarrow E^{\bullet}$  one defines **right derived functors** 

$$(\mathbf{R}^{i}F)(A) := H^{i}(F(E^{\bullet})) \qquad (i \ge 0).$$

**Proposition** (Properties of derived functors).

- (1)  $L_0F \cong F$ .
- (2) The functors  $L_iF$  are independent of the chosen projective resolution.
- (3) Each  $L_iF$  is additive.

Proof.

- (1) If  $0 \to P_1 \to P_0 \twoheadrightarrow A \to 0$  is exact with  $P_j$  projective, then applying F gives an exact sequence  $F(P_1) \to F(P_0) \to F(A) \to 0$ , and hence  $H_0(F(P_{\bullet})) \cong F(A)$ .
- (2) Any two resolutions of *A* are connected by a chain homotopy equivalence, and applying *F* preserves homotopies, so the induced maps on homology are isomorphisms.
- (3) For a morphism  $f: A \to A'$  and chosen projective resolutions, any lift  $f_{\bullet}$  of f induces a chain map whose effect on homology defines  $L_iF(f)$ . In particular,  $f_{\bullet} + g_{\bullet}$  lifts f + g.

**Remark.** For any projective object P and any  $i \ge 1$  one has  $L_i F(P) = 0$ .

In the category of *R*-modules there are two canonical examples.

#### **Examples:**

(1) Given  $M \in \text{Ob}(\underline{R\text{-Mod}})$ , the functor  $h_M := \text{Hom}(M, -)$  is left exact. Then

$$\operatorname{Ext}_R^i(M,N) := R^i h_M(N).$$

(2) Given  $M \in Ob(\underline{R\text{-Mod}})$ , the functor  $t_M := M \otimes -$  is left exact. Then

$$\operatorname{Tor}_{i}^{R}(M,N):=L^{i}t_{M}(N).$$

**Theorem** (Derived long exact sequence). Let  $0 \to A' \xrightarrow{i} A \xrightarrow{q} A'' \to 0$  be a short exact sequence in an abelian category  $\underline{A}$ , and let  $F : \underline{A} \to \underline{B}$  be a right exact additive functor. Assume  $\underline{A}$  has enough projectives. Then there is a natural long exact sequence

$$\cdots \longrightarrow L_n F(A') \xrightarrow{u_n} L_n F(A) \xrightarrow{v_n} L_n F(A'') \xrightarrow{\partial_n} L_{n-1} F(A') \longrightarrow$$

$$\cdots \longrightarrow L_1 F(A'') \xrightarrow{\partial_1} L_0 F(A') \longrightarrow L_0 F(A) \longrightarrow L_0 F(A'') \to 0.$$

The maps  $u_n$  and  $v_n$  are induced by i and q, and  $\partial_n$  is the connecting morphism. The construction is functorial in morphisms of short exact sequences.

*Proof.* We outline a proof sketch. Choose projective resolutions  $P'_{\bullet} \to A'$  and  $P''_{\bullet} \to A''$ . By the Horseshoe Lemma there exists a short exact sequence of complexes

$$0 \longrightarrow P'_{\bullet} \longrightarrow P_{\bullet} \longrightarrow P''_{\bullet} \longrightarrow 0,$$

with degreewise splittings  $P_n \cong P'_n \oplus P''_n$ . Applying F yields a short exact sequence of complexes

$$0 \longrightarrow F(P'_{\bullet}) \longrightarrow F(P_{\bullet}) \longrightarrow F(P''_{\bullet}) \longrightarrow 0,$$

so passing to homology gives the desired long exact sequence in which  $H_n(F(P_{\bullet})) = L_nF(-)$ . Naturality follows from the functoriality of the horseshoe and the connecting morphism in homology.

**Example.** Let  $0 \to A' \xrightarrow{i} A \xrightarrow{q} A'' \to 0$  be a short exact sequence of left *R*-modules and *M* a right *R*-module. Then there is a long exact sequence

$$\ldots \to \operatorname{Tor}_1^R(M,A') \to \operatorname{Tor}_1^R(M,A) \to \operatorname{Tor}_1^R(M,A'') \to M \otimes A' \to M \otimes A \to M \otimes M'' \to 0.$$

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**Proposition** (Dimension shifting). Let F be right exact. Let

$$0 \longrightarrow K_{m+1} \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be an exact sequence with  $P_i$  projective for  $i=0,\ldots,m$ . Denote syzygies  $K_0:=A$  and, for  $i\geq 0$ ,  $K_{i+1}:=\ker(d_i:P_i\to K_i)$ . Then:

(1) For  $i \ge m + 2$ , there are natural isomorphisms

$$L_iF(A) \simeq L_{i-1}F(K_1) \simeq \cdots \simeq L_{i-m-1}F(K_{m+1}).$$

(2) There is a short exact sequence

$$0 \longrightarrow L_{m+1}F(A) \longrightarrow F(K_{m+1}) \longrightarrow F(P_m) \longrightarrow F(K_m) \longrightarrow 0.$$

*Proof.* For the short exact sequence  $0 \to K_{i+1} \to P_i \to K_i \to 0$  with  $P_i$  projective, the long exact sequence of left derived functors gives  $L_jF(P_i)=0$  for  $j\geq 1$  and isomorphisms  $L_jF(K_i)\simeq L_{j-1}F(K_{i+1})$  for  $j\geq 2$ . Iterating, starting from  $0\to K_1\to P_0\to A\to 0$  yields the chain in (1). For (2), apply the long exact sequence to  $0\to K_{m+1}\to P_m\to K_m\to 0$  to obtain

$$0 \to L_1F(K_m) \to F(K_{m+1}) \to F(P_m) \to F(K_m) \to 0$$

then identify  $L_1F(K_m) \simeq L_{m+1}F(A)$  by (1). Consider the short exact sequences

$$0 \to K_{m+2} \to P_{m+1} \xrightarrow{d_{m+1}} K_{m+1} \to 0$$

and

$$0 \to K_{m+1} \xrightarrow{i} P_m \xrightarrow{d_m} K_m \to 0.$$

Applying *F* and using  $L_iF(P) = 0$  for projective *P*, get the commutative diagram with exact rows

$$F(P_{m+2}) \longrightarrow \ker F(d_{m+1}) \longrightarrow \ker F(i) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$F(P_{m+2}) \longrightarrow F(P_{m+1}) \longrightarrow F(K_{m+1}) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow^{F(d_{m+1})} \qquad \downarrow^{F(i)}$$

$$0 \longrightarrow F(P_m) \stackrel{\sim}{\longrightarrow} F(P_m) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow$$

$$F(K_m) \stackrel{\sim}{\longrightarrow} \operatorname{coker} F(i)$$

Snake Lemma yields an isomorphism  $\ker F(i) \cong L_1F(K_m)$ , and by iterating the isomorphisms  $L_iF(K_r) \cong L_{i-1}F(K_{r+1})$  one gets  $\ker F(i) \cong L_{m+1}F(A)$ , which finishes the proof.

**Definition.** An object Q is called F-acyclic if  $L_iF(Q) = 0$  for all  $i \ge 1$ .

**Definition.** An *F*-acyclic resolution of an object *A* is a left resolution

$$\cdots \longrightarrow O_2 \longrightarrow O_1 \longrightarrow O_0 \longrightarrow A \longrightarrow 0$$

with each  $Q_i$  F-acyclic.

**Example.** Let  $F(-) := M \otimes_R (-)$ . A left R-module N is F-acyclic iff  $\operatorname{Tor}_i^R(M,N) = 0$  for all  $i \geq 1$ . The following are equivalent:

- (1) N is flat.
- (2)  $\operatorname{Tor}_1^R(M, N) = 0$  for all  $M \in \operatorname{\underline{Mod-R}}$ .
- (3)  $\operatorname{Tor}_{i}^{R}(M, N) = 0$  for all  $M \in \operatorname{\underline{Mod-R}}$  and all  $i \geq 1$ .

**Proposition.** *If*  $Q_{\bullet} \to A$  *is an F-acyclic resolution, then for every*  $n \ge 0$ 

$$L_nF(A) \simeq H_n(F(Q_{\bullet})).$$

*Proof.* Set  $K_i := \ker(d_i : Q_i \to Q_{i-1})$  for  $i \ge 1$  and  $K_0 := A$ . For each short exact sequence

$$0 \to K_{i+1} \to Q_i \to K_i \to 0$$

with  $Q_i$  being F-acyclic, the long exact sequence of left derived functors yields isomorphisms

$$L_n F(K_i) \cong L_{n-1} F(K_{i+1})$$

for all  $n \ge 2$  and an exact segment

$$0 \longrightarrow L_1F(K_i) \longrightarrow F(K_{i+1}) \longrightarrow F(Q_i) \longrightarrow F(K_i) \longrightarrow 0.$$

Iterating gives  $L_nF(A) \cong L_1F(K_{n-1})$ . For i = 0 the segment reads

$$0 \longrightarrow L_1F(A) \longrightarrow F(K_1) \xrightarrow{F(j)} F(Q_0) \longrightarrow F(A) \longrightarrow 0,$$

which identifies

$$L_1F(A) \cong \ker F(j) \cong H_1(F(O_{\bullet})),$$

and the result follows after shifting the indices

$$L_nF(A)\cong L_1F(K_{n-1})\cong H_1\bigg(F\big(\ldots\to Q_{n-1}\to K_{n-1}\to 0\big)\bigg)\cong H_n(F(Q_\bullet)).$$

#### 6.5 Ext and Tor

Let *R* be a ring and *A*, *B* left *R*-modules. If  $P_{\bullet} \to A$  is a projective resolution, define

$$\operatorname{Tor}_{i}^{R}(A,B) := H_{i}(P_{\bullet} \otimes_{R} B) \qquad (i \geq 0).$$

Also if  $Q_{\bullet} \to B$  is a projective resolution, let

$$tor_i^R(A, B) \cong H_i(A \otimes_R Q_{\bullet}) \qquad (i \ge 0).$$

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**Theorem** (Balancing Tor). For all left R-modules A, B and all  $n \ge 0$  there is a natural isomorphism

$$\operatorname{Tor}_n^R(A,B) \cong \operatorname{tor}_n^R(A,B).$$

*Proof.* Let  $P_{\bullet} \to A$  and  $Q_{\bullet} \to B$  be projective resolutions and set  $K_i := \ker(P_i \to P_{i-1})$  with  $K_0 = A$ , and  $V_i := \ker(Q_i \to Q_{i-1})$  with  $V_0 = B$ . By dimension shifting,

$$\operatorname{Tor}_n^R(A,B) \cong \operatorname{Tor}_1^R(K_i,B)$$
 and  $\operatorname{tor}_n^R(A,B) \cong \operatorname{tor}_1^R(A,V_j)$   $(i+j=n,i,j\geq 1).$ 

Consider the short exact sequences  $0 \to K_i \to P_{i-1} \to K_{i-1} \to 0$  and  $0 \to V_j \to Q_{j-1} \to V_{j-1} \to 0$ . Tensoring element-wise gives a commutative  $3 \times 3$  diagram with exact rows and columns

$$K_{i} \otimes V_{j} \longrightarrow P_{i-1} \otimes V_{j} \longrightarrow K_{i-1} \otimes V_{j} \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$K_{i} \otimes Q_{j-1} \longrightarrow P_{i-1} \otimes Q_{j-1} \longrightarrow K_{i-1} \otimes Q_{j-1} \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$K_{i} \otimes V_{j-1} \longrightarrow P_{i-1} \otimes V_{j-1} \longrightarrow K_{i-1} \otimes V_{j-1} \longrightarrow 0$$

The Snake Lemma identifies  $\operatorname{Tor}_1^R(K_{i-1}, V_j)$  with the kernel of the top horizontal map and also identifies  $\operatorname{tor}_1^R(K_i, V_{j-1})$  with the kernel of the left vertical map; exactness in the middle square shows these kernels coincide, hence

$$\operatorname{Tor}_1^R(K_i, V_{i-1}) \cong \operatorname{tor}_1^R(K_i, V_{i-1})$$

Applying dimension shifting back yields  $\operatorname{Tor}_n^R(A, B) \cong \operatorname{tor}_n^R(A, B)$  for all  $n \geq 0$ .

**Theorem** (Ext via resolutions). For left R-modules A, B and  $n \ge 0$ , if  $B \to E^{\bullet}$  is an injective resolution and  $P_{\bullet} \to A$  is a projective resolution, then

$$\operatorname{Ext}_R^n(A,B) = H^n(\operatorname{Hom}(A,E^{\bullet})) \cong H^n(\operatorname{Hom}(P_{\bullet},B)).$$

#### 6.6 Dimensions

**Definition.** For  $M \in \underline{R\text{-Mod}}$ , the **projective dimension**  $\operatorname{pd}_R(M)$  is the minimal integer n such that there exists a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

The **injective dimension**  $id_R(M)$  is defined dually via an injective resolution

$$0 \to M \to I^0 \to \cdots \to I^n \to 0.$$

**Example.** Over  $R = \mathbb{Z}$  and for  $M = \mathbb{Z}/n$ , one has  $pd_R(M) = 1$ , since there is a projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow M \longrightarrow 0.$$

**Proposition.** *If* R *is a PID, then*  $pd_R(M) \leq 1$  *for all* R-modules M, and dually  $id_R(M) \leq 1$  *for all* M.

**Theorem.** Let M be a left R-module and  $d \ge 0$ . The following are equivalent:

- (1)  $\operatorname{pd}_{R}(M) \leq d$ .
- (2)  $\operatorname{Ext}_R^n(M, N) = 0$  for all n > d and all N.
- (3)  $\operatorname{Ext}_{R}^{d+1}(M, N) = 0$  for all N.
- (4) If  $0 \to K_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$  is exact with each  $P_i$  projective, then  $K_d$  is projective.

*Proof.* Implications  $4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$  are clear. We show  $3 \Rightarrow 4$ . By dimension shifting,

$$\operatorname{Ext}_R^{d+1}(M,N) \cong \operatorname{Ext}_R^1(K_d,N)$$

for all N. Hence  $\operatorname{Ext}_R^1(K_d, -) = 0$ , and the lemma below shows that  $K_d$  is projective.

**Lemma.** A left R-module P is projective if and only if  $\operatorname{Ext}^1_R(P, N) = 0$  for all N.

Proof. The forward implication is clear. Now, given a short exact sequence

$$0 \to N' \to N \xrightarrow{g} N'' \to 0$$

apply  $\operatorname{Hom}_R(P, -)$  to get an exact row

$$0 \to \operatorname{Hom}(P, N') \to \operatorname{Hom}(P, N) \to \operatorname{Hom}(P, N'') \to \operatorname{Ext}^1_R(P, N') = 0.$$

Thus  $\operatorname{Hom}(P, N) \to \operatorname{Hom}(P, N'')$  is an epimorphism, so P is projective.

A dual statement of the theorem also holds for injective dimensions.

**Theorem.** For a left module N and integer  $d \ge 0$ , the following are equivalent:

- (1)  $id_R(N) \leq d$ .
- (2)  $\operatorname{Ext}_{R}^{n}(M, N) = 0$  for all n > d and all M.
- (3)  $\operatorname{Ext}_{R}^{d+1}(M, N) = 0$  for all M.
- (4) If  $0 \to N \to E^0 \to \cdots \to E^{d-1} \to K^d \to 0$  is exact with each  $E_i$  injective, then  $K_d$  is injective.

**Lemma.** A left R-module N is injective if and only if  $\operatorname{Ext}^1_R(R/I, N) = 0$  for all left ideals I.

*Proof.* Follows from the long exact sequence for  $I \to R \to R/I$  and F = Hom(-, N):

$$\operatorname{Hom}(R,N) \to \operatorname{Hom}(I,N) \to \operatorname{Ext}^1_R(R/I,N) \to 0.$$

**Theorem.** *The following numbers are equal:* 

- (1)  $\sup\{\operatorname{id}_R N \mid N \in \underline{R\text{-Mod}}\}.$
- (2)  $\sup\{\operatorname{pd}_R M \mid M \in \underline{R\text{-Mod}}\}.$
- (3)  $\sup\{pd_R M \mid M \text{ is finitely generated}\}.$

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- (4)  $\sup\{\operatorname{pd}_R I \mid I \leq R \text{ a left ideal}\}.$
- (5)  $\sup\{d \mid \exists M, N \in \underline{R}\text{-Mod } with \ \operatorname{Ext}_R^d(M, N) \neq 0\}.$

*Proof.* From the theorems on projective and injective dimensions it follows that (1) = (5) = (2). Moreover,  $(2) \ge (3) \ge (4)$  is immediate. If  $d := (4) < \infty$  and  $N \in \underline{R\text{-Mod}}$ , take

$$0 \to N \to E^0 \to \cdots \to E^d \to K \to 0.$$

to be an injective resolution of N. For every left ideal  $I \leq R$ , dimension shifting gives

$$\operatorname{Ext}_R^{d+1}(R/I,N) \cong \operatorname{Ext}_R^1(R/I,K).$$

Since  $\operatorname{pd}_R(R/I) \leq d$ , the left side vanishes, hence  $\operatorname{Ext}^1_R(R/I,K) = 0$  for all I. As we showed, this forces K injective, so  $\operatorname{id}_R N \leq d$ . Thus  $(1) \leq (4)$ , and all five numbers are equal.

**Definition.** This common value is called the **left global dimension** of *R*, and is denoted lgl *R*.

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**Definition.** For a right *R*-module *M*, the **flat dimension**  $fd_R(M)$  is the least  $d \in \mathbb{Z}_{\geq 0}$  such that there exists an exact sequence

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with each  $F_i$  flat; if no such d exists, set  $fd_R(M) = \infty$ .

**Lemma.** Fix  $d \ge 0$ . For a right R-module M, the following are equivalent:

- (1)  $fd_R(M) \leq d$ .
- (2)  $\operatorname{Tor}_{i}^{R}(M, N) = 0$  for all i > d and all  $N \in \operatorname{\underline{Mod-R}}$ .
- (3)  $\operatorname{Tor}_{d+1}^{R}(M, N) = 0$  for all  $N \in \operatorname{\underline{Mod-R}}$ .
- (4) If  $0 \to K_d \to F_{d-1} \to \cdots \to F_0 \to M \to 0$  is exact with each  $F_i$  flat, then  $K_d$  is flat.

**Theorem.** The following numbers are equal:

- (1)  $\sup\{\operatorname{fd}_R(M)\mid M\in\underline{R\operatorname{-Mod}}\}.$
- (2)  $\sup\{fd_R(R/I) \mid I \leq R \text{ a right ideal}\}.$
- (3)  $\sup\{\operatorname{fd}_R(N)\mid N\in \operatorname{\underline{Mod-}R}\}.$
- (4)  $\sup\{fd_R(R/I) \mid I \leq R \text{ a left ideal}\}.$
- (5)  $\sup\{d \mid \exists M \in \underline{R}\text{-Mod}, N \in \underline{\text{Mod-}R} \text{ with } \operatorname{Tor}_d^R(M,N) \neq 0\}.$

**Definition.** The **weak global dimension** of R is the common value above, and is denoted wgl(R).

### 7 Delta Functors

# 7.1 Definition and Basic Properties

Fix abelian categories  $\underline{A}$  and  $\underline{B}$ .

**Definition** (Delta Functor). A **(cohomological) delta functor**  $F = \{F^n, \delta_C^n\} : \underline{A} \to \underline{B}$  is:

- (1) a collection of additive functors  $F^n : \underline{A} \to \underline{\mathcal{B}}$  for  $n \ge 0$ ;
- (2) for any short exact sequence  $0 \to A \to B \to C \to 0$ , a collection  $\delta_{\mathbb{C}}^n : F^n(C) \to F^{n+1}(A)$  for  $n \ge 0$  of morphisms in  $\underline{\mathcal{B}}$ .

Satisfying the following axioms:

(i) For any short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , the following is exact:

$$0 \to F^0(A) \to F^0(B) \to F^0(C) \xrightarrow{\delta_C^0} F^1(A) \to F^1(B) \to F^1(C) \xrightarrow{\delta_C^1} \cdots$$

(ii) For any commutative diagram with exact rows:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

the diagram

$$F^{n}(C) \xrightarrow{\delta_{C}^{n}} F^{n+1}(A)$$

$$\downarrow \qquad \qquad \downarrow$$

$$F^{n}(C') \xrightarrow{\delta_{C'}^{n}} F^{n+1}(A')$$

commutes, i.e.  $\delta^n$  is natural.

**Remark.**  $F^{\bullet} \in \text{Lex}(\underline{\mathcal{A}}, \underline{\mathcal{B}}).$ 

**Example.**  $\{R^iF\}$  is a  $\delta$ -functor.

**Definition** (Morphism of Delta Functors). A morphism  $t:(F^n, \delta_F^n) \to (G^n, \delta_G^n)$  of δ-functors is a collection of natural transformations  $t^n: F^n \to G^n$  for  $n \ge 0$  such that for any short exact sequence  $0 \to A \to B \to C \to 0$  in  $\underline{\mathcal{A}}$ , the diagram

$$F^{n}(C) \xrightarrow{\delta_{F}^{n}} F^{n+1}(A)$$

$$\downarrow^{t^{n}} \qquad \downarrow^{t^{n+1}}$$

$$G^{n}(C) \xrightarrow{\delta_{G}^{n}} G^{n+1}(A)$$

commutes.

**Definition** (Homological Delta Functor). A **homological**  $\delta$ -functor is a functor

$$K^{\bullet}: \operatorname{Lex}(\underline{\mathcal{A}}, \underline{\mathcal{B}}) \to \delta\operatorname{-Func}(\underline{\mathcal{A}}, \underline{\mathcal{B}}).$$

**Definition** (Effaceability; Grothendieck). Let  $F : \underline{A} \to \underline{B}$  be an additive fuctor.

- (1) *F* is **effaceable** if for every  $A \in \underline{A}$  there exists a monomorphism  $u : A \hookrightarrow E$  such that F(u) = 0.
- (2) *F* is **coeffaceable** if for every  $A \in \underline{A}$  there exists an epimorphism  $v : P \rightarrow A$  such that F(v) = 0.

**Lemma.** Assume that  $\underline{A}$  has enough injectives. Then F is effaceable if and only if F(I) = 0 for all  $I \in \text{Inj}(\underline{A})$ .

*Proof.* (⇒) Given injective *I*, choose a monomorphism  $u : I \hookrightarrow E$  with F(u) = 0. Since *I* is injective, *u* splits, hence  $E \simeq I \oplus I'$ . Applying *F*, the map F(u) identifies with the inclusion  $F(I) \to F(I) \oplus F(I')$ , so 0 = F(u) forces  $\mathrm{id}_{F(I)} = 0$  and thus F(I) = 0. (⇐) For any *A* pick a monomorphism  $i : A \hookrightarrow I$  with *I* injective. Then F(I) = 0 by assumption, hence F(i) = 0, proving effaceability.

### 7.2 Universal $\delta$ -functors

**Definition.** A δ-functor  $F = (F^n, \delta_F^n) : \underline{A} \to \underline{\mathcal{B}}$  is **universal** if for every δ-functor  $G = (G^n, \delta_G^n)$  and natural transformation  $t : F^0 \to G^0$  there exists a unique morphism of δ-functors  $\{t^n\}_{n \geq 0} : F \to G$  with  $t^0 = t$ .

**Lemma.** If a universal  $\delta$ -functor exists, it is unique up to a unique isomorphism of  $\delta$ -functors.

**Theorem.** If a cohomological  $\delta$ -functor  $(F^n, \delta_F^n)$  is effaceable in all degrees  $n \geq 1$ , then F is universal.

*Proof.* Given  $t^0: F^0 \to G^0$ , construct  $t^n$  by induction on n. Suppose  $t^k$  are defined for  $k \le n$ . For each A choose a short exact sequence  $0 \to A \xrightarrow{i} B \to C \to 0$  with  $F^{n+1}(i) = 0$  (effaceability). Exactness gives a factorization

$$F^n(C) \longrightarrow \operatorname{coker} F^n(i) \xrightarrow{\overline{\delta}_F^n} F^{n+1}(A).$$

By functoriality of cokernels,  $t_B^n$  and  $t_C^n$  induce  $z_A^n$ : coker  $F^n(i) \to \operatorname{coker} G^n(i)$ . Define  $t_A^{n+1}$  as the unique map making the following diagram commute:

$$F^{n}(B) \longrightarrow F^{n}(C) \longrightarrow \operatorname{coker} F^{n}(i) \xrightarrow{\bar{\delta}_{F}^{n}} F^{n+1}(A)$$

$$\downarrow t_{B}^{n} \qquad \downarrow t_{C}^{n} \qquad \downarrow z_{A}^{n} \qquad \downarrow t_{A}^{n+1}$$

$$G^{n}(B) \longrightarrow G^{n}(C) \longrightarrow \operatorname{coker} G^{n}(i) \xrightarrow{\bar{\delta}_{G}^{n}} G^{n+1}(A)$$

Naturality of connecting morphisms shows this  $t_A^{n+1}$  is well defined and natural in A; uniqueness follows by exactness and the same diagram chase.

**Corollary.** If  $F: \underline{A} \to \underline{B}$  is left exact additive, then its right derived functors  $R^nF$  with the standard connecting morphisms form a universal  $\delta$ -functor.

### 7.3 Universal Coefficients Theorem

**Theorem** (Künneth formula). Let  $P = (\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots)$  be a chain complex of flat right R-modules such that the images  $d(P_n) \subseteq P_{n-1}$  are flat for all  $n \in \mathbb{Z}$ . For every  $n \in \mathbb{Z}$  and every  $M \in R$ -Mod there is a natural short exact sequence

$$0 \longrightarrow H_n(P) \otimes_R M \longrightarrow H_n(P \otimes_R M) \longrightarrow \operatorname{Tor}_1^R(H_{n-1}(P), M) \longrightarrow 0. \tag{*}$$

*Proof.* By the long exact sequence of Tor for  $0 \to Z_n \to P_n \to d(P_n) \to 0$  and the flatness of  $P_n$  and  $d(P_n)$ , each  $Z_n := \ker(d_n)$  is flat. Hence there is a short exact sequence of complexes

$$0 \longrightarrow Z_{\bullet} \longrightarrow P_{\bullet} \longrightarrow dP_{\bullet} \longrightarrow 0$$

where the differentials on  $Z_{\bullet}$  and  $dP_{\bullet}$  are zero. Tensoring with M remains exact and yields a long exact sequence in homology

$$H_{n+1}(dP_{\bullet}\otimes_R M) \to H_n(Z_{\bullet}\otimes_R M) \to H_n(P_{\bullet}\otimes_R M) \to H_n(dP_{\bullet}\otimes_R M).$$

Since the outer complexes have zero differentials, one identifies

$$H_n(Z_{\bullet} \otimes_R M) \simeq Z_n \otimes_R M, \quad H_n(dP_{\bullet} \otimes_R M) \simeq d(P_n) \otimes_R M, \quad H_{n+1}(dP_{\bullet} \otimes_R M) \simeq d(P_{n+1}) \otimes_R M.$$

The short exact sequence

$$0 \to d(P_{n+1}) \to Z_n \to H_n(P) \to 0$$

identifies the left term with  $\operatorname{Tor}_1^R(H_n(P), M)$  and gives an exact segment

$$0 \to \operatorname{Tor}_1^R(H_n(P), M) \to Z_n \otimes_R M \to H_n(P \otimes_R M) \to d(P_n) \otimes_R M \to 0.$$

Finally, the short exact sequence  $0 \to d(P_n) \to Z_{n-1} \to H_{n-1}(P) \to 0$  identifies

$$\operatorname{coker}(Z_n \otimes_R M \to H_n(P \otimes_R M)) \simeq \operatorname{Tor}_1^R(H_{n-1}(P), M).$$

**Corollary.** *If*  $R = \mathbb{Z}$ , the sequence  $(\star)$  splits (non-naturally); hence

$$H_n(P \otimes_{\mathbb{Z}} M) \simeq H_n(P) \otimes_{\mathbb{Z}} M \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M).$$

*Proof.* Over  $\mathbb{Z}$ , subgroups of free abelian groups are free; thus each  $d(P_n) \subseteq P_{n-1}$  is free, and there exists a decomposition  $P_{n-1} \cong \mathbb{Z}^{I_{n-1}} \oplus d(P_n)$ . Passing to tensors,  $d(P_n) \otimes M$  is a direct summand of  $P_{n-1} \otimes M$ . Modding out  $Z_n \otimes M$  and  $\ker(d_n \otimes 1_M)$  by the common image  $d_{n+1} \otimes 1_M$  shows that  $H_n(P) \otimes M$  is a direct summand of  $H_n(P \otimes M)$ ; the complement is  $\operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M)$  by  $(\star)$ .

**Definition.** Let  $P \in \underline{\mathrm{Ch}}(\underline{\mathrm{Mod-}R})$  be a chain complex of right R-modules and  $Q \in \underline{\mathrm{Ch}}(\underline{R}\text{-}\underline{\mathrm{Mod}})$  a chain complex of left R-modules. Define the tensor product complex  $(P \otimes_R Q, d)$  by

$$(P \otimes_R Q)_n = \bigoplus_{p+q=n} P_p \otimes_R Q_q,$$

with differential on homogeneous tensors given by

$$d(a \otimes b) = d_P(a) \otimes b + (-1)^p a \otimes d_Q(b), \qquad a \in P_p, \ b \in Q_q.$$

**Theorem** (Künneth formula for complexes). *If each*  $P_n$  *and*  $d(P_n) \subseteq P_{n-1}$  *is flat (as right R-modules) for all* n, then for every n there is a natural short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \longrightarrow H_n(P \otimes_R Q) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R (H_p(P), H_q(Q)) \longrightarrow 0.$$

*If R is a PID, this sequence (noncanonically) splits.* 

# 7.4 Topological Application of the Künneth Formula

Let *X* be a topological space. A **singular** *n***-simplex** in *X* is a continuous map  $\sigma : \Delta^n \to X$  from the standard simplex  $\Delta^n = \langle e_0, \dots, e_n \rangle$  (the convex hull of the vertices  $e_0, \dots, e_n$ ).

**Definition** (Singular chains). The group of **singular** *n***-chains** is the free abelian group on singular *n*-simplices,

$$S_n(X) := \mathbb{Z}\langle \sigma \mid \sigma : \Delta^n \to X \rangle \simeq \bigoplus_{\sigma} \mathbb{Z}[\sigma].$$

For a generator corresponding to  $\sigma$ , write  $[p_0, \ldots, p_n] := [\sigma(e_0), \ldots, \sigma(e_n)]$ . The boundary map  $\partial: S_n(X) \to S_{n-1}(X)$  is

$$\partial[p_0, \dots, p_n] = \sum_{k=0}^n (-1)^k [p_0, \dots, \widehat{p_k}, \dots, p_n] = \sum_{k=0}^n (-1)^k [\sigma \circ d^k(e_0, \dots, \widehat{e_k}, \dots, e_n)],$$

where  $d^k: \Delta^{n-1} \hookrightarrow \Delta^n$  is the *k*-th face inclusion.

For an abelian group *M*, **homology with coefficients** in *M* is defined by

$$H_n(X, M) := H_n(S_{\bullet}(X) \otimes_{\mathbb{Z}} M), \qquad M \in \underline{Ab}.$$

By the universal coefficients theorem,

$$H_n(X,M) \simeq H_n(X,\mathbb{Z}) \otimes_{\mathbb{Z}} M \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),M)$$
 (e.g.  $M = \mathbb{Z}/n$ ).

**Theorem** (Eilenberg–Zilber). *There is a natural chain homotopy equivalence* 

$$S_{\bullet}(X \times Y) \simeq S_{\bullet}(X) \otimes_{\mathbb{Z}} S_{\bullet}(Y),$$

whence an isomorphism  $H_n(X \times Y, \mathbb{Z}) \simeq H_n(S_{\bullet}(X) \otimes S_{\bullet}(Y))$ .

Combining with Künneth, for all spaces *X*, *Y* one obtains the (integral) Künneth decomposition

$$H_n(X \times Y) \simeq \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(Y) \oplus \bigoplus_{p+q=n-1} \operatorname{Tor}_1^{\mathbb{Z}} (H_p(X), H_q(Y)).$$

**Theorem** (Universal Coefficient Theorem). Let P be a chain complex of projective right R-modules with  $P_n$  and  $d(P_n)$  projective for all n. For any left R-module M there is a (noncanonically) split short exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^1(H_{n-1}(P), M) \longrightarrow H^n(\operatorname{Hom}_R(P, M)) \longrightarrow \operatorname{Hom}_R(H_n(P), M) \longrightarrow 0.$$

#### APPLICATION

For a topological space *X* and an abelian group *M*, define singular cohomology with coefficients by

$$H^n(X,M) := H^n(\operatorname{Hom}_{\mathbb{Z}}(S_{\bullet}(X),M)).$$

Then the universal coefficient theorem over Z gives a (noncanonical) decomposition

$$H^n(X,M) \simeq \operatorname{Hom}(H_n(X,\mathbb{Z}),M) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),M).$$

# 7.5 Ext<sup>1</sup> and Extensions

#### **Definitions:**

(1) For objects A, C in an abelian category  $\underline{A}$ , an **extension** of A by C is a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0.$$

Write  $E = (A \xrightarrow{i} B \xrightarrow{p} C)$ .

(2) A morphism  $r = (\alpha, \beta, \gamma) : E \to E'$  between  $E = (A \to B \to C)$  and  $E' = (A' \to B' \to C')$  is a commutative diagram with exact rows

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$0 \longrightarrow A' \xrightarrow{i'} B' \xrightarrow{p'} C' \longrightarrow 0$$

(in particular,  $p'\beta = \gamma p$  and  $\beta i = i'\alpha$ ).

(3) Two extensions of the same A, C are **equivalent**,  $E \sim E'$ , if there exists an isomorphism of extensions  $(id_A, \beta, id_C) : E \to E'$  (equivalently,  $\beta : B \xrightarrow{\sim} B'$ ). Denote by Ext(C, A) the set of equivalence classes of extensions of A by C.

#### **FUNCTORIALITY**

Given  $E \in \operatorname{Ext}(C, A)$  and a morphism  $\gamma : C' \to C$ , there exists a unique class  $\gamma^* E \in \operatorname{Ext}(C', A)$  together with a morphism of extensions  $(\operatorname{id}_A, \beta, \gamma) : \gamma^* E \to E$  fitting into a pullback square

$$0 \longrightarrow A \longrightarrow B \times_C C' \longrightarrow C' \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \gamma$$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Dually, for  $\alpha : A \to A'$  there exists a unique class  $\alpha_* E \in \text{Ext}(C, A')$  with a morphism of extensions  $(\alpha, \beta, \text{id}_C) : E \to \alpha_* E$  obtained by the pushout

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \qquad \downarrow \qquad \parallel$$

$$0 \longrightarrow A' \longrightarrow A' \sqcup_{A} B \longrightarrow C \longrightarrow 0.$$

#### **SUM OF EXTENSIONS**

For  $E_1$ ,  $E_2 \in \text{Ext}(C, A)$ , define their **sum** by the following formula:

$$E_1 + E_2 := \nabla_{A*} (\Delta_C^*(E_1 \oplus E_2)) \in \text{Ext}(C, A),$$

where  $\Delta_C : C \to C \oplus C$  is the diagonal and  $\nabla_A : A \oplus A \to A$  the codiagonal.

The zero element is the split extension and  $-E = (-id_A)_*E$ .

#### **ADDITIVITY**

For all  $\alpha : A \rightarrow A'$  and  $\gamma : C' \rightarrow C$ :

$$\alpha_*(E_1 + E_2) = \alpha_* E_1 + \alpha_* E_2, \qquad (E_1 + E_2) \circ \gamma = E_1 \circ \gamma + E_2 \circ \gamma.$$

Write  $E \circ \gamma := \gamma^* E$ . The operations are additive in the morphisms:

$$(\alpha_1 + \alpha_2)_*E = (\alpha_1)_*E + (\alpha_2)_*E$$
,  $E \circ (\gamma_1 + \gamma_2) = E \circ \gamma_1 + E \circ \gamma_2$ .

Let *E* be an extension  $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ . Choose a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \longrightarrow 0.$$

Pick a lift  $\alpha: P_0 \to B$  with  $p \circ \alpha = \epsilon$ . Since  $p \alpha d_1 = 0$ , there is a unique map  $\delta_1: P_1 \to A$  with  $i \delta_1 = \alpha d_1$ . Because  $d_1 d_2 = 0$ , one has  $\delta_1 d_2 = 0$ , so  $\delta_1$  is a 1–cocycle in  $\operatorname{Hom}_R(P_{\bullet}, A)$ . Put

$$\Psi([E]) := [\delta_1] \in H^1(\operatorname{Hom}_R(P_{\bullet}, A)) = \operatorname{Ext}^1_R(C, A).$$

**Lemma.** The map  $\Psi$  is well-defined.

*Proof.* If  $\alpha'$  is another lift with  $p\alpha' = \epsilon$ , then  $\alpha' - \alpha = is$  for a unique  $s : P_0 \to A$ . The corresponding cocycles satisfy  $\delta'_1 - \delta_1 = s \, d_1$ , a coboundary. Hence  $[\delta'_1] = [\delta_1]$  in Ext<sup>1</sup>, so  $\Psi$  is well defined.

Let  $[d_1] \in \operatorname{Ext}^1(C, A)$  be represented by a cocycle  $d_1 : P_1 \to A$  with  $d_1d_2 = 0$ , where

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_0} P_0 \longrightarrow C \longrightarrow 0$$

is a projective resolution of *C*. Put  $B_2 := \operatorname{im}(d_2)$  and let  $\bar{d}_1 : P_1/B_2 \to A$  be induced by  $d_1$ . Define

$$\Theta([d_1]) := \left\lceil \bar{d}_1 : 0 \longrightarrow P_1/B_2 \longrightarrow P_0 \longrightarrow C \longrightarrow 0 \right\rceil,$$

that is, the pushout of  $0 \to P_1/B_2 \to P_0 \to C \to 0$  along  $\bar{d}_1$ . We argue that  $\Theta$  is well-defined.

**Lemma.** If  $d'_1 = d_1 + s d_2$  for some  $s : P_0 \to A$ , then  $\Theta([d'_1])$  is equivalent to  $\Theta([d_1])$ .

*Proof.* Let  $\bar{d}'_1: P_1/B_2 \to A$  be induced by  $d'_1$ . Consider the cocartesian squares

The universal property of pushouts applied to the pair of maps  $(\mathrm{id}_A,\mathrm{id}_{P_0})$  and  $(\mathrm{id}_A-s\,p,\mathrm{id}_{P_0})$  (where  $p:P_0 \twoheadrightarrow C$ ) yields mutually inverse isomorphisms  $B_{d_1} \stackrel{\sim}{\longrightarrow} B_{d_1'}$  commuting with the structure maps to A and C. Hence the bottom short exact rows

$$0 \longrightarrow A \longrightarrow B_{d_1} \longrightarrow C \longrightarrow 0$$
,  $0 \longrightarrow A \longrightarrow B_{d'_1} \longrightarrow C \longrightarrow 0$ 

represent the same class in Ext(C, A).

**Proposition** (w/o proof). *The maps*  $\Psi$  *and*  $\Theta$  *are inverse bijections.*