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This is an excerpt that contains only the low-level proof for 4.1 c) and d). However, it would be nice if someone could provide a a rigorous proof of the same statement using the duality theorems.

Exercise 4.1 (Equalizing strategies)

Let $N = \{1, 2\}$ be a set of players, $S_1 := \{T, M, B\}$, $S_2 := \{L, C, R\}$ their sets of pure strategies, and $u_1, u_2 : S_1 \times S_2 \to \mathbb{R}$ their utility functions given by the following matrix:

$$\begin{bmatrix} & L & C & R \\ T & (3,-3) & (-3,3) & (0,0) \\ M & (2,-2) & (6,-6) & (4,-4) \\ B & (2,-2) & (5,-5) & (6,-6). \end{bmatrix}$$

Let's write $U := (u_1(x, y))_{x \in S_1, y \in S_2}$. Since this is a zero-sum game, the analogon of the matrix U for player 2 is $-U^T$.

a) We want to find a mixed strategy for player 1 such that the expected gain is the same for all pure strategies of player 2. Let's denote this mixed strategy by a S_1-valued random variable X_1. That is, we want to show that there exists a constant $c_1 \in \mathbb{R}$ and numbers $\mathbb{P}[X_1 = x]$ for $x \in S_1$ such that for all $y \in S_2$ it holds:

$$c_1 = \mathbb{E}[u_1(X_1, y)] = \sum_{x \in S_1} \mathbb{P}[X_1 = x] u_1(x, y).$$

Keeping in mind that it must hold that $\sum_x \mathbb{P}[X_1 = x] = 1$, we obtain the following linear equation:

$$\begin{bmatrix} u_1(T,L) & u_1(M,L) & u_1(B,L) & -1 \\ u_1(T,C) & u_1(M,C) & u_1(B,C) & -1 \\ u_1(T,R) & u_1(M,R) & u_1(B,R) & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{P}[X_1 = T] \\ \mathbb{P}[X_1 = M] \\ \mathbb{P}[X_1 = B] \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

which we write more compactly as

$$\begin{bmatrix} U^T & -1_{3\times1} \\ 1_{1\times3} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0_{3\times1} \\ 1 \end{bmatrix}$$

with the obvious meaning of p_1. This can be solved with something like Octave/Matlab as follows:

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U = [3,-3,0;2,6,4; 2,5,6];
o = [1;1;1];
first = [U',-o;o',0];
second = [-U, -o;o',0];
rhs = [0;0;0;1];
% The result for part a)
disp(first \ rhs);
% The result for part b)
disp(second \ rhs);
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The result is:

$$p_1^T = \begin{bmatrix} 0.4 & 0.6 & 0 \end{bmatrix} \qquad c_1 = 2.4.$$

b) A completely symmetric derivation for player 2 leads to the equation

$$\begin{bmatrix} -U & -1_{3\times 1} \\ 1_{1\times 3} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0_{3\times 1} \\ 1 \end{bmatrix}$$

with the solution

$$p_2^T = \begin{bmatrix} 0.88 & 0.08 & 0.04 \end{bmatrix} \qquad c_2 = -2.4.$$

c) We want to show that the equalizing strategies X_1 and X_2 (with the properties and probabilities as above) are optimal for both players.

First, notice the following property (resulting from what has been called “multilinearity” in the lecture). If Y_2 is some mixed strategy of the second player, then it holds:

$$\mathbb{E}[u_1(X_1, Y_2)] = \sum_{y \in S_2} \mathbb{P}[Y_2 = y]\mathbb{E}[u_1(X_1, y)] = \sum_{y \in S_2} \mathbb{P}[Y_2 = y]c_1 = c_1.$$

In particular, we obtain just the constant c_1 if we minimize over all possible mixed strategies of player 2:

$$\min_{Y_2} \mathbb{E}[u_1(X_1, Y_2)] = c_1.$$

A symmetric statement holds for c_2. This allows us to make estimations for Maxmin-values for both players. Here are two inequalities for Maxmin-values of player 1:

$$\begin{aligned}
\max_{Y_1} \min_{Y_2} \mathbb{E}[u_1(Y_1, Y_2)] &\geq \min_{Y_2} \mathbb{E}[u_1(X_1, Y_2)] = \min_{Y_2} c_1 = c_1 \\
\max_{Y_1} \min_{Y_2} \mathbb{E}[u_1(Y_1, Y_2)] &= \max_{Y_1} \min_{Y_2} \mathbb{E}[-u_2(Y_1, Y_2)] \\
&\leq \max_{Y_1} \mathbb{E}[-u_2(Y_1, X_2)] \\
&= \max_{Y_1} (-c_2) \\
&= -c_2
\end{aligned}$$

In particular, if we have the situation that $c_2 = -c_1$, these two inequalities combine into a single equality:

$$\max_{Y_1} \min_{Y_2} \mathbb{E}[u_1(Y_1, Y_2)] = c_1.$$

Exchanging the indices 1 and 2 yields a completely symmetric result for player 2.

In parts a) and b) we have found out that for the concrete example we have $c_1 = -c_2 = 2.4$, so we can conclude that the Maxmin-value for player 1 is 2.4 and the Maxmin-value for player 2 is -2.4, which by definition makes the equalizing strategies X_1 and X_2 *optimal*.

d) Now we want to consider a two-player zero-sum game where the players 1 and 2 have more possible actions. We want to show that if both players have equalizing strategies, these strategies are optimal.

Suppose $S_1 = \{1 \dots n\}$ and $S_2 = \{1 \dots m\}$ for some natural numbers n, m. Let U be a $n \times m$ matrix defined as above. Suppose there are equalizing strategies X_1 and X_2, that is, there exist probability vectors $p_1 \in \mathbb{R}^n$ and $p_2 \in \mathbb{R}^m$ such that the following holds:

$$\begin{bmatrix} U^T & -1_{m\times 1} \\ 1_{1\times n} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -U & -1_{n\times 1} \\ 1_{1\times m} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix}.$$

In c) we have shown that it is sufficient to establish $c_2 = -c_1$ in order to prove the optimality (the argument did not depend on the dimension of the matrix U in any way). It holds:

$$\begin{aligned} c_2 &= \begin{bmatrix} p_2^T & c_2 \end{bmatrix} \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} p_2^T & c_2 \end{bmatrix} \begin{bmatrix} U^T & -1_{m\times 1} \\ 1_{1\times n} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} \\ &= -\left(\begin{bmatrix} -U & -1_{n\times 1} \\ 1_{1\times m} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ c_2 \end{bmatrix} \right)^T \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} \\ &= -\begin{bmatrix} 0_{1\times n} & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} \\ &= -c_1 \end{aligned}$$

Hence, the equalizing strategies are optimal. ■

e) Finally, we want to consider a counterexample. Consider the following two-player zero-sum game:

$$\begin{bmatrix} & L & R \\ T & (1,-1) & (1,-1) \\ B & (-1,1) & (-1,1) \end{bmatrix}$$

Here, the first player has an equalizing strategy with $p_1 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$. With this strategy, the first player would on average win nothing, no matter what the strategy of the second player is.

However, the optimal strategy would obviously be the pure strategy T. Since the decision of the second player can not affect the outcome in any way, he can take either L or R as optimal strategy. The value of the game is therefore $(1, -1)$, and not 0, if both play rationally, the first player wins $(+1)$ and the second loses (-1).

However, this is not a contradiction to d), because here the second player does not have an equalizing strategy: no matter how he weights L and R, the outcome in the row T is always (-1), the outcome in the row B is always $(+1)$, and $(-1) \neq (+1)$.