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Exercise 5.1 (Compactness of certain families of measures) Let S be a finite set. Consider the family of all probability measures on S:

$$\Delta(S) := \left\{ \sigma : S \to [0,1] \;\middle|\; \sum_{s \in S} \sigma(s) = 1 \right\}.$$

We interpret it as a subset of the finitely-dimensional space $\mathbb{R}^S$, which can be thought of as the set of all $\mathbb{R}$-vectors indexed by elements of S. It is (unnaturally) isomorphic to $\mathbb{R}^{|S|}$ (one has to pick some enumeration of S), so all theorems that hold for the canonical finite-dimensional real vector space $\mathbb{R}^n$ (for $n = |S|$) also hold for $\mathbb{R}^S$. In particular, the Heine-Borel theorem holds.

We want to show that $\Delta(S)$ is compact. For all $\sigma \in \Delta(S)$ it holds:

$$\|\sigma\|_\infty = \max_{s \in S} |\sigma(s)| \leq 1,$$

so $\Delta(S)$ is bounded.

To show that it is closed, consider the summation function:

$$F : \mathbb{R}^S \to \mathbb{R}, \qquad F(\sigma) := \sum_{s \in S} \sigma(s).$$

We assume that it is known that F is continuous. The single-point set $\{1\}$ is closed, so it's preimage $F^{-1}(\{1\})$ is closed. The unit cube $[0,1]^S$ is closed as a finite product of closed sets. Therefore, $\Delta(S)$ is closed as intersection of two closed sets:

$$\Delta(S) = [0,1]^S \cap F^{-1}(\{1\})$$

By the Heine-Borel theorem, $\Delta(S)$ is compact. ■

Exercise 5.2 (Nash equilibria by Kakutani's fixed point theorem) We want to use Kakutani's fixed point theorem[1] to prove the existence of Nash equilibria in strategic games with mixed strategies.

Let n be a number of players, S_i finite sets of pure strategies for each $i = 1 \dots n$, $u_i : \prod_i S_i \to \mathbb{R}$ utility functions, and

$$U_i(\sigma) \equiv U_i(\sigma_1, \dots, \sigma_n) := \mathbb{E}[u_i(\sigma_1, \dots, \sigma_n)]$$

the expected utilities for mixed strategies $\sigma_i \in \Sigma_i := \Delta(S_i)$.

[1] Notice that the statement is different from what was given in the problem statement. It's unclear what the canonical topology on $\mathfrak{P}(X)$ should be.

(Kakutani) Let $X \subset \mathbb{R}^n$ nonempty, compact and convex set. Let $f : X \to \mathfrak{P}(X)$ such that $f(x) \neq \varnothing$ is convex for all $x \in X$. Let the graph of f

$$\Gamma_f = \{(x, y) \in X \times X \mid x \in X, y \in f(x)\}$$

be closed in X^2. Then f has a fixed point $x_0 \in X$ in the sense that $x_0 \in f(x_0)$. ■

a) Let $X := \Sigma := \prod_{i=1}^n \Sigma_i$. As we have seen in 5.1, each Σ_i is compact and convex. By $(n-1)$-fold application of 4.2 a) we see that the product X is also convex. To see that X is compact, we can either use Heine-Borel (notice that finite products of closed spaces are closed and finite products of bounded subsets of some $\mathbb{R}^n_i$ are again bounded), or nuke the problem with the disproportionally general Tychonoff's theorem.
b) Now consider the *best-responses function*

$$BR_i : \Sigma \to \mathfrak{P}(\Sigma_i), \quad BR_i(\sigma) := \operatorname{argmax}_{\theta \in \Sigma_i} U_i(\sigma_{-i}, \theta),$$
$$BR : \Sigma \to \mathfrak{P}(\Sigma), \quad BR(\sigma) := \prod_{i=1}^{n} BR_i$$

where argmax is interpreted as a set-function that can return multiple maxima. We want to show that $BR_i(\sigma)$ fulfill the first condition in Kakutani's theorem. By exercise 5.1 the set Σ_i is compact, and U_i is continuous, therefore the partially applied function $U_i(\sigma_{-1}, -)$ has at least one maximum on Σ_i for arbitrary σ. This shows that $BR_i(\sigma)$ is a nonempty set.

The convexity of the sets $BR_i(\sigma)$ is a simple consequence of the multilinearity of the functions U_i. Let $x, y \in BR_i(\sigma)$ and $t \in [0, 1]$. Let $M := \max_\theta U_i(\sigma_{-1}, \theta)$. It holds:

$$U_i(\sigma_{-1}, (t-1)x + ty) = (1-t)U_i(\sigma_{-1}, x) + tU_i(\sigma_{-1}, y) = (1-t)M + tM = M,$$

and therefore by definition $(1-t)x + ty \in BR_i(\sigma)$. Since we already know from 4.2 b) that products of convex sets are again convex, we conclude that $BR(\sigma) = \prod_{i=1}^n BR_i(\sigma)$ is also convex (and nonempty).
c) Now we show that the graph Γ_{BR} is closed.

Let $((\sigma^n, r^n))_n$ be a convergent sequence in Γ_{BR}. Let $(\sigma^*, r^*) := \lim_{n\to\infty}(\sigma^n, r^n)$ be it's limit. We have to show that the limit also lies in Γ_{BR}. Notice the following facts:

(1) All U_i are continuous

(2) Since Σ is compact, U_i are even *uniformly* continuous, from which it immediately follows that the function $\sigma \mapsto \max_{\rho \in \Sigma_i} U_i(\sigma_{-i}, \rho)$ is also continuous.

(3) Instead of considering the ill-behaved and discontinuous argmax function, we can focus on the $\max$ function, since it holds:

$$r_i \in BR_i(\sigma) \quad \Leftrightarrow \quad U_i(\sigma_{-i}, r_i) = \max_{\rho \in \Sigma_i} U_i(\sigma_{-i}, \rho).$$

Using all this, we compute:

$$\begin{aligned}
U_i(\sigma^*_{-i}, r^*_i) &\overset{\text{def }\sigma^*,r^*}{=} U_i\left(\lim_{n\to\infty}(\sigma^n_{-i}, r^n_i)\right) \\
&\overset{(1)}{=} \lim_{n\to\infty} U_i(\sigma^n_{-i}, r^n_i) \\
&\overset{\text{def }\Gamma_{BR}}{=} \lim_{n\to\infty} \max_{\rho\in\Sigma_i} U_i(\sigma^n_{-i}, \rho) \\
&\overset{(2)}{=} \max_{\rho\in\Sigma_i} U_i\left(\lim_{n\to\infty} \sigma^n_{-i}, \rho\right) \\
&\overset{\text{def }\sigma^*}{=} \max_{\rho\in\Sigma_i} U_i(\sigma^*_{-i}, \rho).
\end{aligned}$$

This is by (3) equivalent to $r^*_i \in BR_i(\sigma^*)$. Since this holds for all $i = 1, \ldots, n$, we conclude that $(\sigma^*, r^*) \in \Gamma_{BR}$. Since this holds for all convergent series, the graph Γ_{BR} is closed.

d) From the fixed point theorem of Kakutani it now follows that there must be a fixed point $\sigma^\dagger \in \Sigma$ such that

$$\sigma^\dagger \in BR(\sigma^\dagger),$$

which is by definition the same as to say that for all players i and all other possible responses $r_i \in \Sigma_i$ it holds:

$$U_i(\sigma^\dagger) \geq U_i(\sigma^\dagger_{-i}, r_i),$$

that is: for all players there is no profitable deviation from the strategy $\sigma^\dagger_i$, this is exactly the definition of a Nash-Equilibrium. ■

Exercise 5.2 (Evolutionary Stable States) Consider the following symmetric game:

$$\begin{bmatrix} & D & H \\ D & (2,2) & (1,3) \\ H & (3,1) & (7,7) \end{bmatrix}$$

We claim that there is only one symmetric Nash-Equilibrium, and that it is evolutionary stable.

Let p denote the probability for the pure strategy D, then $(1-p)$ is the probability for H (because of that, it is enough to specify only one probability, so we denote strategies simply by a single real number from $[0,1]$). The utility for the first "individual"-player is as follows (it's the same for the second "population"-player):

$$U_i\,(p,p) = \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} p \\ 1-p \end{bmatrix} = 5p^2 - 10p + 7$$

The first derivative of this expression disappears for $p = 1$, therefore all the extrema lie on the boundary of the unit interval. It holds (not surprisingly);

$$U_i(p=0, p=0) = 7 \quad U_i(p=1, p=1) = 2,$$

therefore the only local maximum is at $p = 0$, this is where the single symmetric Nash-Equilibrium is.

Now assume that we perturb the strategy $p = 0$ by an $\varepsilon > 0$ and obtain a strategy $\varepsilon \equiv (\varepsilon, 1-\varepsilon)$. For any other strategy $q \in [0,1]$ it holds:

$$U_1(q,\varepsilon) = \left[\begin{array}{c} q, 1-q \end{array}\right] \left[\begin{array}{cc} 2 & 1 \\ 3 & 7 \end{array}\right] \left[\begin{array}{c} \varepsilon \\ 1-\varepsilon \end{array}\right] = 5q\varepsilon - 6q - 4\varepsilon + 7$$

The derivative by d/dq is $5\varepsilon - 6$, therefore there are no extrema inside of the interval $[0,1]$. On the boundary $\{0,1\}$ it holds:

$$U_1(0,\varepsilon) = 7 - 4\varepsilon, U_1(1,\varepsilon) = 1 - 3\varepsilon,$$

the first expression is always larger than the second, therefore the strategy $p = 0$ is always the best response to all other perturbed strategies ε. This shows that $p = p_D = 0$ is an evolutionary stable equilibrium.