Exercise 6.1 (Properly mixed strategies) Let S be finite or countable set of pure strategies (with at least two elements), let $\Sigma := \Delta(S)$ be the mixed strategies. We call a strategy σ *properly mixed* if there does not exist an $s \in S$ such that $\sigma(s) = 1$ and $\sigma(s') = 0$ for all other $s' \neq s$.

Now let $\sigma \in \Sigma$ be an arbitrary strategy. We want to show that there exists a sequence $(\sigma_n)_n$ of properly mixed strategies with $\lim_{n\to\infty} \sigma_n = \sigma$.

We show a slightly stronger statement that there exists a sequence of σ_n with $\sigma_n(s) > 0$ for *all* $s \in S$.

Suppose that we can find a strategy $\gamma \in \Sigma$ with the property $\gamma(s) > 0$ for all $s \in S$. Define the sequence as follows:

$$\sigma_n := \frac{1}{n}\gamma + \left(1 - \frac{1}{n}\right)\sigma.$$

For all $s \in S$ it holds:

$$\sigma_n(s) \geq \frac{1}{n}\gamma(s) > 0,$$

and in particular, σ_n is properly mixed.

This sequence indeed converges to σ (e.g. in the $\|-\|_\infty$-norm, or actually in any norm $\|-\|$):

$$\|\sigma_n - \sigma\| = \left\|\frac{1}{n}\gamma - \frac{1}{n}\sigma\right\| \leq \frac{1}{n}\left(\|\gamma\| + \|\sigma\|\right) = \frac{\text{const}}{n} \overset{n\to\infty}{\longrightarrow} 0,$$

that is, $\lim_{n\to\infty} \sigma_n = \sigma$.

The remaining question is whether we can obtain a γ as above. For S finite, we can construct γ as follows:

$$\gamma(s) := \frac{1}{|S|}.$$

If S is countably infinite, we can choose some bijection ψ between S and $\mathbb{N}$ and define γ as follows:

$$\gamma(s) := 2^{-\psi(s)},$$

this is indeed a probability distribution, because:

$$\sum_{s\in S}\gamma(s) = \sum_{n\in\mathbb{N}}\gamma\left(\psi^{-1}(s)\right) = \sum_{n\in\mathbb{N}} 2^{-n} = 1.$$

In both cases we obtain a γ as required for the construction of the sequence of the properly mixed strategies. ■

Exercise 6.2 (“Perfect” Nash-Equilibria) The concept of “perfect nash-equilibria” does not seem to occur anywhere except the few slides of the lecture. Even the 1k+ pages book on game theory does not mention it (?). Link to some literature would be highly appreciated.

Exercise 6.3 (Evolutionary Stable Strategies) Let $N = \{1, 2\}$ be the set of players, S a finite set of strategies (with at least two elements) and $u_1 : \Sigma(S)^2 \to [0, \infty]$ a payoff function. The payoff for the second player is assumed to be symmetric: $u_2(x, y) = u_1(y, x)$ (we don't need this fact here, but the whole model does not make any sense otherwise).
a) Let $x^* \in S$ be a *strict* symmetric Nash-Equilibrium, that is, for all $x \neq x^*$ it holds:

$$u_1(x, x^*) < u_1(x^*, x^*).$$

Then x^* is an evolutionary stable strategy, that is: for all $x \neq x^*$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it holds:

$$(1 - \varepsilon)u_1(x, x^*) + \varepsilon u_1(x, x) = (1 - \varepsilon)u_1(x^*, x^*) + \varepsilon u_1(x^*, x).$$

Proof: Fix some $x \neq x^*$. Let $\|u_1\|_\infty := \max_{s\in\Sigma(S)} |u_1(s)|$ denote the maximum payoff. We choose the ε_0 as follows:

$$\varepsilon_0 := \frac{u_1(x^*, x^*) - u_1(x, x^*)}{4\,\|u_1\|_\infty},$$

this choice will becomes obvious after one looks at the following inequality. For all $\varepsilon < \varepsilon_0$ it now holds:

$$\begin{aligned}
&u_1(x, x^*) + \varepsilon\,(u_1(x, x) - u_1(x, x^*) + u_1(x^*, x^*) - u_1(x^*, x)) \\
&\leq u_1(x, x^*) + \varepsilon\,(4\,\|u_1\|_\infty) \\
&< u_1(x, x^*) + \varepsilon_0\,(4\,\|u_1\|_\infty) \\
&= u_1(x, x^*) + (u_1(x^*, x^*) - u_1(x, x^*)) \\
&= u_1(x^*, x^*)
\end{aligned}$$

This is exactly equivalent to the definition of ESS (the previous expression arises from the next one, again: the original thought process goes in the opposite direction of the proof):

$$(1 - \varepsilon)u_1(x, x^*) + \varepsilon u_1(x, x) < (1 - \varepsilon)u_1(x^*, x^*) + \varepsilon u_1(x^*, x).$$

■

Remark: The sense of this exercise is not to just say "that's true, because that's what the book says", because the book says (quote): "verify!" (Theorem 5.52).

b) Let σ^* be an ESS and let σ' be a symmetric Nash-Equilibrium that is also a best response to σ^*. Then $\sigma^* = \sigma'$.
Proof: This is a corollary (or rather a partial paraphrasing) of the theorem 5.52. Suppose for the sake of contradiction that $\sigma' \neq \sigma^*$. By theorem 5.52 applied to the ESS σ^*

one of the following two conditions must hold:

$$
\begin{aligned}
u_1(\sigma', \sigma^*) &< u_1(\sigma^*, \sigma^*) \\
u_1(\sigma', \sigma^*) = u_1(\sigma^*, \sigma^*) \wedge u_1(\sigma', \sigma') &< u_1(\sigma^*, \sigma').
\end{aligned}
$$

If the first condition holds, then σ' is not a best response to σ^*. If the second condition holds, then σ' is not a Nash-Equilibrium (because σ^* is a possible profitable deviation from σ'). Both cases yield a contradiction. Therefore it must hold: $\sigma' = \sigma^*$. ■

c) d) Omitted.