

Exercise 4.1 (Equalizing strategies)

Let $N = \{1, 2\}$ be a set of players, $S_1 := \{T, M, B\}$, $S_2 := \{L, C, R\}$ their sets of pure strategies, and $u_1, u_2 : S_1 \times S_2 \rightarrow \mathbb{R}$ their utility functions given by the following matrix:

$$\begin{bmatrix} & L & C & R \\ T & (3, -3) & (-3, 3) & (0, 0) \\ M & (2, -2) & (6, -6) & (4, -4) \\ B & (2, -2) & (5, -5) & (6, -6) \end{bmatrix}$$

Let's write $U := (u_1(x, y))_{x \in S_1, y \in S_2}$. Since this is a zero-sum game, the analogon of the matrix U for player 2 is $-U^T$.

a) We want to find a mixed strategy for player 1 such that the expected gain is the same for all pure strategies of player 2. Let's denote this mixed strategy by a S_1 -valued random variable X_1 . That is, we want to show that there exists a constant $c_1 \in \mathbb{R}$ and numbers $\mathbb{P}[X_1 = x]$ for $x \in S_1$ such that for all $y \in S_2$ it holds:

$$c_1 = \mathbb{E}[u_1(X_1, y)] = \sum_{x \in S_1} \mathbb{P}[X_1 = x] u_1(x, y).$$

Keeping in mind that it must hold that $\sum_x \mathbb{P}[X_1 = x] = 1$, we obtain the following linear equation:

$$\begin{bmatrix} u_1(T, L) & u_1(M, L) & u_1(B, L) & -1 \\ u_1(T, C) & u_1(M, C) & u_1(B, C) & -1 \\ u_1(T, R) & u_1(M, R) & u_1(B, R) & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{P}[X_1 = T] \\ \mathbb{P}[X_1 = M] \\ \mathbb{P}[X_1 = B] \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

which we write more compactly as

$$\begin{bmatrix} U^T & -1_{3 \times 1} \\ 1_{1 \times 3} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0_{3 \times 1} \\ 1 \end{bmatrix}$$

with the obvious meaning of p_1 . This can be solved with something like Octave/Matlab as follows:

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U = [3, -3, 0; 2, 6, 4; 2, 5, 6];
o = [1; 1; 1];
first = [U', -o; o', 0];
second = [-U, -o; o', 0];
rhs = [0; 0; 0; 1];
% The result for part a)
disp(first \ rhs);
% The result for part b)
disp(second \ rhs);
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The result is:

$$p_1^T = \begin{bmatrix} 0.4 & 0.6 & 0 \end{bmatrix} \quad c_1 = 2.4.$$

b) A completely symmetric derivation for player 2 leads to the equation

$$\begin{bmatrix} -U & -1_{3 \times 1} \\ 1_{1 \times 3} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0_{3 \times 1} \\ 1 \end{bmatrix}$$

with the solution

$$p_2^T = \begin{bmatrix} 0.88 & 0.08 & 0.04 \end{bmatrix} \quad c_2 = -2.4.$$

c) We want to show that the equalizing strategies X_1 and X_2 (with the properties and probabilities as above) are optimal for both players.

First, notice the following property (resulting from what has been called “multilinearity” in the lecture). If Y_2 is some mixed strategy of the second player, then it holds:

$$\mathbb{E}[u_1(X_1, Y_2)] = \sum_{y \in S_2} \mathbb{P}[Y_2 = y] \mathbb{E}[u_1(X_1, y)] = \sum_{y \in S_2} \mathbb{P}[Y_2 = y] c_1 = c_1.$$

In particular, we obtain just the constant c_1 if we minimize over all possible mixed strategies of player 2:

$$\min_{Y_2} \mathbb{E}[u_1(X_1, Y_2)] = c_1.$$

A symmetric statement holds for c_2 . This allows us to make estimations for Maxmin-values for both players. Here are two inequalities for Maxmin-values of player 1:

$$\begin{aligned} \max_{Y_1} \min_{Y_2} \mathbb{E}[u_1(Y_1, Y_2)] &\geq \min_{Y_2} \mathbb{E}[u_1(X_1, Y_2)] = \min_{Y_2} c_1 = c_1 \\ \max_{Y_1} \min_{Y_2} \mathbb{E}[u_1(Y_1, Y_2)] &= \max_{Y_1} \min_{Y_2} \mathbb{E}[-u_2(Y_1, Y_2)] \\ &\leq \max_{Y_1} \mathbb{E}[-u_2(Y_1, X_2)] \\ &= \max_{Y_1} (-c_2) \\ &= -c_2 \end{aligned}$$

In particular, if we have the situation that $c_2 = -c_1$, these two inequalities combine into a single equality:

$$\max_{Y_1} \min_{Y_2} \mathbb{E}[u_1(Y_1, Y_2)] = c_1.$$

Exchanging the indices 1 and 2 yields a completely symmetric result for player 2.

In parts a) and b) we have found out that for the concrete example we have $c_1 = -c_2 = 2.4$, so we can conclude that the Maxmin-value for player 1 is 2.4 and the Maxmin-value for player 2 is -2.4 , which by definition makes the equalizing strategies X_1 and X_2 *optimal*.

d) Now we want to consider a two-player zero-sum game where the players 1 and 2 have more possible actions. We want to show that if both players have equalizing strategies, these strategies are optimal.

Suppose $S_1 = \{1 \dots n\}$ and $S_2 = \{1 \dots m\}$ for some natural numbers n, m . Let U be a $n \times m$ matrix defined as above. Suppose there are equalizing strategies X_1 and X_2 , that is, there exist probability vectors $p_1 \in \mathbb{R}^n$ and $p_2 \in \mathbb{R}^m$ such that the following holds:

$$\begin{bmatrix} U^T & -1_{m \times 1} \\ 1_{1 \times n} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0_{m \times 1} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -U & -1_{n \times 1} \\ 1_{1 \times m} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0_{n \times 1} \\ 1 \end{bmatrix}.$$

In c) we have shown that it is sufficient to establish $c_2 = -c_1$ in order to prove the optimality (the argument did not depend on the dimension of the matrix U in any way). It holds:

$$\begin{aligned} c_2 &= \begin{bmatrix} p_2^T & c_2 \end{bmatrix} \begin{bmatrix} 0_{m \times 1} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} p_2^T & c_2 \end{bmatrix} \begin{bmatrix} U^T & -1_{m \times 1} \\ 1_{1 \times n} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} \\ &= - \left(\begin{bmatrix} -U & -1_{n \times 1} \\ 1_{1 \times m} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ c_2 \end{bmatrix} \right)^T \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} \\ &= - \begin{bmatrix} 0_{1 \times n} & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ c_1 \end{bmatrix} \\ &= -c_1 \end{aligned}$$

Hence, the equalizing strategies are optimal. ■

e) Finally, we want to consider a counterexample. Consider the following two-player zero-sum game:

$$\begin{bmatrix} & L & R \\ T & (1, -1) & (1, -1) \\ B & (-1, 1) & (-1, 1) \end{bmatrix}$$

Here, the first player has an equalizing strategy with $p_1 = [0.5 \ 0.5]^T$. With this strategy, the first player would on average win nothing, no matter what the strategy of the second player is.

However, the optimal strategy would obviously be the pure strategy T . Since the decision of the second player can not affect the outcome in any way, he can take either L or R as optimal strategy. The value of the game is therefore $(1, -1)$, and not 0, if both play rationally, the first player wins $(+1)$ and the second loses (-1) .

However, this is not a contradiction to d), because here the second player does not have an equalizing strategy: no matter how he weights L and R , the outcome in the row T is always (-1) , the outcome in the row B is always $(+1)$, and $(-1) \neq (+1)$.

Exercise 4.2 (Convexity)

a) Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be two convex sets. We want to show that $A \times B$ is again convex.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ be two arbitrary points and $t \in [0, 1]$. Since A is convex, $a_1(1-t) + a_2t \in A$. Since B is convex, $b_1(1-t) + b_2t \in B$. Therefore it holds:

$$(a_1, b_1)(1-t) + (a_2, b_2)t = (a_1(1-t) + a_2t, b_1(1-t) + b_2t) \in A \times B.$$

■

b) Let S be a finite set. Let

$$\Delta(S) := \left\{ \sigma : S \rightarrow [0, 1] \mid \sum_{s \in S} \sigma(s) = 1 \right\}.$$

We want to show that $\Delta(S)$ is convex.

Let $\sigma, \mu \in \Delta(S)$ and $t \in [0, 1]$ arbitrary. For all $s \in S$ it holds:

$$[(1-t)\sigma + t\mu](s) \leq (1-t) \cdot 1 + t \cdot 1 = 1,$$

and it is even more obvious that the value is nonnegative, so the range of the function $[(1-t)\sigma + t\mu]$ is contained in $[0, 1]$.

Furthermore, it holds:

$$\begin{aligned} \sum_{s \in S} [(1-t)\sigma + t\mu](s) &= \sum_{s \in S} (1-t)\sigma(s) + t\mu(s) \\ &= (1-t) \sum_{s \in S} \sigma(s) + t \sum_{s \in S} \mu(s) \\ &= (1-t) \cdot 1 + t \cdot 1 \\ &= 1, \end{aligned}$$

therefore $[(1-t)\sigma + t\mu]$ is again in $\Delta(S)$. ■

Notice that we have never actually used the fact that S is finite. Moreover, the statement holds for measures on arbitrary sigma-algebras, but one has to work a little bit to make some sense of the “summation” (e.g. by replacing it with the Lebesgue-integral).

Exercise 4.3 (Symmetric game) Consider the following game of players 1 and 2:

$$\begin{bmatrix} & L & C & R \\ T & 0, 0 & 7, 6 & 6, 7 \\ M & 6, 7 & 0, 0 & 7, 6 \\ B & 7, 6 & 6, 7 & 0, 0 \end{bmatrix}.$$

a) We want to find an equilibrium of mixed strategies and show that it is unique. Since the situation is completely symmetric, we focus on the first player.

Suppose the first player chooses a mixed strategy X with a probability vector $p = (p_T, p_M, p_B)$. Suppose that the second player chooses the same probability vector for his strategy Y (in the same order for L, C, R). This gives us an upper bound for the Min-value for the first player playing with strategy p :

$$\mathbb{E}[u_1(X, Y)] = p^T \begin{bmatrix} 0 & 7 & 6 \\ 6 & 0 & 7 \\ 7 & 6 & 0 \end{bmatrix} p = 13(p_1p_2 + p_2p_3 + p_1p_3).$$

This expression obviously¹ has $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as it's only maximum. The value of the game is in this case $(6 + 7)/3 = \frac{13}{3}$.

b) If the both players could cooperate, the first player could deviate to the pure strategy T , and the second player could respond with C , which would lead to the result $(7, 6)$, which is more than $(4.\bar{3}, 4.\bar{3})$. However, if the second player fails to respond with C , and chooses L instead, then the first player wins nothing at all, so player 1 might prefer to keep his randomized strategy in order to mitigate the risks. This is not an example of “non-constructive egoism / exploitation”, but rather an example where the second player could simply *fail* to respond with the answer that would be best for both, so it's rather a “BoS”-type example.

¹Ok, it might be not immediately obvious, to prove it rigorously one would have to optimize this function on a triangle, with gradients and Lagrange-multipliers and stuff like that... It's a bit tedious and is rather calculus than game-theory. It's very similar to the proof of inequalities of geometric and arithmetic means, but can also be seen by induction and some shuffling of the parentheses