AM 201 APPLIED MATHEMATICS - II

Sol 1
(i) Given : $u = \frac{1}{2}\log(x^2+y^2)$

Differentiating partially w.r.to x and y,

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{2x}{x^2+y^2}\right] = \frac{x}{x^2+y^2} \;\&\; \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1)-2(x)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{---} \; (1)$$

and, $$\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{2y}{x^2+y^2}\right] = \frac{y}{x^2+y^2} \;\&\; \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{---} \; (2)$$

Adding (1) and (2)

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\therefore$ u is harmonic function.

Harmonic Conjugate →

$$dv = \left[\frac{\partial v}{\partial x}\right]dx + \left[\frac{\partial v}{\partial y}\right]dy$$

$$dv = \left[-\frac{\partial u}{\partial y}\right]dx + \left[\frac{\partial u}{\partial x}\right]dy$$

Integrating we get,

$$v = \int_{y=\text{const.}} \left[-\frac{\partial u}{\partial y}\right] dx + \int_{\text{independent of } x} \left[\frac{\partial u}{\partial x}\right] dy + c$$

$$\Rightarrow v = -\int_{y=\text{const}} \left[\frac{y}{x^2+y^2}\right] dx + (\text{No term free from } x) + c$$

$$v = -\tan^{-1}\left[\frac{x}{y}\right] + c$$

Thus $\boxed{v = -\tan^{-1}\left[\frac{x}{y}\right] + c}$

↳ This is harmonic conjugate of u.

Sol 1

(ii) By definition, A linear transformation is a function $T: R^n \rightarrow R^m$ which satisfies

1) $T(x+y) = T(x) + T(y) \quad \forall\ x, y \in R^n$
2) $T(cx) = cT(x) \quad \forall\ x \in R^n$ & $c \in R$

Here we have,

$$T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2, \quad T: R^3 \rightarrow R^1$$

By definition to check if T is a linear transformation we check for $T(cx)$

$$\therefore T(cx_1, cx_2, cx_3) = c^2x_1^2 + c^2x_2^2 + c^2x_3^2$$

$$\Rightarrow T(cx_1, cx_2, cx_3) = c^2(x_1^2 + x_2^2 + x_3^2)$$

$$\Rightarrow T(cx) = c^2T(x) \Rightarrow \boxed{T(cx) \neq cT(x)}$$

we say, here, T is not linear transformation

Sol 2

(i) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \cdots$

$$a_n = \frac{x^n}{n.(n+1)}$$

if there excites an N so that $n \geq N$

if ~~L~~ $L<1$ then the series converges

if $L>1$ then the series diverges

if $L=1$ then the Ratio test is inconclusive

where, $L = \lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right|$

Applying Ratio test

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^n+1}{(n+1)(n+2)}}{\frac{x^n}{n(n+1)}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{nx}{(n+2)} \right|$$

By applying limit

$$L = \lim_{n\to\infty} \left| \frac{x}{(1+\frac{2}{n})} \right|$$

$\therefore L = |x|$

for convergence

$L<1$, $|x|<1$

Hence, series converges for $\Rightarrow -1 \leq x \leq 1$

i.e series converges for $x \leq 1$ and diverges for $x>1$

Sol2

(ii) $f(z) = u(x,y) + iv(x,y)$

when

given $\Rightarrow v(u,y) = e^{-x}(x\sin y - y\cos y)$

diffrentiating with respect to x and y respectivly

$V_x = e^{-x}(\sin y - x\sin y + y\cos y)$

$V_y = e^{-x}(x\cos y + y\sin y - \cos y)$

Again diffrentiating with respect to x and y respectivly

$V_{xx} = e^{-x}(-2\sin y + u\sin y - y\cos y)$

$V_{yy} = e^{-x}(2\sin y - x\sin y + y\cos y)$

Since $V_{xx} + V_{yy} = 0$

the function is harmonic

as $u(x,y)$ is conjugate of $v(x,y)$

Then u and v satisfy equations

$u_x = V_y$ and $u_y = -V_x$

$V_x = -u_y = e^{-x}(\sin y - u\sin y + y\cos y)$ —(1)

integrating (1) with respect to y keeping x constant

$u(x,y) = -e^{-x}(x\cos y + y\sin y) + \phi(x)$ —(2)

where $\phi(x)$ is a function of x only

Differentiating (2) with respect to x and using relation

$~~v_x~~$ $U_x = V_y$

$$-e^{-x}(\cos y - x\cos y - y\sin y) + \cancel{\phi(x)}\ \phi'(x)$$

$$= e^{-x}(x\cos y + y\sin y - \cos y)$$

$$\Rightarrow \phi'(x) = 0$$

i.e

$$\phi u = C_1$$

where C_1 is constant

$$\Rightarrow u(x,y) = -e^{-x}(x(\cos y) + y(\sin y)) + C_1$$

Sol3

(i)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 5 & 9 & -1 \end{bmatrix}$$

Let A be a characterstic root of a square matrix

If $|A - \lambda I| = 0$

then

$$\left| \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 5 & 9 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 2 & -1-\lambda & 0 \\ 5 & 9 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 - 2\lambda^2 + \lambda + 2 = 0$$

$\boxed{\text{roots are } \lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 2}$

Hence these are eigen values

For eigen vectors

$$\lambda = 1 \quad \begin{bmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \\ 5 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-x_1 + x_2 = 0$$
$$5x_1 + 9x_2 - 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 9 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -1 & 0 \\ 5 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ 5 & 9 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{-x_2}{2} = \frac{x_3}{-14}$$

Null space of matrix $= \begin{bmatrix} -2 \\ -2 \\ -14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$

For $\lambda = -1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$y_1 + y_2 = 0$; $2y_1 = 0$; $y_1 = 0$; $y_2 = 0$; $y_3 = 1$

Null space of matrix $= \cancel{\{0\,0\,1\}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = -2$ $\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 5 & 9 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0$

$$2z_1 + z_2 + 0z_3 = 0$$
$$5z_1 + 9z_2 + z_3 = 0$$

$$\frac{z_1}{\begin{vmatrix} 1 & 0 \\ 9 & 1 \end{vmatrix}} = \frac{-z_2}{\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 2 & 1 \\ 5 & 9 \end{vmatrix}}$$

$$\frac{z_1}{1} = \frac{z_2}{-2} = \frac{z_3}{13}$$

null space of matrix is $\begin{bmatrix} 1 \\ -2 \\ 13 \end{bmatrix}$

So eigen vectors are

$$\begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \\ 13 \end{bmatrix}$$

Sol3

(ii) a) P(Red card) ⇒

Total Cards = 52

Red cards = 26

$$P(\text{Red Card}) = \frac{26}{\text{Total Cards}} = \frac{26}{52} = \boxed{\frac{1}{2}}$$

b) P(a Club) ⇒ $\frac{\text{No of Clubs}}{\text{Total No of Cards}}$

$$\Rightarrow \frac{13}{52} = \boxed{\frac{1}{4}}$$

c) P(one of Count cards (jack, queen, king))

$$\Rightarrow \frac{\text{No of Count cards [jack, queen, king]}}{\text{Total No of Cards}}$$

$$\Rightarrow \frac{12}{52} \Rightarrow \boxed{\frac{3}{13}}$$

Sol 4

i) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow |A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$$

According to Cayley - Hamilton theorem

$$\lambda = A$$

$$-A^3 + 3A^2 - A + 3 = 0$$

verify : $-A^3 + 3A^2 - A + 3I = 0$

$$-\begin{bmatrix} -1 & 10 & 12 \\ +1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} + 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For A^{-1}

$$-A^3 A^{-1} + 3A^2 A^{-1} - AA^{-1} + 3IA^{-1} = 0$$

$$-A^2 I + 3AI - I + 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 3A + I$$

$$A^{-1} = \frac{1}{3}\left[A^2 - 3A + I\right]$$

$$A^{-1} = \frac{1}{3}\left[\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -1 & -2/3 & 4/3 \\ 1 & 1/3 & -2/3 \\ -1 & 0 & 1 \end{bmatrix}}$$

Sol 4

ii) a) $|z| < 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{(1+z)} - \frac{1}{3+z}\right]$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{2}(3+z)^{-1}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left[1 + \frac{z}{3}\right]^{-1}$$

$$= \frac{1}{2}\left(1 - z + z^2 - z^3 + \dots\right) - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right]$$

$$= \left[\frac{1}{2} - \frac{1}{6}\right] + \left[\frac{-z}{2} + \frac{z}{18}\right] + \left[\frac{z^2}{2} - \frac{z^2}{54}\right] + \left[\frac{-z^3}{2} + \frac{z^3}{162}\right] + \dots$$

$$\boxed{= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \dots}$$

Hence the required expansion.

b) $0 < |z+1| < 2$

$|z+1| > 0$ and $|z+1| < 2$

$$\frac{|z+1|}{2} < 1$$

$$f(z) = \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{z+1+2}\right] \cancel{= \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{2+z+1}\right]}$$

$$= \frac{1}{2}\left[\frac{1}{1+z} - (2+z+1)^{-1}\right]$$

Taylor's and Laurent's series

$$= \frac{1}{2(1+z)} - \frac{1}{2\cdot 2}\left[1 + \frac{z+1}{2}\right]^{-1}$$

$$= \frac{1}{2(1+z)} - \frac{1}{4}\left[1 - \frac{(z+1)}{2} + \frac{(z+1)^2}{4} - \cdots\right]$$

$$\boxed{= \frac{1}{2(1+z)} - \frac{1}{4} + \frac{(z+1)}{8} - \frac{(z+1)^2}{16} + \cdots}$$

Hence, the required expansion.

Sol 5

i) Total fuses $= 20$; defective $= 5 = P(D)$
Non defective $= 15 = P(N)$

a) P(at most one defective)

~~Required ⇒ P(D, N, N)~~

$P(N, N, N) + P(N, N, D)$

$$\Rightarrow \frac{15}{20} \times \frac{14}{19} \times \frac{13}{18} + \frac{15}{20} \times \frac{14}{19} \times \frac{5}{18}$$

$$\Rightarrow \frac{2730}{6840} + \frac{1050}{6840}$$

$$\Rightarrow \frac{273}{684} + \frac{105}{684} = \frac{378}{684} \Rightarrow \frac{189}{342}$$

$$= \frac{63}{114} = \boxed{\frac{21}{38}}$$

b) P(exactly 3 defective)

$P(D, D, D)$

$$\Rightarrow \frac{5}{20} \times \frac{4}{19} \times \frac{3}{18} = \boxed{\frac{1}{114}}$$

c) P(No Defective)

$$P(N, N, N) \Rightarrow \frac{15 \times 14 \times 13}{20 \times 19 \times 18} = \boxed{\frac{91}{228}}$$

Ans 5

ii) Test the conv. of series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{3}\right)^3 x^3 + \cdots$$

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\lim_{x \to \infty} (u_n)^{1/n} = \lim_{x \to \infty} \left[\frac{(n+1)x}{(n+3)}\right]$$

$$= \lim_{x \to \infty} \left[\frac{(1+1/n)x}{(1+2/n)}\right] = x$$

$\therefore$ By cauchi's root test series

conv. if $x < 1$ and div if $x > 1$

Test fails for $x = 1$

$\Rightarrow$ $n = 1$, $u_n = \left(\frac{n+1}{n+2}\right)^n$

$$\therefore \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{(1+1/n)^n}{\left[(1+2/n)^{n/2}\right]^2}$$

$$= \frac{e}{e^2} = \frac{1}{e} \neq 0$$

given series is div.

$\therefore$ " " " conv. if $x < 1$ and

div if $x \geq 1$