Parallel Singular Value Decomposition

Introduction

Singular Value Decomposition

The singular value decomposition of a matrix A is the factorization of A into the product of three matrices $A = UDV^T$ where the columns of U and V are orthonormal and the matrix D is diagonal with positive real entries. The SVD is useful in many tasks. Here we mention one example. In many applications, the data matrix A is close to a matrix of low rank and it is useful to find a low rank matrix which is a good approximation to the data matrix .

Jacobi method

Jacobi algorithm was initially proposed by Jacobi C.G.J and is used to compute eigenvalues and eigenvectors. Based on Jacobi's algorithm, a fast and stable method was proposed by Golub-Kahan to decompose a matrix, rectangular or square, in its sub-matrices, as shown in

$$\left[U, S, V^T\right] = \text{Jacobi\_SVD}\ (M, \varepsilon)$$

where *M* shows sensitivity encoding matrix, *ε* is the tolerance level for SVD algorithm to converge. *U* and *V* are complex unitary matrices which contain left-singular and right-singular vectors of matrix *M,* respectively, whereas *S* is a diagonal matrix containing the positive singular values of *M,* as represented in

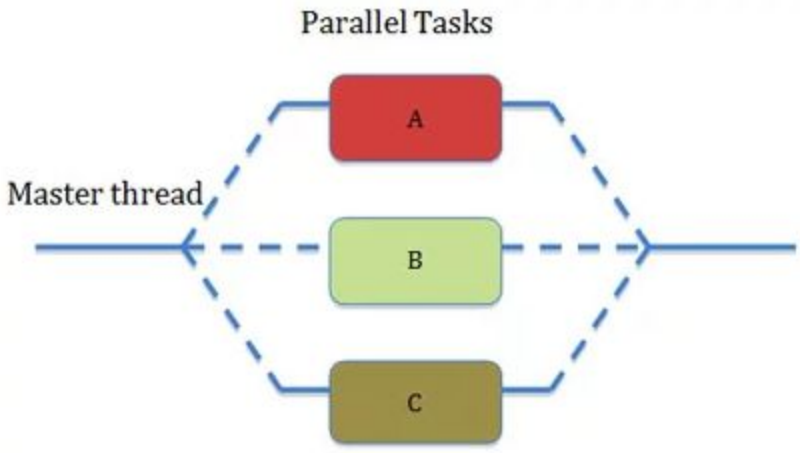
$$S = \begin{bmatrix} \sigma_x & 0 & \cdots & 0 \\ 0 & \sigma_{x-1} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_1 \end{bmatrix}$$

The diagonal values of *S* can be sorted in a descending order as:

$$\sigma_x > \sigma_{x-1} > \sigma_{x-2} > \cdots > \sigma_1$$

OpenMP

It is a directive based parallel programming model. OpenMP program is essentially a sequential program augmented with compiler directives to specify parallelism.



Jacobi SVD Algorithm

The pseudo-code of Jacobi SVD algorithm is given below. The matrix *U* in the algorithm is initialized by an input matrix *M,* while an initial guess of the desired eigenvectors (*V*) is taken as identity matrix.

repeat
  for all i<j
    $\alpha = \sum_{k=1}^{n} U_{ki}^2$
    $\beta = \sum_{k=1}^{n} U_{kj}^2$
    $\gamma = \sum_{k=1}^{n} U_{ki}\, U_{kj}$
    $\zeta = \frac{(\beta - \alpha)}{2\gamma}$ (Compute Jacobi rotation)
    $t = signum \frac{(\zeta)}{(|\zeta| + \sqrt{1+\zeta^2})}$
    $c = \frac{1}{\sqrt{1+t^2}}$
    $s = ct$
    for all k<n (update matrix U)
      $t = U_{ki}$
      $U_{ki} = ct - sU_{kj}$
      $U_{kj} = st + cU_{kj}$
    endfor
    for all k < n (update matrix *V*)
      $t = Vki$
      $Vki = ct - s\, Vkj$
      $Vkj = st + c\, Vkj$
    endfor
  endfor
until all $\frac{|c|}{\sqrt{\alpha\beta}} \leq \epsilon$

This algorithm repeatedly selects pairs of $i < j$ from rows and columns of matrix MM^T. *N*-dimensional linear product space R_{kl} is given below. Jacobi rotation is applied on the 2D linear subspace of R_{kl}. The Jacobi rotation process is repeated until the

convergence criterion ε is achieved. Orthogonal updates are applied during each iteration in Jacobi algorithm to make the matrix *M* more diagonal than it was in the previous iteration. As a result of the diagonal update, off-diagonal elements are small enough and can be replaced by zero where R_{kl} is an identity matrix except for *c* placed in the diagonal and *s* which are symmetrically placed off the diagonal. The entries of *c* and *s* can be computed by the algorithm. The singular values (*S*) are implicitly generated at convergence, and the right (*V*) and left (*U*) singular vectors are recovered by multiplying all the Jacobi rotations together.

The pseudo-inverse of matrix *M* with Jacobi SVD can be calculated using Eq. (10). The non-zero diagonal elements of *S* are inverted and multiplied with *V* and U^T to get the inverse of the rectangular matrix, *M*:

$$M^{-1} = V \times S^{-1} \times U^T$$

Conclusion

Jacobi SVD algorithm is numerically stable and provides a good solution for large matrices to compute SVD . Jacobi SVD algorithm is computationally fast algorithm and is well-suited for implementation on parallel processors.

Team Members

Pranjali Ingole **(2018201029)**

Kratika Kothari **(2018201060)**

Shafiya Naaz **(2018201062)**