Notes **PHYS225 - Intermediate E&M**

Lecture 9

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1 Overview

1.1 Last Time

- Capacitance $Q = CV$
- Laplace's equation $\nabla^2 V = 0$
- Method of images

1.2 Today

- Review conclusions of the method of images solution to Laplace and induced charge
- Separation of variables
- Wave guides (without the waves)

2 Laplace's equation

Recall: we can use any method to solve Laplace's equation as long as the solution satisfies the boundary conditions.

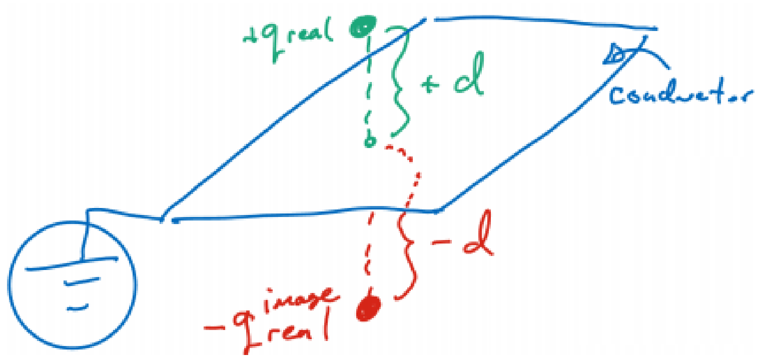


Figure 1: Boundary conditions are $V(z=0) = 0, V(z=\infty) = 0$

We wrote down the solution as

$$\boxed{V = \left(\frac{1}{4\pi\epsilon_0}\right)\left[\frac{+q_{\text{real}}}{(x^2+y^2+(z-d)^2)^{1/2}} + \frac{-q^{\text{image}}_{\text{real}}}{(x^2+y^2+(z+d)^2)^{1/2}}\right]}$$

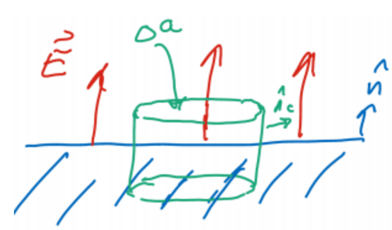
$\longrightarrow$ But what about the surface itself?

Remember that for a conductor

$$\boxed{\vec{E}^{\text{surf}} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

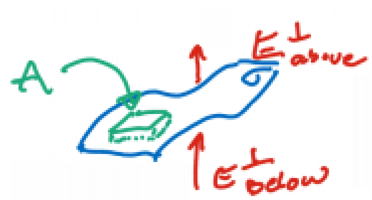
and therefore:

$$-\partial_n V = \frac{\sigma}{\epsilon_0} \longrightarrow \boxed{\sigma = -\epsilon_0 \partial_n V}$$



More generally:

$$\oint_S \vec{E}\cdot \,\mathrm{d}\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \Rightarrow \boxed{E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}}$$



For our image charge problem, this implies evaluating the partial derivative in the $\hat{z}$ direction

$$\begin{aligned}\sigma &= -\epsilon_0 \partial_z V|_{z=0} \\ &= -\frac{1}{4\pi}\left[\frac{-q(z-d)}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{-q(z+d)}{(x^2+y^2+(z+d)^2)^{1/2}}\right]\end{aligned}$$

So we end up with total surface charge density

$$\boxed{\sigma = -\frac{qd}{2\pi}\left(x^2+y^2+d^2\right)^{-3/2}}$$

2.1 Boundary Conditions

Taking the above general expression for the surface charge

$$\boxed{\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

$$(-\partial_n V_{\text{above}}) - (-\partial_n V_{\text{below}}) = \frac{\sigma}{\epsilon_0}\hat{n}$$

$$\boxed{\partial_n V_{\text{above}} - \partial_n V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{n}}$$

with the discontinuity boundary conditions for the surface charge above.

Now, using Stoke's law

$$\oint_C \vec{E}\cdot \,\mathrm{d}\vec{\ell} = \int_S \vec{\nabla}\times\vec{E}\cdot\,\mathrm{d}\vec{a} = 0$$

where $\mathrm{d}\vec{a}$ can either be pointing into or out of the page. We can write down more boundary conditions

$$\boxed{E^{\parallel}_{\text{above}} - E^{\parallel}_{\text{below}} = 0} \quad \text{and} \quad \boxed{V_{\text{above}} - V_{\text{below}} = 0}$$

3 Separation of Variables

Consider what form the solution to

$$\nabla^2 V = 0$$

might take in a Cartesian system where we know how V, σ are constrained on the boundaries (and think back to the expressions that we just derived above)

$$\nabla^2 V = \partial_x^2 V + \partial_y^2 V + \partial_z^2 V = 0$$

Assume for a second that

$$\boxed{V = V(x, y, z) = f(x)g(y)h(z)}$$

But: why would we do this? Because the principle of superposition, coupled with the fact that we will expression the solution using a complete set of orthogonal functions that will allow us to extend the solution to complex cases. Common sets of functions

- Sines and cosines
- Bessel functions
- Hermite polynomials
- Chebyshev polynomials
- Legendre polynomials
- Spherical harmonics

So let's go back to the issue at hand.

$$\nabla^2 V = \partial_x^2 f(gh) + \partial_y^2 g(fh) + \partial_z^2 h(fg) = 0$$

Divide through by fgh:

$$\Rightarrow \frac{1}{f}\partial_x^2 f + \frac{1}{g}\partial_y^2 g + \frac{1}{h}\partial_z^2 h = 0$$

where each of those terms must be constants since they are all canceling each other out. Expanding this out, we find

$$\left.\begin{aligned} \tfrac{1}{f}\partial_x^2 f &= \alpha^2 \\ \tfrac{1}{g}\partial_y^2 g &= \beta^2 \\ \tfrac{1}{h}\partial_z^2 h &= \gamma^2 \end{aligned}\right\} \alpha^2 + \beta^2 = -\gamma^2$$

and this should immediately remind you of an exponential

$$\partial_x(e^{\alpha x}) = \alpha e^{\alpha x} \rightarrow \partial_x^2(e^{\alpha x}) = \alpha^2 e^{\alpha x}$$

As a result, we have

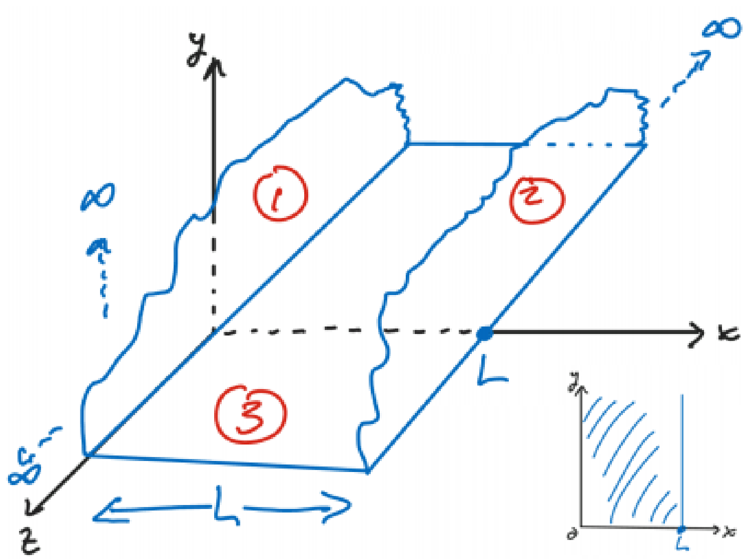
$$\boxed{\begin{aligned} f(x) &= a_1 e^{\alpha x} + a_2 e^{-\alpha x} \\ g(y) &= b_1 e^{\beta y} + b_2 e^{-\beta y} \\ h(z) &= c_1 e^{\gamma z} + c_2 e^{-\gamma z} \end{aligned}}$$

which means our potential is written as the complete set of orthogonal functions until we apply boundary conditions to simplify it

$$\boxed{V(x,y,z) = \sum_{\alpha,\beta,\gamma} \left[a_1 e^{\alpha x} + a_2 e^{-\alpha x}\right] \left[b_1 e^{\beta y} - b_2 e^{-\beta y}\right] \left[c_1 e^{\gamma z} + c_2 e^{-\gamma z}\right]}$$

3.1 Conducting slot

Let's take an explicit example: the conducting slot. This is a rectangular conducting "box" with no lid.



First, let's write down our boundary conditions!

1. $V(0, y, z) = 0 (x = 0)$
2. $V(L, y, z) = 0 (x = L)$
3. $V(x, \infty, z) = 0 (y = \infty)$
4. $V(x, 0, z) = f(x) (y = 0)$

Note: since infinite in z, this is really a 2D problem! $\gamma = 0$

So if we start writing out the general solution here in 2D, we have

$$V(x,y) = \sum_{\alpha^2+\beta^2=0} \left[A_1 e^{\alpha x} + A_2 e^{-\alpha x}\right] \left[b_1 e^{\beta y} + b_2 e^{-\beta y}\right]$$

where I have implicitly set

$$A_1 = a_1 \left[c_1(0) + c_2(0)\right]$$
$$A_2 = a_2 \left[c_1(0) + c_2(0)\right]$$

Next, we apply the condition that

$$\alpha^2 + \beta^2 = 0 \Rightarrow \alpha^2 = -\beta^2$$
$$\boxed{\alpha = i\beta}$$

This means our potential is now

$$V(x,y) = \sum_{\beta} \left[A_1 e^{i\beta x} + A_2 e^{-i\beta x}\right] \left[b_1 e^{\beta y} + b_2 e^{-\beta y}\right]$$

Take another look at the boundary conditions. Note that the third boundary condition states that we want $V \to 0$ as $y \to \infty$ which $\Rightarrow b_1 = 0$. So rewrite potential again taking this into account

$$V(x,y) = \sum_{\beta>0} b_2 \left[A_1 e^{i\beta x} + A_2 e^{-i\beta x}\right] e^{-\beta y}$$

And since $b_2 A_1$ and $b_2 A_2$ are just constants

$$V(x,y) = \sum_{\beta>0} \left[A_\beta e^{i\beta x} + B_\beta e^{-i\beta x}\right] e^{-\beta y}$$

Now, we can apply the boundary conditions for one side panel (first boundary condition) $V(0,y) = 0$

$$V(0,y) = \sum_{\beta>0} \underbrace{\left[A_\beta + B_\beta\right]}_{A_\beta + B_\beta = 0} e^{-\beta y} = 0$$

Then again, we rewrite our potential

$$V(x,y) = \sum_{\beta>0} A_\beta \left[e^{i\beta x} - e^{-i\beta x}\right] e^{-\beta y} = \sum_{\beta>0} A_\beta \left[2i \sin(\beta x)\right] e^{-\beta y}$$

Now, we apply the boundary condition for the other side

$$V(L,y) = \sum_{\beta>0} 2i A_\beta \sin(\beta L) e^{-\beta y} = 0$$

Noticing that the condition on sin implies

$$\beta L = n\pi \longrightarrow \beta = \frac{n\pi}{L}$$

so we obtain

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n\pi y}{L}\right)}$$