CALCULATIONS

Launch Pressure:

- 70 psi

Mass Flow Rate:

$m_{fr} = A \times cd \times \sqrt{(2 \times \rho \times \Delta P)}$

- A is the area of the nozzle; $A = \pi r^2$; $r = 0.0125$ m
- cd is the coefficient of discharge; cd = .98
- ρ is the density of water; $\rho = 998\ \mathrm{kg}/\mathrm{m}^3$
- ΔP is the change in pressure; $\Delta P = P_b - P_a$, where P_b is the launch pressure in the bottle and P_a is the atmospheric pressure.
- $1\ \text{psi} = 6.8948 \times 10^3\ \mathrm{N}/\mathrm{m}^2$ ✓

$$\Delta P = P_b - P_a = 70 - 14.7\ \text{psi} = 55.3\ \text{psi}$$

$$A = \pi r^2 = \pi (0.0125)^2 = 4.909 \times 10^{-4}\ \mathrm{m}^2$$

$$\Delta P = 55.3\ \text{psi} \times \frac{6.8948 \times 10^3\ \mathrm{N}/\mathrm{m}^2}{1\ \text{psi}}$$

3.8128

$$\Delta P = 3.7879 \times 10^5\ \mathrm{N}/\mathrm{m}^2$$

$$m_{fr} = A \times cd \times \sqrt{(2 \times \rho \times \Delta P)}$$

$$= (4.909 \times 10^{-4}\ \mathrm{m}^2)(.98)\sqrt{[(2)(998\ \frac{\mathrm{kg}}{\mathrm{m}^3})(3.7879 \times 10^5\ \frac{\mathrm{N}}{\mathrm{m}^2})]}$$

3.8128

$$m_{fr} = 13.23\ \mathrm{kg}/\mathrm{s}$$

13.21

Exit Velocity:

$$V_{exit} = \frac{m_{fr}}{(\rho \times A)}$$

- m_{fr} is the mass flow rate initially calculated

$$V_{exit} = \frac{m_{fr}}{(\rho \times A)}$$

13.27

$$= \frac{13.23 \frac{kg}{s}}{(998 \frac{kg}{m^3})(4.909 \times 10^{-4}\ m^2)}$$

$$V_{exit} = 27.0\ m/s$$

27.09

Thrust of Rocket:

$F_T = m_{fr} \times V_{exit}$

- m_{fr} is the mass flow rate initially calculated
- V_{exit} is the exit velocity of the rocket

$$F_T = m_{fr} \times V_{exit}$$

$$= (13.23 \frac{kg}{s})(27.0 \frac{m}{s})$$

13.27 27.1

$$F_T = 357.21 \frac{kg \cdot m}{s^2} = 357.21\ N$$

$$F_T = 357.21\ N$$

359.62

Mass of Water:

$m_{H_2O} = \rho \times V_{H_2O}$

- V_{H_2O} is the volume of water in the soda can; $V_{H_2O} = \pi r^2 h$ where r is the radius of the soda can and h is the height of the can; r = 0.0325 m and h=0.12 m
- ρ is the density of water; $\rho = 998\ kg/m^3$

$$V_{H_2O} = \pi r^2 h$$

$$= \pi(0.0325\ m)^2(0.12\ m)$$

$$V_{H_2O} = 3.982 \times 10^{-4}\ m^3$$

$$m_{H_2O} = \rho \times V_{H_2O}$$

$$= (998 \frac{kg}{m^3})(3.982 \times 10^{-4}\ m^3)$$

$$m_{H_2O} = 0.3974\ kg$$

FORCES

The forces on the rocket in the following calculations are broken into components; x-component and y-component. This has to be done in order to calculate the net force. The net force on the rocket is equal to the vector sum of the x and y forces. Since multiple forces act in each direction, some against one another, the forces must first be separated by components. After this, the net force on the rocket can be calculated.

Opposing Forces: Horizontal (x-direction)

$F_x = F_T\cos(\theta) - F_d$

- F_T is the thrust of the rocket; $F_T = 357.21$ N
- F_d is the drag force of the wind on the rocket; $F_d \approx 0$
- θ is the angle of elevation of rocket at launch; $\theta = 75°$

$$\begin{aligned} F_x &= F_T\cos(\theta) - F_d\cos(\theta) \\ &= (357.21\text{ N})\cos(75°) - 0 \\ F_x &= 92.45\text{ N} \end{aligned}$$

Opposing Forces: Vertical (y-direction)

$F_y = F_T\sin(\theta) - F_g$

- F_T is the thrust of the rocket; $F_T = 357.21$ N
- F_g is the force of gravity on the rocket; $F_g = \overline{m} \times g$
- $\overline{m}$ is the average mass of the empty rocket bottle and the mass of the water; $\overline{m} = 0.2835$ kg

ballast

- g is the acceleration of gravity on earth; $g = 9.81\ ^{m}/_{s^2}$
- θ is the angle of elevation of rocket at launch; $\theta = 75°$

$$\begin{aligned} F_y &= F_T\sin(\theta) - F_g \\ &= (357.21\text{ N})\sin(75°) - (0.2835\text{ kg})(9.81\ \frac{m}{s^2}) \\ F_y &= 342.26\text{ N} \end{aligned}$$

Net Force:

$F_{net} = \bar{F}_x + \bar{F}_y = \sqrt{(F_x^2 + F_y^2)}$

- F_x is the sum of the forces in the x-direction
- F_y is the sum of the forces in the y-direction

$$\begin{aligned} F_{net} &= \sqrt{(F_x^2 + F_y^2)} \\ &= \sqrt{[(92.45\text{ N})^2 + (342.26\text{ N})^2]} \\ F_{net} &= 354.53\text{ N} \end{aligned}$$

Net Acceleration of Rocket:

$F_{net} = \overline{m} \times a$

- $\overline{m}$ is the average mass of the empty rocket bottle and the mass of the water; $\overline{m}$=0.2835 kg
- F_{net} is the net force on the rocket initially calculated

$$F_{net} = \overline{m} \times a$$

$$a = \frac{F_{net}}{\overline{m}}$$

$$= \frac{354.53 \text{ N}}{0.2835 \text{ kg}} = \frac{354.53 \frac{\text{kg} \cdot \text{m}}{\text{s}^2}}{0.2835 \text{ kg}}$$

$$a = 1251 \text{ m}/\text{s}^2$$

Range:

$$R = \frac{V_{rocket}{}^2 \times \sin(2\theta)}{g}$$

- V_{rocket} is the velocity of the rocket; $V_{rocket} = a \times t_i$
- t_i is the time when all the water is expelled from the rocket; $t_i = \frac{m_{H_2O}}{m_{fr}}$
- θ is the angle of elevation of rocket at launch; $\theta = 75°$

$$t_i = \frac{m_{H_2O}}{m_{fr}}$$

$$= \frac{0.3974 \text{ kg}}{13.23 \frac{\text{kg}}{\text{s}}}$$

$$t_i = 0.03 \text{ s}$$

$$V_{rocket} = a \times t_i$$

$$= (1251 \frac{\text{m}}{\text{s}^2})(0.03 \text{ s})$$

$$V_{rocket} = 37.53 \text{ m}/\text{s}$$

$$R = \frac{V_{rocket}{}^2 \times \sin(2\theta)}{g}$$

$$= \frac{(37.53 \frac{\text{m}}{\text{s}})^2 \sin(2 \times 75°)}{9.81 \frac{\text{m}}{\text{s}^2}}$$

$$R = 71.79 \text{ m}$$

Final Time:

$$t_f = \frac{R}{V_{rocket_x}} + 2t_i$$

- R is the range of the rocket initially calculated

- V_{rocket_x} is the x-component of the velocity of the rocket; $V_{rocket_x} = V_{rocket}\cos(\theta)$. The range of the rocket is measured in the x-direction, so only the component of the velocity of the rocket that is in the x-direction is needed. Therefore, V_{rocket_x} is necessary.
- t_i is time it takes water to expel water from rocket; use $2t_i$ to add time for rocket to expel water and to account for the symmetry inherent in projectile motion. The time it takes for an object to go up is equal to the time it takes for the same object to come back down. Since t_i added for the rocket going up, it has to be added again for rocket coming back down.
- θ is the angle of elevation of rocket at launch; $\theta = 75°$

$$V_{rocket_x} = V_{rocket}\cos(\theta)$$

$$= (37.53\,\frac{m}{s})\cos(75°)$$

$$V_{rocket_x} = 9.71\,{}^{m}/_{s}$$

$$t_f = \frac{71.79\text{ m}}{9.71\,\frac{m}{s}} + 2(0.03\text{ s})$$

$$t_f = 7.45\text{ s}$$

Mass Flow Rate:

The thrust of the rocket is the most important variable that needs to be found. We found that the thrust is a product of the exit velocity and the mass flow rate of the bottle rocket, and the exit velocity can be calculated by finding the mass flow rate of the water in the rocket. Seeing this, the first calculation that we decided to calculate was the mass flow rate of the water out of the nozzle of the bottle. This was done using the equation

$$m_{\dot{t}} = A \times cd \times \sqrt{(2 \times \rho \times \Delta P)}$$

Where ρ is the density of the fluid (water), ΔP is the change in pressure (calculated from the known launch pressure and standard atmospheric pressure), and A is the area of the opening from which the water exits (determined by basic geometry - the shape of a nozzle is a circle, therefore its area is defined as $A = \pi r^2$, where r is the radius of the nozzle). Plugging in these values, we find the rate at which the mass of water is flowing out of the rocket.

Exit Velocity:

Having calculated the mass flow rate, we can then find the exit velocity of the water. For a propulsion device, the mass flow rate is equal to the product of a liquid's density, exit velocity, and the area of the opening that the fluid exits through. As the mass flow rate was previously calculated, the value could be divided by the water's density and area of the nozzle.

Thrust of Rocket:

Now, the thrust of the rocket can be calculated. The rocket thrust is equal to the product of the mass flow rate and the exit velocity of the bottle. The thrust of the rocket is crucial in determining the hang time of the rocket.

Mass of Water:

For the competition, the water put into the bottle is equal to the amount of water that can fit into a soda can. Finding the dimensions of a soda can, the volume of water placed inside of the rocket can be estimated. A soda can is roughly cylindrical in shape, so the equation for the volume of a cylinder was used to estimate the volume of the water used to propel the rocket.

Opposing Forces: Horizontal (x-direction)

To calculate net the horizontal force acting on the rocket, we must first add all of the forces acting in the x-direction: the horizontal component of thrust of the rocket and the drag force of the air acting horizontally in the opposite direction (which we approximate to be zero). The rocket is launched at an angle of 75 degrees above the horizontal, allowing it to have both a horizontal and vertical component. The horizontal component is equal to the magnitude of the thrust of the rocket multiplied by the cosine of the angle of elevation.

Opposing Forces: Vertically (y-direction)

To calculate the net vertical force acting on the rocket, we must first determine all of the forces acting in the y-direction: the vertical component of thrust of the rocket and the drag force of the air acting vertically (which we approximate to be zero). The rocket is launched at an angle of 75 degrees above the horizontal, allowing it to have both a horizontal and vertical component. The vertical component is equal to the magnitude of the thrust of the rocket multiplied by the sine of the angle of elevation.

Net Force:

To calculate the net force acting on the rocket, the vector sum of the components of the net force was found.

Net Acceleration:

Having calculated the net force, we can now calculate the net acceleration of the rocket from the knowledge that force is equal to the mass of an object multiplied by its acceleration. This was done by finding the average mass of the empty bottle rocket and the mass of the water (as it moves through the air, water is dispelled and the mass is changing, so an average mass was the optimal solution).

Velocity of Rocket:

The velocity of the rocket once the water has been completely dispelled must be calculated. Up until that point, the velocity is changing as the mass of the rocket is also changing. To calculate this value, we referred to the fact that velocity was equal to the product of acceleration and time. To find the velocity of the bottle once all of the water was dispelled, we first needed to find how long it took for the water to dispel. By dividing the mass of water by the mass flow rate, we were able to calculate this value. This was then substituted into our velocity equation, where we calculated the velocity of the rocket.

Peak Time:

We then calculated the time it took for the rocket to reach the peak of its trajectory. This was done as the velocity at the peak of projectile motion is purely in the horizontal direction (the y-component is equal to zero). Knowing the initial and final velocities' y-components, we were able to determine how long it would take the rocket to reach its peak after all the water has dispelled.

Final Time:

One of the principles in projectile motion is that the time it takes to go up is equal to the time it takes to go down (given no outside interferences). Adding the time it took for the water to dispel and the time it took for the empty rocket to reach its peak, we calculated the time the rocket spent in the air for exactly half of its trajectory. By doubling this value (trip up and trip down), we effectively determined the total time the rocket would spend in the air.