


Determination of the beam asymmetry Σ in η - and η' -photoproduction using Bayesian statistics

Master thesis for the CBELSA/TAPS collaboration

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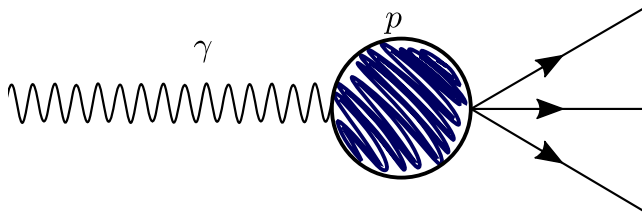
September 22, 2022

Setting the scene

The Standard Model of Particle Physics

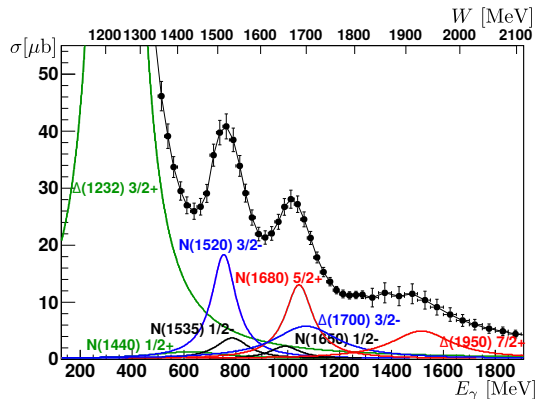
- ▶ matter consists of 12 (anti-) *fermions*
- ▶ quarks interact via *strong interaction*
- ▶ form bound states: mesons (e.g. $q\bar{q}$) and baryons (e.g. qqq)

baryon spectroscopy (photoproduction) gives insight in strong interaction



Setting the scene

observe resonances N^*/Δ^* in the cross sections $\sigma(\gamma p \rightarrow pM)$



Total cross section $\sigma(\gamma p \rightarrow p\pi^0)$ [Wunderlich et al. 2017]

→goal: (help to) identify contributing resonances as strong bound states!

1. Theoretical basics

2. Experimental Setup

3. Results

Determination of Σ_η using BAYESIAN statistics

Determination of $\Sigma_{\eta'}$

4. Conclusion

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Theoretical basics I

- ▶ resonances are broad, overlapping, require complicated partial-wave-analysis (PWA)
- ▶ constraints for the analysis can be derived from polarization observables
- ▶ ultimate goal: "complete experiment"; unambiguous, model-independent PWA solution \rightarrow several single and double polarization observables needed

Beam-target polarization observables

photon	target polarization			
		x	y	z
unpolarized	σ_0	-	T	-
linearly polarized	$-\Sigma$	H	$-P$	$-G$
circularly polarized	-	F	-	$-E$

[Sandorfi et al. 2011]

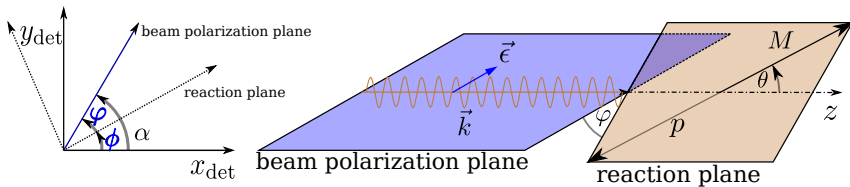
Theoretical basics I

Beam asymmetry Σ

$$\frac{d\sigma}{d\Omega}(E_\gamma, \cos\theta, \varphi) = \frac{d\sigma}{d\Omega_0}(E_\gamma, \cos\theta) \cdot \left[1 - p_\gamma^{\text{lin}} \Sigma \cos(2\varphi)\right]$$

polarization angle $\varphi = \alpha - \phi$ with $\alpha = \alpha^\perp/\parallel = \pm 45^\circ$, polarization degree p_γ^{lin}

[Sandorfi et al. 2011]



Definition of the polarization angle

Theoretical basics II

- ▶ polarization observables are input for further analysis
- ▶ idea: increase amount of information gained from results using BAYESIAN inference

BAYES' theorem

$$p(\theta|y) \propto p(y|\theta) \cdot p(\theta)$$
$$\textit{posterior} \propto \textit{likelihood} \cdot \textit{prior}$$

parameters θ and data y .

[Gelman et al. 2014]

Theoretical basics II

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BAYES' theorem

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Theoretical basics II

$$p(\theta|y) \propto p(y|\theta) \cdot p(\theta)$$

- prior $p(\theta)$ and likelihood $p(y|\theta)$ can be specified once the model is known
→ gain *distributions* $p(\theta|y)$ instead of point estimates with error bars

BAYESIAN parameter inference

For each parameter $\theta_n \in \theta$ we gain *marginal posteriors*

$$p(\theta_n|y) = \int d\theta_1 \cdots \int d\theta_{n-1} \int d\theta_{n+1} \cdots \int d\theta_N p(\theta_1 \dots \theta_N | y).$$

usually approximated using MARKOV-Chain Monte Carlo (MCMC) draws $\theta^{(s)}$

[Sivia and Skilling 2005]

Theoretical basics II

- ▶ all BAYESIAN fits performed using the *Python* frontend of *Stan*



[Stan development team 2022; Hoffman and Gelman 2014]

Theoretical basics II

- ▶ all BAYESIAN fits performed using the *Python* frontend of *Stan*
- ▶ MCMC-sampling: adaptive HAMILTONIAN Monte-Carlo (HMC), i.e. No-U-Turn-Sampling (NUTS)
 - ▶ generate samples $\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(S)}$ where each $\theta^{(t)}$ depends only on $\theta^{(t-1)}$
 - ▶ simulate draws from the posterior by updating at point t such that the posterior increases (importance sampling)



[Stan development team 2022; Hoffman and Gelman 2014]

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- ▶ diagnosing convergence of MCMC:
 - ▶ potential scale reduction statistic $1.00 \lesssim \hat{R} \lesssim 1.01$
 - ▶ Monte Carlo standard error (MCSE) ‘small’,
e.g. for mean μ $\text{MCSE} = \frac{\text{std}(\{\theta^{(s)}\})}{S}$



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e.g. for mean μ $\text{MCSE} = \frac{\text{std}(\{\theta^{(s)}\})}{S}$
- ▶ goodness of fit: posterior predictive checks (PPC)
 - ▶ $p(y^{\text{rep}}|y) = \int d\theta p(y^{\text{rep}}|\theta)p(\theta|y)$
 - ▶ in practice: $y^{\text{rep},(s)} \sim p(y|\theta^{(s)})$



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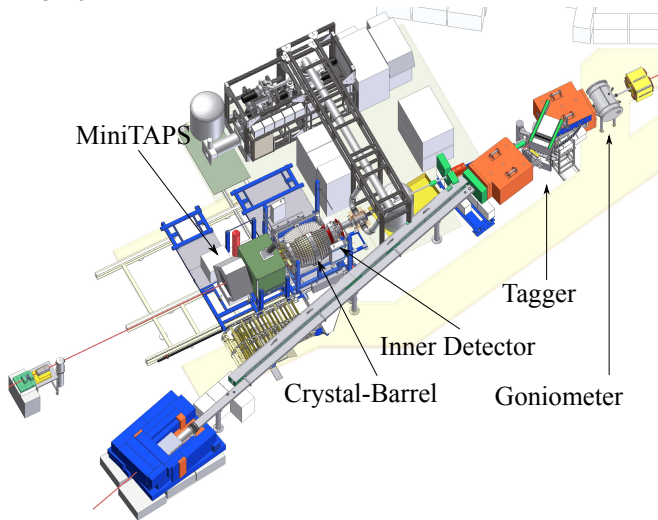
Determination of Σ_η using BAYESIAN statistics

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CBELSA/TAPS experiment

- ▶ generate photon beam from accelerated electrons via bremsstrahlung, with $E_\gamma \leq 3.2 \text{ GeV}$
- ▶ photon beam impinges on liquid hydrogen target:
 $\gamma p \rightarrow pX \rightarrow pM$
- ▶ $M = \pi^0/\eta/\eta'/\dots$
- ▶ data set:
July-October 2013,
1065 h beam time



Overview of the experimental area, adapted from [Walther 2021]

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Confirming pre-published results of Σ_η

- ▶ polarization observables are needed for different final states ($\pi^0, \eta, \eta', \dots$)
- ▶ high precision measurement of beam asymmetry for η production recently published [F. Afzal et al. 2020]
- ▶ goal: confirm results using BAYESIAN fitting methods

Confirming pre-published results for Σ_η

Event selection (η)

analysis performed in 11x12 bins of $(E_\gamma, \cos \theta)$ for $\gamma p \rightarrow p\eta \rightarrow p\gamma\gamma$ by [F. Afzal et al. 2020]

Methods

Remember: $\frac{d\sigma}{d\Omega}(E_\gamma, \cos \theta, \varphi) = \frac{d\sigma}{d\Omega_0}(E_\gamma, \cos \theta) \cdot \left[1 - p_\gamma^{\text{lin}} \Sigma \cos(2\varphi)\right]$

binned fit to event yield asymmetries:

$$A(E_\gamma, \theta, \phi) = \frac{N^\perp(E_\gamma, \theta, \phi) - N^\parallel(E_\gamma, \theta, \phi)}{p_\gamma^\parallel N^\perp(E_\gamma, \theta, \phi) + p_\gamma^\perp N^\parallel(E_\gamma, \theta, \phi)} = \Sigma(E_\gamma, \theta) \cos\left(2\left(\alpha^\parallel - \phi\right)\right)$$

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remember: $\frac{d\sigma}{d\Omega}(E_\gamma, \cos \theta, \varphi) = \frac{d\sigma}{d\Omega_0}(E_\gamma, \cos \theta) \cdot \left[1 - p_\gamma^{\text{lin}} \Sigma \cos(2\varphi)\right]$

unbinned maximum likelihood fit:

$$\tilde{p}(\phi, \Sigma) = \frac{\left(1 + p_\gamma \Sigma \cos\left(2\left(\alpha^\parallel - \phi\right)\right)\right) \cdot \epsilon(\phi)}{C}$$

Confirming pre-published results for Σ_η

applying Bayesian approach to event yield asymmetries:

- assume GAUSSIAN errors, i.e.

$$A(\phi) = \Sigma \cos \left(2 \left(\alpha^\parallel - \phi \right) \right) + \epsilon$$

where $\epsilon \sim \mathcal{N}(0, \sigma)$

- likelihood $p(A|\Sigma)$ of given by

$$y_i \sim \mathcal{N} \left(\Sigma \cos \left(2 \left(\alpha^\parallel - \phi_i \right) \right), \sigma_i \right) \quad \mathcal{L} = \prod_i p(y_i|\theta)$$

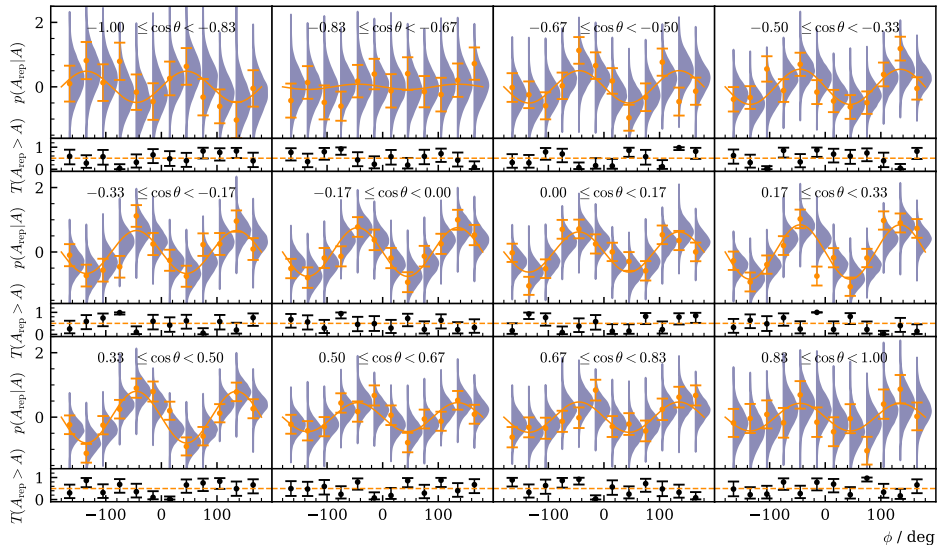
- prior:

$$p(\Sigma) \sim \mathcal{N}(0, 1)_{[-1, 1]}$$

Sample from posterior $p(\Sigma|A) \propto p(A|\Sigma) \cdot p(\Sigma)$!

Confirming pre-published results for Σ_η

PPC for *binned* BAYESIAN fit:



Confirming pre-published results for Σ_η

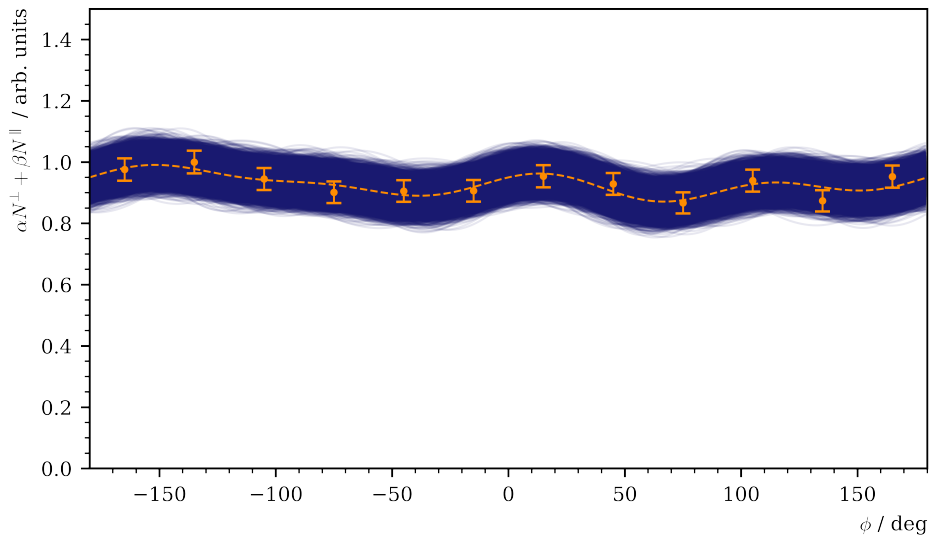
applying **Bayesian approach to unbinned fit:**

- ▶ event based likelihood given by product of all single-event likelihoods
- ▶ assign priors for all fit parameters (18 in total)
- ▶ truncate beam asymmetry to allowed region $[-1, 1]$
- ▶ perform toy Monte Carlo experiments

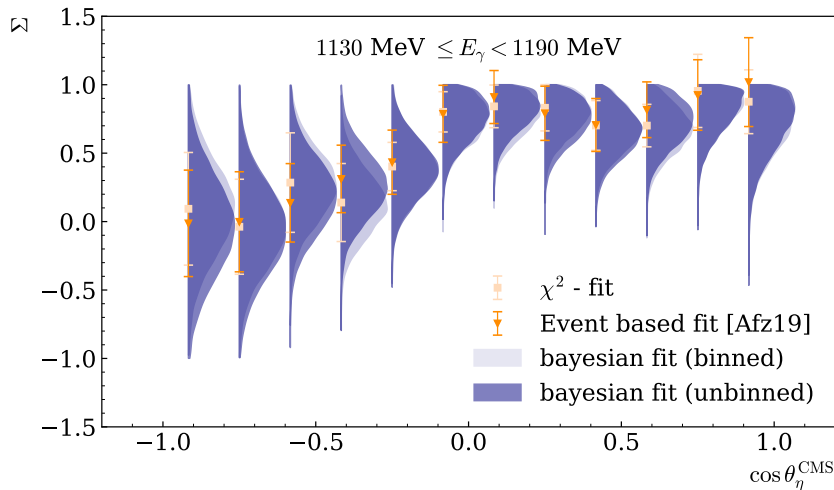
Sample from posterior !

Confirming pre-published results for Σ_η

PPC for *unbinned* BAYESIAN fit:

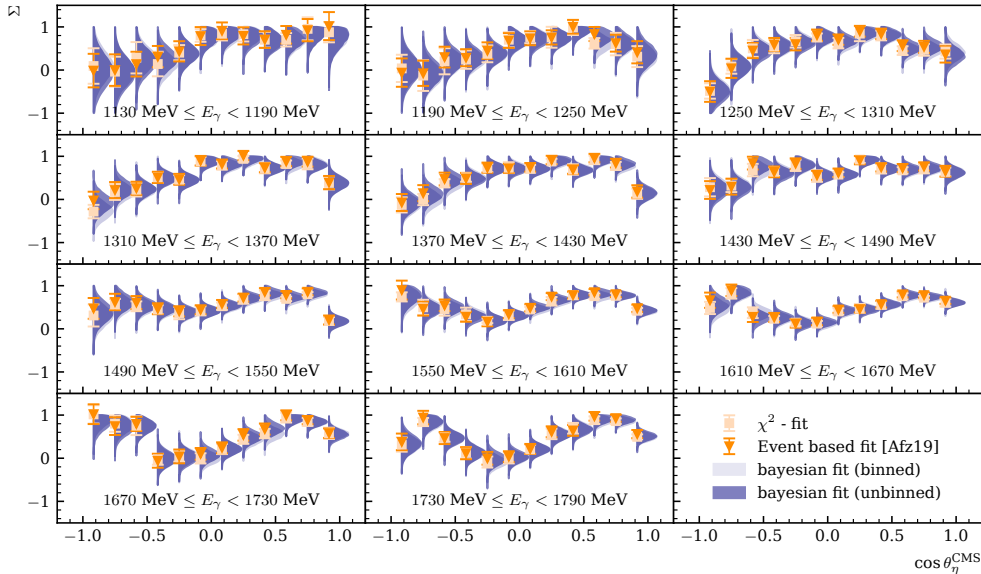


Confirming pre-published results for Σ_η



Distinct advantage: sample only in physically allowed parameter space

Confirming pre-published results for Σ_η



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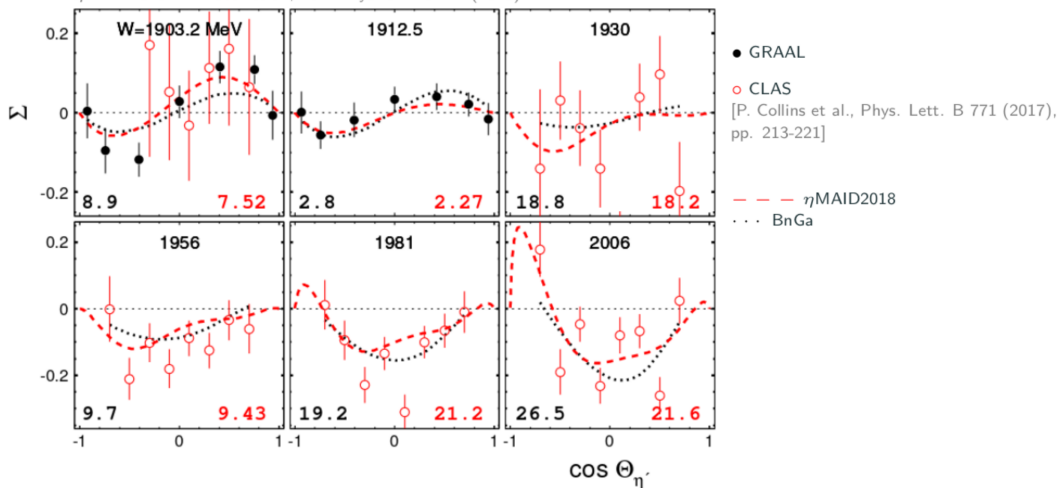
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Determining the beam asymmetry in η' photoproduction

existing database:

η MAID2018: L. Tiator et al., Eur. Phys. J. A 54.12 (2018)



Determining the beam asymmetry in η' photoproduction

First, perform event selection regarding η' photoproduction

η'	
Decay mode	Branching ratio
$\pi^+\pi^-\eta$	42.6%
$\rho^0\gamma(\rightarrow \pi^+\pi^-\gamma)$	28.9% (28.9%)
$\pi^0\pi^0\eta(\rightarrow 6\gamma)$	22.8% (8.8%)
$\omega\gamma(\rightarrow \pi^+\pi^-\pi^0\gamma/\pi^0\gamma\gamma)$	2.52% (2.2%/0.21%)
$\gamma\gamma$	2.3%

[Workman et al. 2022]

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[Workman et al. 2022]

Determining the beam asymmetry in η' photoproduction

event selection:

analysis performed in 3x6 bins of $(E_\gamma, \cos \theta)$ for $\gamma p \rightarrow p\eta' \rightarrow p\gamma\gamma$,

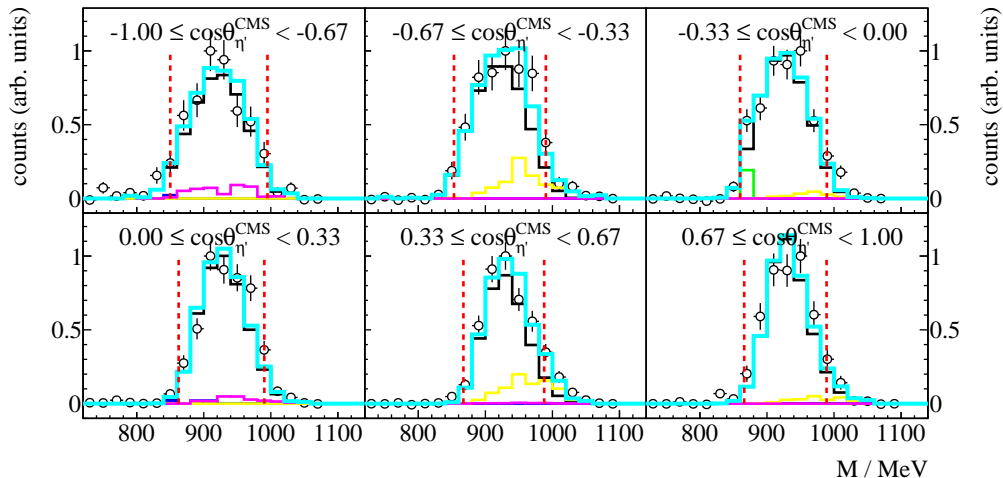
$E_\gamma \in [1500, 1800]$ MeV

$$p_\gamma + p_p = p_{\eta'} + p_{\text{recoil}} = \underbrace{p_{\gamma_1} + p_{\gamma_2}}_{=p_{\eta'}} + p_{\text{recoil}}$$

$$p_\gamma + p_p = p_{\eta'} + p_X = \underbrace{p_{\gamma_1} + p_{\gamma_2}}_{=p_{\eta'}} + p_X$$

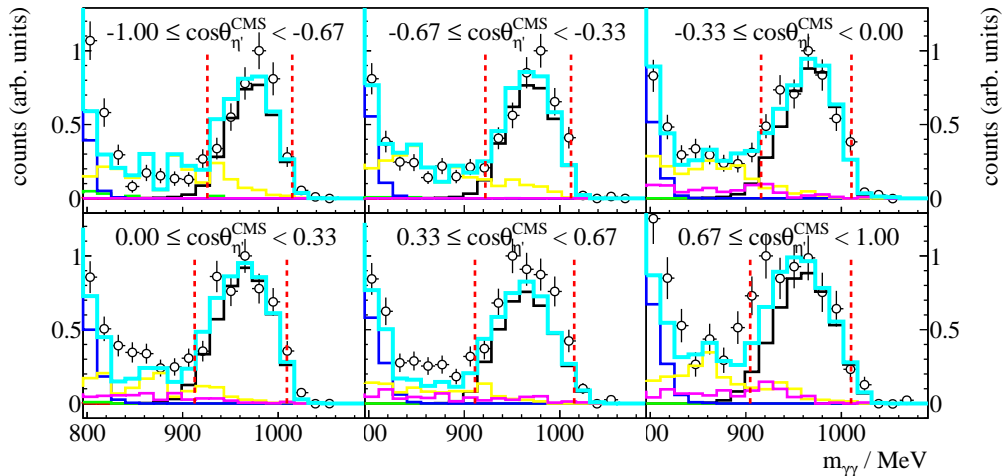
- ▶ one charged, two uncharged detector hits in coincidence with beam γ
- ▶ Coplanarity $\Delta\phi = \phi_{\eta'} - \phi_{\text{recoil}} \stackrel{!}{=} 180^\circ$
- ▶ Polar angle $\theta_X \stackrel{!}{=} \theta_{\text{recoil}}$
- ▶ Missing mass $m_X \stackrel{!}{=} m_p$
- ▶ Invariant mass $m_{\gamma\gamma} \stackrel{!}{=} m_{\eta'}$

Determining the beam asymmetry in η' photoproduction



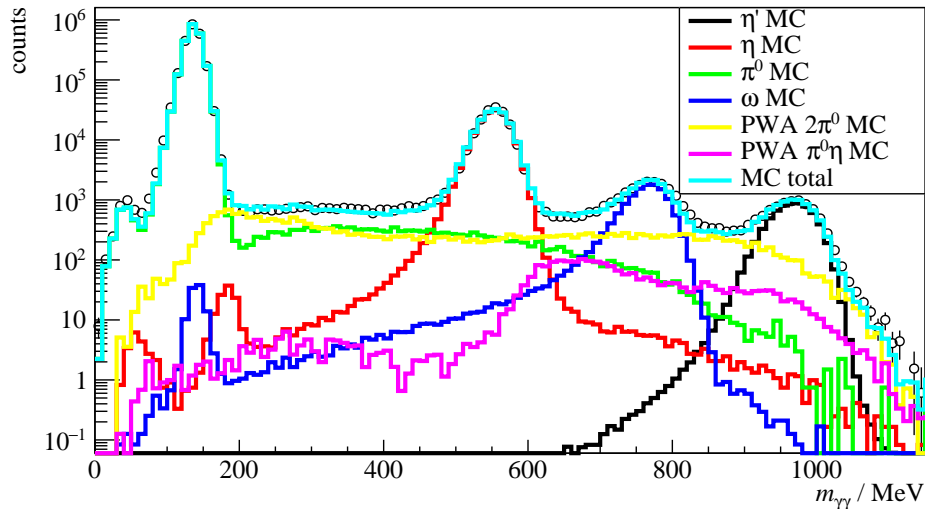
data points: m_X , MC histograms: total, $p\eta'$, $p2\pi^0$, $p\pi^0\eta$

Determining the beam asymmetry in η' photoproduction



data points: $m_{\gamma\gamma}$, data points: m_X , MC histograms: total, $p\eta'$, $p2\pi^0$, $p\pi^0\eta$, $p\omega$

Determining the beam asymmetry in η' photoproduction

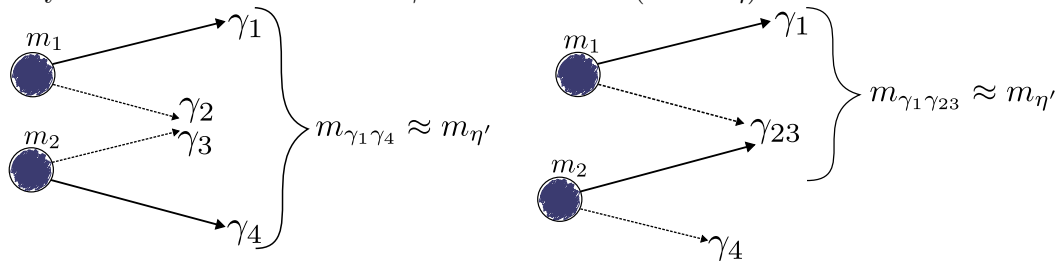


Determining the beam asymmetry in η' photoproduction

why these contributions from 4γ final states $2\pi^0$ (and $\pi^0\eta$)??

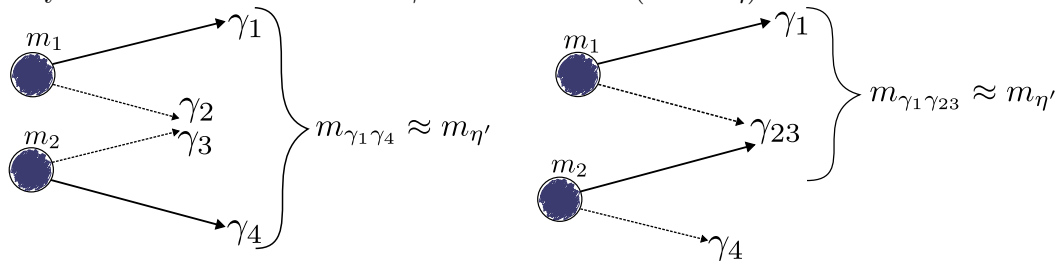
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acceptance for $2\pi^0$ events is almost vanishing $A(E_\gamma, \cos\theta) < 2 \cdot 10^{-3}$, yet

$$R = \frac{\sigma_{2\pi^0} \cdot \text{BR}_{2\pi^0 \rightarrow \gamma\gamma} \cdot \tilde{A}_{2\pi^0}}{\sigma_{\eta'} \cdot \text{BR}_{\eta' \rightarrow \gamma\gamma} \cdot \tilde{A}_{\eta'}} = \frac{5 \text{ } \mu\text{b} \cdot 0.9765 \cdot 2 \cdot 10^{-3}}{1 \text{ } \mu\text{b} \cdot 0.023 \cdot 0.61} \approx 0.7!$$

explains high background contributions of up to 45% (background to total)

[Workman et al. 2022; Crede et al. 2009; Dieterle et al. 2020]

Determining the beam asymmetry in η' photoproduction

How do we get rid of background contributions?

- real photons mimic a two-photon final state, no sensible additional cuts have been found

Determining the beam asymmetry in η' photoproduction

How do we get rid of background contributions?

- ▶ real photons mimic a two-photon final state, no sensible additional cuts have been found
- ▶ main background from $2\pi^0$ photoproduction
- ▶ beam asymmetry for this reaction determined by [Mahlberg 2022]
- ▶ correct estimates for Σ according to the amount of background

$$\Sigma^{\text{meas}} = (1 - \delta) \cdot \Sigma_{\eta'} + \delta \Sigma_{2\pi^0}$$

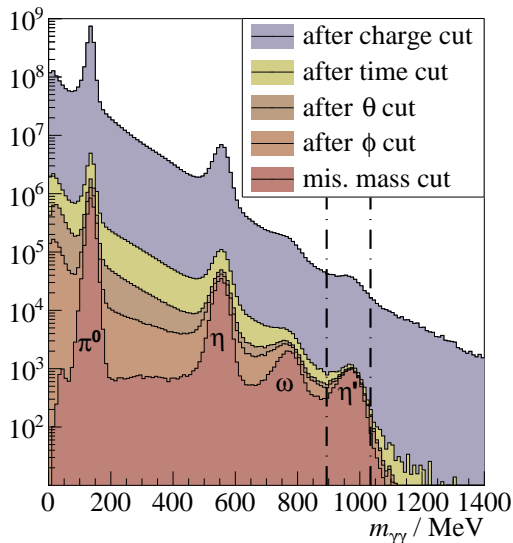
Determining the beam asymmetry in η' photoproduction

total: $\sim 8000 \eta' \rightarrow \gamma\gamma$ events

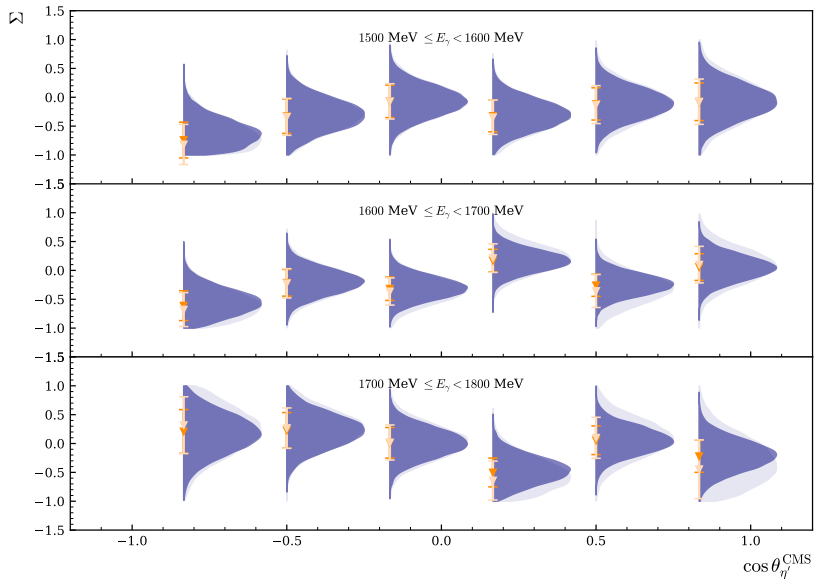
- ▶ perform unbinned fit as maximum likelihood fit and BAYESIAN fit
- ▶ BAYESIAN fit: modify likelihood

$$\tilde{p}(\phi, \Sigma) \rightarrow \tilde{p}\left(\phi, (1 - \delta)\Sigma_1 + \delta\Sigma_2^{\text{true}}\right)$$
$$\Sigma_2^{\text{true}} \sim \mathcal{N}(\Sigma_2^{\text{meas}}, \tau)$$

- ▶ unbinned maximum likelihood fit: shift point estimates *after fit*



Determining the beam asymmetry in η' photoproduction



Determining the beam asymmetry in η' photoproduction

Systematic error: $\Delta\Sigma_{\eta'}^{\text{sys}} = \sqrt{\left(\frac{\Delta p_\gamma}{p_\gamma} \Sigma_{\eta'}\right)^2 + (\Delta\Sigma_{\eta'})^2}$.

- polarization degree

$$\frac{\Delta p_\gamma}{p_\gamma} = \begin{cases} 0.05 & E_\gamma < 1600 \text{ MeV}, \\ 0.08 & \text{otherwise} \end{cases}$$

- background contributions

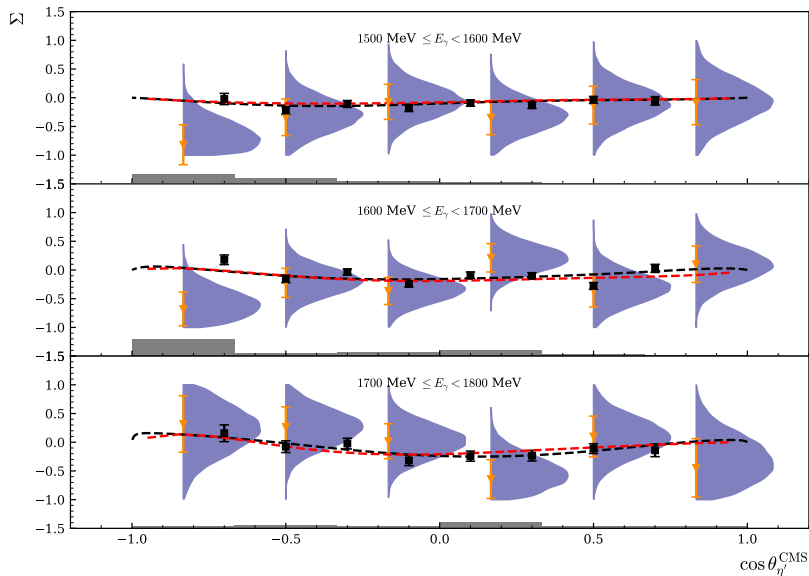
$$\Sigma^{\text{meas}} = (1 - \delta_1 - \delta_2) \cdot \Sigma_{\eta'} + \delta_1 \Sigma_{2\pi^0} + \delta_2 \Sigma^{\text{r bkg}},$$

thus:

$$\Delta\Sigma_{\eta'} = \max \left[\left| \frac{\Sigma^{\text{meas}} - \delta_1 \cdot \Sigma_{2\pi^0} \pm \delta_2 \cdot 1}{1 - \delta_1 - \delta_2} - \frac{\Sigma^{\text{meas}} - \delta \cdot \Sigma_{2\pi^0}}{1 - \delta} \right| \right]$$

[F. N. Afzal 2019; Eberhardt 2012]

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Conclusion

Summary

- ▶ Σ extracted for η and η' final state
- ▶ η results obtained with BAYESIAN fit agree with previous results
- ▶ η' results agree with previous measurements

Outlook

- ▶ use posterior *distributions* for PWA calculations
- ▶ increase precision of existing data , i.e. investigate $\eta' \rightarrow \pi^0 \pi^0 \eta$
- ▶ measurement of observables near η' production threshold

BACKUP & REFERENCES

Full PDF for unbinned maximum likelihood fit

$$\begin{aligned}
 -\ln \mathcal{L} = & \sum_{i=1}^n -\ln(p_{\text{prompt}}(\phi_i, p_{\gamma,i}, \Sigma, a_1 \dots a_4, b_1 \dots b_4)) + \\
 & \sum_{j=1}^m -\ln \left(p_{\text{sideband}}(\phi_j, p_{\gamma,j}, \Sigma^{\text{bkg}}, a_1^{\text{bkg}} \dots a_4^{\text{bkg}}, b_1^{\text{bkg}} \dots b_4^{\text{bkg}}) \right)
 \end{aligned}$$

where

$$\begin{aligned}
 p_{\text{prompt}} = & f_{\text{sig}} \cdot \tilde{p}(\phi, p_{\gamma}, \Sigma, a_1 \dots a_4, b_1 \dots b_4) \\
 & + (1 - f_{\text{sig}}) \cdot \tilde{p}(\phi, p_{\gamma}, \Sigma^{\text{bkg}}, a_1^{\text{bkg}} \dots a_4^{\text{bkg}}, b_1^{\text{bkg}} \dots b_4^{\text{bkg}}) \\
 p_{\text{sideband}} = & \tilde{p}(\phi, p_{\gamma}, \Sigma^{\text{bkg}}, a_1^{\text{bkg}} \dots a_4^{\text{bkg}}, b_1^{\text{bkg}} \dots b_4^{\text{bkg}})
 \end{aligned}$$

and

$$\tilde{p}(\phi, \Sigma) = \frac{\left(1 + p_{\gamma} \Sigma \cos \left(2 \left(\alpha^{\parallel} - \phi\right)\right)\right) \cdot \left(\sum_{k=0}^4 a_k \sin(k\phi) + b_k \cos(k\phi)\right)}{1 - \frac{1}{2} a_2 p_{\gamma} \Sigma}$$

Additional theoretical basics

Unpolarized differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\rho \sum_{\text{spins}} |\langle f | \mathcal{F} | i \rangle|^2,$$

where

$$\mathcal{F} = i(\vec{\sigma} \cdot \vec{\epsilon})F_1 + (\vec{\sigma} \cdot \hat{q})(\vec{\sigma} \cdot (\hat{k} \times \vec{\epsilon}))F_2 + i(\vec{\sigma} \cdot \hat{k})(\hat{q} \cdot \vec{\epsilon})F_3 + i(\vec{\sigma} \cdot \hat{q})(\hat{q} \cdot \vec{\epsilon})F_4$$

F_i : complex CGLN Amplitudes




[Chew et al. 1957]

$\frac{d\sigma}{d\Omega} \in \mathbb{R}$, not sufficient to determine \mathcal{F} unambiguously
→ Polarization Observables can be related to F_i





Diagnostics of a BAYESIAN fit

- ▶ \hat{R} : measure of convergence for chains
- ▶ Monte-Carlo-Standard-Error: measure for adequate sample size
- ▶ *posterior predictive checks*: "goodness of fit"





References I

-  Afzal, F. et al. (Oct. 2020). ‘Observation of the $p\eta'$ Cusp in the New Precise Beam Asymmetry Σ Data for $\gamma p \rightarrow p\eta'$ '. In: *Phys. Rev. Lett.* 125 (15), p. 152002. DOI: [10.1103/PhysRevLett.125.152002](https://doi.org/10.1103/PhysRevLett.125.152002). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.125.152002>.
-  Afzal, Farah Noreen (Sept. 2019). ‘Measurement of the beam and helicity asymmetries in the reactions $\gamma p \rightarrow p\pi^0$ and $\gamma p \rightarrow p\eta'$ '. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn.
-  Anisovich, A.V. et al. (2018). ‘Proton- η' interactions at threshold’. In: *Physics Letters B* 785, pp. 626–630. ISSN: 0370-2693. DOI: <https://doi.org/10.1016/j.physletb.2018.06.034>. URL: <https://www.sciencedirect.com/science/article/pii/S0370269318304908>.





References II

-  Chew, G. F. et al. (June 1957). ‘Relativistic Dispersion Relation Approach to Photomeson Production’. In: *Phys. Rev.* 106 (6), pp. 1345–1355. DOI: 10.1103/PhysRev.106.1345. URL: <https://link.aps.org/doi/10.1103/PhysRev.106.1345>.
-  Collins, P. et al. (Mar. 2017). ‘Photon beam asymmetry Σ for η and η' photoproduction from the proton’. In: *Physics Letters B*.
-  Crede, V. et al. (2009). ‘Photoproduction of eta and eta-prime mesons off protons’. In: *Phys. Rev. C* 80, p. 055202. DOI: 10.1103/PhysRevC.80.055202. arXiv: 0909.1248 [nucl-ex].
-  Dieterle, M. et al. (Aug. 2020). ‘Helicity-Dependent Cross Sections for the Photoproduction of π^0 Pairs from Nucleons’. In: *Physical Review Letters* 125.6. DOI: 10.1103/physrevlett.125.062001. URL: <https://doi.org/10.1103/PhysRevLett.125.062001>.




References III

-  Eberhardt, Holger (2012). ‘Bestimmung von Polarisationsobservablen in der π^0 und ω Photoproduktion am Proton mit dem CBELSA/TAPS-Experiment’. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn.
-  Gelman, Andrew et al. (2014). *Bayesian Data Analysis*. Vol. 3. Chapman & Hall/CRC.
-  Hoffman, Matthew D. and Andrew Gelman (2014). ‘The No-U-Turn Sampler: Adaptively Setting Path Lengths in Hamiltonian Monte Carlo’. In: *Journal of Machine Learning Research* 15.47, pp. 1593–1623. URL: <http://jmlr.org/papers/v15/hoffman14a.html>.
-  Mahlberg, Philipp (2022). ‘Thesis in Preparation’. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn.

References IV

-  Sandorfi, A. M. et al. (Apr. 2011). ‘Determining pseudoscalar meson photoproduction amplitudes from complete experiments’. In: *Journal of Physics G: Nuclear and Particle Physics* 38.5, p. 053001. ISSN: 1361-6471. DOI: 10.1088/0954-3899/38/5/053001. URL: <http://dx.doi.org/10.1088/0954-3899/38/5/053001>.
-  Sivia, D.S. and J. Skilling (2005). *Data Analysis – A Bayesian Tutorial*. Vol. 2. Oxford University Press.
-  Stan development team (2022). *Stan Modeling Language Users Guide and Reference Manual*. Vol. 2.29. URL: <https://mc-stan.org>.
-  Tiator, L. et al. (Dec. 2018). ‘Eta and etaprime photoproduction on the nucleon with the isobar model EtaMAID2018’. In: *The European Physical Journal A* 54.12. DOI: 10.1140/epja/i2018-12643-x. URL: <https://doi.org/10.1140%2Fepja%2Fi2018-12643-x>.

References V

-  Walther, Dieter (2021). *Crystal Barrel. A 4π photon spectrometer*. URL: <https://www.cb.uni-bonn.de> (visited on 27/09/2021).
-  Workman, R. L. et al. (2022). ‘Review of Particle Physics’. In: *PTEP 2022*, p. 083C01. DOI: 10.1093/ptep/ptac097.
-  Wunderlich, Y. et al. (May 2017). ‘Determining the dominant partial wave contributions from angular distributions of single- and double-polarization observables in pseudoscalar meson photoproduction’. In: *The European Physical Journal A* 53.5. ISSN: 1434-601X. DOI: 10.1140/epja/i2017-12255-0. URL: <http://dx.doi.org/10.1140/epja/i2017-12255-0>.