

# Module 2, Part 2: Random vectors, covariance, multivariate Normal distribution

TMA4268 Statistical Learning V2022

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# Overview

- Random vectors
- The covariance and correlation matrix
- The multivariate normal distribution

## Random vector

- A random vector  $X_{(p \times 1)}$  is a  $p$ -dimensional vector of random variables. For example
  - Weight of cork deposits in  $p = 4$  directions (N, E, S, W).
  - Factors to predict body fat: bmi, age, weight, hip circumference,...
- Joint distribution function:  $f(x)$ .
- From joint distribution function to marginal (and conditional distributions).

$$f_1(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_p) dx_2 \cdots dx_p$$

- Cumulative distribution (definite integrals!) used to calculate probabilities.
- Independence:  $f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$  and  $f(x_1 | x_2) = f_1(x_1)$ .

## Moments

The moments are important properties of the distribution of  $X$ .  
We will look at:

- E: Mean of random vector and random matrices.
- Cov: Covariance matrix.
- Corr: Correlation matrix.
- E and Cov of multiple linear combinations.

## The Cork deposit data

- Classical multivariate data set from Rao (1948).
- Weighth of bark deposits of  $n = 28$  cork trees in  $p = 4$  directions (N, E, S, W).

```
corkds = as.matrix(read.table("https://www.math.ntnu.no/emner/TMA4268/2019v/data/corkMKB.txt"))  
dimnames(corkds)[[2]] = c("N", "E", "S", "W")  
head(corkds)
```

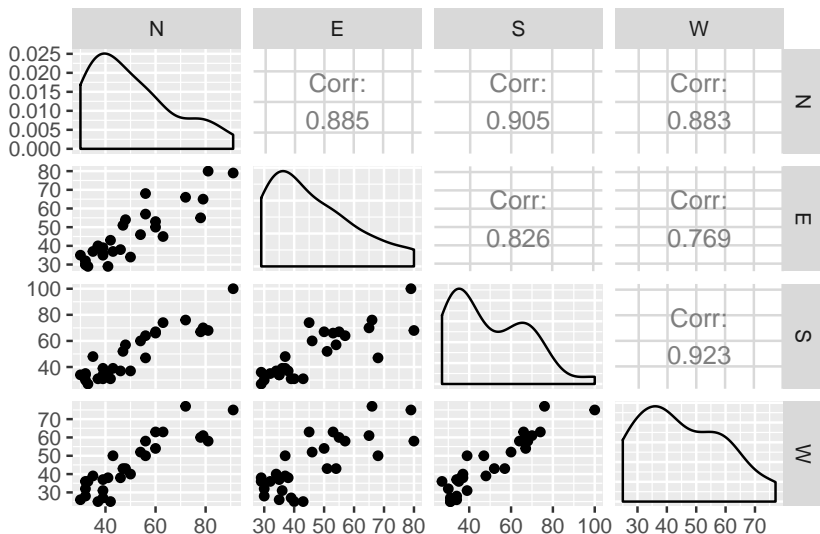
```
##      N  E  S  W  
## [1,] 72 66 76 77  
## [2,] 60 53 66 63  
## [3,] 56 57 64 58  
## [4,] 41 29 36 38  
## [5,] 32 32 35 36  
## [6,] 30 35 34 26
```

```
dim(corkds)
```

```
## [1] 28  4
```

Look at the data (always the first thing to do):

```
library(GGally)
corkds <- as.data.frame(corkds)
ggpairs(corkds)
```



- Here we have a random sample of  $n = 28$  cork trees from the population and observe a  $p = 4$  dimensional random vector for each tree.
- This leads us to the definition of random vectors and a random matrix for cork trees:

$$X_{(28 \times 4)} = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ \vdots & \vdots & \ddots & \vdots \\ X_{28,1} & X_{28,2} & X_{28,3} & X_{28,4} \end{bmatrix}$$

## Rules for means

- Random vector  $X_{(p \times 1)}$  with mean vector  $\mu_{(p \times 1)}$ :<sup>1</sup>

$$X_{(p \times 1)} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}, \text{ and } \mu_{(p \times 1)} = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_p) \end{bmatrix} .$$

- Same rule for random matrices.
- Random matrix  $X_{(n \times p)}$  and random matrix  $Y_{(n \times p)}$ :

$$E(X + Y) = E(X) + E(Y) .$$

(Rules of vector addition)

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<sup>1</sup>Observe that  $E(X_j)$  is calculated from the marginal distribution of  $X_j$  and contains no information about dependencies between  $X_j$  and  $X_k$ ,  $k \neq j$ .



- Random matrix  $X_{(n \times p)}$  and conformable constant matrices  $A$  and  $B$ :

$$E(AXB) = AE(X)B$$

Proof: Board

Q:

- What are the univariate analogue to the formulas on the previous two slides (which you studied in your first introductory course in statistics)?

## The covariance

In the introductory statistics course we defined the covariance

$$\begin{aligned}\rho_{ij} &= \text{Cov}(X_i, X_j) = \text{E}[(X_i - \mu_i)(X_j - \mu_j)] \\ &= \text{E}(X_i \cdot X_j) - \mu_i \mu_j .\end{aligned}$$

- What is the covariance called when  $i = j$ ?
- What does it mean when the covariance is
  - negative
  - zero
  - positive?

Make a scatter plot for negative, zero and positive correlation  
(see also R example).

## Variance-covariance matrix

- Consider random vector  $X_{(p \times 1)}$  with mean vector  $\mu_{(p \times 1)}$ :

$$X_{(p \times 1)} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}, \text{ and } \mu_{(p \times 1)} = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_p) \end{bmatrix}$$

- Variance-covariance matrix  $\Sigma$  (real and symmetric)

$$\begin{aligned} \Sigma &= \text{Cov}(X) = E[(X - \mu)(X - \mu)^T] \\ &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix} = E(XX^T) - \mu\mu^T \end{aligned}$$

- The diagonal elements in  $\Sigma$ ,  $\sigma_{ii} = \sigma_i^2$ , are variances.
- The off-diagonal elements are covariances  
$$\sigma_{ij} = \text{E}[(X_i - \mu_i)(X_j - \mu_j)] = \sigma_{ji}.$$
- $\Sigma$  is called variance, covariance and variance-covariance matrix and denoted both  $\text{Var}(X)$  and  $\text{Cov}(X)$ .

## Exercise: the variance-covariance matrix

Let  $X_{4 \times 1}$  have variance-covariance matrix

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Explain what this means.

## Correlation matrix

Correlation matrix  $\rho$  (real and symmetric)

$$\rho = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}\sigma_{pp}}} \\ \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{1p}}{\sqrt{\sigma_{11}\sigma_{pp}}} & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}\sigma_{pp}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}\sigma_{pp}}} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{12} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1p} & \rho_{2p} & \cdots & 1 \end{bmatrix}$$

$$\rho = (V^{\frac{1}{2}})^{-1}\Sigma(V^{\frac{1}{2}})^{-1}, \text{ where } V^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix}$$



## Exercise: the correlation matrix

Let  $X_{4 \times 1}$  have variance-covariance matrix

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Find the correlation matrix.

**A:**

## Linear combinations

Consider a random vector  $X_{(p \times 1)}$  with mean vector  $\mu = E(X)$  and variance-covariance matrix  $\Sigma = \text{Cov}(X)$ .

The linear combinations

$$Z = CX = \begin{bmatrix} \sum_{j=1}^p c_{1j}X_j \\ \sum_{j=1}^p c_{2j}X_j \\ \vdots \\ \sum_{j=1}^p c_{kj}X_j \end{bmatrix}$$

have

$$E(Z) = E(CX) = C\mu$$

$$\text{Cov}(Z) = \text{Cov}(CX) = C\Sigma C^T$$

**Exercise:** Follow the proof (board) - what are the most important transitions?

## Exercise: Linear combinations

$$X = \begin{bmatrix} X_N \\ X_E \\ X_S \\ X_W \end{bmatrix}, \mu = \begin{bmatrix} \mu_N \\ \mu_E \\ \mu_S \\ \mu_W \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sigma_{NN} & \sigma_{NE} & \sigma_{NS} & \sigma_{NW} \\ \sigma_{NE} & \sigma_{EE} & \sigma_{ES} & \sigma_{EW} \\ \sigma_{NS} & \sigma_{EE} & \sigma_{SS} & \sigma_{SW} \\ \sigma_{NW} & \sigma_{EW} & \sigma_{SW} & \sigma_{WW} \end{bmatrix}$$

Scientists would like to compare the following three *contrasts*: N-S, E+W and (E+W)-(N+S), and define a new random vector  $Y_{(3 \times 1)} = C_{(3 \times 4)} X_{(4 \times 1)}$  giving the three contrasts.

- Write down  $C$ .
- Explain how to find  $E(Y_1)$  and  $\text{Cov}(Y_1, Y_3)$ .
- Use R to find the mean vector, covariance matrix and correlations matrix of  $Y$ , when the mean vector and covariance matrix for  $X$  given below.

```
corkds <- as.matrix(read.table("https://www.math.ntnu.no/emner/TMA4268/2019v/data/corkMKB.txt"))
dimnames(corkds)[[2]] <- c("N", "E", "S", "W")
mu = apply(corkds, 2, mean)
mu
Sigma = var(corkds)
Sigma
```

```
##           N           E           S           W
## 50.53571 46.17857 49.67857 45.17857
##           N           E           S           W
## N 290.4061 223.7526 288.4378 226.2712
## E 223.7526 219.9299 229.0595 171.3743
## S 288.4378 229.0595 350.0040 259.5410
## W 226.2712 171.3743 259.5410 226.0040
(C <- matrix(c(1, 0, -1, 0, 0, 1, 0, 1, -1, 1, -1, 1), byrow = T, nrow = 3))
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0  -1    0
## [2,]    0    1    0    1
## [3,]   -1    1  -1    1
```

```
C %*% Sigma %*% t(C)
```

```
##           [,1]           [,2]           [,3]
## [1,] 63.53439 -38.57672 21.02116
## [2,] -38.57672 788.68254 -149.94180
## [3,] 21.02116 -149.94180 128.71958
```

## The covariance matrix - more requirements?

Random vector  $X_{(p \times 1)}$  with mean vector  $\mu_{(p \times 1)}$  and covariance matrix

$$\Sigma = \text{Cov}(X) = \text{E}[(X - \mu)(X - \mu)^T] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix}$$

- The covariance matrix is by construction symmetric, and it is common to require that the covariance matrix is positive semidefinite. This means that, for every vector  $b \neq 0$

$$b^T \Sigma b \geq 0 .$$

- Why do you think that is?

Hint: Is it possible that the variance of the linear combination  $Y = b^T X$  is negative?

## Random vectors - Single-choice exercise

Quizz on [www.menti.com](http://www.menti.com), code 3967 5773



### Question 1: Mean of sum

$X$  and  $Y$  are two bivariate random vectors with  $E(X) = (1, 2)^T$  and  $E(Y) = (2, 0)^T$ . What is  $E(X + Y)$ ?

- A:  $(1.5, 1)^T$
- B:  $(3, 2)^T$
- C:  $(-1, 2)^T$
- D:  $(1, -2)^T$

## Question 2: Mean of linear combination

$X$  is a 2-dimensional random vector with  $E(X) = (2, 5)^T$ , and  $b = (0.5, 0.5)^T$  is a constant vector. What is  $E(b^T X)$ ?

- A: 3.5
- B: 7
- C: 2
- D: 5

### Question 3: Covariance

$X$  is a  $p$ -dimensional random vector with mean  $\mu$ . Which of the following defines the covariance matrix?

- A:  $E[(X - \mu)^T(X - \mu)]$
- B:  $E[(X - \mu)(X - \mu)^T]$
- C:  $E[(X - \mu)(X - \mu)]$
- D:  $E[(X - \mu)^T(X - \mu)^T]$

## Question 4: Mean of linear combinations

$X$  is a  $p$ -dimensional random vector with mean  $\mu$  and covariance matrix  $\Sigma$ .  $C$  is a constant matrix. What is then the mean of the  $k$ -dimensional random vector  $Y = CX$ ?

- A:  $C\mu$
- B:  $C\Sigma$
- C:  $C\mu C^T$
- D:  $C\Sigma C^T$

### Question 5: Covariance of linear combinations

$X$  is a  $p$ -dimensional random vector with mean  $\mu$  and covariance matrix  $\Sigma$ .  $C$  is a constant matrix. What is then the covariance of the  $k$ -dimensional random vector  $Y = CX$ ?

- A:  $C\mu$
- B:  $C\Sigma$
- C:  $C\mu C^T$
- D:  $C\Sigma C^T$

## Question 6: Correlation

$X$  is a 2-dimensional random vector with covariance matrix

$$\Sigma = \begin{bmatrix} 4 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

Then the correlation between the two elements of  $X$  are:

- A: 0.10
- B: 0.25
- C: 0.40
- D: 0.80

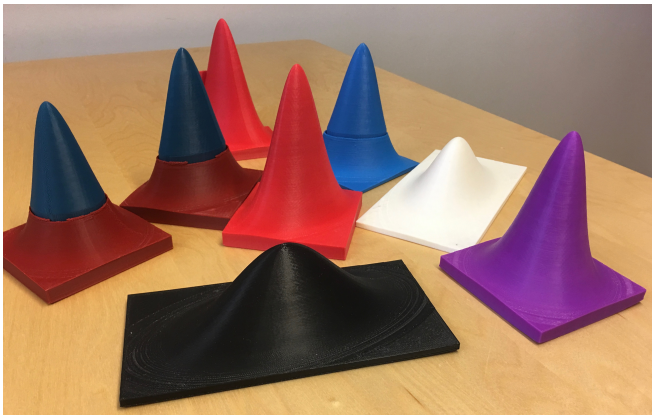


# The multivariate normal distribution

Why is the mvN so popular?

- Many natural phenomena may be modelled using this distribution (just as in the univariate case).
- Multivariate version of the central limit theorem- the sample mean will be approximately multivariate normal for large samples.
- Good interpretability of the covariance.
- Mathematically tractable.
- Building block in many models and methods.





3D multivariate Normal distributions

## The multivariate normal (mvN) pdf

The random vector  $X_{p \times 1}$  is multivariate normal  $N_p$  with mean  $\mu$  and (positive definite) covariate matrix  $\Sigma$ . The pdf is:

$$f(x) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right\}$$

### Questions:

- How does this compare to the univariate version?

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\}$$

- Why do we need the constant in front of the exp?
- What is the dimension of the part in exp?
- What happens if the determinant  $|\Sigma| = 0$  (degenerate case)?

## Four useful properties of the mvN

Let  $X_{(p \times 1)}$  be a random vector from  $N_p(\mu, \Sigma)$ .

1. The graphical contours of the mvN are ellipsoids (can be shown using spectral decomposition).
2. Linear combinations of components of  $X$  are (multivariate) normal
3. All subsets of the components of  $X$  are (multivariate) normal (special case of the above).
4. Zero covariance implies that the corresponding components are independently distributed (in contrast to general distributions).

If you need a refresh, you might find that video useful:

<https://www.youtube.com/watch?v=eho8xH3E6mE>

All of these are proven in TMA4267 Linear Statistical Models.

The result 4 is rather useful! If you have a bivariate normal and observed covariance 0, then your variables are independent.

## Contours of multivariate normal distribution

- Contours of constant density for the  $p$ -dimensional normal distribution are ellipsoids defined by  $x$  such that

$$(x - \mu)^T \Sigma^{-1} (x - \mu) = b$$

where  $b > 0$  is a constant.

- These ellipsoids are centered at  $\mu$  and have axes  $\pm \sqrt{b \lambda_i} e_i$ , where  $\Sigma e_i = \lambda_i e_i$  (eigenvector for  $\lambda_i$ ), for  $i = 1, \dots, p$ .
- To see this the spectral decomposition of the covariance matrix is useful.
- $(x - \mu)^T \Sigma^{-1} (x - \mu)$  is distributed as  $\chi_p^2$ .

Note:

*In M4: Classification the mvN is very important and we will often draw contours of the mvN as ellipses- and this is the reason why we do that.*

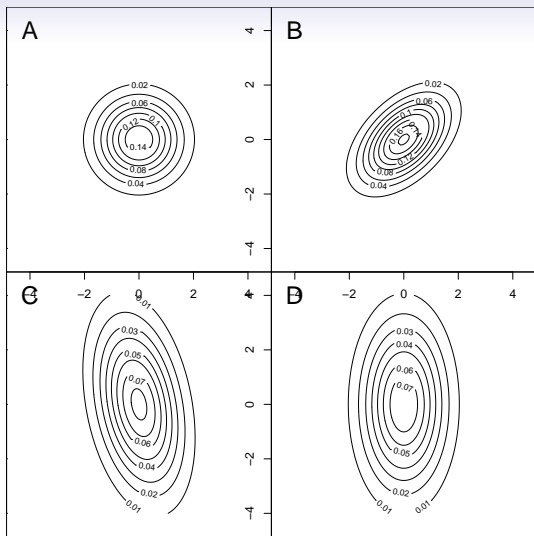
Identify the mvNs from their contours

$$\text{Let } \Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}.$$

The following four figure contours have been generated:

- 1:  $\sigma_x = 1, \sigma_y = 2, \rho = -0.3$
- 2:  $\sigma_x = 1, \sigma_y = 1, \rho = 0$
- 3:  $\sigma_x = 1, \sigma_y = 1, \rho = 0.5$
- 4:  $\sigma_x = 1, \sigma_y = 2, \rho = 0$

**Match the distributions to the figures on the next slide.**





## Multiple choice - multivariate normal

A second quizz on [www.menti.com](http://www.menti.com), code 3967 5773

Choose the correct answer. Let's go!

## Question 1: Multivariate normal pdf

The probability density function is  $(\frac{1}{2\pi})^{\frac{p}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\{-\frac{1}{2}Q\}$  where  $Q$  is

- A:  $(x - \mu)^T \Sigma^{-1} (x - \mu)$
- B:  $(x - \mu) \Sigma (x - \mu)^T$
- C:  $\Sigma - \mu$

## Question 2: Trivariate normal pdf

What graphical form has the solution to  $f(x) = \text{constant}$ ?

- A: Circle
- B: Parabola
- C: Ellipsoid
- D: Bell shape

### Question 3: Multivariate normal distribution

$X_p \sim N_p(\mu, \Sigma)$ , and  $C$  is a  $k \times p$  constant matrix.  $Y = CX$  is

- A: Chi-squared with  $k$  degrees of freedom
- B: Multivariate normal with mean  $k\mu$
- C: Chi-squared with  $p$  degrees of freedom
- D: Multivariate normal with mean  $C\mu$

## Question 4: Independence

Let  $X \sim N_3(\mu, \Sigma)$ , with

$$\Sigma = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

Which two variables are independent?

- A:  $X_1$  and  $X_2$
- B:  $X_1$  and  $X_3$
- C:  $X_2$  and  $X_3$
- D: None – but two are uncorrelated.

## Question 5: Constructing independent variables?

Let  $X \sim N_p(\mu, \Sigma)$ . How can I construct a vector of independent standard normal variables from  $X$ ?

- A:  $\Sigma(X - \mu)$
- B:  $\Sigma^{-1}(X + \mu)$
- C:  $\Sigma^{-\frac{1}{2}}(X - \mu)$
- D:  $\Sigma^{\frac{1}{2}}(X + \mu)$



## Further reading/resources

- Videos on YouTube by the authors of ISL, Chapter 2



# Acknowledgements

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