

$$\begin{aligned}
 & \log(p_k(x)) \\
 &= \log(\pi_k) + \frac{1}{2} \frac{2\mu_k x}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} \\
 & \quad + \text{const} \text{ (w.r.t. } k) \\
 &= \underbrace{x \frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k) + \text{const}}_{=: S_k(x)}
 \end{aligned}$$

For example:  $K=2$ ,  $\pi_1 = \pi_2$   
 Decision boundary

$$S_1(x) = S_2(x)$$

$$\begin{aligned}
 & x \cdot \frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} + \log(0.5) \\
 &= x \cdot \frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} + \log(0.5)
 \end{aligned}$$

$$\Rightarrow X(\cancel{\mu_1} - \mu_2) = \underbrace{\frac{\mu_1^2}{2} - \frac{\mu_2^2}{2}}$$

$$\frac{1}{2}(\mu_1 + \mu_2)(\cancel{\mu_1} - \mu_2)$$

$$\Rightarrow X = \frac{\mu_1 + \mu_2}{2}$$


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$$S_1(x) = S_2(x)$$

$$x \frac{\hat{\mu}_1}{\hat{\sigma}^2} - \frac{\hat{\mu}_1^2}{2\hat{\sigma}^2} + \log(\hat{\Pi}_1) = x \frac{\hat{\mu}_2}{\hat{\sigma}^2} - \frac{\hat{\mu}_2^2}{2\hat{\sigma}^2} + \log(\hat{\Pi}_2)$$

$$\begin{array}{l} \Rightarrow \\ \uparrow \\ \text{reformulate} \end{array} \quad X = \underbrace{\frac{\hat{\mu}_1 + \hat{\mu}_2}{2}} + \underbrace{\hat{\sigma}^2 \frac{\log(\hat{\Pi}_2) - \log(\hat{\Pi}_1)}{(\hat{\mu}_1 - \hat{\mu}_2)}}$$