TMA4268 Statistical Learning

Module 6: Solution sketches

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Recommended exercise 1

For the least square estimator, the solution can be found in the first session here.

For the maximum likelihood estimator, the solution can be found here.

Recommended exercise 2

```
library(ISLR) # Package with data for an Introduction to Statistical
              # Learning with Applications in R
```

```
# Load Credit dataset
data(Credit)

# Check column names
names(Credit)
```

```
##  [1] "ID"        "Income"    "Limit"     "Rating"    "Cards"
##  [6] "Age"       "Education" "Gender"    "Student"   "Married"
## [11] "Ethnicity" "Balance"
```

```
# Check dataset shape
dim(Credit)
```

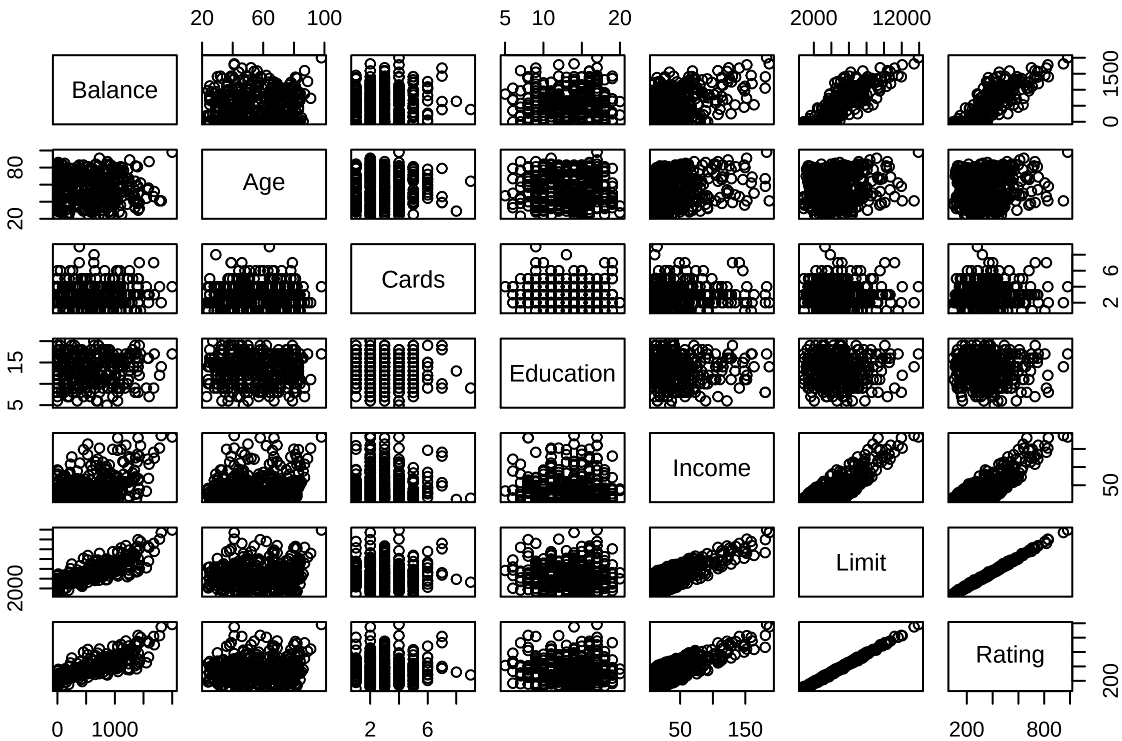
```
## [1] 400  12
```

```
head(Credit)
```

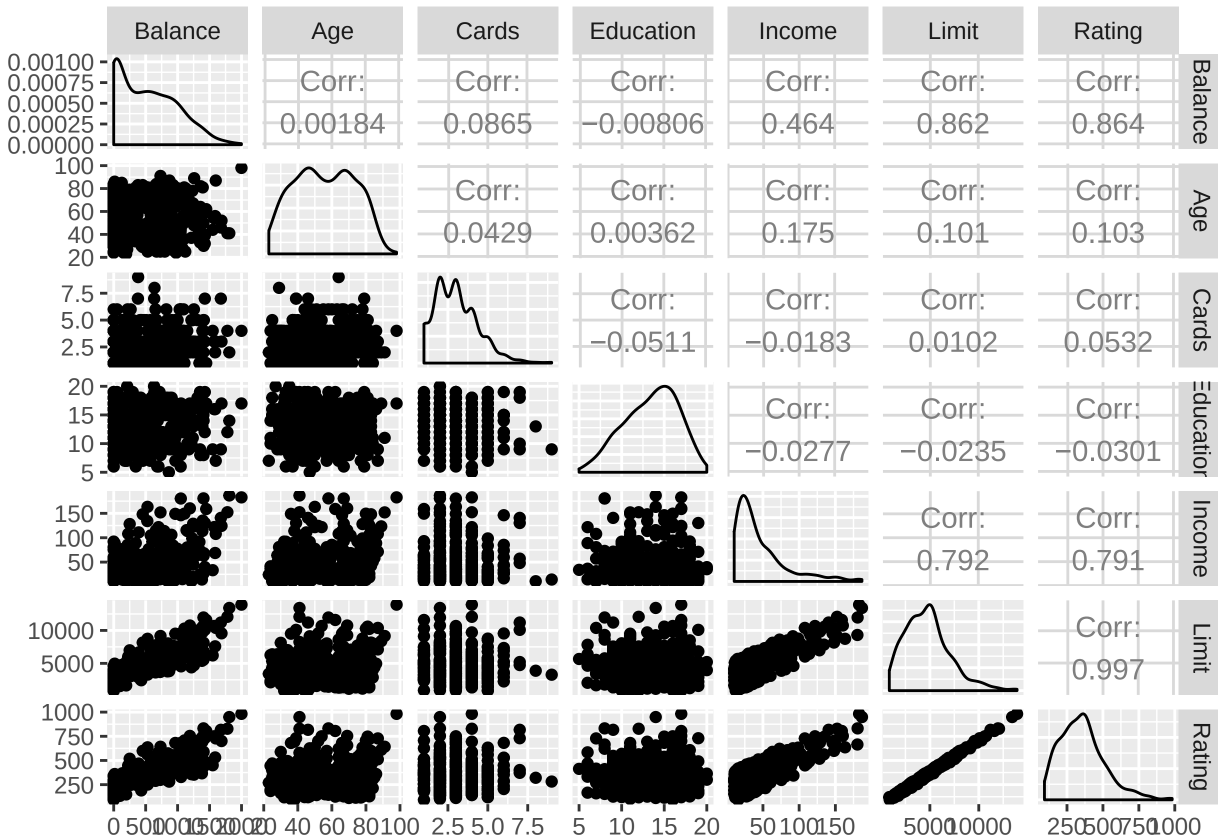
```
##   ID  Income Limit Rating Cards Age Education Gender Student Married
## 1  1  14.891  3606    283     2  34        11   Male      No     Yes
## 2  2 106.025  6645    483     3  82        15 Female     Yes     Yes
## 3  3 104.593  7075    514     4  71        11   Male      No      No
## 4  4 148.924  9504    681     3  36        11 Female      No      No
## 5  5  55.882  4897    357     2  68        16   Male      No     Yes
## 6  6  80.180  8047    569     4  77        10   Male      No      No
##   Ethnicity Balance
## 1 Caucasian     333
## 2     Asian     903
## 3     Asian     580
## 4     Asian     964
## 5 Caucasian     331
## 6 Caucasian    1151
```

```r
# Select variable to plot
pairwise_scatter_data <- Credit[,c("Balance", "Age", "Cards", "Education",
                                   "Income", "Limit", "Rating")]
```

```r
# Simplest possible pairwise scatter plot
pairs(pairwise_scatter_data)
```




```r
# More interesting but slower pairwise plot from package GGally
library(GGally)
ggpairs(data=pairwise_scatter_data)
```



Check here for quick get started to ggpairs

Recommended exercise 3

```
# Exclude 'ID' column
credit_data <- subset(Credit, select=-c(ID))

# Counting the dummy variables as well
credit_data_number_predictors <- 11

# Take a look at the data
head(credit_data)
```

```
##    Income Limit Rating Cards Age Education Gender Student Married
## 1  14.891  3606    283     2  34        11   Male      No     Yes
## 2 106.025  6645    483     3  82        15 Female     Yes     Yes
## 3 104.593  7075    514     4  71        11   Male      No      No
## 4 148.924  9504    681     3  36        11 Female      No      No
## 5  55.882  4897    357     2  68        16   Male      No     Yes
## 6  80.180  8047    569     4  77        10   Male      No      No
##   Ethnicity Balance
## 1 Caucasian     333
## 2     Asian     903
## 3     Asian     580
## 4     Asian     964
## 5 Caucasian     331
```

```
## 6 Caucasian    1151
```

```r
# Summary statistics
summary(credit_data)
```

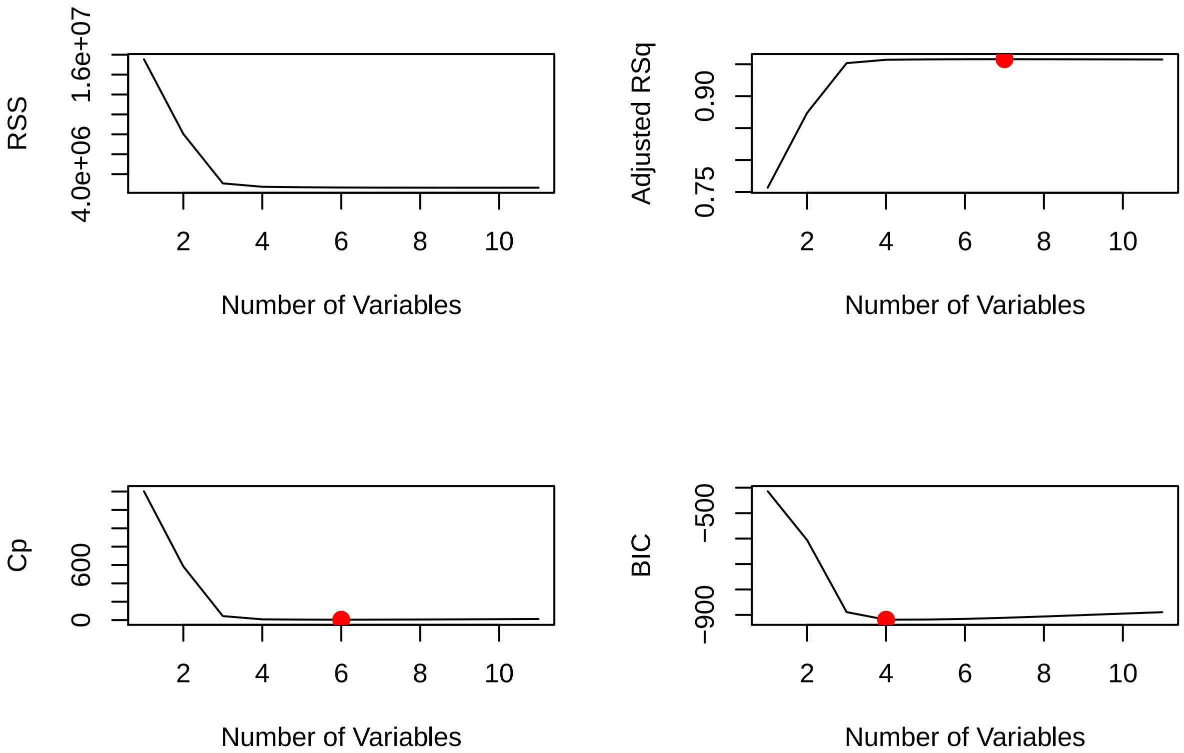
```
##      Income           Limit           Rating          Cards      
##  Min.   : 10.35   Min.   :  855   Min.   : 93.0   Min.   :1.000  
##  1st Qu.: 21.01   1st Qu.: 3088   1st Qu.:247.2   1st Qu.:2.000  
##  Median : 33.12   Median : 4622   Median :344.0   Median :3.000  
##  Mean   : 45.22   Mean   : 4736   Mean   :354.9   Mean   :2.958  
##  3rd Qu.: 57.47   3rd Qu.: 5873   3rd Qu.:437.2   3rd Qu.:4.000  
##  Max.   :186.63   Max.   :13913   Max.   :982.0   Max.   :9.000  
##       Age          Education        Gender     Student   Married  
##  Min.   :23.00   Min.   : 5.00    Male :193   No :360   No :155  
##  1st Qu.:41.75   1st Qu.:11.00   Female:207   Yes: 40   Yes:245  
##  Median :56.00   Median :14.00                                   
##  Mean   :55.67   Mean   :13.45                                   
##  3rd Qu.:70.00   3rd Qu.:16.00                                   
##  Max.   :98.00   Max.   :20.00                                   
##            Ethnicity      Balance       
##  African American: 99   Min.   :   0.00  
##  Asian           :102   1st Qu.:  68.75  
##  Caucasian       :199   Median : 459.50  
##                         Mean   : 520.01  
##                         3rd Qu.: 863.00  
##                         Max.   :1999.00  
```

```r
# Create train and test set indexes
set.seed(1)
train_perc <- 0.75
credit_data_train_index <- sample(1:nrow(credit_data), nrow(credit_data)*train_perc)
credit_data_test_index <- (-credit_data_train_index)

# Create train and test set
credit_data_training <- credit_data[credit_data_train_index, ]
credit_data_testing <- credit_data[credit_data_test_index, ]
```

```r
library(leaps)

# Perform best subset selection using all the predictors and the training data
best_subset_method=regsubsets(Balance~.,credit_data_training,nvmax=credit_data_number_predictors)

# Save summary obj
best_subset_method_summary=summary(best_subset_method)
```

```r
# Plot RSS, Adjusted R^2, C_p and BIC

par(mfrow=c(2,2))
plot(best_subset_method_summary$rss,xlab="Number of Variables",ylab="RSS",type="l")
plot(best_subset_method_summary$adjr2,xlab="Number of Variables",ylab="Adjusted RSq",type="l")
bsm_best_adjr2 = which.max(best_subset_method_summary$adjr2)

points(bsm_best_adjr2,best_subset_method_summary$adjr2[bsm_best_adjr2], col="red",cex=2,pch=20)
plot(best_subset_method_summary$cp,xlab="Number of Variables",ylab="Cp",type='l')
bsm_best_cp=which.min(best_subset_method_summary$cp)
```

```r
points(bsm_best_cp,best_subset_method_summary$cp[bsm_best_cp],col="red",cex=2,pch=20)
bsm_best_bic=which.min(best_subset_method_summary$bic)

plot(best_subset_method_summary$bic,xlab="Number of Variables",ylab="BIC",type='l')
points(bsm_best_bic,best_subset_method_summary$bic[bsm_best_bic],col="red",cex=2,pch=20)
```

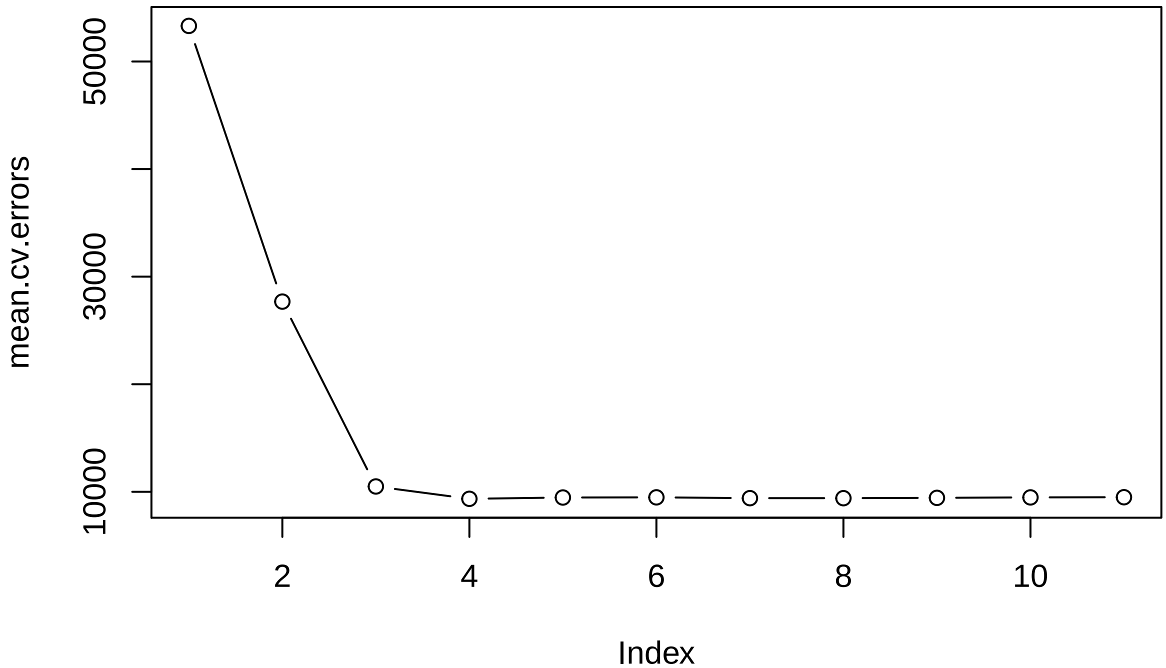



```r
# Create a prediction function to make predictions
# for regsubsets with id predictors included
predict.regsubsets=function(object,newdata,id,...){
  form=as.formula(object$call[[2]])
  mat=model.matrix(form,newdata)
  coefi=coef(object,id=id)
  xvars=names(coefi)
  mat[,xvars]%*%coefi
}

# Create indexes to divide the data between folds
k=10
set.seed(1)
folds=sample(1:k,nrow(credit_data_training),replace=TRUE)
cv.errors=matrix(NA,k,credit_data_number_predictors, dimnames=list(NULL, paste(1:credit_data_number_pre

# Perform CV
for(j in 1:k){
  best_subset_method=regsubsets(Balance~.,data=credit_data_training[folds!=j,],nvmax=credit_data_number_
  for(i in 1:credit_data_number_predictors){
    pred=predict(best_subset_method,credit_data_training[folds==j,],id=i)
    cv.errors[j,i]=mean( (credit_data_training$Balance[folds==j]-pred)^2)
    }
}
```

```
# Compute mean cv errors for each model size
mean.cv.errors=apply(cv.errors,2,mean)
mean.cv.errors
```

```
##         1         2         3         4         5         6         7
## 53308.978 27681.063 10497.276  9349.190  9468.743  9484.566  9410.272
##         8         9        10        11
##  9409.024  9437.443  9480.517  9496.783
```

```
# Plot the mean cv errors
par(mfrow=c(1,1))
plot(mean.cv.errors,type='b')
```




```
# Fit the selected model using the whole training data
# and compute test error

# models selected
number_predictors_selected <- 4

# Create info for lm call
variables <- names(coef(best_subset_method,id=number_predictors_selected))
variables <- variables[!variables %in% "(Intercept)"]
bsm_formula <- as.formula(best_subset_method$call[[2]])
bsm_design_matrix <- model.matrix(bsm_formula,credit_data_training)[, variables]
bsm_data_train <- data.frame(Balance = credit_data_training$Balance, bsm_design_matrix)

# Fit a standard linear model using only the selected
# predictors on the training data
model_best_subset_method <- lm(formula = bsm_formula, bsm_data_train)
summary(model_best_subset_method)
```

```
##
## Call:
```

```
## lm(formula = bsm_formula, data = bsm_data_train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -160.26  -76.81  -11.21   48.15  350.49
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -5.216e+02  1.758e+01 -29.670  < 2e-16 ***
## Income      -7.856e+00  2.651e-01 -29.627  < 2e-16 ***
## Limit        2.706e-01  4.001e-03  67.622  < 2e-16 ***
## Cards        2.426e+01  3.981e+00   6.094 3.43e-09 ***
## StudentYes   4.196e+02  1.782e+01  23.542  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 96.14 on 295 degrees of freedom
## Multiple R-squared:  0.9575, Adjusted R-squared:  0.9569
## F-statistic:  1661 on 4 and 295 DF,  p-value: < 2.2e-16
```

```r
# Make predictions on the test set
bsm_design_matrix_test <- model.matrix(bsm_formula,credit_data_testing)[, variables]
bsm_predictions <- predict(object = model_best_subset_method, newdata = as.data.frame(bsm_design_matrix_

# Compute test squared errors
bsm_squared_errors <- (credit_data_testing$Balance-bsm_predictions)^2
squared_errors <- data.frame(bsm_squared_errors=bsm_squared_errors)


# test MSE
mean(bsm_squared_errors)
```

```
## [1] 12243.75
```

Recommended exercise 4

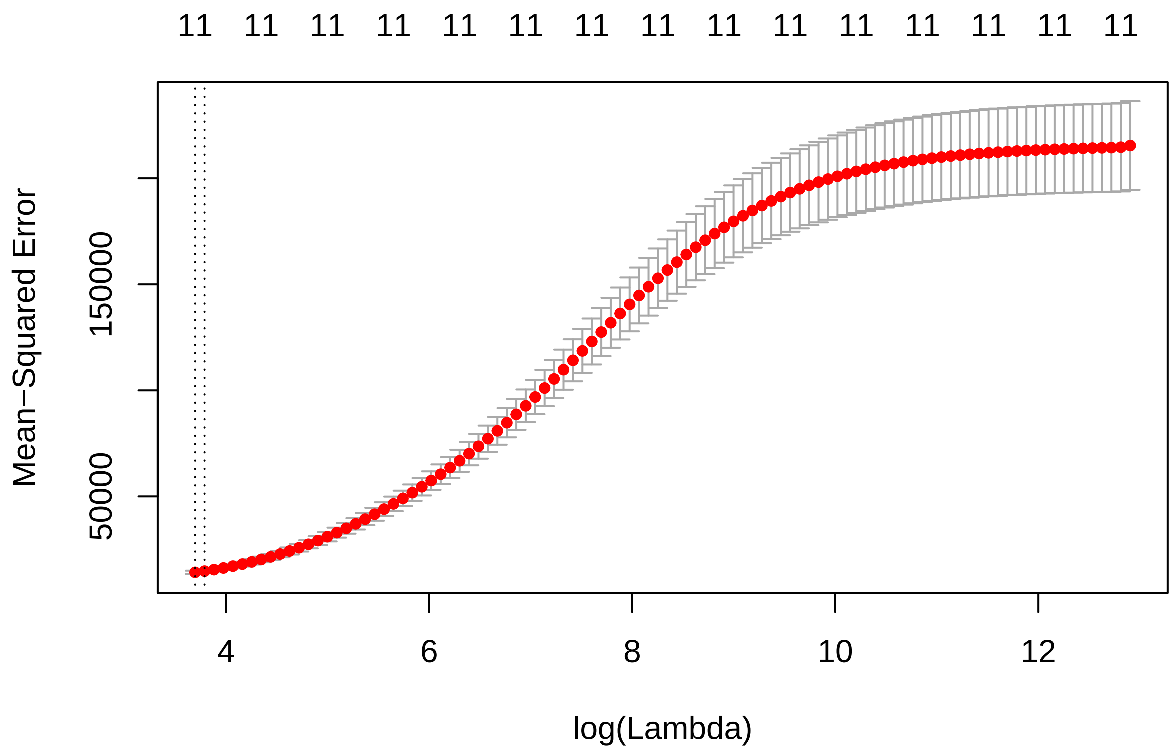
Similar analysis as previous exercise, simply replace Best Subset Selection (`best_subset_method=regsubsets(Balance~.,cred`
by Forward Stepwise Selection (`regfit.fwd=regsubsets(Balance~.,credit_data,nvmax=credit_data_number_predictor`
Backward Stepwise Selection (`regfit.fwd=regsubsets(Balance~.,credit_data,nvmax=credit_data_number_predictors`
and Hybrid Stepwise Selection (`regfit.fwd=regsubsets(Balance~.,credit_data,nvmax=credit_data_number_predictor`

Recommended exercise 5

```r
library(glmnet) # Package Lasso and Elastic-Net Regularized
                # Generalized Linear Models
```

```r
x_train <- model.matrix(Balance~.,credit_data_training)[,-1]
y_train <- credit_data_training$Balance

x_test <- model.matrix(Balance~.,credit_data_testing)[,-1]
y_test <- credit_data_testing$Balance

ridge_mod <- glmnet(x_train,y_train,alpha=0)
```

```r
set.seed(1)
cv.out=cv.glmnet(x_train, y_train,alpha=0)
plot(cv.out)
```



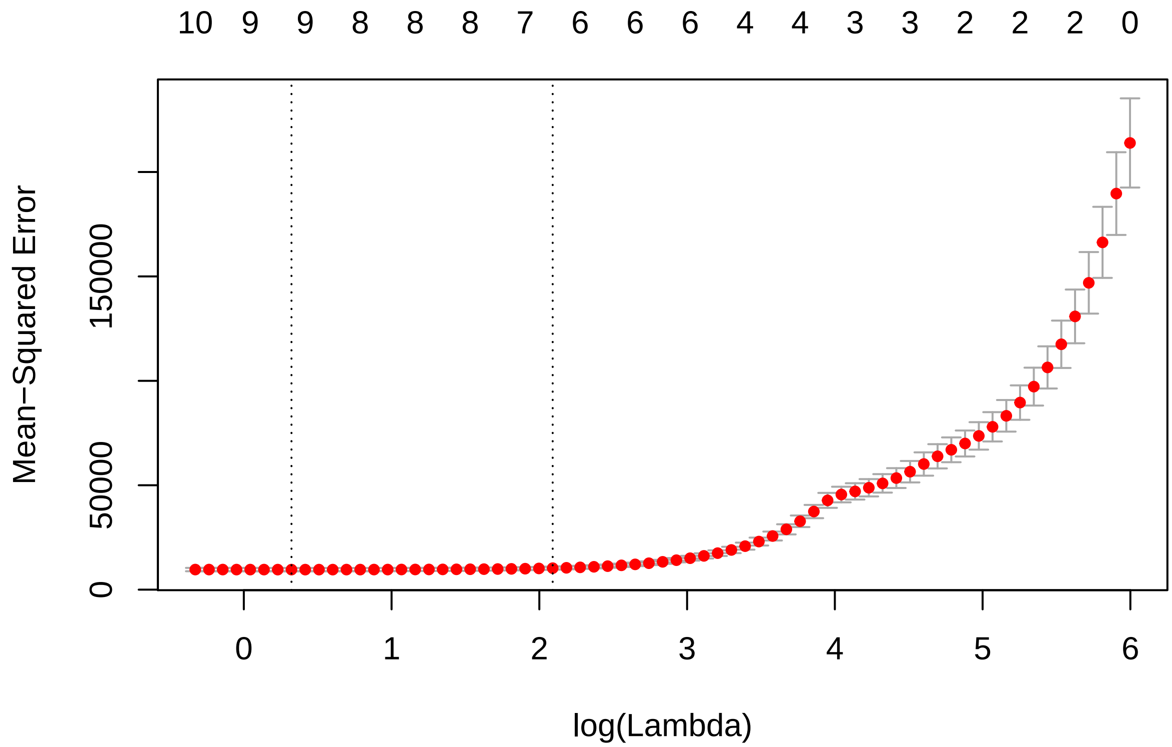

```r
best_lambda_ridge <- cv.out$lambda.min
best_lambda_ridge
```

```
## [1] 40.24862
```

```r
ridge_predictions = predict(ridge_mod,s=best_lambda_ridge,newx=x_test)
ridge_square_errors <- as.numeric((ridge_predictions-y_test)^2)
squared_errors <- data.frame(ridge_square_errors = ridge_square_errors, squared_errors)
```

Recommended exercise 6

```r
lasso_mod <- glmnet(x_train,y_train,alpha=1)

set.seed(1)
cv.out=cv.glmnet(x_train, y_train,alpha=1)
plot(cv.out)
```




```
best_lambda_lasso <- cv.out$lambda.min
best_lambda_lasso
```

```
## [1] 1.380717
```

```
lasso_predictions = predict(lasso_mod,s=best_lambda_lasso,newx=x_test)
lasso_square_errors <- as.numeric((lasso_predictions-y_test)^2)
squared_errors <- data.frame(lasso_square_errors = lasso_square_errors, squared_errors)
```

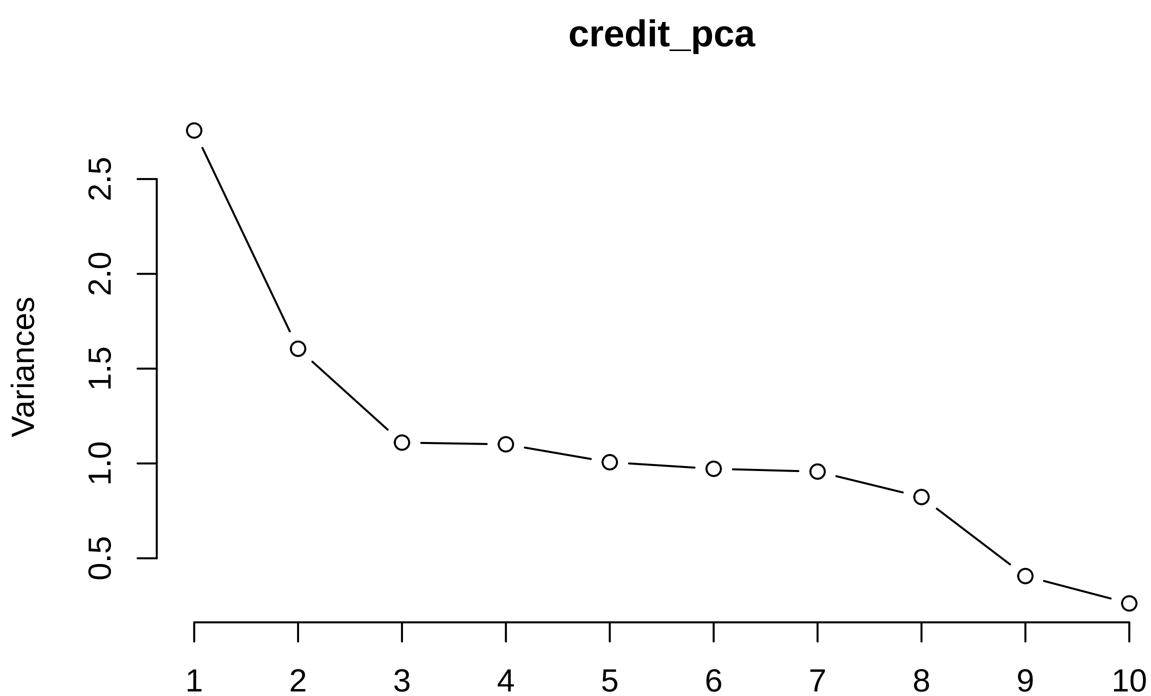
Recommended exercise 7

```
x <- model.matrix(Balance~.,credit_data)[,-1]

credit_pca <- prcomp(x, center = TRUE, scale. = TRUE)

print(credit_pca)
```

```
## Standard deviations (1, .., p=11):
##  [1] 1.66007642 1.26685832 1.05356810 1.04926273 1.00322222 0.98576693
##  [7] 0.97830708 0.90714714 0.63722533 0.51174012 0.04617646
##
## Rotation (n x k) = (11 x 11):
##                       PC1          PC2          PC3           PC4
## Income       -0.542206953  0.029036783 -0.033270648 -6.564051e-05
## Limit        -0.586332930  0.017502630 -0.024351723  4.678929e-02
## Rating       -0.586751867  0.014971105 -0.004630758  3.687909e-02
## Cards        -0.019086978 -0.008549632  0.479005750 -2.720228e-01
## Age          -0.122783390 -0.071116603  0.107188498 -4.787335e-01
## Education     0.026797471  0.096557225 -0.475418336  1.990653e-01
## GenderFemale -0.002519860  0.052811098 -0.334014058 -4.207748e-02
```

```
## StudentYes           0.002276904  0.125422970 -0.618650527 -2.963169e-01
## MarriedYes          -0.026218561  0.094278214  0.125718135  7.389864e-01
## EthnicityAsian       0.032769895  0.696759512  0.105703127  6.686132e-03
## EthnicityCaucasian -0.004070799 -0.686505857 -0.100240068  1.338718e-01
##                            PC5          PC6          PC7          PC8
## Income              -0.02816858  0.02297156 -0.04086888  0.03502243
## Limit                0.02393728  0.06109959  0.02753603 -0.07998103
## Rating               0.03044748  0.04901285  0.06298342 -0.07474080
## Cards                0.07450235 -0.28313105  0.77070237 -0.10917776
## Age                 -0.29468570 -0.58353604 -0.35860755  0.41270188
## Education           -0.58335540 -0.40244676  0.21601791 -0.41794930
## GenderFemale         0.74620452 -0.51375214 -0.10203846 -0.22746095
## StudentYes           0.05874438  0.20236658  0.42777847  0.53366278
## MarriedYes           0.04850438 -0.32419986  0.13571418  0.53676497
## EthnicityAsian       0.02125450  0.01284830 -0.04334986  0.01824866
## EthnicityCaucasian  0.04400214 -0.02306227  0.10322555  0.06987098
##                             PC9         PC10          PC11
## Income              -0.016018928  0.836411394  0.0017092799
## Limit               -0.010697575 -0.379489022  0.7053633132
## Rating              -0.005366527 -0.373834509 -0.7081335719
## Cards                0.005357720  0.059511066  0.0305564113
## Age                 -0.048994454 -0.102540342  0.0005901693
## Education           -0.021973159  0.014172918 -0.0036133922
## GenderFemale         0.014513597  0.027300122  0.0001327203
## StudentYes           0.022068488 -0.032119354  0.0044219212
## MarriedYes           0.119017609 -0.018248384  0.0051766487
## EthnicityAsian      -0.706522468 -0.014783578 -0.0035849536
## EthnicityCaucasian -0.694731116  0.008145839 -0.0004464620
```

```r
plot(credit_pca, type = "l")
```



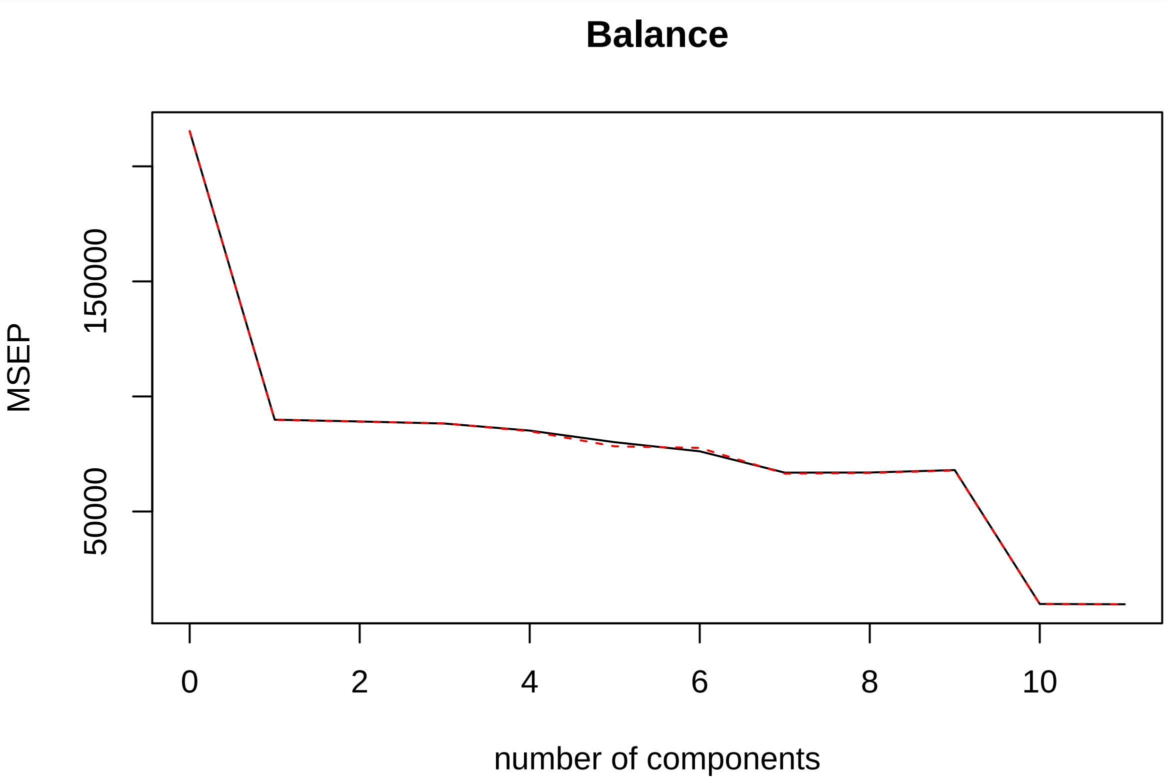
```r
summary(credit_pca)
```

```
## Importance of components:
##                           PC1    PC2    PC3    PC4    PC5     PC6     PC7
## Standard deviation     1.6601 1.2669 1.0536 1.0493 1.0032 0.98577 0.97831
## Proportion of Variance 0.2505 0.1459 0.1009 0.1001 0.0915 0.08834 0.08701
## Cumulative Proportion  0.2505 0.3964 0.4973 0.5974 0.6889 0.77727 0.86427
##                            PC8     PC9    PC10    PC11
## Standard deviation     0.90715 0.63723 0.51174 0.04618
## Proportion of Variance 0.07481 0.03691 0.02381 0.00019
## Cumulative Proportion  0.93908 0.97600 0.99981 1.00000
```

The first PC explain along 25% of the variability in the data. Then the second PC explain an extra 15% of the variability in the data. From the third PC until 8th PC the extra variability explained per PC varies between 7.5% to 10%, dropping to 3.6% on the 9th PCA. So I would likely use 8 PCs for the Credit dataset.

Recommended exercise 8

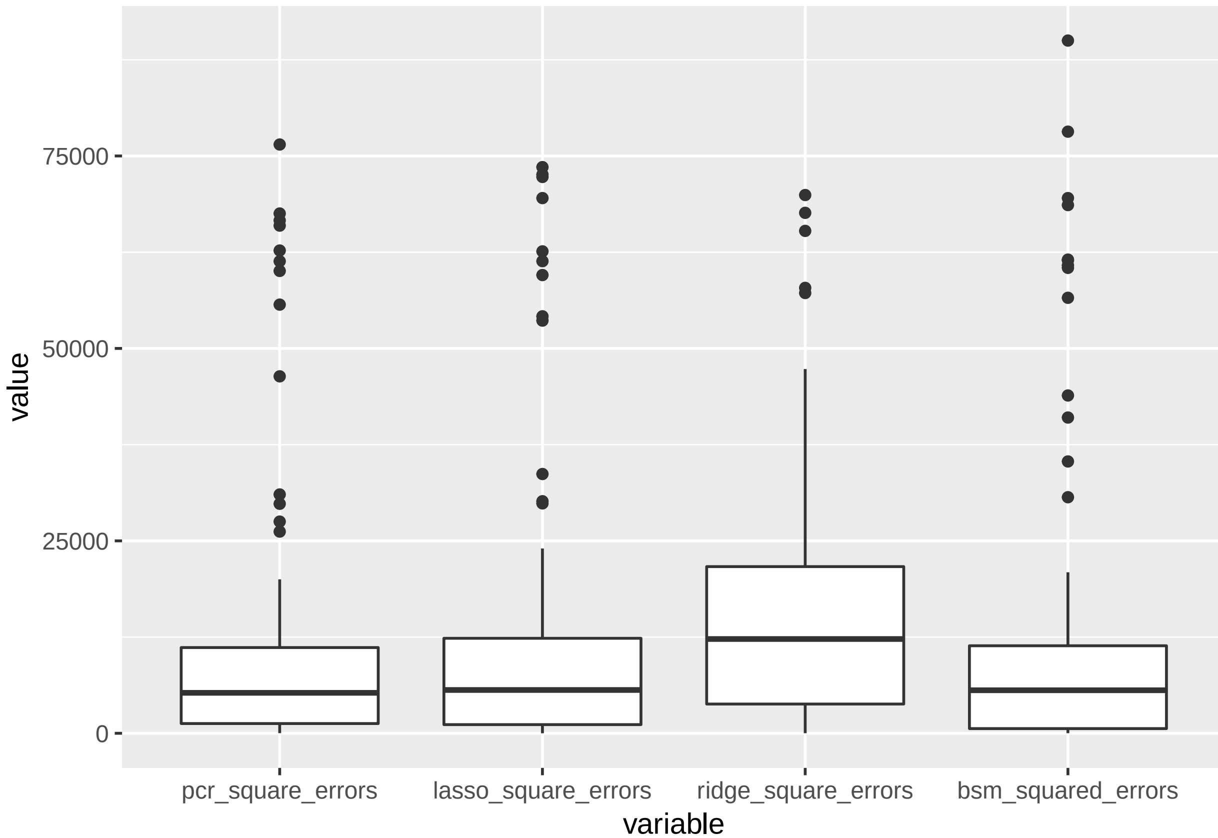
```r
library(pls)

set.seed(1)

pcr_model <- pcr(Balance~., data=credit_data_training,scale=TRUE, validation="CV")
validationplot(pcr_model,val.type="MSEP")
```




```r
pcr_predictions = predict(pcr_model,credit_data_testing,ncomp=10)
pcr_square_errors <- as.numeric((pcr_predictions-credit_data_testing$Balance)^2)
squared_errors <- data.frame(pcr_square_errors = pcr_square_errors, squared_errors)
mean(pcr_square_errors)
```

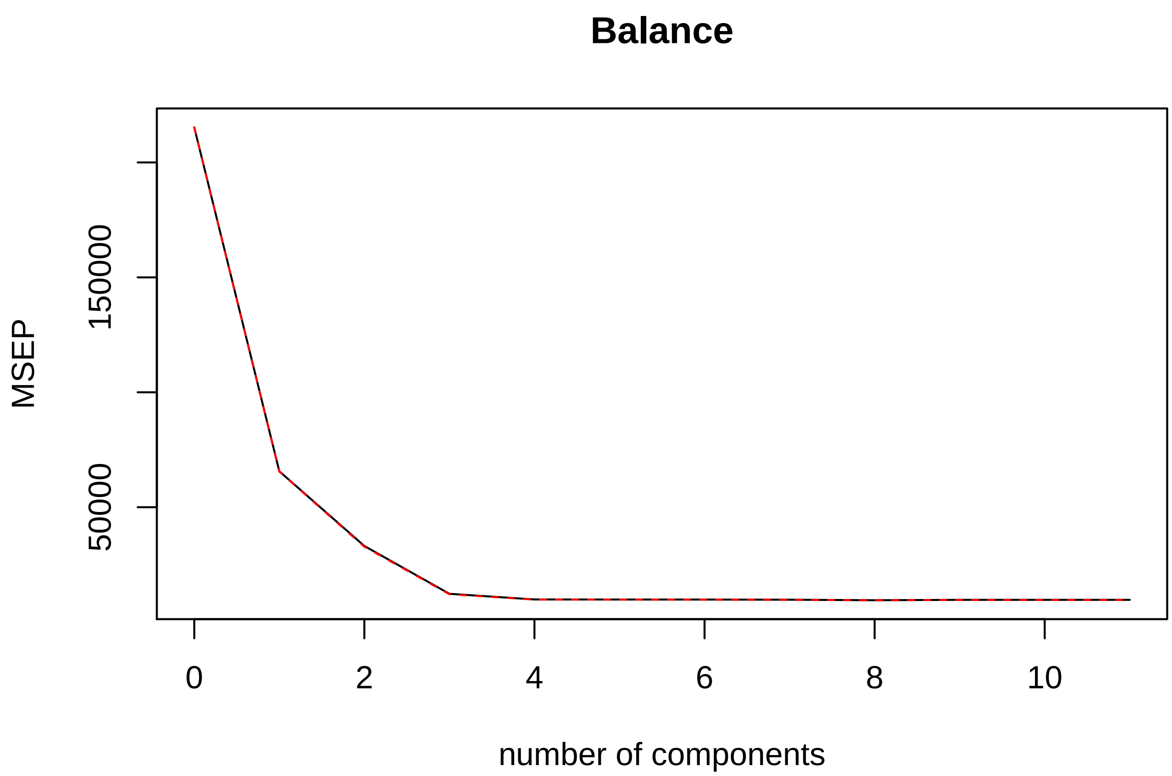
```
## [1] 11578.1
```

```r
library(ggplot2)
library(reshape2)
ggplot(melt(squared_errors)) + geom_boxplot(aes(variable, value))
```



Recommended exercise 9

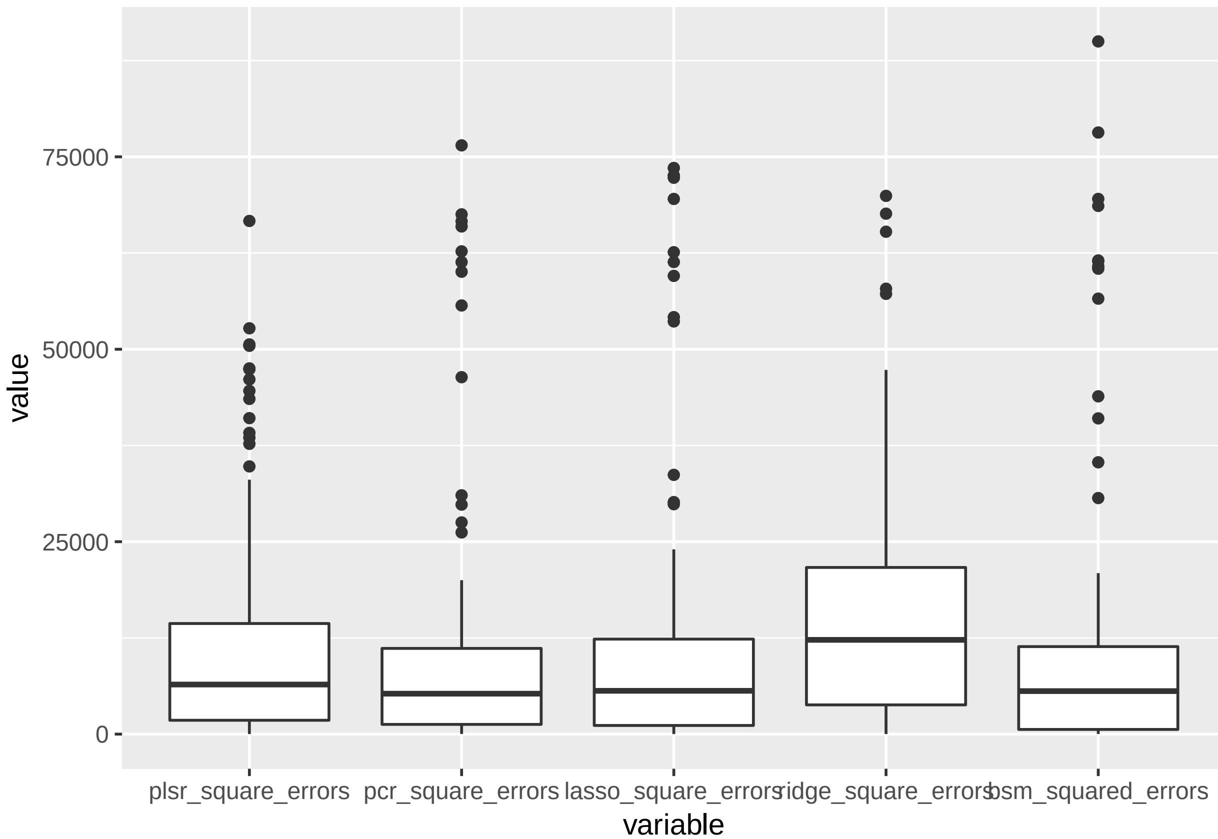
```r
library(pls)

set.seed(1)

plsr_model <- plsr(Balance~., data=credit_data_training,scale=TRUE, validation="CV")
validationplot(plsr_model,val.type="MSEP")
```




```
plsr_predictions = predict(plsr_model,credit_data_testing,ncomp=3)
plsr_square_errors <- as.numeric((plsr_predictions-credit_data_testing$Balance)^2)
squared_errors <- data.frame(plsr_square_errors = plsr_square_errors, squared_errors)
mean(plsr_square_errors)
```

```
## [1] 12476.32
```

```
ggplot(melt(squared_errors)) + geom_boxplot(aes(variable, value))
```




```r
colMeans(squared_errors)
```

```
##  plsr_square_errors   pcr_square_errors lasso_square_errors
##            12476.32            11578.10            12077.15
## ridge_square_errors  bsm_squared_errors
##            15742.83            12243.75
```