Simulation of trajectories of charged particles from solar winds in Earths magnetosphere

Kasper Eikeland

Department of Physics, Norwegian University of Science and Technology
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1 Introduction

The charged particles originating from solar winds ionoze the gas in the upper atmosphere of the Earth, which in turn gives rise to the auroras seen in the nightsky of the Northern hemisphere. (*source?*) In this study we seek to better understand the motion and trajectories of these charged particles subject to the Lorentz force from the magnetic field of the Earth.

To simulate the trajectories of these charged particles, the magnetic field of the Earth is modelled as a dipole field. This model is only sufficiently accurate for distances less than $6R_E$ (cite Baumjohann and Treumann book). A more accurate model would be the double-dipole model of (cite earth magnetic field paper). For the sake of simplicity we compute the trajectories in the dipole model, even for initial positions of magnitude ¿$6R_E$. This is to simulate the particles travelling from the Sun to the Earth. The magnetic north pole is tilted around $11°$ away from the rotational axis of the Earth, which itself is tilted at about $23°$ from the Earth-Sun ecliptic.

According to (Soni et al. 2021) the problem of computing the trajectories for solar wind particles requires higher numerical precision than that of RK4. This is because of numerical instabilities which arise for the trajectories of lower energy protons and electrons. Still, the authors argue that the charged particles from solar winds are accelerated to ~MeV speeds, which make them suitable for simulation using fourth order Runge-Kutta methods. For this study we compute the trajectories of protons with a kinetic energy between 100eV and 10MeV using RK45. The resulting trajectories are validated by considering the conservation of the kinetic energy.

2 Theory

As a magnetic dipole field, the magnetic field of the Earth is given in coordinate-free form (Griffiths 2024) as

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}], \tag{1}$$

where $\boldsymbol{m}$ is the magnetic dipole moment of the Earth, μ_0 is the vacuum permeability and $\boldsymbol{r} = (x, y, z)$ is the position vector. Equation (1) can be written in cartesian coordinates as

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi r^5}\begin{bmatrix} 3x^2 + r^2 & 3xy & 3xz \\ 3xy & 3y^2 + r^2 & 3yz \\ 3xz & 3yz & 3z^2 - r^2 \end{bmatrix}\boldsymbol{m}. \tag{2}$$

At the magnetic equator ($\boldsymbol{r} \perp \boldsymbol{m}$), the magnetic field strength is measured to $B_0 \approx 3.07 \times 10^{-5}$T (Soni et al. 2021). Therefore, the magnitude of the Earths magnetic dipole moment is given by (2) as

$$m = \frac{4\pi}{\mu_0}R_E^3 B_0, \tag{3}$$

where $R_E \approx 6370$km is the radius of the Earth. In this simulation the Earth is placed at the origin of the coordinate system, and the x-axis is along the Sun-Earth axis. The magnetic dipole moment $\boldsymbol{m}$ is rotated $11° + 23°$ around the y-axis.

The relativistic equation of motion for a charged particle of charge q and mass m moving in a magnetic field is given by the Lorentz force

$$\gamma m \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}), \tag{4}$$

where $\boldsymbol{v}$ is the velocity vector of the particle, and γ is the Lorentz factor given as

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{5}$$

where c is the speed of light. For a particle moving only in a magnetic field, γ is constant, as it only depends on the squared magnitude of the velocity, which is unchanged by the Lorentz force (Griffiths 2024). The magnitude of the velocity of the particle is obtained from the kinetic energy of the paricle as

$$v = c\sqrt{1 - \left(\frac{mc^2}{mc^2 + K}\right)^2} \tag{6}$$

where K is the kinetic energy. The position of the particle can be found by integrating the position-velocity relation

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}. \tag{7}$$

Equations (4) and (7) are solved using RK45 (Dormand and Prince 1980).

In a uniform magnetic field, the charged particles will gyrate around the magnetic field lines with a frequency of

$$\omega_g = \frac{q|\boldsymbol{B}|}{\gamma m}, \tag{8}$$

stemming from the cross product in (4). The dipole field (2) however, is non-uniform with a non-zero gradient $\nabla|\boldsymbol{B}|$. This curvature and gradient of the field introduces two additional forms of motion: a periodic bouncing motion along the magnetic field lines, and an azimuthal drift around the axis of the magnetic dipole (Soni et al. 2021). Due to the component of $\boldsymbol{v}$ parallell to $\boldsymbol{B}$ in (4), the particles move along the magnetic field lines. When the particle moves towards a region of higher $|\boldsymbol{B}|$, it reflects. *why?* Thus the particle oscillates between these mirror points along the magnetic field lines. According to (Soni et al. 2021), the gradient is responsible for the azimuthal drift motion. *more info here?*

3 Methods

For this analysis, different trajectories are computed for different initial conditions. The simulations are run for protons of kinetic energy between 100eV and 10MeV. The initial velocities of the protons is then given by $\boldsymbol{v_0} = [v, 0, 0]$ where v is given by equation (6) with $m = m_p$. The initial positions of the protons are given as

$$
\begin{aligned}
x_0 &\in [-2R_E, -12R_E], \\
y_0 &\in [-R_E, R_E], \\
z_0 &\in [-R_E, R_E].
\end{aligned}
$$

The values for x_0, y_0, z_0 are sampled evenly in the given intervals with a sample size of $n_x = 11$, n_y, $n_z = 3$. Thus, for each level of kinetic energy, $n_x \cdot n_y \cdot n_z = 99$ trajectories are computed. All protons are simulated as particles travelling from the Sun to the Earth, i.e. $x_0 < 0$ and $v_y, v_z = 0$. All trajectories are computed for a timespan of 120s.

The simulation code is written in the Python programming language. The equations of motion (4) and (7) are solved using RK45 as implemented in the Python library SciPy (SciPy 2024). Consult the supplementary material (*include?*) for further details.

4 Results

Two-dimensional plots of the magnetic field lines are shown in figure 1 for both the xz-plane at $y = 0$ and the xy-plane at $z = -2R_E$. Observe that the magnetic poles are tilted away from the z-axis.

Figure 2 (a-h) shows the projection onto the xy- and xz-plane of 4 different trajectories computed for different initial conditions. The red circle indicates the Earth. In all cases, the less energetic protons cover less distance in the same time span as the more energetic protons. The trajectories given in figures 2 (a-c) and (e-g) display drift motion of the protons around the axis of the magnetic dipole. Furthermore, the trajectories display both bouncing along the field lines and gyration around the field lines. The gyrofrequency seem to increase as they approach the mirror points, as depicted in figures 2b, 2c, 2f and 2g. Figures 2a and 2e display a higher frequency for the drift motion than that of figures 2b, 2c, 2f and 2g. The bouncing frequency of figure 2b and 2f is evidently higher than that of figures 2c and 2g. Figures 2d and 2h show the trajectory of a proton starting far away from the Earth. In this case, the proton escapes the magnetic field of the Earth. For similar initial conditions however, the motion displayed in figures 2a and 2e is still confined to the Earth's magnetoshpere. *calculation of the frequencies from the data compared to the theoretical calculations?*

To validate the computed trajectories we use the fact that the kinetic energy is conserved for a charged particle only subject to an external magnetic field (Griffiths 2024). Figure 3 shows the deviation of the kinetic energy for 5 different simulations of protons at 100eV, 10keV and 1MeV as a function of time. Notice that the kinetic energy grows with time for the protons with initial position close to the Earth.

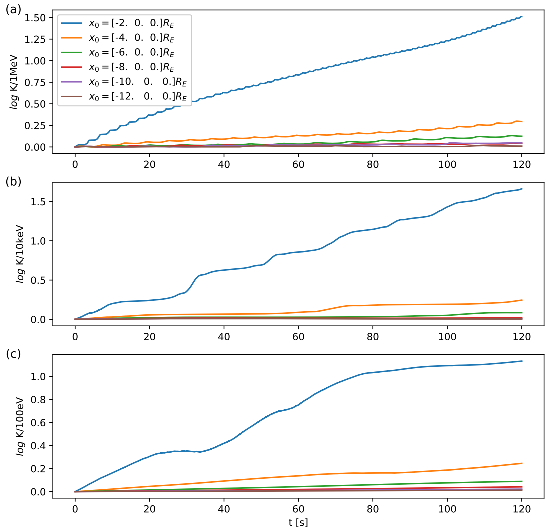


Figure 3: The kinetic energy of the simulated particles as a function of time for different initial conditions. (a): 1MeV protons, (b): 10keV protons, (c): 100eV protons.

5 Discussion

link motion shown in the trajectories to the theory The trajectories for $|\boldsymbol{r}| > 6R_E$ might not make much sense as the dipole model of the magnetic field is not as valid for these distances.

The results from considering the conservation of kinetic energy shows that the kinetic energy grows with time when the protons are close to the Earth. This indicates that there is a need for a higher precision numerical scheme for this case. Moreover, the results show no significant sign of increased numer-

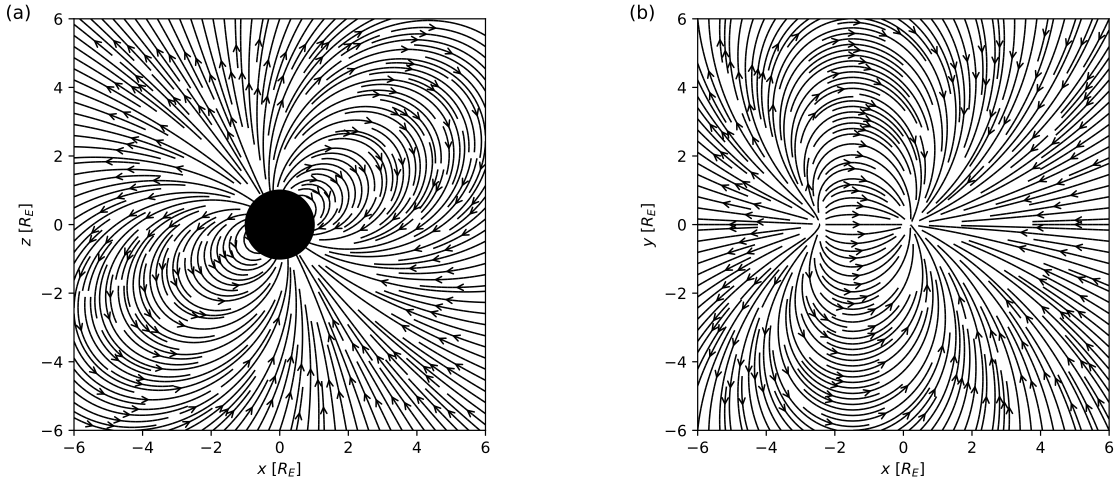
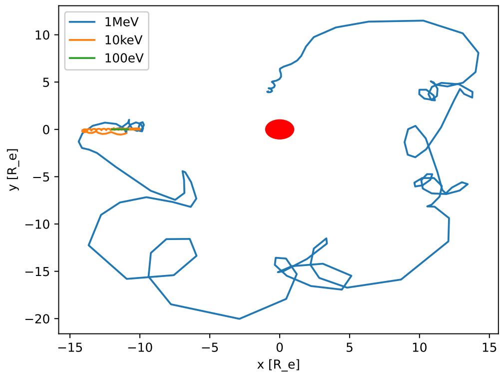


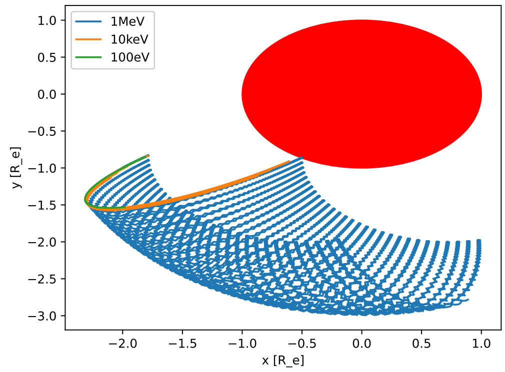
Figure 1: Streamline plots of the magnetic field lines computed via equation (2). (a): The magnetic field lines in the xz-plane at $y = 0$. The black circle indicates the Earths. (b): The magnetic field lines in the xy-plane at $z = -1R_E$. The magnetic equator is tilted around the y-axis towards the sun, i.e in the negative x-direction

ical instability for protons with lower energy. This may be due to the higher precision of RK45 as implemented in SciPy compared to the ordinary RK4 method, as described in (Dormand and Prince 1980). Thus it seems reasonable to apply RK45 for the task of simulating the trajectories of charged particles in Earth's magnetoshpere. *Can it be that the complexity of RK6 is higher than that of RK45, thus giving credit to RK45 as a superior method for this task seeing as it is less complex yet yields competitive results?*

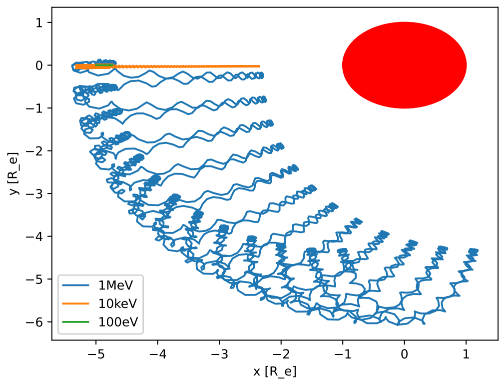
6 Conclusion



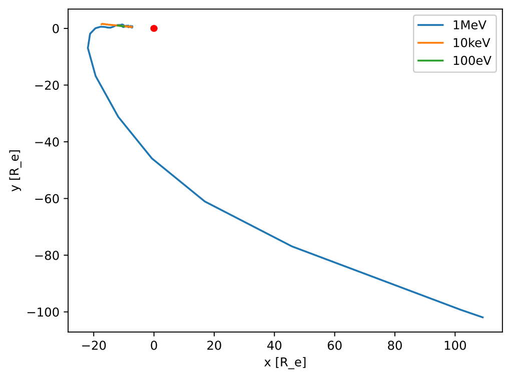
(a) $x_0 = [-12.0, 0.0, -1.0]R_E$, xy-plane.



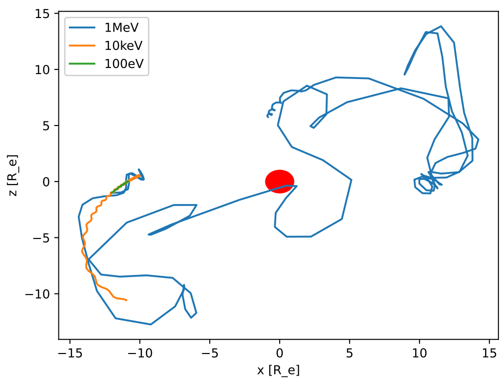
(b) $x_0 = [-2.0, -1.0, 0.0]R_E$, xy-plane.



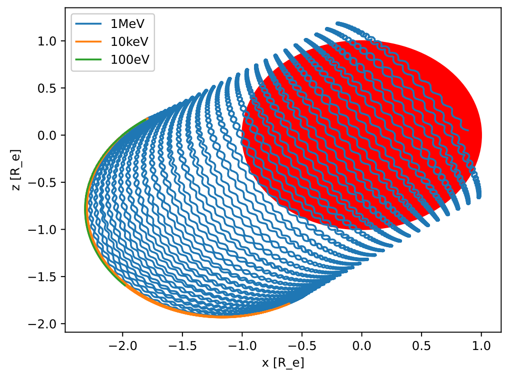
(c) $x_0 = [-5.0, 0.0, -1.0]R_E$, xy-plane.



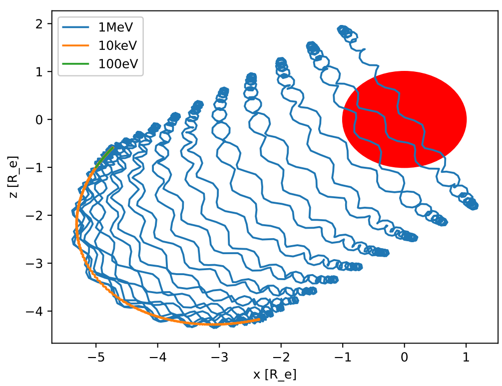
(d) $x_0 = [-12.0, 1.0, 1.0]R_E$, xy-plane.



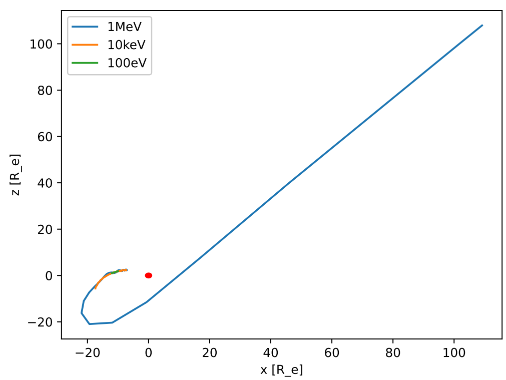
(e) $x_0 = [-12.0, 0.0, -1.0]R_E$, xz-plane.



(f) $x_0 = [-2.0, -1.0, 0.0]R_E$, xz-plane.



(g) $x_0 = [-5.0, 0.0, -1.0]R_E$, xz-plane.



(h) $x_0 = [-12.0, 1.0, 1.0]R_E$, xz-plane.

Figure 2: Trajectories of 100eV, 10keV and 1MeV protons projected onto the xy- and xz-planes. The initial positions of the protons are noted in the subcaptions.

References

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