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► *Prove that the expression (18.112) yields the nth Laguerre polyonmial.*

Evaluating the nth derrivative in (18.112) using Leibnitz‘ theorem, we find

$$\begin{aligned}L_n(x) &= \frac{e^x}{n!}\sum_{r=0}^{n} {}^nC_r, \frac{d^r x^n}{dx^r}\frac{d^{n-r}e^{-x}}{dx^{n-r}} \\ &= \frac{e^x}{n!}\sum_{r=0}^{n}, \frac{n!}{r!(n-r)!}\frac{n!}{(n-r)!}x^{n-r}(-1)^{n-r}e^{-x} \\ &= \sum_{r=0}^{n}(-1)^{n-r}\frac{n!}{r!(n-r)!(n-r)!}x^{n-r}.\end{aligned}$$

Rellabelling the summation using the index $m = n - r$, we obtain

$$L_n(x) = \sum_{m=0}^{n}(-1)^m \frac{n!}{(m!)^2(n-m)!}x^m,$$

which is precisely the expression (18.111) for the nth Laguerre polynomial.◄

Mutual orthogonality

In section 17.4, we noted that Laguerre's equation could be put into Sturm-Liouville form with $p = xe^{-x}, q = 0, \lambda = v$ and $\rho = e^{-x}$, and its natural interval is thus [0,∞]. Since the Laguerre polynomials $L_n(x)$ are solutions of the equation and are regular at the end-points, they must be mutually orthogonal over this interval with respect to the weight function $\rho = e^{-x}$, i.e.

$$\int_0^\infty L_n(x)L_k(x)e^{-x}dx = 0 \qquad \text{if } n \neq k.$$

This result may also be proved directly using the Rodrigues‘ formula (18.112). Indeed, the normalisation, when $k = n$, is most easily found using this method.

► *Show that*

$$I \equiv \int_0^\infty L_n(x)L_n(x)e^{-x}dx = 1. \tag{18.113}$$

Using the Rodrigues‘ formula (18.112), we may write

$$I = \frac{1}{n!}\int_0^\infty L_n(x)\frac{d^n}{dx^n}(x^n e^{-x})dx = \frac{(-1)^n}{n!}\int_0^\infty \frac{d^n L_n}{dx^n}x^n e^{-x}dx,$$

where, in the second equality, we have integrated by parts n times and used the fact that the boundary terms all vanish. When d^nL_n/dx^n is evaluated using (18.111), only the derivative of the $m = n$ term survives and that has the value $[(-1)^n n!n!]/[(n!)^2 0!] = (-1)^n$. Thus we have

$$I = \frac{1}{n!}\int_0^\infty x^n e^{-x}dx = 1,$$

where, in the second equality, we use the expression (18.153) defining the gamma function (see section 18.12).◄