1 Quaternions and RA/DEC/ROLL

Let us define the unit quaternion $\mathbf{q}$ as $\mathbf{q} = (q_r, q_i, q_j, q_k) \equiv q_r + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$ and let us denote the RA, DEC and ROLL angles by α, δ and ρ, respectively. The quaternion $\mathbf{q}$ is an unit quaternion when $\|\mathbf{q}\| = q_r^2 + q_i^2 + q_j^2 + q_k^2 = 1$. Then, the two SO(3) matrices are equivalent:

$$\mathbf{R} \equiv \begin{pmatrix} R_{\mathrm{xx}} & R_{\mathrm{xy}} & R_{\mathrm{xz}} \\ R_{\mathrm{yx}} & R_{\mathrm{yy}} & R_{\mathrm{yz}} \\ R_{\mathrm{zx}} & R_{\mathrm{zy}} & R_{\mathrm{zz}} \end{pmatrix} = \begin{pmatrix} 1-2(q_j^2+q_k^2) & 2(q_iq_j - q_kq_r) & 2(q_iq_k + q_jq_r) \\ 2(q_iq_j + q_kq_r) & 1-2(q_i^2+q_k^2) & 2(q_jq_k - q_iq_r) \\ 2(q_iq_k - q_jq_r) & 2(q_jq_k + q_iq_r) & 1-2(q_i^2+q_j^2) \end{pmatrix} \tag{1}$$

and

$$\begin{aligned}
\mathbf{R} &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sin\delta & 0 & \cos\delta \\ 0 & 1 & 0 \\ -\cos\delta & 0 & \sin\delta \end{pmatrix} \cdot \begin{pmatrix} -\cos\rho & \sin\rho & 0 \\ -\sin\rho & -\cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix} = & (2)\\
&= \begin{pmatrix} \cos\alpha\sin\delta & -\sin\alpha & \cos\alpha\cos\delta \\ \sin\alpha\sin\delta & \cos\alpha & \sin\alpha\cos\delta \\ -\cos\delta & 0 & \sin\delta \end{pmatrix} \cdot \begin{pmatrix} -\cos\rho & \sin\rho & 0 \\ -\sin\rho & -\cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix} = & (3)\\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin\delta\cos\rho & \sin\delta\sin\rho & \cos\delta \\ -\sin\rho & -\cos\rho & 0 \\ \cos\delta\cos\rho & -\cos\delta\sin\rho & \sin\delta \end{pmatrix} = & (4)\\
&= \begin{pmatrix} -\cos\alpha\sin\delta\cos\rho + \sin\alpha\sin\rho & \cos\alpha\sin\delta\sin\rho + \sin\alpha\cos\rho & \cos\alpha\cos\delta \\ -\sin\alpha\sin\delta\cos\rho - \cos\alpha\sin\rho & \sin\alpha\sin\delta\sin\rho - \cos\alpha\cos\rho & \sin\alpha\cos\delta \\ \cos\delta\cos\rho & -\cos\delta\sin\rho & \sin\delta \end{pmatrix} & (5)
\end{aligned}$$

The boresight pointing is then defined by the $z+$ axis, i.e.:

$$\mathbf{p} = \mathbf{R} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{pmatrix} \tag{6}$$

In order to obain $\mathbf{q} = (q_r, q_i, q_j, q_k)$ from the matrix $\mathbf{R}$, one can use the following equation:

$$(q_r, q_i, q_j, q_k) = \left(\frac{\sqrt{1 + R_{\mathrm{xx}} + R_{\mathrm{yy}} + R_{\mathrm{zz}}}}{2}, \frac{R_{\mathrm{zy}} - R_{\mathrm{yz}}}{K}, \frac{R_{\mathrm{xz}} - R_{\mathrm{zx}}}{K}, \frac{R_{\mathrm{yx}} - R_{\mathrm{xy}}}{K} \right) \tag{7}$$

where $K = 2\sqrt{1 + R_{\mathrm{xx}} + R_{\mathrm{yy}} + R_{\mathrm{zz}}}$. This formula is numerically unstable if $q_r \to 0$, or $R_{\mathrm{xx}} + R_{\mathrm{yy}} + R_{\mathrm{zz}} \to -1$, i.e. close to rotations of 180°.

The values of α, δ and ρ can be retrieved using the equations

$$\begin{aligned}
\alpha &= \arg(R_{\mathrm{xz}}, R_{\mathrm{yz}}) = \arg(q_iq_k + q_jq_r, q_jq_k - q_iq_r) & (8)\\
\delta &= \arcsin(R_{\mathrm{zz}}) = \arcsin[q_r^2 + q_k^2 - q_i^2 - q_j^2] = \arg\left[q_r^2 + q_k^2 - q_i^2 - q_j^2, 2\sqrt{(q_i^2+q_j^2)(q_k^2+q_r^2)}\right] & (9)\\
\rho &= \arg(R_{\mathrm{zx}}, -R_{\mathrm{zy}}) = \arg(q_iq_k - q_jq_r, -q_jq_k - q_iq_r) & (10)
\end{aligned}$$