$F: \mathbb{R}^d \to \mathbb{R}^d$ $\quad Z$ — выпуклое мн-во

найти $z^* \in Z$: $\langle F(z^*), z - z^* \rangle \geq 0 \quad \forall z \in Z$

задача вариационного неравенства

$$Z = \mathbb{R}^d \qquad \langle F(z^*); z - z^* \rangle \geq 0 \quad \forall z \in \mathbb{R}^d$$

$$\Updownarrow$$

$$F(z^*) = 0$$

1) $F(z) = \nabla f(z) \qquad \nabla f(z^*) = 0$

минимизация $\min\limits_{z \in \mathbb{R}^d} f(z)$

2) $F(z) = F(x,y) = \begin{pmatrix} \nabla_x f(x,y) \\ -\nabla_y f(x,y) \end{pmatrix} \qquad z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{d_x + d_y}$

$$\langle F(z^*); z - z^* \rangle \geq 0$$

$$\left\langle \begin{pmatrix} \nabla_x f(x^*, y^*) \\ -\nabla_y f(x^*, y^*) \end{pmatrix}; \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \right\rangle \geq 0$$

$$\langle \nabla_x f(x^*, y^*); x - x^* \rangle + \langle -\nabla_y f(x^*, y^*); y - y^* \rangle \geq 0$$

$f(x,y)$ — выпукла по x ($\forall y \; f(\cdot, y)$ — выпукла)

— вогнута по y ($\forall x \; f(x, \cdot)$ — вогнута)

$$[f(x, y^*) - f(x^*, y^*)] + [f(x^*, y^*) - f(x^*, y)] \geq 0$$

$$\forall x, y \in X \times Y$$

- $x = x^*$

(x^*, y^*) : $\forall y \quad f(x^*, y^*) \geq f(x^*, y)$

- $y = y^*$

(x^*, y^*) : $\forall x \quad f(x, y^*) \geq f(x^*, y^*)$

$$\min_{x \in X} \left(\max_{y \in Y} f(x,y) \right) = \max_{y} \min_{x} f(x,y)$$

седловая задача

---

1) $\| F(z_1) - F(z_2) \| \leq L \| z_1 - z_2 \|$ – липш. F с L

2) $\langle F(z_1) - F(z_2); z_1 - z_2 \rangle \geq \mu \| z_1 - z_2 \|^2$

– μ- сильная монотонность

- $\langle \nabla f(z_1) - \nabla f(z_2); z_1 - z_2 \rangle \geq \mu \| z_1 - z_2 \|^2$

$\Updownarrow$

μ- сильная выпуклость

- из μ- сильной выпуклости и μ- сильной вогнутости $f(x,y)$ следует $\langle F(z_2) - F(z_1); z_2 - z_1 \rangle \geq \mu \| z_1 - z_2 \|^2$

$$\left.\begin{aligned} f(x_1, y) &\geq f(x_2, y) + \langle \nabla_x f(x_2, y); x_1 - x_2 \rangle + \frac{\mu}{2} \| x_1 - x_2 \|^2 \\ f(x_2, y) &\geq f(x_1, y) + \langle \nabla_x f(x_1, y); x_2 - x_1 \rangle + \frac{\mu}{2} \| x_1 - x_2 \|^2 \end{aligned}\right\} \text{выпуклость}$$

$$\left.\begin{aligned} f(x, y_1) &\leq f(x, y_2) + \langle \nabla_y f(x, y_2); y_1 - y_2 \rangle + \frac{\mu}{2} \| y_1 - y_2 \|^2 \\ f(x, y_2) &\leq f(x, y_1) + \langle \nabla_y f(x, y_1); y_2 - y_1 \rangle + \frac{\mu}{2} \| y_1 - y_2 \|^2 \end{aligned}\right\} \text{вогнутость}$$

$y = y_2$

$$\langle \nabla_x f(x_2, y_2); x_2 - x_1 \rangle \geq f(x_2, y_2) - f(x_1, y_2) + \frac{\mu}{2} \| x_1 - x_2 \|^2$$

$y = y_1$

$$- \langle \nabla_x f(x_1, y_1); x_2 - x_1 \rangle \geq f(x_1, y_1) - f(x_2, y_1) + \frac{\mu}{2} \| x_1 - x_2 \|^2$$

$x = x_2$

$$\ldots \geq f(x_2, y_1) - f(x_2, y_2) + \frac{\mu}{2} \| y_1 - y_2 \|^2$$

$$-\langle \nabla_y f(x_2,y_2), y_2 - y_1 \rangle \ge f(x_2,y_1) - f(x_2,y_2) + \frac{\mu}{2}\|y_1-y_2\|^2$$

$$\langle \nabla_y f(x_1,y_1); y_2 - y_1 \rangle \ge f(x_1,y_2) - f(x_1,y_1) + \frac{\mu}{2}\|y_1-y_2\|^2$$

$x = x_1$

Суммируя 4 предыдущих:

$$\langle F(z_2) - F(z_1); z_2 - z_1 \rangle \ge \mu \|z_2 - z_1\|^2$$

обратное то же верно!

$$z^{k+1} = z^k - \gamma \nabla f(z^k)$$

$$z^{k+1} = z^k - \gamma F(z^k)$$

L - липш. опер., μ - сильно - мон. F $\quad Z = \mathbb{R}^d$

$$\|z^{k+1} - z^*\|^2 = \|z^k - z^*\|^2 - 2\gamma \langle F(z^k); z^k - z^* \rangle + \gamma^2 \|F(z^k)\|^2$$

$F(z^*) = 0$

$$= \|z^k - z^*\|^2 - 2\gamma \langle F(z^k) - F(z^*); z^k - z^* \rangle + \gamma^2 \|F(z^k) - F(z^*)\|^2$$

L - лип и μ - сильн. мон

$$\le \|z^k - z^*\|^2 - 2\mu\gamma \|z^k - z^*\|^2 + \gamma^2 L^2 \|z^k - z^*\|^2$$

$$= (1 - 2\mu\gamma + \gamma^2 L^2)\|z^k - z^*\|^2$$

$$-2\mu + 2\gamma L^2 = 0 \qquad \gamma = \frac{\mu}{L^2} \quad \left(GD: \gamma = \frac{1}{L}\right)$$

$$O\left(\frac{L^2}{\mu^2} \log\frac{1}{\varepsilon}\right) \qquad \left(GD: O\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right)\right)$$

$$\min_{x\in\mathbb{R}} \max_{y\in\mathbb{R}} \; xy \quad \begin{array}{l} \to \nabla_x f(x,y) = y \\ \to \nabla_y f(x,y) = x \end{array} \qquad \begin{array}{l} \nabla_x f(x^*,y^*) = 0 \\ \nabla_y f(x^*,y^*) = 0 \end{array}$$

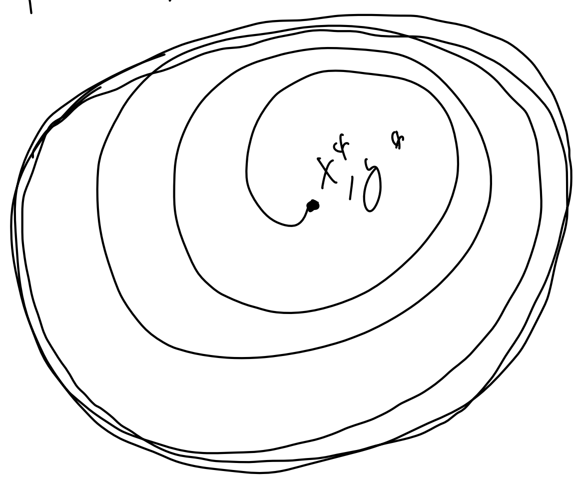
$$x^{k+1} = x^k - \gamma y^k$$
$$y^{k+1} = y^k + \gamma x^k$$

$$x^* = 0$$
$$y^* = 0$$

$$\|z^{k+1} - z^*\|^2 = (x^{k+1} - x^*)^2 + (y^{k+1} - y^*)^2$$
$$= (x^k - \gamma y^k)^2 + (y^k + \gamma x^k)^2$$
$$= (x^k)^2 + \gamma^2 (y^k)^2 - 2\gamma x^k y^k$$
$$(y^k)^2 + \gamma^2 (x^k)^2 + 2\gamma x^k y^k$$
$$= (1+\gamma^2)\left((x^k)^2 + (y^k)^2\right) =$$
$$= (1+\gamma^2)\left((x^k - x^*)^2 + (y^k - y^*)^2\right) =$$
$$= (1+\gamma^2)\,\|z^k - z^*\|^2$$

$$\max_{y\in \bar{Y}} f(x^k, y) - \min_{x \in \bar{X}} f(x, y^k)$$

$\bar{X}, \bar{Y}$ - ограничены



Extra Gradient

$$z^{k+1/2} = z^k - \gamma F(z^k) \qquad Z = \mathbb{R}^d$$

$$z^{k+1} = z^k - \gamma F(z^{k+1/2})$$

$$z^+ = z - y$$

$$\|z^+ - u\|^2 = \|z^+ - z + z - u\|^2$$

$$= \|z-u\|^2 + 2\langle z^+ - z, z - u\rangle + \|z^+ - z\|^2$$

$$= \|z-u\|^2 + 2\langle z^+ - z, z^+ - u\rangle - \|z^+ - z\|^2$$

$$= \|z-u\|^2 + \cancel{2\langle z^+ - (z-y), z^+ - u\rangle}$$

$$- 2\langle y, z^+ - u\rangle - \|z^+ - z\|^2$$

$$= \|z-u\|^2 - 2\langle y, z^+ - u\rangle - \|z^+ - z\|^2$$

---

1) $z^+ = z^{k+1} \quad u = z^* \quad z = z^k \quad y = \gamma F(z^{k+1/2})$

$$\|z^{k+1} - z^*\|^2 = \|z^k - z^*\|^2 - 2\gamma\langle F(z^{k+1/2}); z^{k+1} - z^*\rangle$$

$$- \|z^{k+1} - z^k\|^2$$

+

2) $z^+ = z^{k+1/2} \quad y = \gamma F(z^k) \quad z = z^k \quad u = z^{k+1}$

$$\|z^{k+1/2} - z^{k+1}\|^2 = \|z^k - z^{k+1}\|^2 - 2\gamma\langle F(z^k); z^{k+1/2} - z^{k+1}\rangle$$

$$- \|z^{k+1/2} - z^k\|^2$$

+:

$z^{k+1/2}$

$$\|z^{k+1} - z^*\|^2 = \|z^k - z^*\|^2 - 2\gamma\langle F(z^{k+1/2}); z^{k+1} - z^*\rangle$$

$$+ \qquad -2\gamma\langle F(z^k); z^{k+1/2} - z^{k+1}\rangle - \|z^{k+1/2} - z^k\|^2$$

$$\|z^{k+1/2} - z^{k+1}\|^2$$

$$\|z^{k+1}-z^*\|^2 = \|z^k - z^*\|^2 - 2\gamma\langle F(z^{k+1/2}); z^{k+1/2} - z^*\rangle + \|z^{k+1/2} - z^{k+1}\|^2 - 2\gamma\langle F(z^k) - F(z^{k+1/2}); z^{k+1/2} - z^{k+1}\rangle - \|z^{k+1/2} - z^k\|^2$$

Коши

$$\|z^{k+1}-z^*\|^2 \le \|z^k - z^*\|^2 - 2\gamma\langle F(z^{k+1/2}); z^{k+1/2} - z^*\rangle + \cancel{\|z^{k+1/2} - z^{k+1}\|^2} + \gamma^2\|F(z^k) - F(z^{k+1/2})\| + \cancel{\|z^{k+1} - z^{k+1/2}\|^2} - \|z^{k+1/2} - z^k\|^2$$

Липшицевость

$$\|z^{k+1}-z^*\|^2 \le \|z^k - z^*\|^2 - 2\gamma\langle F(z^{k+1/2}) - F(z^*); z^{k+1/2} - z^*\rangle + \gamma^2 L^2\|z^k - z^{k+1/2}\|^2 - \|z^{k+1/2} - z^k\|$$

сильная - мон.

$$\|z^{k+1}-z^*\|^2 \le \|z^k - z^*\|^2 - 2\gamma\mu\|z^{k+1/2} - z^*\|^2 + \gamma^2 L^2\|z^k - z^{k+1/2}\|^2 - \|z^{k+1/2} - z^k\|$$

$$\|a+b\|^2 \le 2\|a\|^2 + 2\|b\|^2$$

$$-2\|a\|^2 \le -\|a+b\|^2 + 2\|b\|^2$$

$$a = z^{k+1/2} - z^*$$

$$a+b = z^k - z^*$$

$$b = z^k - z^{k+1/2}$$

$$\|z^{k+1}-z^*\|^2 \le \|z^k - z^*\|^2 - \gamma\mu\|z^k - z^*\|^2 + 2\gamma\mu\|z^k - z^{k+1/2}\|^2 + \gamma^2 L^2\|z^k - z^{k+1/2}\|^2 - \|z^{k+1/2} - z^k\|$$

$$= (1-\gamma\mu)\|z^k - z^*\|^2 - (1-2\gamma\mu-\gamma^2L^2)\|z^{k+1/2}-z^k\|^2$$

$$\gamma \le \frac{1}{3L}$$

$$\left.\begin{array}{l} 2\gamma\mu \le \frac{2}{3}\frac{\mu}{L} \le \frac{2}{3} \\ \gamma^2L^2 \le \frac{1}{9} \end{array}\right\} \le 1$$

$$\|z^{k+1}-z^*\|^2 \le (1-\gamma\mu)\|z^k - z^*\|^2$$

$$O\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right)$$

нижняя $\Omega\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right)$ (можно пример седловой)

$$\frac{L}{\sqrt{\mu_x\mu_y}} \qquad \frac{L}{\min(\mu_x;\mu_y)}$$

$$z^{k+1/2} = z^k - \gamma F(z^{k-1/2})$$

$$z^{k+1} = z^k + \gamma F(z^{k+1/2})$$

Past/Optimistic Extra Gradient

$x\,y$

$$z^k = \begin{pmatrix} x^k \\ y^k \end{pmatrix} \qquad F(z^k) = \begin{pmatrix} y^k \\ -x^k \end{pmatrix}$$

$$\langle z^k - z^*, F(z^k) \rangle = 0$$ ортогональны

$$z^k = \begin{pmatrix} x^k \\ y^k \end{pmatrix} \qquad z^{k+1/2} = \begin{pmatrix} x^k - \gamma y^k \\ y^k + \gamma x^k \end{pmatrix} \qquad F(z^{k+1/2}) = \begin{pmatrix} y^k + \gamma x^k \\ -x^k + \gamma y^k \end{pmatrix}$$

$$\langle z^k - z^*, F(z^{k+1/2}) \rangle = x^k y^k + \gamma (x^k)^2 - x^k y^k + \gamma (y^k)^2 \geq 0$$

$$x^T A y$$