Combinatorial geometry - Solutions

1. *[Splitting a square]*

Problem

Each of 9 lines divide a square into two quadrilaterals, whose areas have a proportion of $2:3$ to each other. Prove that there exists a point, where at least three of these lines meet!

Solution - Pidgeonhole

First note that the lines cannot intersect neighbouring sides of a square $ABCD$, because that would result in a triangle and a pentagon. Therefore they intersect the opposite lines. Let the line l intersect sides BC and AD in points M and N, respectively. Trapezia $ABMN$ and $CDNM$ have the same height ($AB = CD$, because $ABCD$ is a square), therefore their surface areas have the same proportion as their median lines (because trapezium surface area = median line $\times$ height). Therefore MN intersects the line segment k that connects the midpoints of BC and AD at a point such that divides k in proportions $2:3$

But in a square there are only 4 points like that, and we have 9 lines, so one of the points will have at least 3 lines by PP.

2. $[Eulers formula for polyhedron]$

Problem

Prove that, for any convex polyhedron $V + F - E = 2$, where V - number of vertices, E - number of edges and F - number of faces of the polyhedron!

Solution

The stereogeometrical part consists of saying that convex polyhedron is equivalent to a planar graph - you demonstrate this by *pulling* one face of the polyhedron so that it expands and becomes exterior part of the graph.

For Eulers Graph formula (same thing) see Graphs 2.

3. *[Splitting a circle]*

Problem

What is the maximum number of areas that a circle is divided into by an inscribed polygon with n sides and all its diagonals?

Solution - Euler's formula, methods of counting, `https://en.wikipedia.org/wiki/Dividing_a_circle_into_areas`

On geometri'sh fact we are going to use is:

Lemma 3.1. When adding a new vertex to an inscribed polygon, it is possible to do it in a way that all newly created diagonals never create a point at which at least three lines meet

Proof. For the opposite to happen, then the new vertex v has to belong to some line that some olde vertex w and some internal point p belongs to. Since the polygon is finite, there are finite combinations of w and p, but there are infinitely many points on the circle, so there will always be some free points that we can choose for the location of v. ■

From this we can deduce that if the number of regions is maximal, then at every internal crossing exactly two chords intersect.

We can view this figure as a planar graph $G = \{V, E\}$, where all chord intersections are vertices. As G is planar, then Euler's formula $|V| - |E| + |F| = 2$ holds. We can use it to calculate

$$|F| = |E| - |V| + 2 \tag{1}$$

$|V|$ consists of n vertices of the original polygon plus the number of internal points where chords intersect. By Lemma we know that those points are defined by two chords and those, in turn, are uniquely defined by selection of 4 of the vertices of the original polygon, therefore $|V| = n + \binom{n}{4}$

$|E|$ consists of several types of edges:

a. Arcs between vertices of polygon - there are exactly n of these

b. Edges between two adjacent vertices of the polygon - there are exactly n of these

c. Edges connecting two interior vertices

d. Edges connecting an interior vertex and a vertex from original polygon

To find the number of type **??**. and **??**. edges, we consider that each interior point is connected to 4 edges. This yields $4\binom{n}{4}$, but this number contains every **??**. edge twice but every **??**. edge once.

To find a number of type **??**. edges, notice that every diagonal of the original polygon contains exactly 2 such edges. Number of diagonals in n vertex polygon is $\binom{n}{2} - n$ and, therefore, number of edges of type **??**. is $2(\binom{n}{2} - n)$

Therefore $|E| = \frac{4\binom{n}{4}+2(\binom{n}{2}-n)}{2} + n + n = 2\binom{n}{4} + \binom{n}{2} + n$.

Inserting all this into (**??**) we get $|F| = \binom{n}{4} + \binom{n}{2} + 2$

Since Eulers formula includes "external" face, then the number of parts of circle is

$$r_G = \binom{n}{4} + \binom{n}{2} + 1 \tag{2}$$

which yields

$$r_G = \frac{n!}{(n-4)!4!} + \frac{n!}{(n-2)!2!} + 1 \tag{3}$$

which can be solved as

$$r_G = \frac{1}{24} n(n^3 - 6n^2 + 23n - 18) + 1 \tag{4}$$

4. [BW1999] Answer:No

Let O denote the centre of the disc, and $P_1, ..., P_6$ the vertices of an inscribed regular hexagon in the natural order (see Figure 1).

If the required partitioning exists, then $\{O\}, \{P_1, P_3, P_5\}$ and $\{P_2, P_4, P_6\}$ are conained in different subsets. Now consider the circles of radius 1 centered in P_1, P_3, and P_5. The circle of radius $1/\sqrt{3}$ centered in O intersects these three circles in the vertices A_1, A_2, A_3 of an equilateral triangle of side length 1. The vertices of this triangle belong to different subsets, but none of them can belong to the same subset as P_1 - a contradiction. Hence the required partitioning does not exist.

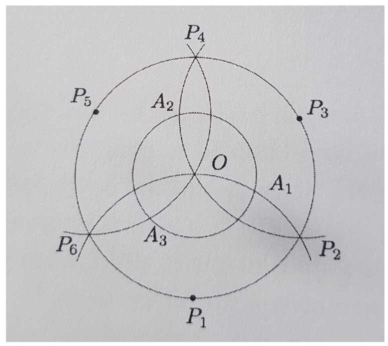

Figure 1

5. [IMO2013PL2] We can start off by imagining the points in their worst configuration. With some trials, we find 2013 lines to be the answer to the worst cases. We can assume the answer is 2013. We will now prove it.

We will **first** prove that the sufficient number of lines required for a *good* arrangement for a configuration consisting of u red points and v blue points, where u is even and v is odd and $u - v = 1$, is v.

Notice that the condition "no three points are co-linear" implies the following: No blue point will get in the way of the line between two red points and vice versa. What this means, is that for any two points A and B of the same color, we can draw two lines parallel to, and on different sides of the line AB, to form a region with only the points A and B in it.

Now consider a configuration consisting of u red points and v blue ones (u is even, v is odd, $u > v$). Let the set of points $S = \{A_1, A_2, ...A_k\}$ be the out-most points of the configuration, such that you could form a convex k-gon, $A_1A_2A_3...A_k$, that has all of the other points within it.

If the set S has at least one blue point, there can be a line that separates the plane into two regions: one only consisting of only a blue point, and one consisting of the rest. For the rest of the blue points, we can draw parallel lines as mentioned before to split them from the red points. We end up with v lines.

If the set S has no blue points, there can be a line that divides the plane into two regions: one consisting of two red points, and one consisting of the rest. For the rest of the red points, we can draw parallel lines as mentioned before to split them from the blue points. We end up with $u - 1 = v$ lines.

Now we will show that there are configurations that can not be partitioned with less than v lines.

Consider the arrangement of these points on a circle so that between every two blue points there are at least one red point (on the circle).

There are no less than $2v$ arcs of this circle, that has one end blue and other red (and no other colored points inside the arc) - one such arc on each side of each blue point. For a line partitioning to be good, each of these arcs have to be crossed by at least one line, but one line can not cross more than 2 arcs on a circle - therefore, this configuration can not be partitioned with less than v lines!

Our proof is done, and we have our final answer: 2013.

6. $[IMO2017PL3]$

There is no such strategy for the hunter.

If there were, that would mean that hunter's strategy would work, no matter how the rabbit moved or where the radar pings R_n' appeared.We will show the opposite - with bad luck from radar pings, there is no strategy for the hunter that guarantees that the distance stays below 100 in 10^9 rounds.

So, let d_n be the distance between the hunter and the rabit after n rounds. Of course, if $d_n \geq 100$ for any $n < 10^9$, we are done (rabbit just keeps moving straight away from the hunter), so we assume $d_n < 100$.

We will show that, while $d_n < 100$, no matter the strategy hunter follows, rabbit has a way of increasing d_n^2 by at least $\frac{1}{2}$ every 200 rounds (as long as radar pings favor the rabbit). This way d_n^2 will reach 10^4 in less than $2 \cdot 10^4 \cdot 200 = 4 \cdot 10^6 < 10^9$ rounds and the rabbit wins.

Suppose the hunter is at H_n and the rabit is at R_n. Suppose even that the rabbit *reveals* its position at this moment to the hunter (this allows us to ignore information from all previous pings). Let r be the line H_nR_n and Y_1 and Y_2 be points which are 1 unit away from r and 200 units away from R_n as in Figure 2:

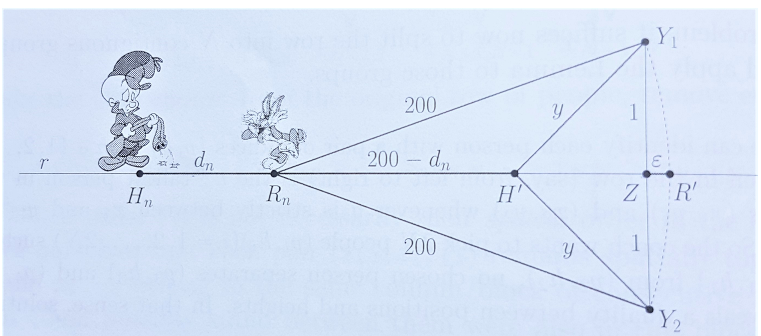

Figure 2

Rabbits plan is to simply choose one of the points Y_1 and Y_2 and hop 200 rounds straight toward it. Since all the hops stays within 1 unit distance of r, it is possible that all radar pings stay on r. In particular, in this case the hunter has no way of knowing whether the rabit chose Y_1 or Y_2.

Looking at such pings (all on r), hunter has no better strategy than to move straight along line r to H'. No matter what he does, after 200 moves he will be located on or left of H'. If he would end up above r, he would be even further from Y_2, and similarly, if he would venture below r, he would be farther from Y_1. In other words, no matter what strategy the hunter follows, he can never be sure his distance to the rabbit will be less than $y \overset{\text{def}}{=} H'Y_1 = H'Y_2$ after these 200 rounds.

To estimate y^2 we take Z as the midpoint of segment Y_1Y_2, we take R' as a point 200 units to the right of R_n and we define $\varepsilon = ZR'$ (note that $H'R' = d_n$). Then

$$y^2 = 1 + (H'Z)^2 = 1 + (d_n - \varepsilon)^2 = d_n^2 - 2\varepsilon d_n + \varepsilon^2 + 1 \tag{5}$$

where

$$\varepsilon = 200 - R_nZ = 200 - \sqrt{200^2 - 1} = \frac{1}{200 + \sqrt{200^2 - 1}} > \frac{1}{400}$$

Squaring both sides of $\varepsilon = 200 - \sqrt{200^2 - 1}$ we can get $\varepsilon^2 + 1 = 400\varepsilon$, so from (1) we get

$$y^2 = d_n^2 - 2\varepsilon d_n + \varepsilon^2 + 1 = d_n^2 + \varepsilon(400 - 2d_n) \tag{6}$$

Since $\varepsilon > \frac{1}{400}$ and $d_n < 100$, then from (2) follows $y^2 > d_n^2 + \frac{1}{2}$. So, as we claimed, with this list of radar pings, no matter what the hunter does, the rabbit might achieve $d_{n+200}^2 > d_n^2 + \frac{1}{2}$.

The wabbit wins.