Graphs 3 - Solutions

1. *[BW1994PL19]*

Problem

The Wonder Island Intelligence Service has 16 spies in Tartu. Each of them watches on some of his colleagues. It is known that if spy A watches on spy B, then B does not watch on A. Moreover, any 10 spies can numbered in such a way that the first spy watches on the second, the second watches on the third, ..., the tenth watches on the first.

Prove that any 11 spies can also be numbered is a similar manner!

Solution

We call spies A and B *neutral* to each other if neither A watches B nor B watches A. For each spy A_1 denote a_1, b_1 and c_1 the number of spies that watches on A_i, the number of spies A_i watches and the number of spies neutral to A_i. Clearly we have

$$a_i + b_i + c_i = 15$$

And, we have

$$a_i + c_i \leq 8$$

$$b_i + c_i \leq 8$$

because if any of the these two inequalities did not hold, we could select 10 spies that could not be cyclically watching each other. Adding together these inequalities and equality we get that $c_i \leq 1$, that means that for any A_i number of spies neutral to A_i is 0 or 1

Now suppose the opposite, that there is a a group of 11 spies that cannot be numbered as required. Let B be an arbitrary spy in this group. Name other 10 spies in this group $C_1, C_2, \ldots, C_{10}$ so that C_1 watches $C_2, \ldots, C_{10}$ watches C_1. Suppose, that none of these $C_1, \ldots, C_{10}$ is neutral to B. Some of the spies from C group must watch B (because $b_B \leq 8$), and as it is cyclical, say C_1 watches B.

Can B watch C_2? No, because then we could insert B into the cycle, between C_1 and C_2 and get a cycle with 11 spies, which we assumed does not exist. So C_2 must also watch B. But by the same argument, C_3 must watch B, C_4 must watch B, ..., C_{10} must watch B. Which is impossible, since $a_B \leq 8$.

Therefore one of our assumptions must be false. First assume, that assumption of not having neutrals among C_i is false, but that is equivalent to saying that between any 11 spies each must have exactly one neutral, which is impossible by parity of vertice powers.

Therefore our second assumption - that there are such 11 spies that cannot be ordered in a cycle, exist - is false, and any 11 can be ordered in a cycle.

2. *[BW2010PL7]* There are some cities in a country; one of them is the capital. For any two cities A and B there is a direct flight from A to B and a direct flight from B to A, both having the same price. Suppose that all round trips with exactly one landing in every city have the same total cost.

Prove that all round trips that miss the capital and with exactly one landing in every remaining city cost the same!

3. *[IMO2007PL3]* In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room!

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4. *[IMO2013SLC3]* A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

a. If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

b. At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled!

5. *[IMO2015SLC7]* In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies.

Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part!