$\binom{10}{4}$	The number of ways of choosing a team of four people out of a room of 10 people (where the order that we pick the people does not matter).
$P(10,4)$	The number of ways of choosing a team of four people out a room of 10 people, where the order *does* matter (for example, we choose the four people, and then designate a team captain, co-captain, mascot, and manager).
$5\cdot\binom{13}{5}$	The number of ways you can choose a team of five people chosen from a room of 13 people, where you are designating the captain.
$\binom{17}{8}\binom{10}{2}$	The number of different teams of eight girls and two boys, chosen from a pool of 17 girls and 10 boys.
$\binom{17}{4}+\binom{17}{3}$	The number of ways of picking a team of three *or* four people, chosen from a room of 17 people.
$\binom{17}{10}=\binom{17}{7}$	Each selection of 10 winners from a group of 17 is simultaneously a selection of seven losers from this group.

6.1.13 *The Symmetry Identity.* Generalizing the last example, observe that for all integers n,r with $n\geq r\geq 0$, we have

$$\binom{n}{r}=\binom{n}{n-r}.$$

This can also be verified with algebra, using the formula from 6.1.11, but the combinatorial argument used above is much better. The combinatorial argument shows us *why* it is true, while algebra merely shows us *how* it is true.

Pascal's Triangle and the Binomial Theorem

6.1.14 *The Summation Identity.* Here is a more sophisticated identity with binomial coefficients: For all integers n,r with $n\geq r\geq 0$,

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}.$$

Algebra can easily verify this, but consider the following combinatorial argument: Without loss of generality, let $n=17, r=10$. Then we need to show *why*

$$\binom{17}{10}+\binom{17}{11}=\binom{18}{11}.$$

Let us count all 11-member committees formed from a group of 18 people. Fix one of the 18 people, say, "Erika." The 11-member committees can be broken down into two mutually exclusive types: those with Erika and those without. How many include Erika? Having already chosen Erika, we are free to chose 10 more people from the remaining pool of 17. Hence there are $\binom{17}{10}$ committees that include Erika. To count the committees without Erika, we must choose 11 people, but again out of 17, since we need to remove Erika from the original pool of 18. Thus $\binom{17}{11}$ committees exclude Erika. The total number of 11-member committees is the sum of the number of committees with Erika plus the number without Erika, which establishes the equality. The argument certainly works if we replace 17 and 10 with n and r (but it is easier to follow the reasoning with concrete numbers). ■

6.1.15 Recall **Pascal's Triangle**, which you first encountered in Problem 1.3.17 on page 10. Here are the first few rows. The elements of each row are the sums of pairs of adjacent elements of the prior row (for example, $10 = 4 + 6$).

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's Triangle contains all of the binomial coefficients: Label the rows and columns, starting with zero, so that, for example, the element in row 5, column 2 is 10. In general, the element in row n, column r will be equal to $\binom{n}{r}$. This is a consequence of the summation identity (6.1.14) and the fact that

$$\binom{n}{0} = \binom{n}{n} = 1$$

for all n. Ponder this carefully. It is very important.

6.1.16 When we expand $(x+y)^n$, for $n = 0, 1, 2, 3, 4, 5$, we get

$$
\begin{array}{rcl}
(x+y)^0 & = & 1, \\
(x+y)^1 & = & x+y, \\
(x+y)^2 & = & x^2+2xy+y^2, \\
(x+y)^3 & = & x^3+3x^2y+3xy^2+y^3, \\
(x+y)^4 & = & x^4+4x^3y+6x^2y^2+4xy^3+y^4, \\
(x+y)^5 & = & x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.
\end{array}
$$

Certainly it is no coincidence that the coefficients are exactly the elements of Pascal's Triangle. Indeed, in general it is true that the coefficient of $x^r y^{n-r}$ in $(x+y)^n$ is equal to $\binom{n}{r}$. You should be able to explain why by thinking about what happens when you multiply $(x+y)^k$ by $(x+y)$ to get $(x+y)^{k+1}$. You should see the summation identity in action and essentially come up with an induction proof.

6.1.17 *The Binomial Theorem.* Formally, the **binomial theorem** states that, for all positive integers n,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

Expanding the sum out gives the easier-to-read formula

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

(Finally we see why $\binom{n}{r}$ is called a binomial coefficient!)

6.1.18 *A Combinatorial Proof of the Binomial Theorem.* We derived the binomial theorem above by observing that the coefficients in the multiplication of the polynomial $(x+y)^k$ by $(x+y)$ obeyed the summation formula. Here is a more direct "combinatorial" approach, one where we think about how multiplication takes place in order to understand *why* the coefficients are what they are. Consider the expansion of, say,

$$(x+y)^7 = \underbrace{(x+y)(x+y)\cdots(x+y)}_{7 \text{ factors}}.$$

When we begin to multiply all this out, we perform "FOIL" with the first two factors, getting (before performing any simplifications)

$$(x^2+yx+xy+y^2)(x+y)^5.$$

To perform the next step, we multiply each term of the first factor by x, and then multiply each by y, and then add them up, and then multiply all that by $(x+y)^4$. In other words, we get (without simplifying)

$$(x^3+yx^2+xyx+y^2x+x^2y+yxy+xy^2+y^3)(x+y)^4.$$

Notice how we can read off the "history" of each term in the first factor. For example, the term xy^2 came from multiplying x by y and then y again in the product $(x+y)(x+y)(x+y)$. There are a total of $2 \times 2 \times 2 = 8$ terms, since there are two "choices" for each of the three factors. Certainly this phenomenon will continue as we multiply out all seven factors and we will end up with a total of 2^7 terms.

Now let us think about combining like terms. For example, what will be the coefficient of x^3y^4? This is equivalent to determining how many of the 2^7 unsimplified terms contained 3 xs and 4 ys. As we start to list these terms,

$$xxxyyyy,\ xxyxyyy,\ xxyyxyy, \ldots$$

we realize that counting them is just the "Mississippi" problem of counting the permutations of the word "XXXYYYY." The answer is $\frac{7!}{3!4!}$, and this is also equal to $\binom{7}{3}$.

6.1.19 Ponder 6.1.18 carefully, and come up with a general argument. Also, work out the complete multiplications for $(x+y)^n$ for n up to 10. If you have access to a computer, try to print out Pascal's Triangle for as many rows as possible. Whatever you do, become very comfortable with Pascal's Triangle and the binomial theorem.

Strategies and Tactics of Counting

When it comes to strategy, combinatorial problems are no different from other mathematical problems. The basic principles of wishful thinking, penultimate step, make it easier, etc., are all helpful investigative aids. In particular, careful experimentation with small numbers is often a crucial step. For example, many problems succumb to a three-step attack: experimentation, conjecture, proof by induction. The strategy of recasting is especially fruitful: to counteract the inherent dryness of counting, it helps to *visualize* problems creatively (for example, devise interesting "combinatorial arguments") and look for hidden symmetries. Many interesting counting problems involve very imaginative multiple viewpoints, as you will see below.

But mostly, combinatorics is a tactical game. You have already learned the fundamental tactics of multiplication, division, addition, permutations, and combinations. In the sections below, we will elaborate on these and develop more sophisticated tactics and tools.

Problems and Exercises

6.1.20 Find a combinatorial explanation for the following facts or identities.

(a) $\binom{2n}{2} = 2\binom{n}{2} + n^2.$

(b) $\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}.$

6.1.21 Define $d(n)$ to be the number of divisors of a positive integer n (including 1 and n).

(a) Show that if

$$n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$$

is the prime factorization of n, then

$$d(n) = (e_1+1)(e_2+1)\cdots(e_t+1).$$

For example, $360 = 2^3 3^2 5^1$ has $(3+1)(2+1)(1+1) = 24$ distinct divisors.

(b) Complete the solution of the Locker problem (Problem 2.2.3), which we began on page 29.

6.1.22 Use the binomial theorem and the algebraic tactic of substituting convenient values to prove the following identities (for all positive integers n):

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$

6.1.23 Show that the total number of subsets of a set with n elements is 2^n. Include the set itself and the empty set.

6.1.24 Prove the identities in 6.1.22 again, but this time using combinatorial arguments.

6.1.25 (Russia 1996) Which are there more of among the natural numbers between 1 and 1,000,000: numbers that can be represented as a sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

6.1.26 (AIME 1996) Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

6.1.27 Eight boys and nine girls sit in a row of 17 seats.

(a) How many different seating arrangements are there?

(b) How many different seating arrangements are there if all the boys sit next to each other and all the girls sit next to each other?

(c) How many different seating arrangements are there if no child sits next to a child of the same sex?

6.1.28 *How Many Marriages?*

(a) In a traditional village, there are 10 boys and 10 girls. The village matchmaker arranges all the marriages. In how many ways can she pair off the 20 children? Assume (the village is traditional) that all marriages are heterosexual (i.e., a marriage is a union of a male and a female; male-male and female-female unions are not allowed).

(b) In a not-so-traditional village, there are 10 boys and 10 girls. The village matchmaker arranges all the marriages. In how many ways can she pair off the 20 children, if homosexual marriages (male-male or female-female) as well as heterosexual marriages are allowed?

(c) In another not-so-traditional village, there are 10 boys and 10 girls, and the village matchmaker arranges all the marriages, allowing, as in (b), same-sex marriages. In addition, the matchmaker books 10 **different** honeymoon trips for each couple, choosing from 10 different destinations (Paris, London, Tahiti, etc.) In how many ways can this be done? Notice that now, you need to count not only who is married to who, but where these couples get to go for their honeymoon.

6.1.29 If you understood the binomial theorem, you should have no trouble coming up with a **multinomial** theorem. As a warm-up, expand $(x+y+z)^2$ and $(x+y+z)^3$. Think about what make the coefficients what they are. Then come up with a general formula for $(x_1+x_2+\cdots+x_n)^r$.

6.2 Partitions and Bijections

We stated earlier that combinatorial reasoning is largely a matter of knowing exactly when to add, multiply, subtract, or divide. We shall now look at two tactics, often used in tandem. **Partitioning** is a tactic that deliberately focuses our attention on addition, by breaking a complex problem into several smaller and simpler pieces. In contrast, the **encoding** tactic attempts to count something in one step, by first producing a **bijection** (a fancy term for a 1-to-1 correspondence) between each thing we want to count and the individual "words" in a simple "code." The theory behind these tactics is quite simple, but mastery of them requires practice and familiarity with a number of classic examples.

Counting Subsets

A **partition** of a set S is a division of S into a union of nonempty *mutually exclusive* (pairwise disjoint) sets. We write

$$S = A_1 \cup A_2 \cup \cdots \cup A_r, \quad A_i \cap A_j = \emptyset, \ i \neq j.$$

Another notation that is sometimes used is the symbol $\sqcup$ to indicate "union of pairwise disjoint sets." Thus we could write

$$S = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r = \bigsqcup_{i=1}^{r} A_i$$

to indicate that the set S has been partitioned by the A_i.

Recall that $|S|$ denotes the cardinality (number of elements) of the set S. Obviously if S has been partitioned by the A_i, we must have

$$|S| = |A_1| + |A_2| + \cdots + |A_r|,$$

since there is no overlapping.

This leads to a natural combinatorial tactic: Divide the thing that we want to count into mutually exclusive and easy-to-count pieces. We call this tactic **partitioning**. For