IMO SL list 3

1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \dots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \dots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

2. We have $n \geq 2$ lamps $L_1, \dots, L_n$ in a row, each of them being either *on* or *off* . Every second we simultaneously modify the state of each lamp as follows:

- if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off;
- otherwise, L_i is switched on.

Initially all the lamps are off except the leftmost one which is on.

a. Prove that there are infinitely many integers n for which all the lamps will eventually be off.

b. Prove that there are infinitely many integers n for which the lamps will never be all off.

3. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

4. There are given 2^{500} points on a circle labeled $1, 2, \dots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.