Pigeonhole 2 - Solutions

1. *[Naval Warfare]* Battlefield is a 10×10 squares grid and Admiral needs to position following ships on this grid: one carrier - rectangle of 1×4 grid squares; two battleships 1×3; three cruisers 1×2; and four frigates 1×1. Ships have to match cells on grid, they may not touch each other, but are allowed to touch sides of the grid. Prove that

a. If Admiral starts with larger ships and proceeds to smaller, it can always be done

b. If Admiral starts with some smaller ships, before placing larger ones, it can happen that she can not place all the ships on the battlefield

Solution

In the second case it is possible to arrive at an example (see Figure 1)

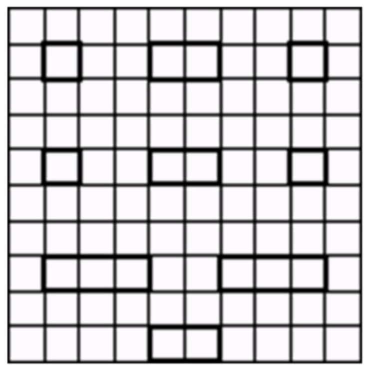

Figure 1: Nowhere to put carrier!

For the first part of the problem it is important to note that it is not enough that we will always be able to deploy the last ship - it can happen that we would be able to deploy the last one, but not one of the previous (bigger) ones!

Nevertheless it could be beneficial to reason from this direction. How would we prove that all the frigates can be deployed? We can visualise 16 locations like this:

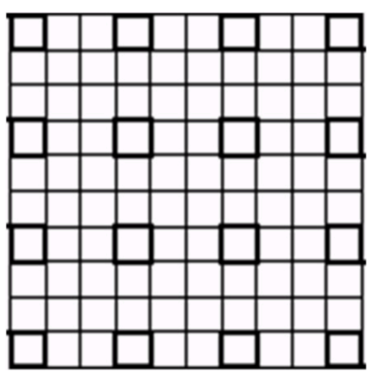

Figure 2: Grid for Frigate

These marked location have a property that any deployed ship can touch no more than 2 of these locations and any deployed frigate can touch no more than 1 of these locations. Since the maximum we have deployed so far is 3 frigates plus 6 larger ships, no more than $3 + 6 \times 2 = 15$ locations are touched. Therefore we can deploy a frigate in the untouched location!

Now, working backwards, we also need to prove that it is possible to deploy all the cruisers and all the battleships. We do it with a very similar argument (sans frigate exception) with locations seen in Figures 3 and 4

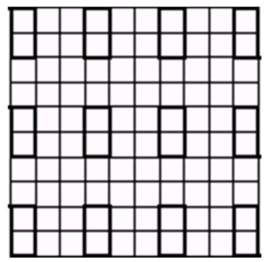

Figure 3: Grid for cruisers

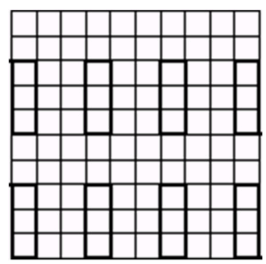

Figure 4: Grid for battleships

2. *[Elusive freedom of choice]* Johnny had to choose 372 distinct numbers out of set $\{1, 2, ..., 1200\}$ with a property that no two chosen numbers have a difference of $4, 5$, or 9.

Prove that Johnny chose number 600!

Solution

This proof starts with observation, that $9 = 4 + 5$.

Lemma 2.1. lEmmy

Proof. Proof ■

3. *[Convoluted treasure hunt]* Miervaldis and Visvaldis are playing a following game. Large circle is drawn on an Euclidean field (plane). Miervaldis has hidden treasures in n points within the circle. Visvaldis wishes to find all treasures.

One guess for Visvaldis is to choose a point on a field (within or without the circle) and Miervaldis announces distance from this point to the nearest unrecovered treasure or, if this distance is zero, uncovers the treasure. Visvaldis is capable of marking point in this plane, drawing straight lines and even using a (sufficiently large) compass.

Can Visvaldis find all the treasures with no more than $(n+1)^2$ guesses?

4. *[Splitting a square]* Each of 9 lines divide a square into two quadrilaterals, whose areas have a proportion of $2 : 3$ to each other. Prove that there exists a point, where at least three of these lines meet!

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5. *[Order out of randomness]* Numbers $\{1, 2, 3, ..., 101\}$ are written in a line in some random order. Prove, that it is possible to erase 90 of them so, that remaining 11 are ordered in either ascending or descending order!

6. *[Mindblowing triangles]* Cells of an infinite grid are all colored in one of three colors each. Prove that it is possible to find a right triangle whose vertices are cells of the same color!