Graphs 2 - Solutions

1. *[Traffic in Emirate]*

Problem

An Emirate has 8 cities. Sultan wants to build such a road system, that for any two cities it is possible to travel from one to another passing through no more than one other city. Also he wishes that no more than k roads leave each city. For what values of k this is possible?

Solution

If $k = 2$ then, consider some city A. One can travel directly to no more than 2 other cities, call them B_1, B_2. From those one can travel further, but to no more than 4 cities - C_1, C_2, C_3, C_4. So we have accounted for 7 of the 8 cities and the last city is not reachable from A wiith no more than two edges.

If $k = 3$ it is doable by drawing an octogon with all long diagonals.

2. *[Returning to cliques]*

Problem

Prove that between 6 random people there always exist at least 2 cliques (groups of three that either all know each other, or all do not know each other)!

Solution - double counting, counting of ends

Consider it as a full two-colored graph. We call a *corner* any unordered pair of edges that belong to one vertex. There are $5 * 4/2 = 10$ corners on each vertex, so, $6 * 10 = 60$ corners in all graph.

Call a corner *monocolored* if its two edges are of the same color, and *polycolored* otherwise.

A monocolored triangle has 3 monocolored corners, while policolored triangle has 2 polycolored and 1 monocolored corner. There are a total of $6 * 5 * 4/6 = 20$ different triangles in a full 6-graph (either count combintations of vertices or divide number of corners by 3).

Each vertex has at least 4 monocolored corners - if there are 4 or more same color edges on one vertex, then there is at least 6 monocolored corners, and if there are 3 and 2 same colored edges, then its exactly 4.

Therefore there are at least $6 * 4 = 24$ monocolored corners in the graph. We know that there is one monocolored triangle, so that uses 3 monocolored corners. We are left with 21 monocolored corners to distribute between 19 remaining trinagles - by Pidgeonhole Principle, there will one more triangle T that has more than 1 monocolored edge, but that means that that T has to be monocolored. QED.

3. *[Eulers formula for planar graphs]*

Problem

Prove that, for any planar (planar graph is a graph which can be drawn on a paper so that edges do not intersect) connected graph $V + F - E = 2$, where V - number of vertices, E - number of edges and F - number of faces (areas of plane bounded by edges of the graph, including the infinite area around the graph)!

Solution - induction, invariant, algorithm

Prove by induction.

Induction base

There is only one kind of connected graph with $V = 2$ and it has one edge and only the external face - formula holds.

Induction hypothesis

Eulers formula holds for graphs of size $k - 1$ (number of vertices) and smaller.

Induction step

We will show several transformations that has Eulers formula as invariant:

a. If a graph has a leaf, we can remove that leaf and the edge, decreasing both number of edges and number of vertices by 1; formula holds.

b. If a graph has an internal face that is a polygon with more than 3 vertices, we can add one *diagonal* - an edge between some pair of non-connected vertices on this cycle. We are increasing E and F by exactly 1, so formula holds.

c. If a graph has a triangular face f that shares a border e with exterior face, we can remove e, thus merging f and exterior face. We are decreasing both E and F by 1 by doing so, therefore, formula still holds. Note, that if there were no leaves in a graph, the result will not be disconnected, but may contain leaves.

Now, to our algorithm: we check, if there is a leaf in the graph. If yes, remove it (by *a.*), and we have reduced graph to size $k-1$ where formula holds by our inductive assumption. If there is no leaves, the graph must have some cycles. If any faces larger than 3 vertices exist, we reduce those by applying *b.* Lastly we continuously apply *c.* until a leaf appears in a graph. Once a leaf appears, we remove it by *a.* and we are done (we have reduced the graph to size $k-1$ while holding Eulers formula invariant.

4. *[City of Labirinth]*

Problem

In the city of Labirinth exactly three streets meet at every crossroads. Besides that, all streets are paved in one of the three colors so that streets of every color meet at every crossroad.

Only three streets leave the city. Prove, that the streets leaving the city all have different colors!

Solution - counting ends, double counting

Name the colors 1, 2 and 3 and count the end of the streets. Say, c_i will be the number of i-th color streets leaving the city, n will be the number of crossroads in the city. Then number of ends-of-roads are $n+c_1$, $n+c_2$ and $n+c_3$. All these three numbers have to be even (because each road has two ends) there fore all c_i are of the same parity. But since we are given that $c_1+c_2+c_3=3$, and they are of same parity, then it has to be that $c_1=c_2=c_3=1$

5. *[Mapping the Empire]*

Problem

Empire consists of many countries. It is known that no country borders with more than k others (that includes possible enclaves). Emperor has ordered such a map of Empire to be produced that no two bordering countries are of the same color (and enclaves are of the same color as countries they belong to). Prove that $k+1$ color is enough for this!

Solution - induction

Transform into a graph, countries are vertices, edges are between countries that are neighbours on the map. Powers of vertices are limited by k. Now we need to color the vertices in no more than $k+1$ color so that no edge connects two vertices of the same color.

Do this by induction on number of vertices in graph.

Base For one vertice graph property holds.

Induction hypothesis The property holds for any graph of size n.

Induction step Prove that it holds for $n+1$. Take random graph G (with property of having max vertice power of k) with number of vertices $n+1$. Take one vertice A from G and temporarily hide it and all edges leaving it. What is left is a graph G' of size n and max vertex power of k. By our hypothesis, the property holds in G', so we proceed and color vertices of G' as required. Now we unhide A and its edges.

By property of the graph, A is connected to no more than k other vertices, so we can always choose a color out of $k+1$ available to color A.

6. *[Nightmare Hydra]*

Problem

Hydra consists of only heads and necks (each neck connect exactly two heads). One mighty sword strike can sever all necks that connect any head A. Unfortunately, if this is done, new necks immediately grow from A to all heads that A was not previously connected to.

Hydra is considered defeated, if it is divided into two disconnected parts. What is the minimal number of mighty sword swings to defeat Hydra with a hundread necks?

Solution

Answer is 10.

First, to prove that 10 swings is enough to kill any 100 neck Hydra.

Assume the opposite, that there is a Hydra H such that it can not be killed by 10 swings.

Consider its random head A. It has to be connected to at least $10+1=11$ other heads, otherwise we could hit all the head it is connected to in 10 swings and be done with it. There also has to be at least 10 heads that A is not connected to, because otherwise we could have hit A itself and then the heads it is connected to after inversion. Therefore total number of heads is at least $1+11+10=22$.

Also, if the H can not be killed with 10 swings, then each head of H has to have at least 11 necks, so number of necks is at least $\frac{22\times 11}{2} > 100$ which is a contradiction, since we only have a 100 neck Hydra. Therefore, H can be killed by 10 swings.

Now to show that there exist a 100 neck Hydra such that it can not be killed by 9 swings. We draw our Hydra like this - two columns of 10 heads each, each head from left column is connected to each head in the right column (therefore we have exactly 100 necks)

How to see that it can not be killed by any 9 swings? First, note that it makes no sense to hit one head two or more times, since after each second swing we get the same situation we had before. So, we assume that 9 different heads were hit once each. Each column has 10 heads, so there will be some heads in each column that were not hit. We call these sets $L_{untouched}$ and $R_{untouched}$, and the heads that were hit in each column we call $L_{touched}$ and $R_{touched}$. One of last two sets can be empty, but that does not impact the following argument:

The Hydra is still connected, because all heads in $L_{untouched}$ is still connected to all heads in $R_{untouched}$, but all heads in all $L_{touched}$ is now connected to all heads in $L_{untouched}$. Same for $R_{untouched}$ and $R_{touched}$.