Pigeonhole 1 solutions/comments

1. *[Hobbits]*
Well, this is the definition of the Pigeonhole Principle - if n pigeons (or hobbits) all go into m holes and $n > m$, then there exists a hole with at least 2 pigeons in it.

2. *[More hobbits]*
There should be a hole with at least 3 hobbits, right?

3. *[Even more hobbits]*
We might get to the *Generalization 1*, that if $n = km + l$, then there exists a hole with at least $k + 1$ pidgeon.

4. *[Football problem]*
In total $10m + 1$ players have moved to Toland, so if they are the pigeons and m teams are the holes, then by Generalization 1 there exists a team with at least 11 players in Toland.

5. *[Amount of friends]*
There can be 5 possible number of friends someone can have in such group - from 0 to 4. However, if a group contains a person, who is friend with everyone else, then the group can

6. *[Unique differences]*
There are 14 possible differences between such numbers - from 1 to 14. These will be our holes. What should be the pidgeons? Well, the pairs of given numbers! However 8 numbers make 28 pairs[1], which means that Pigeonhole does not give us what we need, immediately.

A little attention on the 'hole' 14, however, helps - it becomes clear that 14 as a difference between two distinct numbers that do not exceed 15 can only be obtained from one pair of numbers - 1 and 15. So we say, that max 1 pair goes into hole 14, which leaves 27 pairs and 13 holes, and by Pigeonhole principle, we are done.

7. *[Too many kings]* An important (potentially new) point - In "**what is maximum/minimum**" type of problems, there are two parts - show that n works and show that can be more/less than n.

Here we first show that there cant be more than 16 kings on a board (not attacking each other), and then we show that 16 is possible.

We divide the chess board into 16 non-overlapping squares $2 * 2$. Obviously, each $2 * 2$ square can not contain more than 1 king.

Now, $2 * 2$ squares will be our holes, kings - pidgeons and by Pigeonhole follows that if we have more than 16 kings on board, then at least one $2*2$ square will contain at least 2 kings, which will be a problem.

Finally, to show that 16 is achievable, we can put a king in upper-left corner of each $2 * 2$ square and it works.

8. *[Too much money]* This is very similar to previous problem, just with no grid to anchor on. We divide $1m * 1m$ into 25 non-overlapping squares $20cm * 20cm$. By Pigeonhole, one of these will contain at least 3 coins. We cover this square with our paper.

[1] a good warmup/clue task for tactic of *playing around and solving easier problems*

9. *[Existance of cliques]*

This one is a classic and we will come back to it in Graph Theory (also see *Ramsay numbers*).

Consider a person A. By Pigeonhole, he either has at least 3 acquaintances in the group, or at least 3 people he does not know - the 5 remaining people are the pigdeons and two 'buckets' - 'people A knows' and 'people A does not know' - are the holes.

Without loss of generality (maybe this is a place to talk about "**without loss of generality**" construction) assume that A knows at least 3 people - B, C, D. Now there are two possibilities - any of B, C, D know each other or they don't. If former, and, say, B and D know each other, then we have found our triplet - A, B, D who all know each other. If latter - none of B, C, D know each other - we have found our triplet who do not know each other.

10. *[Last digits of 3^x]*

A nod to how Pigeonhole can appear outside combinatorics - for example, Number Theory.

Tactical note - use *Penultimate step* - number ends with 001 if it is congruent to 1 mod 1000, i.e. $3^k - 1$ divides by 1000. If we multiply both sides by 3^l it still holds (since 3^x never divides by 1000). Therefore $3^l(x^k - 1) = 3^{k+l} - 3^l$. We do not proceed directly here, but rather oberve, that we have transformed what we need into a statement that difference of two powers of 3 is divisible by 1000 (tactical name for this - *midpoint/step*)

Now, difference of two numbers divides by 1000 iff these two numbers have same remainder, when dividing by 1000. Do such two powers of 3 exist? Of course, by Pigeonhole! Holes are the possible remainders - 1000 of them, but the pidgeons are powers of 3 - an infinite number! (this is our second building block)

Putting it together, we say - let 3^n and 3^m be two powers such that they have same remainder, when dividing by 1000. Without loss of geerality assume $n > m$. Then $3^n - 3^m$ divides by 1000 evenly. But $3^n - 3^m = 3^m(3^{n-m} - 1)$. First part divides only by powers of 3, so $3^{n-m} - 1$ must be dividing by 1000 and therefore 3^{n-m} ends with 001.

11. *[College problems]*

This illustrates two things - *hard work/perserverance* tactic and the idea that 'holes' can be quite abstract objects.

We number courses 1..5 and we describe each student as a subset of $\{1, 2, 3, 4, 5\}$. There are 32 possible subsets[2]. To use Pigeonhole principle (*penultimate step* tactic) we would need to divide all possible subsets into 10 groups so that if we take two subsets from one group, one will be subset of another.

Then we just use *hard work* to construct these groups. For example, here within each group each subset is subset of all subsets to the right:

1. $[\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}]$
2. $[\{2\}, \{2, 5\}, \{1, 2, 5\}, \{1, 2, 3, 5\}]$
3. $[\{3\}, \{1, 3\}, \{1, 3, 4\}, \{1, 3, 4, 5\}]$
4. $[\{4\}, \{1, 4\}, \{1, 2, 4\}, \{1, 2, 4, 5\}]$
5. $[\{5\}, \{1, 5\}, \{1, 3, 5\}]$
6. $[\{2, 4\}, \{2, 4, 5\}, \{2, 3, 4, 5\}]$
7. $[\{3, 4\}, \{3, 4, 5\}]$
8. $[\{3, 5\}, \{2, 3, 5\}]$
9. $[\{4, 5\}, \{1, 4, 5\}]$
10. $[\{2, 3\}, \{2, 3, 4\}]$

Note - since each subset of size 2 has to be in its own group, there can not be less than 10 such groups!

[2] a warmup/intro/side problem- count the number of subsets

12. *Squares out of chaos]*

We start with some basic number theory - each number can be uniquely expressed as a multiplication of its prime factors raised to some power - in our case a) we are limited to fist 9 prime factors: $n = p_1^{r_1} \cdot p_2^{r_2} \cdot ... \cdot p_9^{r_9}$.

For the multiplication to be a perfect square, the sum of powers for all respective prime factors has to be an even number. I.e. if $a = p_1^{r_1^{(a)}} \cdot p_2^{r_2^{(a)}} \cdot ... \cdot p_9^{r_9^{(a)}}$, $b = p_1^{r_1^{(b)}} \cdot p_2^{r_2^{(b)}} \cdot ... \cdot p_9^{r_9^{(b)}}$ etc, then for $a \cdot b \cdot c \cdot d$ to be perfect square, all of sums $r_1^{(a)} + r_1^{(b)} + r_1^{(c)} + r_1^{(d)}$; $r_2^{(a)} + r_2^{(b)} + r_2^{(c)} + r_2^{(d)}$; ...; $r_9^{(a)} + r_9^{(b)} + r_9^{(c)} + r_9^{(d)}$ have to be even. Since we are interested only in the parity of these sums, we are interested only in the parities of $r_i^{(j)}$.

How many possible combinations of parities of powers of prime factors can a number have under condition a)? It's $2^9 = 512$.[3] Since we have 2018 numbers, by Pigeonhole principle, there will exist 4 such numbers that their 'footprints' - the combination of parities of powers of prime factors - will be identical. These will be the numbers we were looking for in condition a).

Now, in condition b) we have 10 possible prime factors, so the number of possible 'footprints' is $2^{10} = 1024$ and Pidgeonhole principle in its standard form does not help us.

However we can make a following observation: either one of these 'footprints' have at least 4 numbers (in which case we have the numbers we were looking for) or there exist at least 2 'footprints' with at least 2 numbers in each (try to formalize this in Bonus problem). In latter case, we take two pairs of numbers with pairwise identical 'footprints' and it is enough for sums of powers to be even, therefore giving us the numbrs we were looking for.

13. *[Bonus problem 1]*

This is an open-ended problem. Some results could be:

- There are least $\lceil \frac{n}{c} \rceil$ non empty cells
- If $(n - o \cdot m) = k(c - o) + l$ for some $o, k \in \mathbb{Z}^*$, $l \in \mathbb{N}$, then exist at least $(k + 1)$ holes with at least $o + 1$ pidgeons

14. *[Bonus problem 2]*

Lets prove the basic form:*If n pigeons all go into m holes and $n > m$, then there exists a hole with at least 2 pigeons in it.*

Proof by contradiction (this could be new for someone): assume the opposite, that there is no hole with more than one pigeon. Then maximum amount of pigeons in the holes can be counted as $m * 1 = m$ pigeons - a contradiction with the fact that $n > m$. Therefore our assumption was incorrect and there is at least one hole with at least two pigeons.

[3] warmup/side problem - count the number of binary numbers of lenght n