DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY OF MUNICH

Master's Thesis in Informatics

Cloud Supercomputing with the ExaHyPE-Engine

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Wolken Hochleistungsrechnen mit der ExaHyPE-Engine

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I confirm that this master's thesis in informatics is my own work and I have documented all sources and material used.

Munich, January , 2019 Lukas Krenz

Acknowledgments

Quote!

(Booktitle)

Acknowledgements here.

Abstract

Abstract here!

Contents

CHAPTER 1
Introduction

Introduction here.

CHAPTER 2

Methods

Methods here.

Organize into chapters!

2.1. Conservative Form

Standard (hyperbolic) form of conservation law

$$\frac{\partial}{\partial_t} Q + \nabla \cdot F(Q) = S(\boldsymbol{x}, t, Q) \tag{2.1}$$

extend to:

$$\frac{\partial}{\partial_t} Q + \nabla \cdot F(Q, \nabla Q) = S(\boldsymbol{x}, t, Q) \tag{2.2}$$

2.2. The ADER-DG Method

We describe the arbitrary derivative discontinous Galerkin (ADER-DG) method in this chapter.

We discuss the solution of a hyperbolic conservation law equation (2.2) with domain Ω and boundary $\partial\Omega$. In the discontinous Galerkin (DG) framework, we approximate this solution in the broken space Ω.

The solution is described by an interpolating polynomial in each cell. This choice of basis (called *nodal* basis) allows us to easily compute integrals over cells using Gaussian quadrature. In detail, we use a the Lagrange interpolation polynomials.

How do we compute the solution?

Actually write this section. . .

2.3. Compressible Euler

Vector of conserved quantities:

$$Q = (\rho, \rho \boldsymbol{v}, \rho E) \tag{2.3}$$

Flux:

$$F(Q) = \begin{pmatrix} \rho\boldsymbol{v} \\ \boldsymbol{v} \otimes \rho\boldsymbol{v} + p\mathbf{I} \\ \boldsymbol{v} \cdot (\mathbf{I}\rho E + \mathbf{I}p) \end{pmatrix} \tag{2.4}$$

$$p = (\gamma - 1)\left(\rho E - 0.5\left(\boldsymbol{v} \cdot \rho\boldsymbol{v}\right)\right) \tag{2.5}$$

2.4. Compressible Navier Stokes

Following [Dum10] by Dumbser.

The full compressible Navier Stokes equations with heat transfer can be described by

$$F(Q, \nabla Q) = \begin{pmatrix} \rho\boldsymbol{v} \\ \boldsymbol{v} \otimes \rho\boldsymbol{v} + \mathbf{I}p + \boldsymbol{\sigma}(Q, \nabla Q) \\ \boldsymbol{v} \cdot (\mathbf{I}\rho E + \mathbf{I}p + \boldsymbol{\sigma}(Q, \nabla Q)) - \kappa\nabla T \end{pmatrix}, \tag{2.6}$$

where p, $\boldsymbol{\sigma}$ and $(\kappa\nabla T)$ denote the pressure, stress tensor and heat flux respectively. Temperature is denoted by T.

We use the equation of state of an ideal gas

$$p = (\gamma - 1)\left(\rho E - 0.5\left(\boldsymbol{v} \cdot \rho\boldsymbol{v}\right)\right). \tag{2.7}$$

The temperature T relates to pressure and density by the ideal gas law

$$\frac{p}{\rho} = RT, \tag{2.8}$$

where R is the specific gas constant.

Write more text about constants, cite idolikecfd or sth

$$c_v = \frac{1}{\gamma - 1} R \tag{2.9}$$

$$c_p = \frac{\gamma}{\gamma - 1} R \tag{2.10}$$

$$R = c_p - c_v \tag{2.11}$$

$$\gamma = \frac{c_p}{c_v} \tag{2.12}$$

Heat conduction coefficient κ

$$\kappa = \frac{\mu\gamma}{Pr} \frac{1}{\gamma - 1} R \tag{2.13}$$

with R gas constant and Pr Prandtl number

The viscous effects are modeled by the stress tensor

$$\boldsymbol{\sigma}(Q, \nabla Q) = \left(2/3\mu\nabla \cdot \boldsymbol{v}\right) - \mu\left(\nabla(\boldsymbol{v}) + \nabla(\boldsymbol{v})^T\right). \tag{2.14}$$

2.4.1. Boundary conditions

To close the system we need to impose boundary conditions.

For some scenarios we use Cauchy boundary conditions. In most cases, we would like to impose periodic boundary conditions, due to the inner workings of ExaHyPE this is not possible. Instead we use the analytical solution of our problems at the boundary, imposing both value and gradients of the conservative variables. Note that this leads to an error when our problem does not posses an exact analytical solution. This is the case for test cases that are analytical solutions to the incompressible Navier Stokes equations but do not satisfy the compressible equation set.

As a physical boundary condition we limit ourselves to the no-slip boundary condition, where we assume that the fluid has a velocity of zero near the wall. We enfore this by setting

Check if this is the correct physical description!

$$\rho^o = \rho^i, \tag{2.15}$$
$$\rho \boldsymbol{v}^o = -\rho \boldsymbol{v}^i, \tag{2.16}$$
$$\rho E^0 = \rho E^i, \tag{2.17}$$
$$(\nabla Q)^o = (\nabla Q)^i, \tag{2.18}$$

where a superscript of o and i denotes the values outside and inside of the boundary respectively.

2.4.2. Diffusive Flux stuff

Max eigenvalue of convective part $|\lambda_c^{max}|$ i.e. of $(\partial \mathbf{F}/\partial \mathbf{Q}) \cdot \mathbf{n}$, and viscous part $|\lambda_v^{max}|$ i.e. of $(\partial \mathbf{F}/\partial\left(\nabla \mathbf{Q} \cdot \mathbf{n}\right)) \cdot \mathbf{n}$ is

$$|\lambda_c^{max}|| = \|\boldsymbol{v}\| + c \tag{2.19}$$
$$|\lambda_v^{max}|| = \max\left(\frac{4}{3}\frac{\mu}{\rho}, \frac{\gamma\mu}{\Pr\rho}\right) \tag{2.20}$$

with speed of sound $c = \sqrt{\gamma R T}$

Maximum timestep (CFL)

$$\Delta t = \frac{\mathrm{CFL}}{2N+1} \frac{h}{|\lambda_c^{max}| + 2|\lambda_v^{max}|\frac{2N+1}{h}} \tag{2.21}$$

with N polynomial order and h characteristic length scale of elements

Numerical flux:

$$G(q_h^-, \nabla q_h^-; g_h^+, \nabla q_h^+) \cdot \mathbf{n} = \frac{1}{2}\left(F(q_h^+, \nabla q_h^+) + F(q_h^-, \nabla q_h^-)\right) - \frac{1}{2} s_{\max}(q_h^+ - q_h^-) \tag{2.22}$$

with

$$s_{max} = \max\left(|\lambda_c(q_h^-)|, |\lambda_c(q_h^+)\right) + 2\eta \max\left(|\lambda_\nu(q_h^-)|, |\lambda_\nu(q_h^+)\right) \tag{2.23}$$

and

$$\eta = \frac{N+1}{h} \tag{2.24}$$

2.5. Scenarios

2.5.1. Clouds

Air is initially at rest and in hydrostractic balance

$$\frac{\partial p(z)}{\partial z} = -\rho(z)g. \tag{2.25}$$

Inserting equation (2.8) leads to the ODE

$$\frac{\partial p(z)}{\partial z} = -\frac{p(z)}{RT}g, \tag{2.26}$$

which can be solved by seperation of variables. This results in the equation

$$p(z) = c \exp\left(-\frac{gz}{RT}\right), \tag{2.27}$$

where the constant of integration c can be found by considering the case

$$p(0) = c. \tag{2.28}$$

The scenario is described in terms of potential temperature θ

$$\theta = T\frac{p_0}{p}^{R/c_p}, \tag{2.29}$$

where $p_o = 10 \times 10^5$ Pa is the reference pressure. Solving for T leads to

$$T = \theta(\frac{p}{p_0})^{R/c_p}. \tag{2.30}$$

Note: Above approach leads to wrong atmosphere, rather do (with constant potential temperature): Equation of state with potential temperature:

$$\rho(z) = \frac{p_0^{\frac{c_p - c_\nu}{c_p}} p^{\frac{c_\nu}{c_p}}(z)}{R\theta} \tag{2.31}$$

Hydrostatic equilibrium with potential temperature:

$$\frac{d}{dz}p(z) = -\frac{g p_0^{\frac{R}{c_p}} p^{\frac{1}{\gamma}}(z)}{R\theta} \tag{2.32}$$

Solution for pressure with initial condition of $p(0) = p_0$:

$$p(z) = \left(C - \frac{C}{\gamma} - \frac{g p_0^{\frac{R}{c_p}} z}{R\theta} + \frac{g p_0^{\frac{R}{c_p}} z}{R\gamma\theta} \right)^{\frac{\gamma}{\gamma-1}}, \tag{2.33}$$

with constant of integration

$$C = \frac{\left(\gamma^{\frac{\gamma}{\gamma-1}} p_0\right)^{\frac{\gamma-1}{\gamma}}}{\gamma - 1}, \tag{2.34}$$

which can be obtained by setting $p(0) = p_0$.

2.6. Convergence

We first define the L_p norms for $p > 0$ by

$$\|f(x)\| = \left(\int_K |f(x)|^p \, d\mu \right)^{1/p}. \tag{2.35}$$

We start by a known analytical solution $f(x)$ and compare it to our approximation $\hat{f}(x)$. Let Q and $\hat{Q}$ denote the analytical and approximate solution at node. We start by computing the point-wise error, and compute the cell-wise error from this. Observe that for our broken space Ω

$$\|f(x)\| = \sum_{K \in \Omega} \|f_K(x)\|_p. \tag{2.36}$$

We are now ready to integrate the error for each cell. The output contains the position and conserved variables of each point. Each cell consists of $(N+1)^d$ nodes, each associated with a quadrature weight w_i. The quadrature weights are associated with the reference cuboid. We thus need to map our element to the reference element, before we can compute the integral.

Let V denote the volume (or area) of each cell. For example, in three dimensions $V = \Delta x \, \Delta y \, \Delta z$. The final result is then

$$\|f_K(x)\| = \left(V \sum_i |f(\mathbf{x}_i)|^p w_i \right)^{1/p}, \tag{2.37}$$

where we used the mapping to the reference triangle (integration by substitution).

Following [Dum10], the scenario can be described in primitive variables by setting

$$p = p_0 \cos(\mathbf{k}\mathbf{x} - \omega t) + p_b, \tag{2.38}$$

$$\rho = \rho_0 \sin(\mathbf{k}\mathbf{x} - \omega t) + \rho_b, \tag{2.39}$$

$$\mathbf{v} = \mathbf{v}_0 \sin(\mathbf{k}\mathbf{x} - \omega t). \tag{2.40}$$

We set the constants to $\left(p_0 = 0.1, p_b = \gamma^{-1}, \rho_0 = 0.5, \rho_b = 1, \mathbf{v}_0 = 0.25(1,1)^\intercal, \mathbf{k} = \pi/5(1,1)^\intercal\right)$.

We derive a source term by inserting this solution into the PDE using the symbolic math toolkit *SymPy*.

CHAPTER 3
Conclusion

Conclusion here.

APPENDIX A

Placeholder…

Bibliography

[Dum10] M. Dumbser. "Arbitrary high order PNPM schemes on unstructured meshes for the compressible Navier–Stokes equations." In: *Computers & Fluids* 39.1 (2010), pp. 60–76 (see pp. 3, 7).