DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY OF MUNICH

Master's Thesis in Informatics

Cloud Supercomputing with the ExaHyPE-Engine

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Wolken Hochleistungsrechnen mit der ExaHyPE-Engine

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I confirm that this master's thesis in informatics is my own work and I have documented all sources and material used.

Munich, January , 2019 Lukas Krenz

Acknowledgments

Quote!

(Booktitle)

Acknowledgements here.

Abstract

Abstract here!

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CHAPTER 1

Introduction

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CHAPTER 2

Methods

Methods here.

Organize into chapters!

2.1. Conservative Form

Standard (hyperbolic) form of conservation law

$$\frac{\partial}{\partial_t} Q + \nabla \cdot F(Q) = S(\boldsymbol{x}, t, Q) \tag{2.1}$$

extend to:

$$\frac{\partial}{\partial_t} Q + \nabla \cdot F(Q, \nabla Q) = S(\boldsymbol{x}, t, Q) \tag{2.2}$$

2.2. Compressible Euler

Vector of conserved quantities:

$$Q = (\rho, \rho\boldsymbol{v}, \rho E) \tag{2.3}$$

Flux:

$$F(Q) = \begin{pmatrix} \rho\boldsymbol{v} \\ \boldsymbol{v} \otimes \rho\boldsymbol{v} + p\mathbf{I} \\ \boldsymbol{v} \cdot (\mathbf{I}\rho E + \mathbf{I}p) \end{pmatrix} \tag{2.4}$$

$$p = (\gamma - 1)\left(\rho E - 0.5\left(\boldsymbol{v} \cdot \rho\boldsymbol{v}\right)\right) \tag{2.5}$$

2.3. Compressible Navier Stokes

Following [Dum10] by Dumbser.

Flux:

$$F(Q, \nabla Q) = \begin{pmatrix} \rho\boldsymbol{v} \\ \boldsymbol{v} \otimes \rho\boldsymbol{v} + \mathbf{I}p + \boldsymbol{\sigma}(Q, \nabla Q) \\ \boldsymbol{v} \cdot (\mathbf{I}\rho E + \mathbf{I}p + \boldsymbol{\sigma}(Q, \nabla Q)) - \kappa\nabla T \end{pmatrix} \tag{2.6}$$

With: Pressure p

$$p = (\gamma - 1)\left(\rho E - 0.5\left(\boldsymbol{v} \cdot \rho \boldsymbol{v}\right)\right) \tag{2.7}$$

Temperature T

$$\frac{p}{\rho} = RT \tag{2.8}$$

Heat conduction coefficient κ

$$\kappa = \frac{\mu\gamma}{\Pr}\frac{1}{\gamma - 1}R \tag{2.9}$$

with R gas constant and Pr Prandtl number

Sutherland's viscosity law:

$$\mu(T) = \mu_0 \left(\frac{T}{T_0}\right)^{\beta} \frac{T_0 + C}{T + C} \tag{2.10}$$

with $\beta = 1.5$, $C =$ const., reference temperature T_0 and reference viscosity $\mu_=$. Equal to

$$\lambda = \frac{\mu_0(T_0 + C)}{T_0^{\beta}} \tag{2.11}$$

$$\mu(T) = \lambda \frac{T^{\beta}}{T + C}, \tag{2.12}$$

where λ is constant for a given fluid.

stress tensor

$$\boldsymbol{\sigma}(Q, \nabla Q) = \left(2/3 \mu \nabla \cdot \boldsymbol{v}\right) - \mu\left(\nabla(\boldsymbol{v}) + \nabla(\boldsymbol{v})^{T}\right) \tag{2.13}$$

Max eigenvalue of convective part $|\lambda_c^{max}|$ i.e. of $(\partial \mathbf{F}/\partial \mathbf{Q}) \cdot \boldsymbol{n}$, and viscous part $|\lambda_v^{max}|$ i.e. of $(\partial \mathbf{F}/\partial\left(\nabla \mathbf{Q} \cdot \boldsymbol{n}\right)) \cdot \boldsymbol{n}$ is

$$|\lambda_c^{max}|| = \|\boldsymbol{v}\| + c \tag{2.14}$$

$$|\lambda_v^{max}|| = \max\left(\frac{4}{3}\frac{\mu}{\rho}, \frac{\gamma\mu}{\Pr\rho}\right) \tag{2.15}$$

with speed of sound $c = \sqrt{\gamma RT}$

Maximum timestep (CFL)

$$\Delta t = \frac{\mathrm{CFL}}{2N + 1} \frac{h}{|\lambda_c^{max}| + 2|\lambda_v^{max}| \frac{2N+1}{h}} \tag{2.16}$$

with N polynomial order and h characteristic length scale of elements

Numerical flux:

$$G(q_h^-, \nabla q_h^-; g_h^+, \nabla q_h^+) \cdot \boldsymbol{n} = \frac{1}{2}\left(F(q_h^+, \nabla q_h^+) + F(q_h^-, \nabla q_h^-)\right) - \frac{1}{2} s_{max}(q_h^+ - q_h^-) \tag{2.17}$$

with

$$s_{\max} = \max\left(|\lambda_c(q_h^-)|, |\lambda_c(q_h^+)\right) + 2\eta \max\left(|\lambda_v(q_h^-)|, |\lambda_v(q_h^+)\right) \tag{2.18}$$

and

$$\eta = \frac{N+1}{h} \tag{2.19}$$

2.4. Scenarios

2.4.1. Clouds

Air is initially at rest and in hydrostractic balance

$$\frac{\partial p(z)}{\partial z} = -\rho(z) g. \tag{2.20}$$

Inserting equation (2.8) leads to the ODE

$$\frac{\partial p(z)}{\partial z} = -\frac{p(z)}{RT} g, \tag{2.21}$$

which can be solved by seperation of variables. This results in the equation

$$p(z) = c \exp\left(-\frac{gz}{RT}\right), \tag{2.22}$$

where the constant of integration c can be found by considering the case

$$p(0) = c. \tag{2.23}$$

The scenario is described in terms of potential temperature θ

$$\theta = T \frac{p_0}{p}^{R/c_p}, \tag{2.24}$$

where $p_o = 10 \times 10^5$ Pa is the reference pressure. Solving for T leads to

$$T = \theta(\frac{p}{p_0})^{R/c_p}. \tag{2.25}$$

Note: Above approach leads to wrong atmosphere, rather do (with constant potential temperature): Equation of state with potential temperature:

$$\rho(z) = \frac{p_0^{\frac{c_p - c_v}{c_p}} p^{\frac{c_v}{c_p}}(z)}{R\theta} \tag{2.26}$$

Hydrostatic equilibrium with potential temperature:

$$\frac{d}{dz}p(z) = -\frac{g p_0^{\frac{R}{c_p}} p^{\frac{1}{\gamma}}(z)}{R\theta} \tag{2.27}$$

Solution for pressure with initial condition of $p(0) = p_0$:

$$p(z) = \left(C - \frac{C}{\gamma} - \frac{g p_0^{\frac{R}{c_p}} z}{R\theta} + \frac{g p_0^{\frac{R}{c_p}} z}{R\gamma\theta} \right)^{\frac{\gamma}{\gamma-1}}, \tag{2.28}$$

with constant of integration

$$C = \frac{\left(\gamma^{\frac{\gamma}{\gamma-1}} p_0 \right)^{\frac{\gamma-1}{\gamma}}}{\gamma - 1}, \tag{2.29}$$

which can be obtained by setting $p(0) = p_0$.

2.5. Convergence

We first define the L_p norms for $p > 0$ by

$$\|f(x)\| = \left(\int_K |f(x)|^p \, d\mu \right)^{1/p}. \tag{2.30}$$

We start by a known analytical solution $f(x)$ and compare it to our approximation $\hat{f}(x)$. Let Q and $\hat{Q}$ denote the analytical and approximate solution at node. We start by computing the point-wise error, and compute the cell-wise error from this. Observe that for our broken space Ω

$$\|f(x)\| = \sum_{K \in \Omega} \|f_K(x)\|_p. \tag{2.31}$$

We are now ready to integrate the error for each cell. The output contains the position and conserved variables of each point. Each cell consists of $(N+1)^d$ nodes, each associated with a quadrature weight w_i. The quadrature weights are associated with the reference cuboid. We thus need to map our element to the reference element, before we can compute the integral.

Let V denote the volume (or area) of each cell. For example, in three dimensions $V = \Delta x \, \Delta y \, \Delta z$. The final result is then

$$\|f_K(x)\| = \left(V \sum_i |f(\mathbf{x_i})|^p w_i \right)^{1/p}, \tag{2.32}$$

where we used the mapping to the reference triangle (integration by substitution).

Following [Dum10], the scenario can be described in primitive variables by setting

$$p = p_0 \cos(\mathbf{k}\mathbf{x} - \omega t) + p_b, \tag{2.33}$$

$$\rho = \rho_0 \sin(\mathbf{k}\mathbf{x} - \omega t) + \rho_b, \tag{2.34}$$

$$\mathbf{v} = \mathbf{v}_0 \sin(\mathbf{k}\mathbf{x} - \omega t). \tag{2.35}$$

We set the constants to $\left(p_0 = 0.1, p_b = \gamma^{-1}, \rho_0 = 0.5, \rho_b = 1, \mathbf{v}_0 = 0.25(1,1)^\intercal, \mathbf{k} = \pi/5(1,1)^\intercal\right)$.

We derive a source term by inserting this solution into the PDE using the symbolic math toolkit *SymPy*.

CHAPTER 3

Conclusion

Conclusion here.

APPENDIX A

Placeholder…

Bibliography

[Dum10] M. Dumbser. "Arbitrary high order PNPM schemes on unstructured meshes for the compressible Navier–Stokes equations." In: *Computers & Fluids* 39.1 (2010), pp. 60–76 (see pp. 2, 6).