Perfect Binary Trees

Weiss Book Chapter 4.2/4.3

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Lecture slides were prepared based on Douglas W. Harder's notes (**dwharder@alumni.uwaterloo.ca).**

Outline

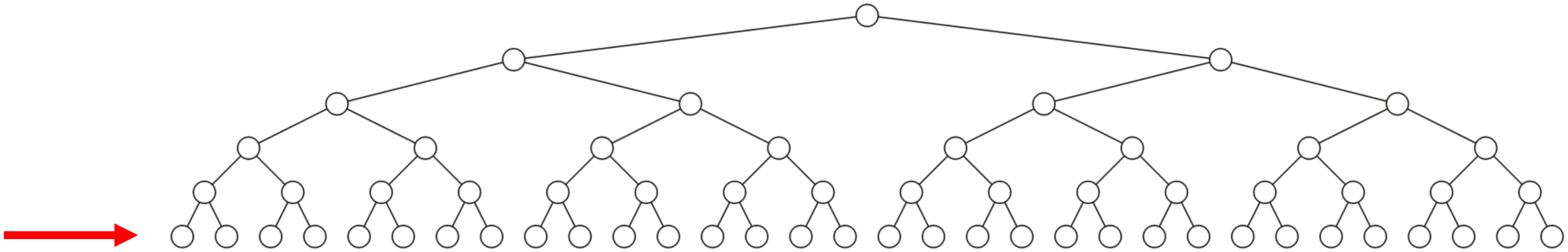
Introducing perfect binary trees

- Definitions and examples
- Number of nodes
- Logarithmic height
- Number of leaf nodes
- Applications

Definition

Standard definition:

– **A perfect binary tree** of height h is a binary tree where
  - All leaf nodes have the same depth h
  - All non-leaf nodes are full



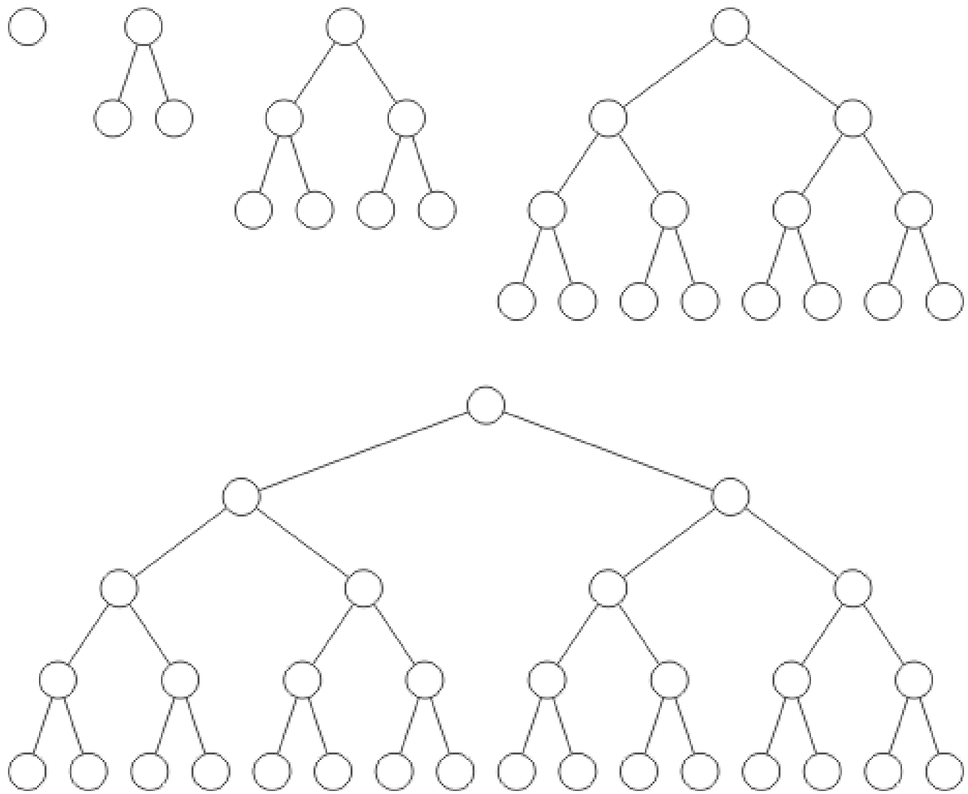
Definition

Recursive definition:

- A binary tree of height $h = 0$ is perfect
- A binary tree with height $h\ (> 0)$ is a perfect if both sub-trees are perfect binary trees of height $h - 1$

Examples

Perfect binary trees of height $h = 0, 1, 2, 3$ and 4



Theorems

Four theorems that describe the properties of perfect binary trees:

- A perfect tree has Q1 nodes
- The height is Q2
- There are Q3 leaf nodes
- The average depth of a node is Q4

These theorems determine the optimal run-time properties of operations on binary trees

$2^{h+1} - 1$ Nodes

Theorem:

A perfect binary tree of height h has $2^{h+1} - 1$ nodes

Proof:

We will use mathematical induction:

1. Show that it is true for $h = 0$
2. Assume it is true for an arbitrary h
3. Show that the truth for h implies the truth for $h + 1$

$2^{h+1} - 1$ Nodes

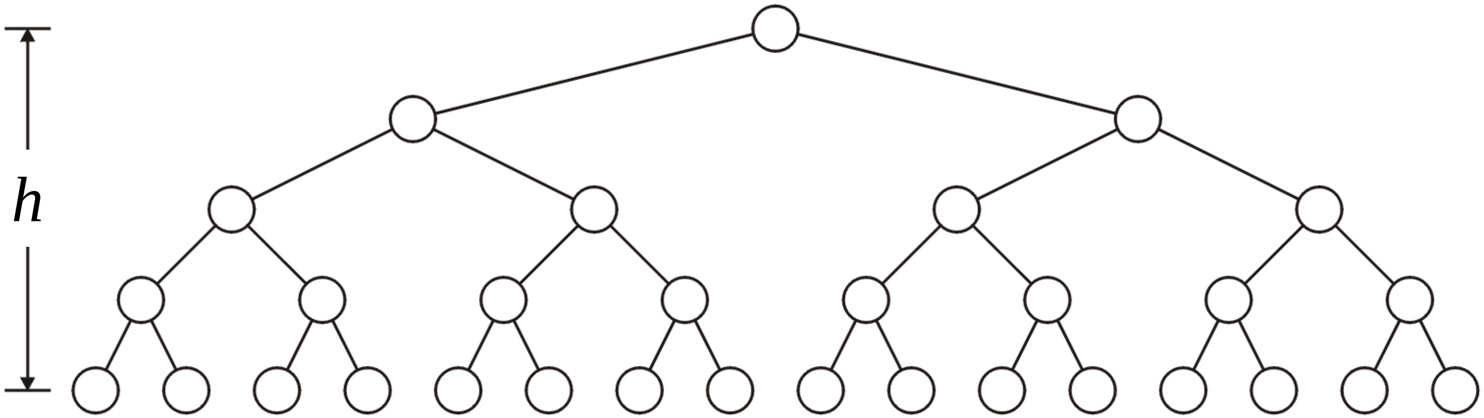
The base case:

- When $h = 0$ we have a single node $n = 1$
- The formula is correct: $2^{0+1} - 1 = 1$

$2^{h+1} - 1$ Nodes

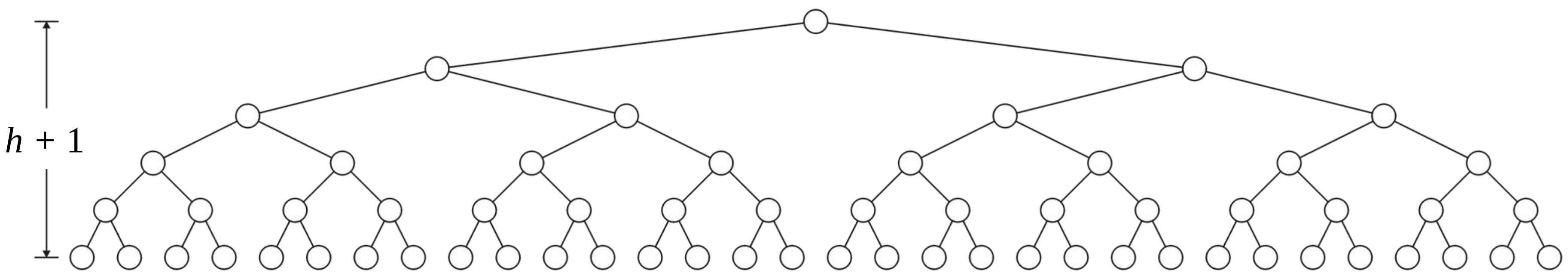
The inductive step:

- Assume that if the height of the tree is h, the number of nodes is $n = 2^{h+1} - 1$



$2^{h+1} - 1$ Nodes

We must show that a tree of height $h + 1$ has
$n = 2^{(h+1)+1} - 1 = 2^{h+2} - 1$ nodes

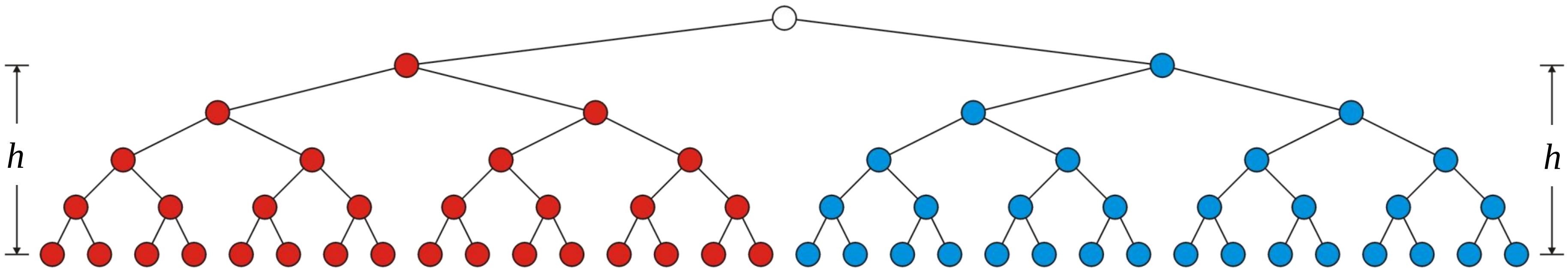


$2^{h+1} - 1$ Nodes

Using the recursive definition, both sub-trees are perfect trees of height h

- By assumption, each sub-tree has $2^{h+1} - 1$ nodes
- Therefore the total number of nodes is

$$(2^{h+1} - 1) + 1 + (2^{h+1} - 1) = 2^{h+2} - 1$$



Logarithmic Height

Theorem

A perfect binary tree with n nodes has height $\lg(n + 1) - 1$

Proof

Solving $n = 2^{h+1} - 1$ for h:

$$n + 1 = 2^{h+1}$$

$$\lg(n + 1) = h + 1$$

$$h = \lg(n + 1) - 1$$

Logarithmic Height

Lemma:

$\lg(n + 1) - 1 = \Theta(\ln(n))$

Proof:

$$\lim_{n\to\infty} \frac{\lg(n+1)-1}{\ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\ln(2)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(n+1)\ln(2)} = \lim_{n\to\infty} \frac{1}{\ln(2)} = \frac{1}{\ln(2)}$$

2^h Leaf Nodes

Theorem:

A perfect binary tree with height h has 2^h leaf nodes

Proof (by induction):

When $h = 0$, there is $2^0 = 1$ leaf node.

Assume that a perfect binary tree of height h has 2^h leaf nodes.

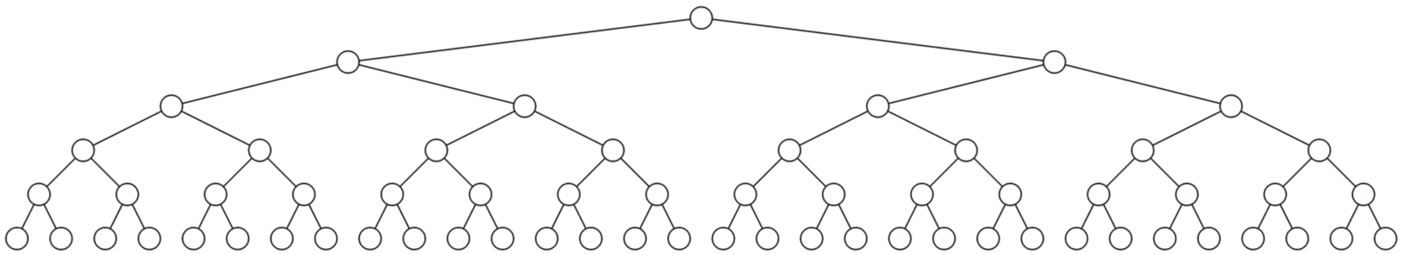
Then observe that both sub-trees of a perfect binary tree of height $h + 1$ have 2^h leaf nodes.

Consequence: Over half all nodes are leaf nodes:

$$\frac{2^h}{2^{h+1}-1} > \frac{1}{2}$$

The Average Depth of a Node

The average depth of a node in a perfect binary tree is



Depth	Count
0	1
1	2
2	4
3	8
4	16
5	32

Sum of the depths

$$\frac{\sum_{k=0}^{h} k2^k}{2^{h+1}-1} = \frac{h2^{h+1} - 2^{h+1} + 2}{2^{h+1}-1} = \frac{h(2^{h+1}-1) - (2^{h+1}-1) + 1 + h}{2^{h+1}-1}$$

$$= h - 1 + \frac{h+1}{2^{h+1}-1} \approx h - 1 = \Theta\left(\ln(n)\right)$$

Number of nodes

The Average Depth of a Node

$$
\begin{array}{rrl}
 & 2\cdot\sum_{k=0}^{h} k\cdot 2^{k} = & 1\cdot 2^{2} + \cdots + (h-1)2^{h} + h\cdot 2^{h+1} \\
- & \sum_{k=0}^{h} k\cdot 2^{k} = & 1\cdot 2^{1} + 2\cdot 2^{2} + \cdots + h\cdot 2^{h} \\
\hline
\end{array}
$$

$$
\begin{aligned}
\sum_{k=0}^{h} k\cdot 2^{k} &= h\cdot 2^{h+1} - \sum_{k=1}^{h} 2^{k} \\
&= h\cdot 2^{h+1} - \frac{2(2^{h}-1)}{2-1} \\
&= h\cdot 2^{h+1} - 2^{h+1} + 2 \quad \blacksquare
\end{aligned}
$$

Applications

Perfect binary trees are considered to be the *ideal* case

- The height and average depth are both $\Theta(\ln(n))$

We will attempt to find trees which are as close as possible to perfect binary trees for efficient operations

Summary

We have defined perfect binary trees and discussed:

- The number of nodes: $n = 2^{h+1} - 1$
- The height: $\lg(n + 1) - 1$
- The number of leaves: 2^h
- Half the nodes are leaves
  - Average depth is $\Theta(\ln(n))$
- It is an ideal case