ECE430.217 Data Structures

# Topological Sort

**Textbook: Weiss Chapter 9.2**

**Byoungyoung Lee**
**https://compsec.snu.ac.kr**
**byoungyoung@snu.ac.kr**

Lecture slides were prepared based on Douglas W. Harder's notes (**dwharder@alumni.uwaterloo.ca).**

# Topological Sort

In this topic, we will discuss:

- Motivations
- Review the definition of a directed acyclic graph (DAG)
- Describe a topological sort and applications
- Kahn’s algorithm

# Motivation

Given a set of tasks with dependencies,

is there an order in which we can complete the tasks?

Dependencies form a partial ordering

- A partial ordering on a finite number of objects can be represented as a directed acyclic graph (DAG)

# Restriction of paths in DAGs

**Claim:**

In a DAG, given two different vertices $v_j$ and $v_k$,
there cannot both be a path from $v_j$ to $v_k$ and a path from $v_k$ to $v_j$

**Proof by contradiction:**

Assume otherwise; thus there exists two paths:

$$(v_j, v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_k)$$

$$(v_k, v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_j)$$

From this, we can construct the path

$$(v_j, v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_k, v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_j)$$

This path is a cycle, but it is assumed a DAG

$\therefore$ contradiction

# Definition of topological sorting

**Definition:**

A topological sorting of the vertices in a DAG is an ordering

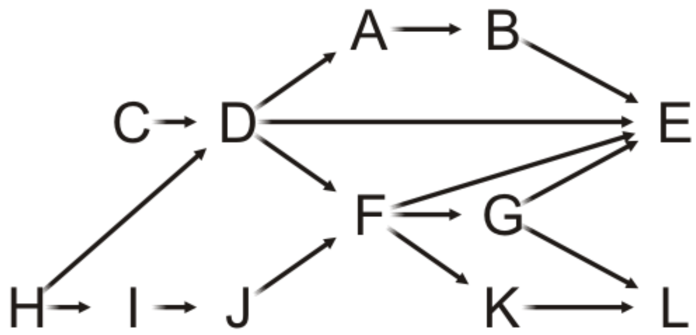
$$v_1, v_2, v_3, \ldots, v_{|V|}$$

such that $v_j$ should appear before $v_k$ if there is a path from $v_j$ to $v_k$

**Example:**

Given this DAG, a topological sort is

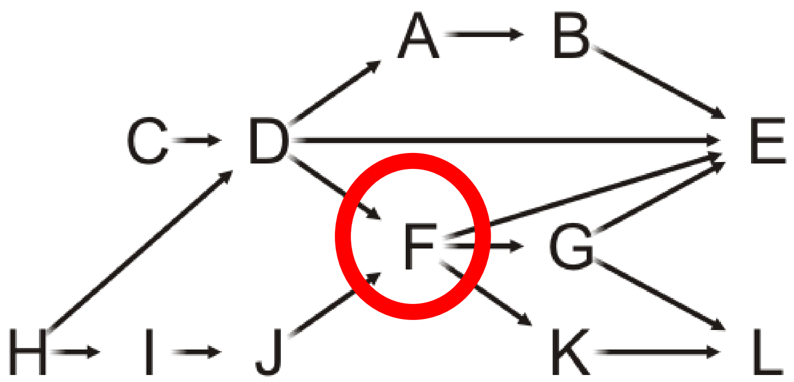
H, C, I, D, J, A, F, B, G, K, E, L



# Example

For example, there are paths from **H, C, I, D**, and **J** to **F**, so all these must come before **F** in a topological sort

**H**, **C**, **I**, **D**, **J**, A, **F**, B, G, K, E, L




Clearly, this sorting need not be unique

# Applications #1

When you are getting ready for a date

You must wear the following:

- jacket, shirt, briefs, socks, tie, etc.

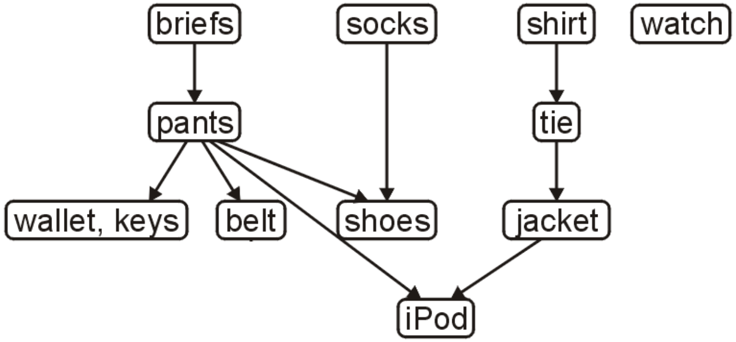
There are certain constraints:

- the pants really should go on after the briefs,
- socks are put on before shoes

http://www.idealliance.org/proceedings/xml03/slides/mansfield&otkunc/Paper/03-02-04.html

# Applications #1

The following is a task graph for getting dressed:




One topological sort is:

briefs, pants, wallet, keys, belt, socks, shoes, shirt, tie, jacket, iPod, watch

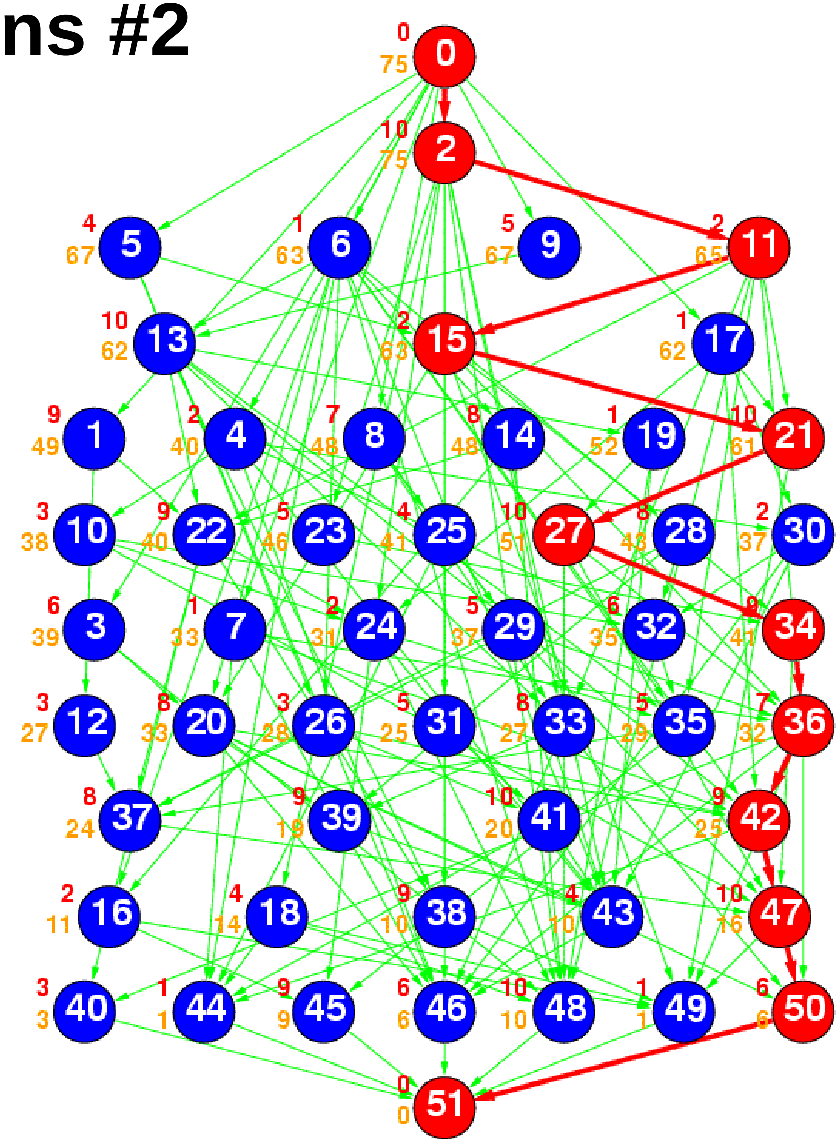
A more reasonable topological sort is:

briefs, socks, pants, shirt, belt, tie, jacket, wallet, keys, iPod, watch, shoes

# Applications #2

The following is a DAG representing a number of tasks

- The green arrows represent dependencies
- The numbering indicates a topological sort of the tasks




Ref: The Standard Task Graph
http://www.kasahara.elec.waseda.ac.jp/schedule/

# Idea: How to Solve Topological Sort

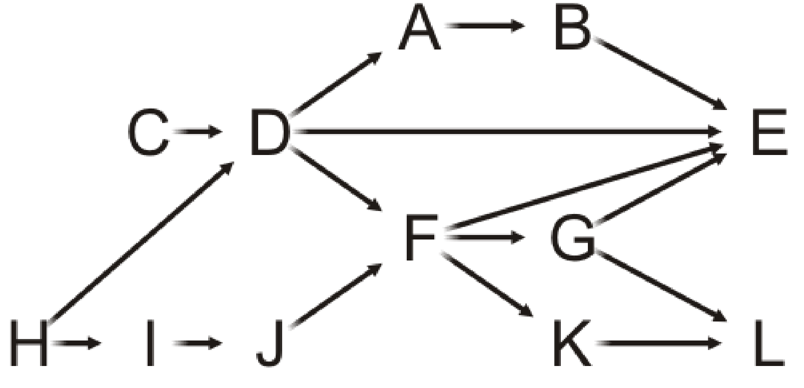
**Idea:**

- Given a DAG $V$, make a copy $W$ and iterate:
  - Find a vertex $v$ in $W$ with in-degree zero
  - Let $v$ be the next vertex in the topological sort
  - Continue iterating with the vertex-induced sub-graph $W \setminus \{v\}$

# Idea: Example

On this graph, iterate the following $|V| = 12$ times

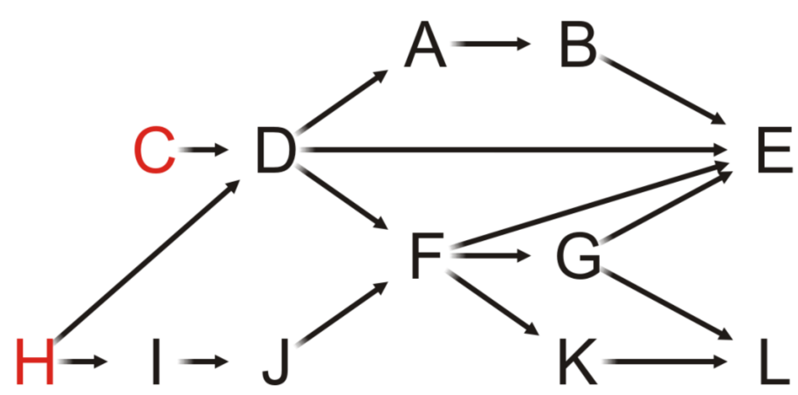
- Choose a vertex $v$ that has in-degree zero
- Let $v$ be the next vertex in our topological sort
- Remove $v$ and all edges connected to it



# Idea: Example

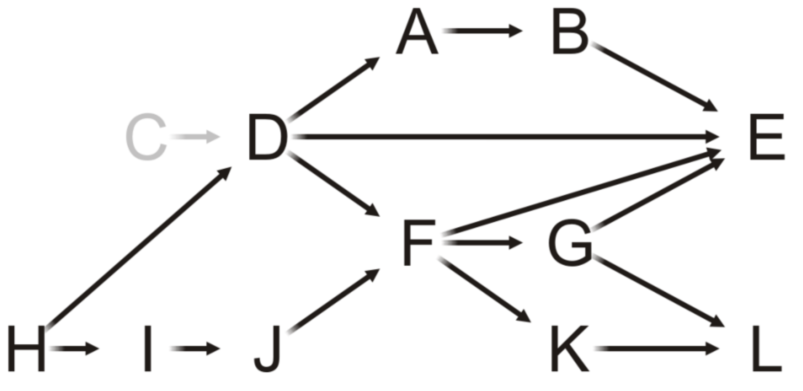
Let's step through this algorithm with this example

- Which task can we start with? Either C or H
- Let's start with Task C



# Idea: Example

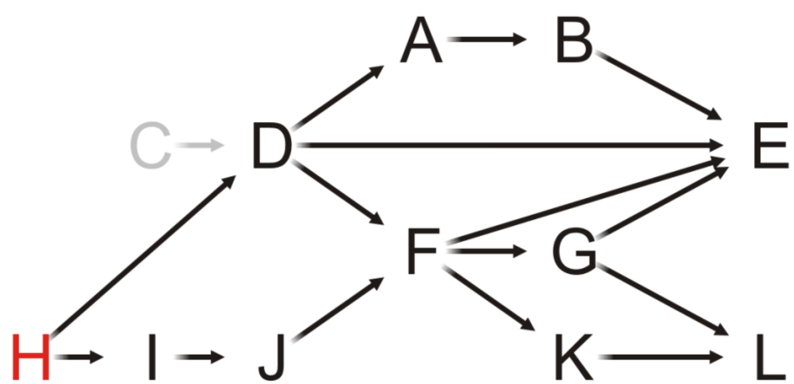
Having completed Task C, which vertices have in-degree zero?




C

# Idea: Example

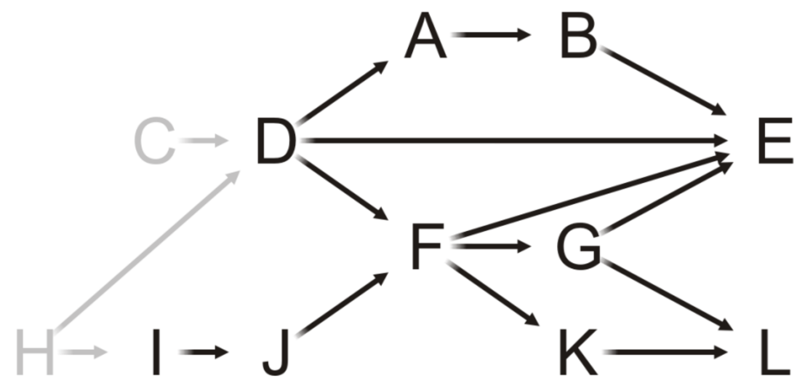
Only Task H can be completed, so we choose it




C

# Idea: Example

Having removed H, what is next?

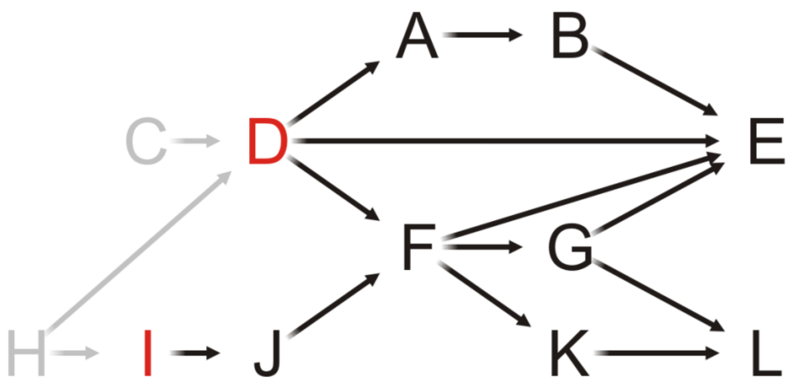



C, H

# Idea: Example

Both Tasks D and I have in-degree zero

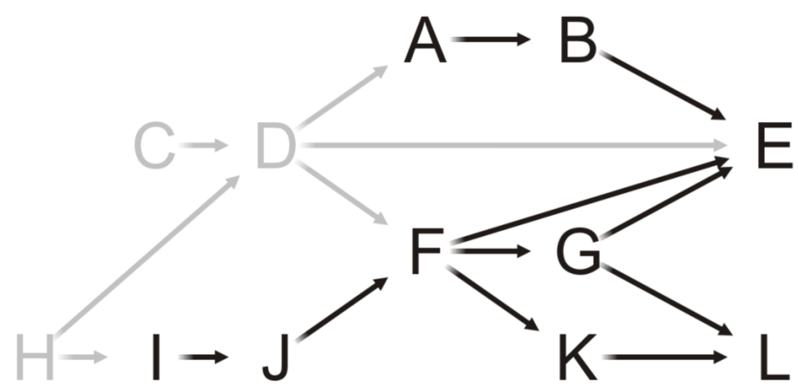
– Let us choose Task D




C, H

# Idea: Example

We remove Task D, and now?

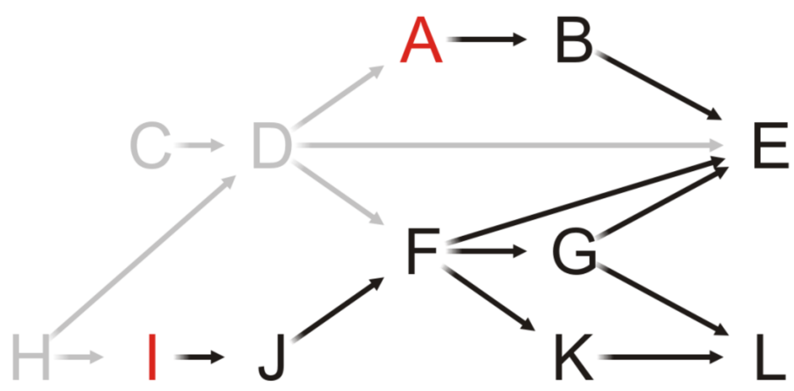



C, H, D

# Idea: Example

Both Tasks A and I have in-degree zero

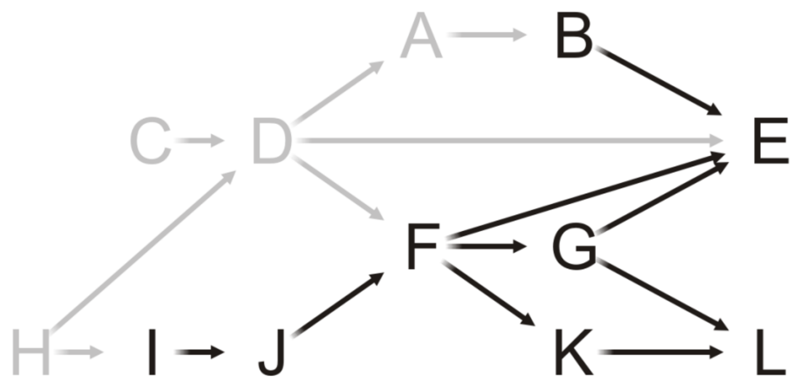
– Let's choose Task A




C, H, D

# Idea: Example

Having removed A, what now?

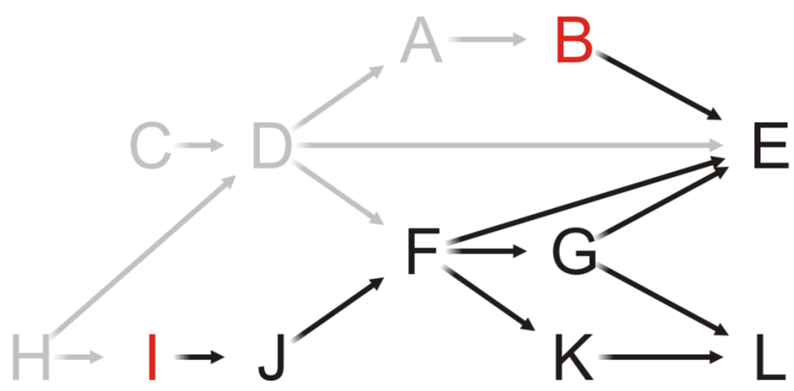



C, H, D, A

# Idea: Example

Both Tasks B and I have in-degree zero

– Choose Task B

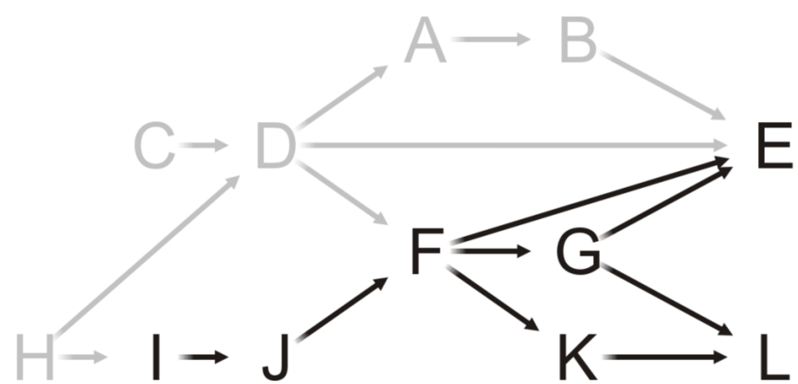



C, H, D, A

# Idea: Example

Removing Task B, we note that Task E still has an in-degree of two

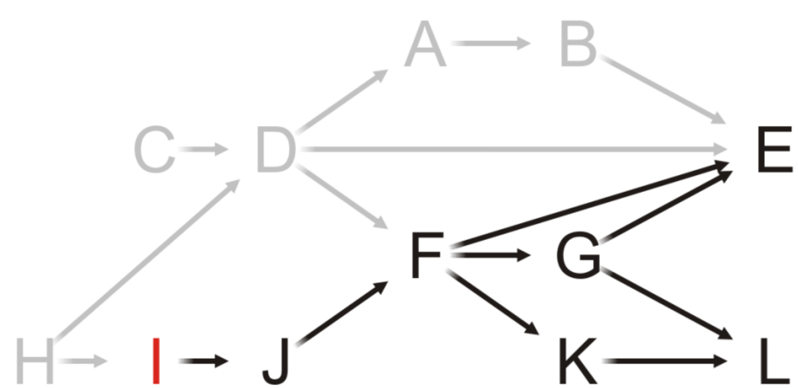
- Next?




C, H, D, A, B

# Idea: Example

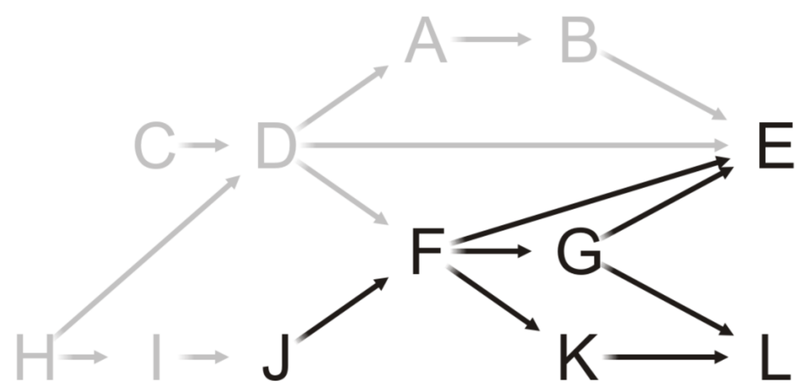
As only Task I has in-degree zero, we choose it




C, H, D, A, B

# Idea: Example

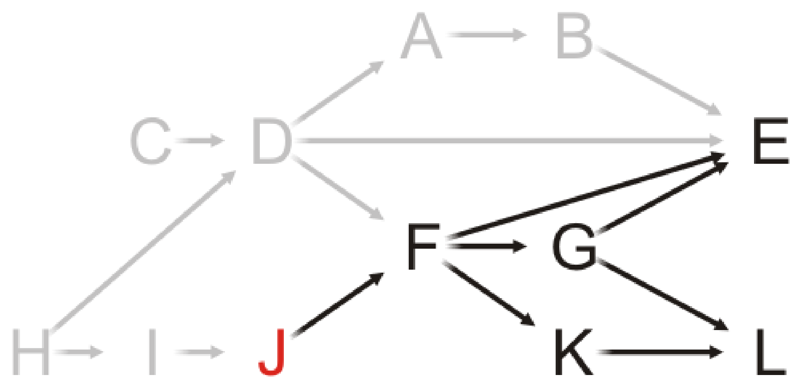
Having completed Task I, what now?




C, H, D, A, B, I

# Idea: Example

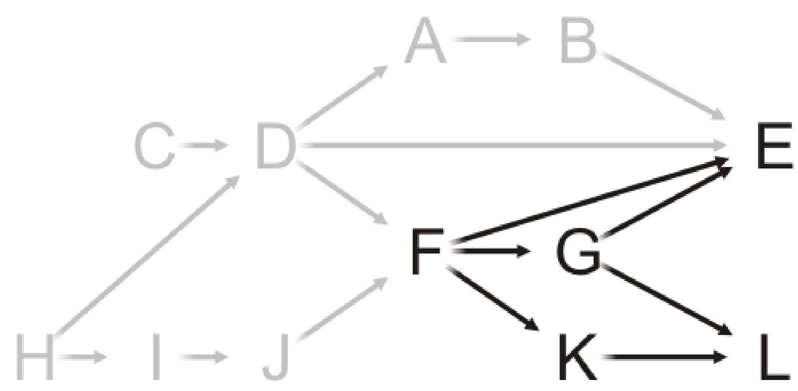
Only Task J has in-degree zero: choose it




C, H, D, A, B, I

# Idea: Example

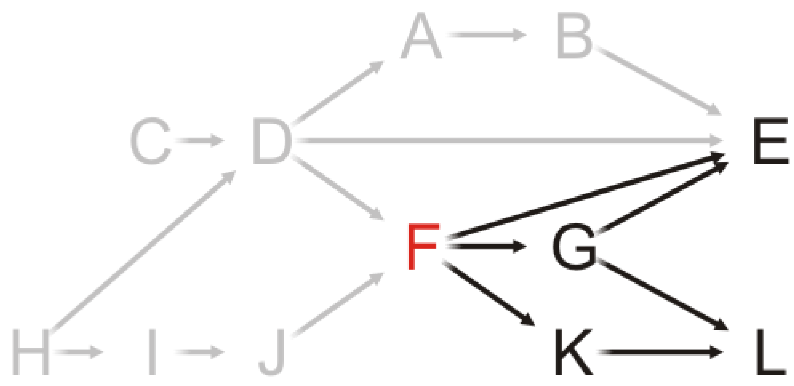
Having completed Task J, what now?




C, H, D, A, B, I, J

# Idea: Example

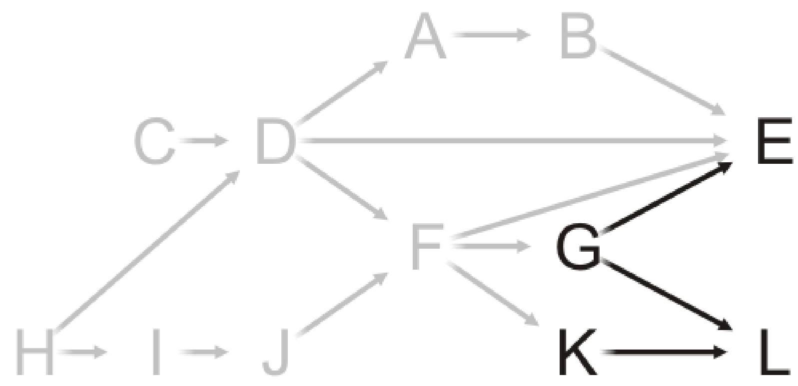
Only Task F can be completed, so choose it




C, H, D, A, B, I, J

# Idea: Example

What choices do we have now?

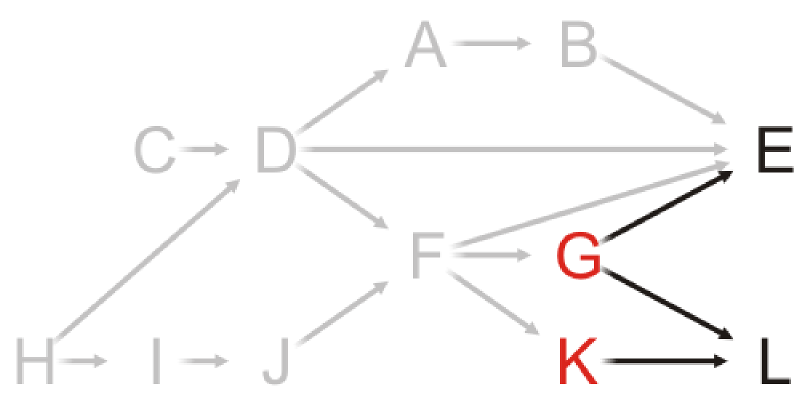



C, H, D, A, B, I, J, F

# Idea: Example

We can perform Tasks G or K

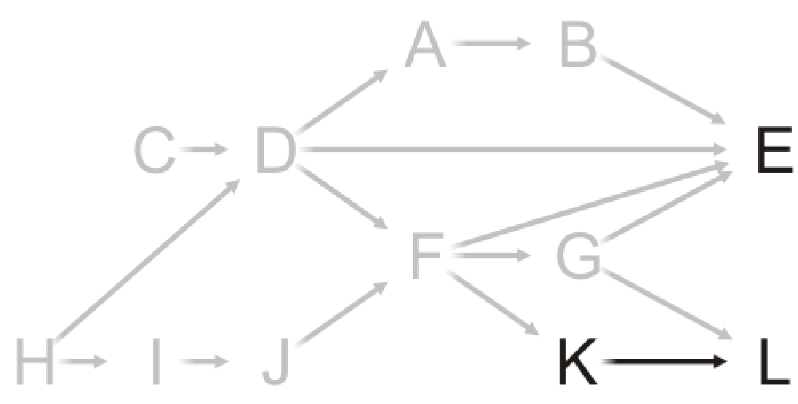
- Choose Task G




C, H, D, A, B, I, J, F

# Idea: Example

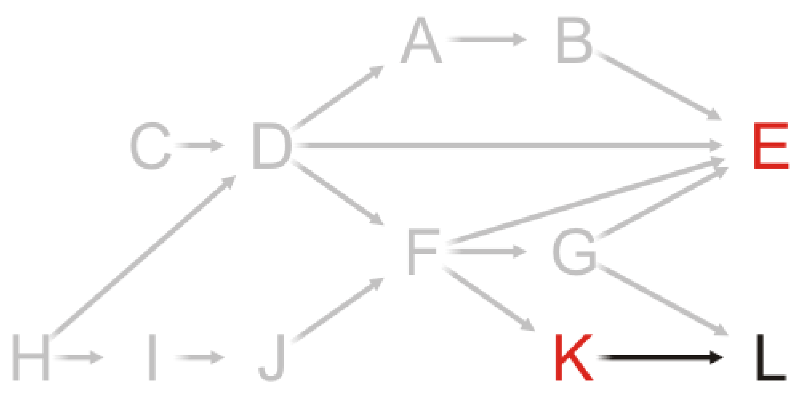
Having removed Task G from the graph, what next?




C, H, D, A, B, I, J, F, G

# Idea: Example

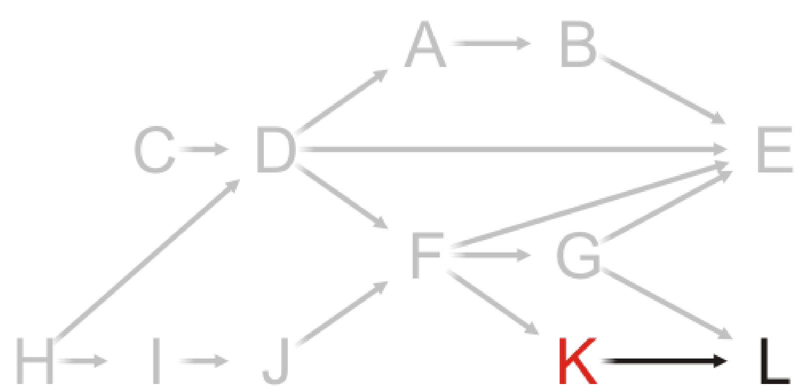
Choosing between Tasks E and K, choose Task E




C, H, D, A, B, I, J, F, G

# Idea: Example

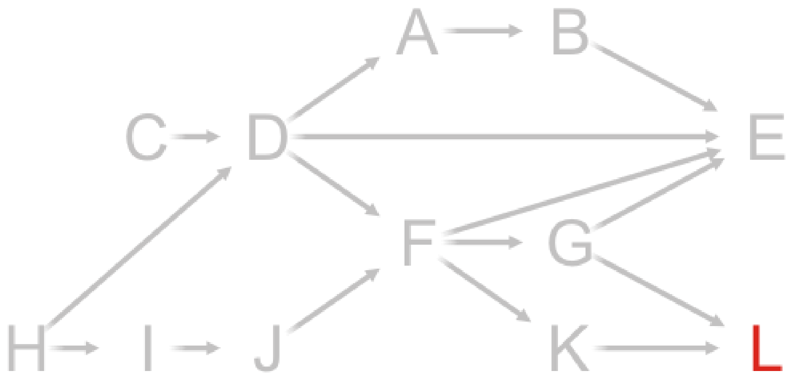
At this point, Task K is the only one that can be run




C, H, D, A, B, I, J, F, G, E

# Idea: Example

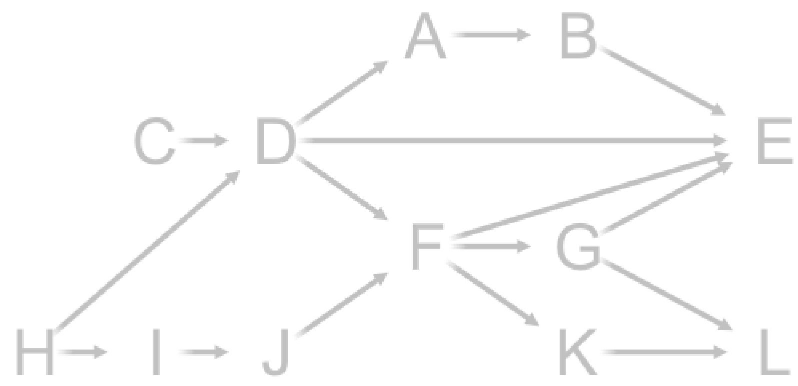
And now that both Tasks G and K are complete,
we can complete Task L




C, H, D, A, B, I, J, F, G, E, K

# Idea: Example

There are no more vertices left

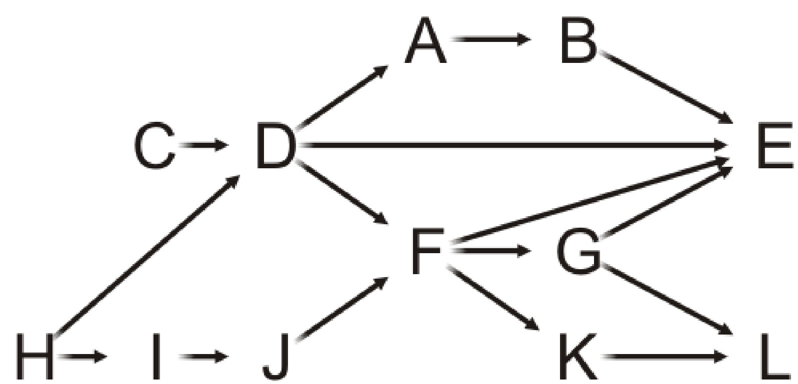



C, H, D, A, B, I, J, F, G, E, K, L

# Idea: Example

Thus, one possible topological sort would be:

C, H, D, A, B, I, J, F, G, E, K, L

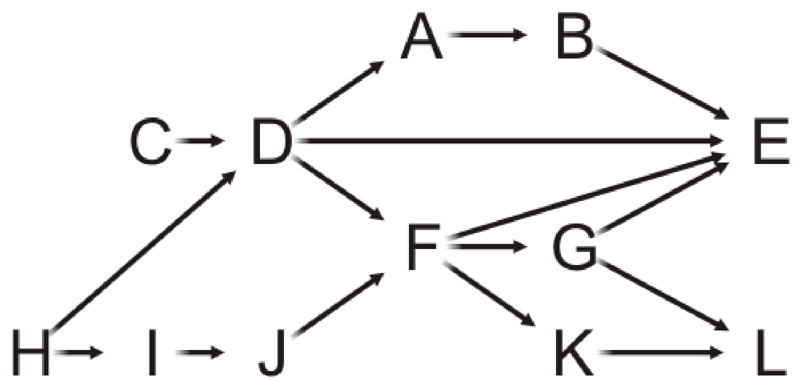


# Idea: Example

Note that topological sorts need not be unique:

C, H, D, A, B, I, J, F, G, E, K, L

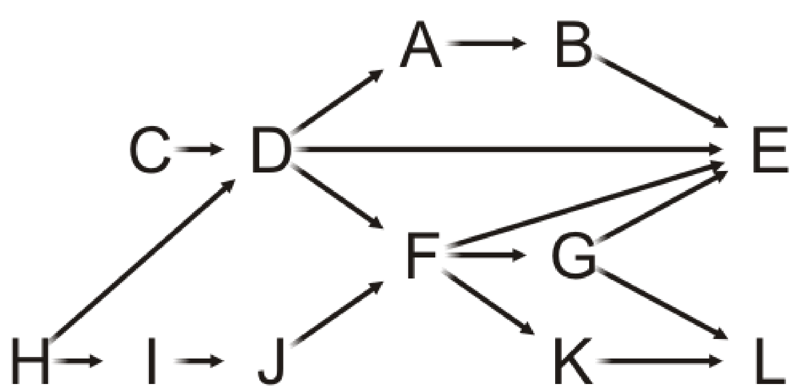
H, I, J, C, D, F, G, K, L, A, B, E



# Kahn's Algorithm

- **Kahn's algorithm** solves the topological sort problem

- **Step #1: Preparing In-degree array**
  - Construct an array, maintaining the in-degrees of each vertex
  - Requires $\Theta(|V|)$ memory




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| D | 2 |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

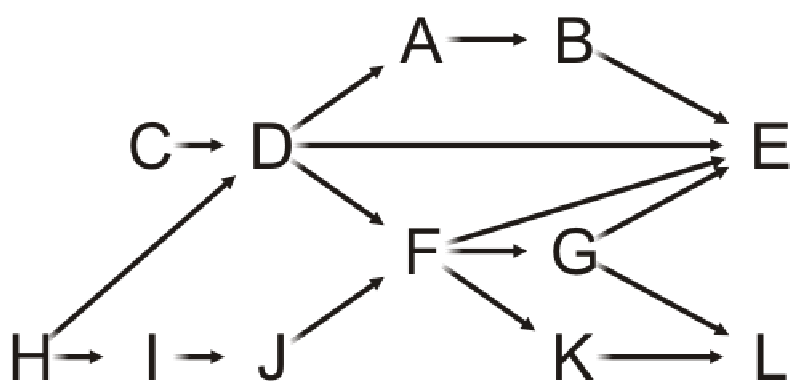
# Kahn’s Algorithm

- **Step #2: Enumerating in-degree array**
  - #2-A. Prepare an empty queue
  - #2-B. Enqueue all the vertices with the in-degree of zero
  - #2-C. While the queue is not empty
    - Dequeue a vertex
    - Add this vertex to the sequence of topological sort
    - Decrement the in-degree of all its neighboring vertices
    - Enqueue the neighboring vertices with the in-degree of zero

# Example: Kahn's Algorithm

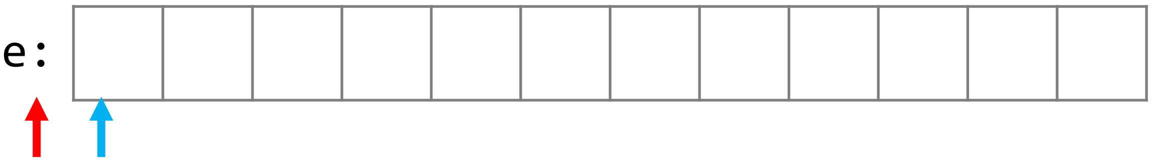
With the previous example, we initialize:

- The array of in-degrees
- The queue




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| D | 2 |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

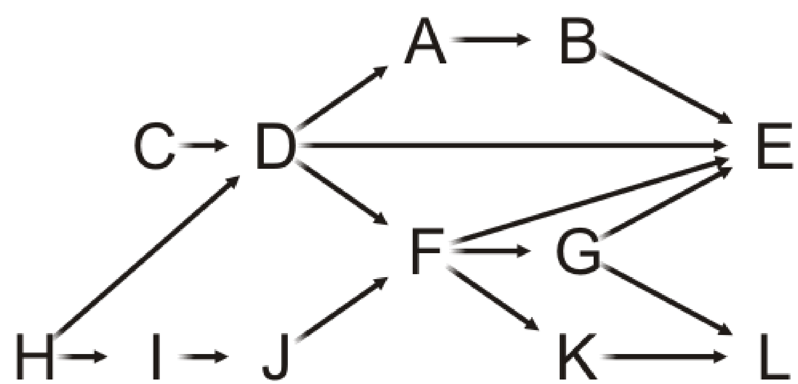
Queue:



The queue is empty

# Example: Kahn's Algorithm

Stepping through the table, push all source vertices into the queue




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| D | 2 |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

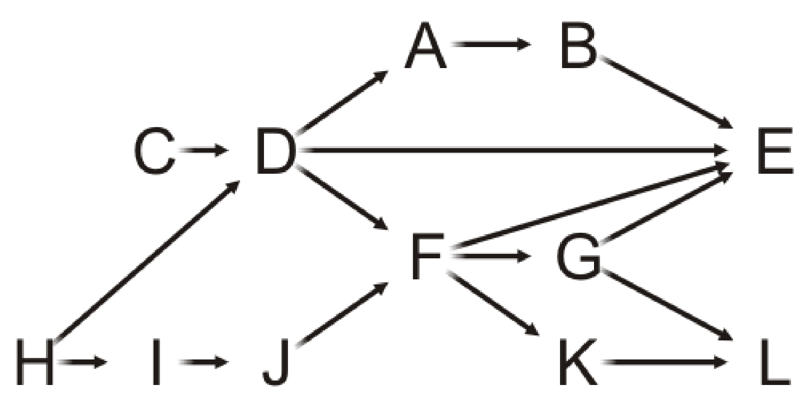
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | |

The queue is empty

# Example: Kahn's Algorithm

Stepping through the table, push all source vertices into the queue




| | |
|---|---|
| A | 1 |
| B | 1 |
| **C** | **0** |
| D | 2 |
| E | 4 |
| F | 2 |
| G | 1 |
| **H** | **0** |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

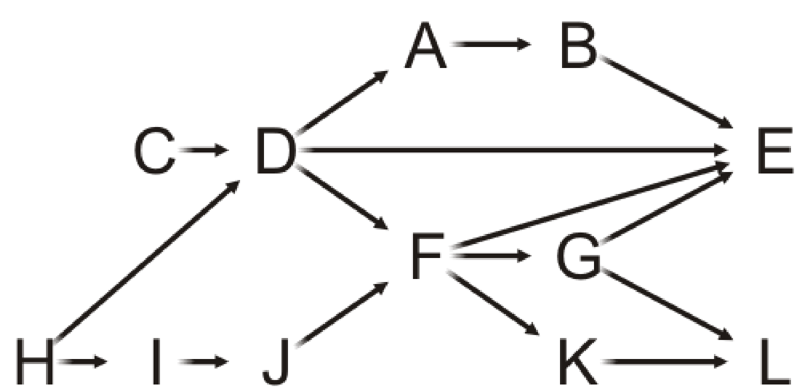
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **C** | **H** | | | | | | | | | | |

The queue is empty

# Example: Kahn’s Algorithm

Pop the front of the queue




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| D | 2 |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

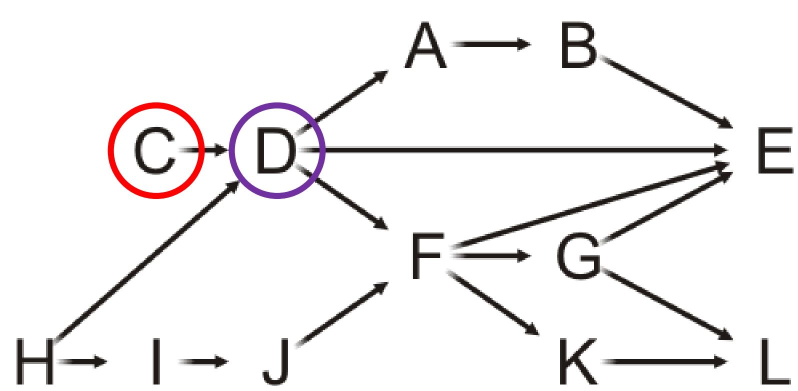
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- C has one neighbor: D




| | |
|---|---|
| A | 1 |
| B | 1 |
| **C** | **0** |
| **D** | **2** |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

Queue:

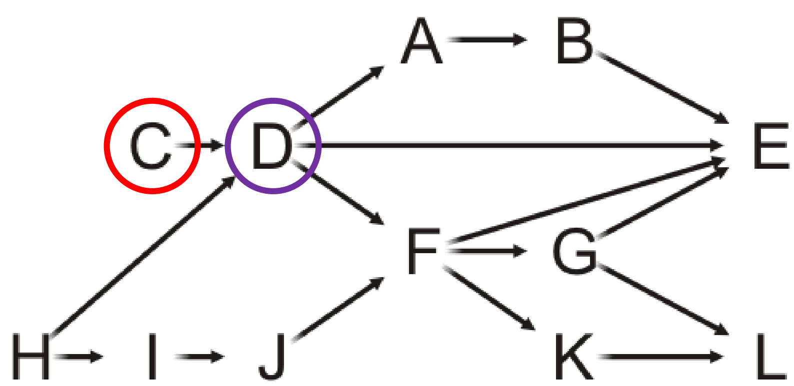
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- C has one neighbor: D
- Decrement in-degree of D

| | |
|---|---|
| A | 1 |
| B | 1 |
| **C** | **0** |
| **D** | **1** |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

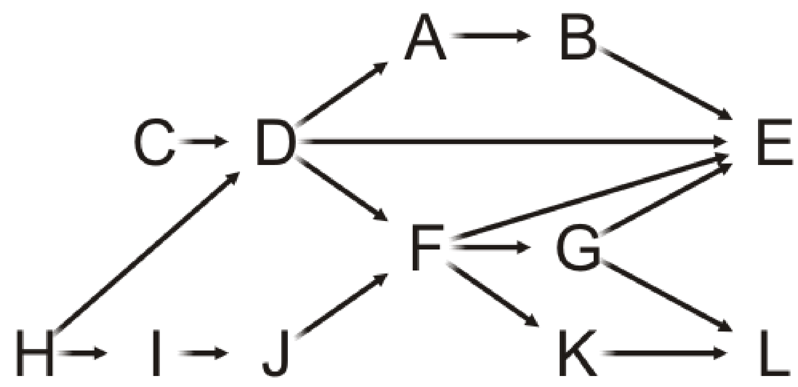



Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue




Queue:

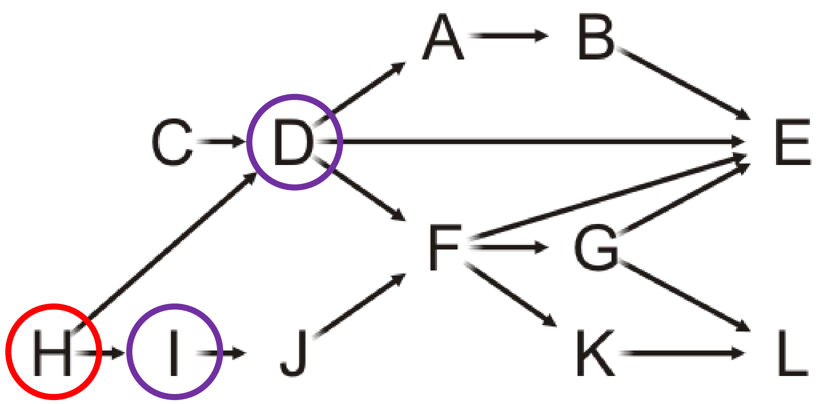
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| D | 1 |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 1 |
| J | 1 |
| K | 1 |
| L | 2 |

# Example: Kahn’s Algorithm

Pop the front of the queue

– H has two neighbors: D and I




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| **D** | **1** |
| E | 4 |
| F | 2 |
| G | 1 |
| **H** | **0** |
| **I** | **1** |
| J | 1 |
| K | 1 |
| L | 2 |

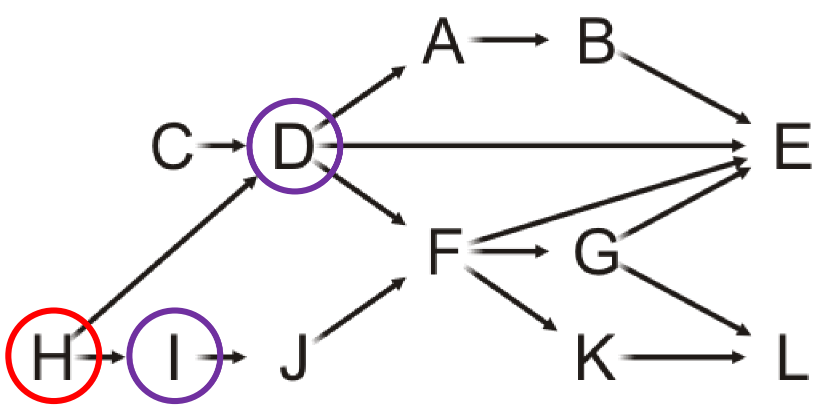
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

# Example: Kahn’s Algorithm

Pop the front of the queue

- H has two neighbors:  D and I
- Decrement their in-degrees




Queue:

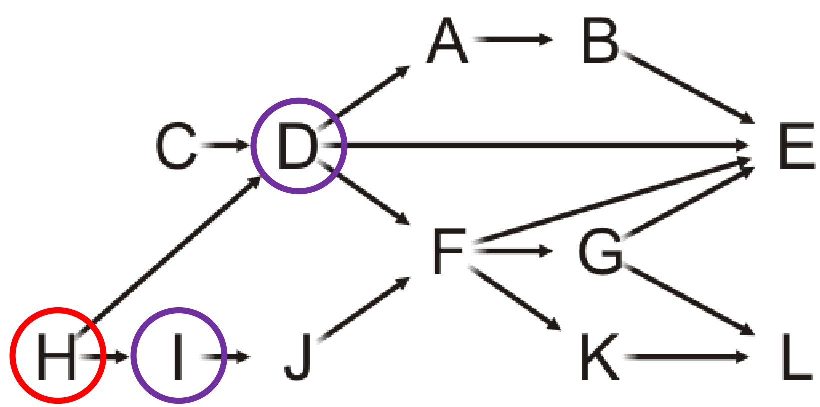
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| **D** | **0** |
| E | 4 |
| F | 2 |
| G | 1 |
| **H** | **0** |
| **I** | **0** |
| J | 1 |
| K | 1 |
| L | 2 |

# Example: Kahn’s Algorithm

Pop the front of the queue

- H has two neighbors: D and I
- Decrement their in-degrees
  - Both are decremented to zero, so push them onto the queue




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| **D** | **0** |
| E | 4 |
| F | 2 |
| G | 1 |
| **H** | **0** |
| **I** | **0** |
| J | 1 |
| K | 1 |
| L | 2 |

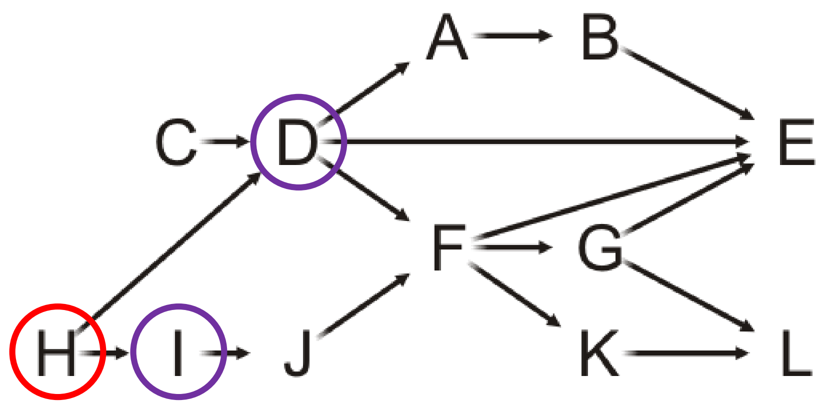
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | | | | | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- H has two neighbors: D and I
- Decrement their in-degrees
  - Both are decremented to zero, so push them onto the queue



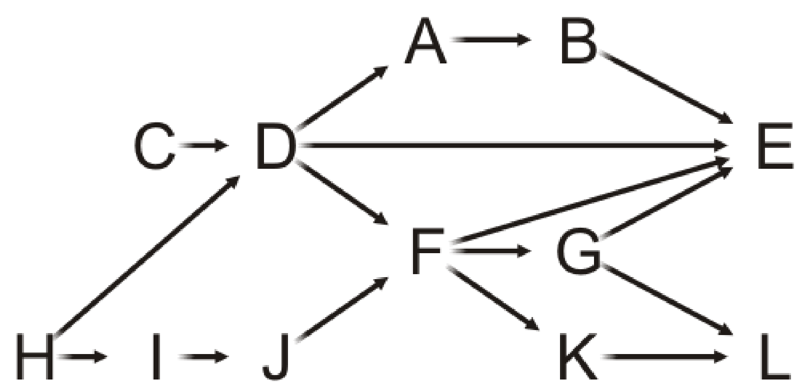

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | | | | | | | | |

| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| **D** | **0** |
| E | 4 |
| F | 2 |
| G | 1 |
| **H** | **0** |
| **I** | **0** |
| J | 1 |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue




| | |
|---|---|
| A | 1 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 4 |
| F | 2 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 1 |
| K | 1 |
| L | 2 |

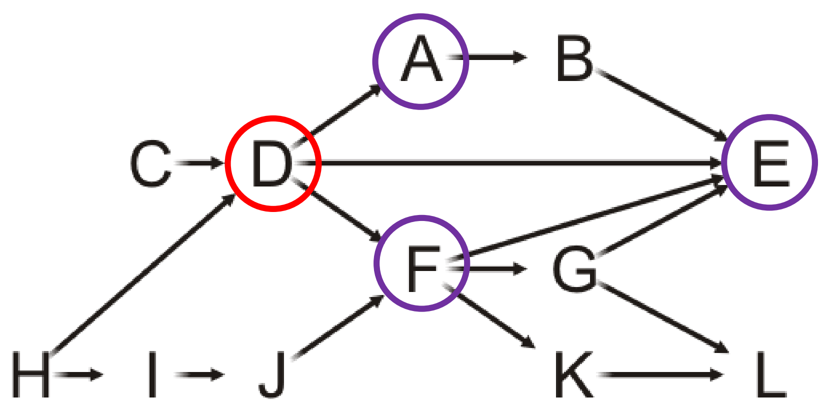
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | | | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- D has three neighbors: A, E and F




Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | | | | | | | | |

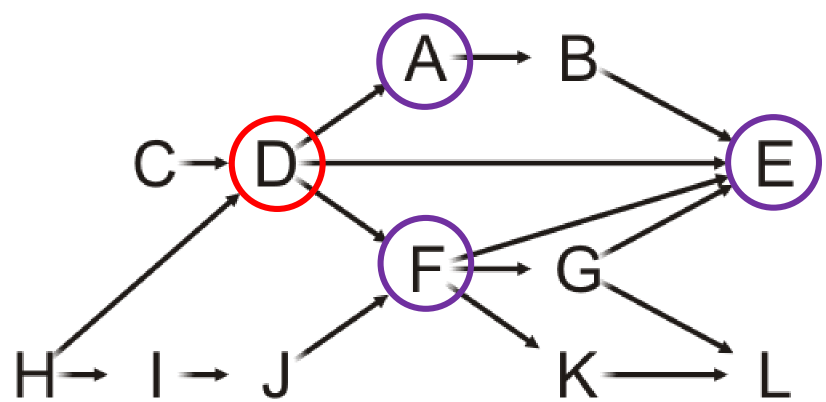
| | |
|---|---|
| **A** | **1** |
| B | 1 |
| C | 0 |
| **D** | **0** |
| **E** | **4** |
| **F** | **2** |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 1 |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue

- D has three neighbors: A, E and F
- Decrement their in-degrees

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 1 |
| K | 1 |
| L | 2 |



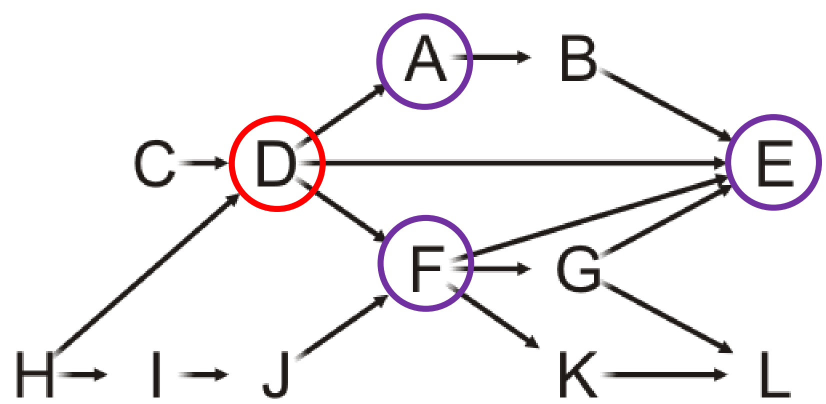

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | | | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- D has three neighbors: A, E and F
- Decrement their in-degrees
  - A is decremented to zero, so push it onto the queue



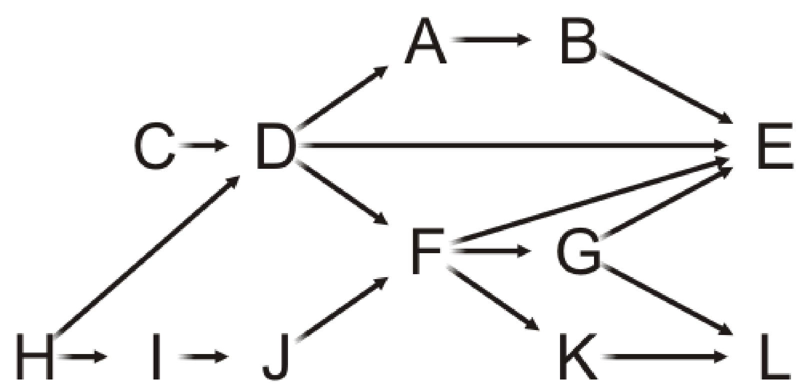

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 1 |
| K | 1 |
| L | 2 |

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | | | | | | | |

# Example: Kahn’s Algorithm

Pop the front of the queue




Queue:

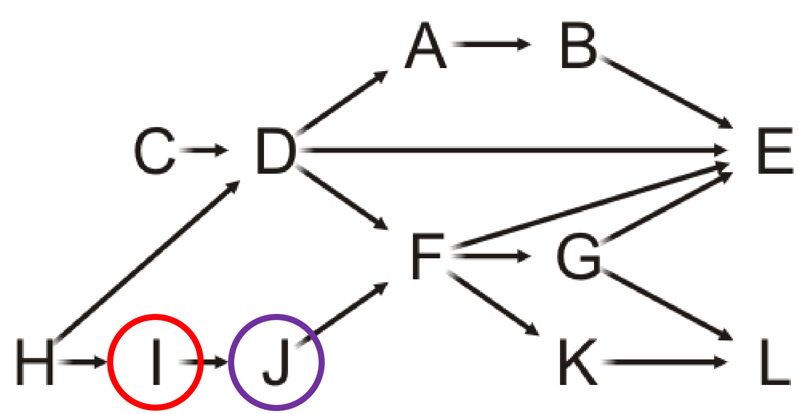
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | | | | | | | |

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 1 |
| K | 1 |
| L | 2 |

# Example: Kahn’s Algorithm

Pop the front of the queue

– I has one neighbor: J




Queue:

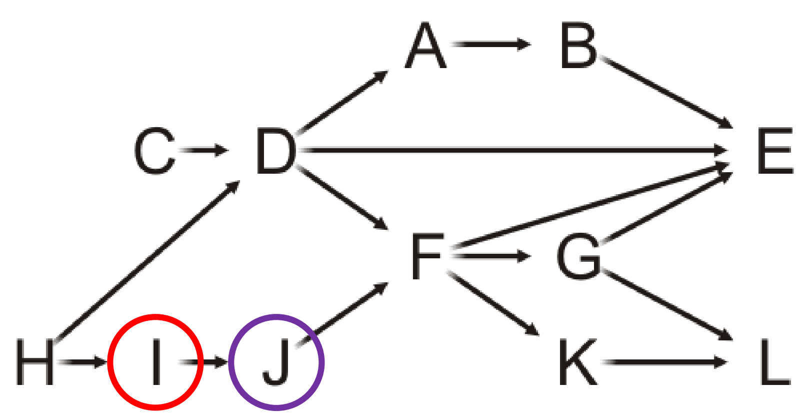
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | | | | | | | |

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| **I** | **0** |
| **J** | **1** |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue

- I has one neighbor: J
- Decrement its in-degree




Queue:

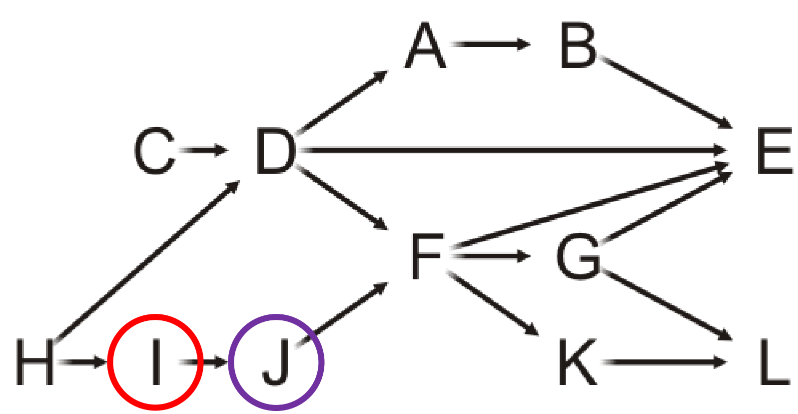
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | | | | | | | |

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| **I** | **0** |
| **J** | **0** |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue

- I has one neighbor: J
- Decrement its in-degree
  - J is decremented to zero, so push it onto the queue



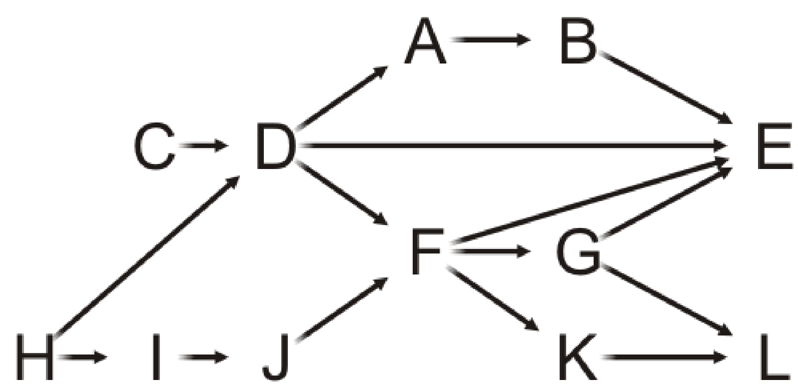

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| **I** | **0** |
| **J** | **0** |
| K | 1 |
| L | 2 |

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | | | | | | |

# Example: Kahn’s Algorithm

Pop the front of the queue



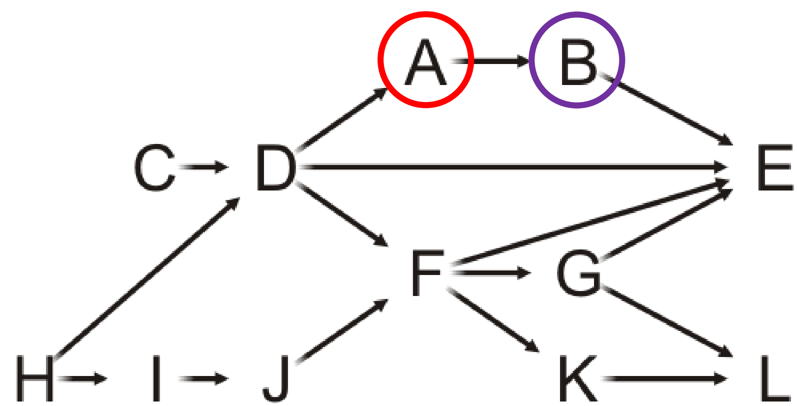

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Queue: | C | H | D | I | A | J | | | | | | |

| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

# Example: Kahn’s Algorithm

Pop the front of the queue

- A has one neighbor: B




| | |
|---|---|
| A | 0 |
| B | 1 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

Queue:

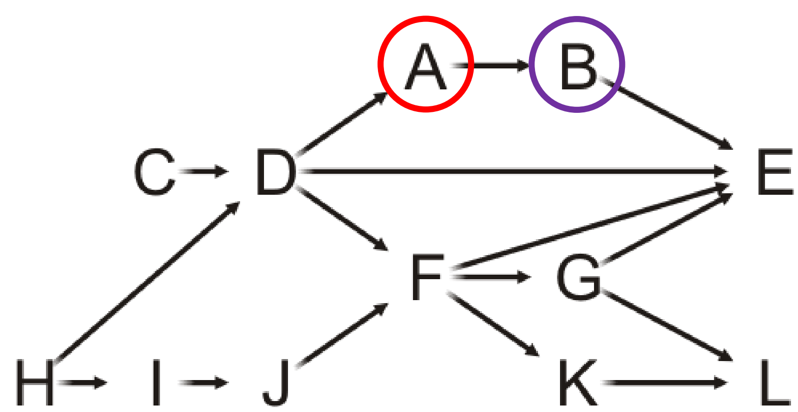
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- A has one neighbor: B
- Decrement its in-degree

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |



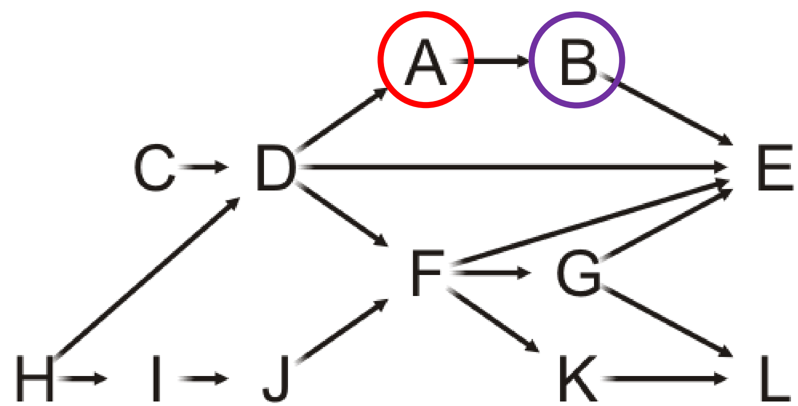

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- A has one neighbor: B
- Decrement its in-degree
  - B is decremented to zero, so push it onto the queue



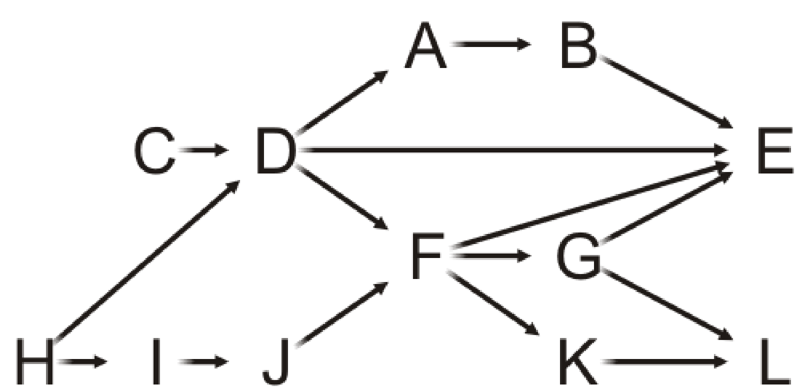

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | | | | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue




Queue:

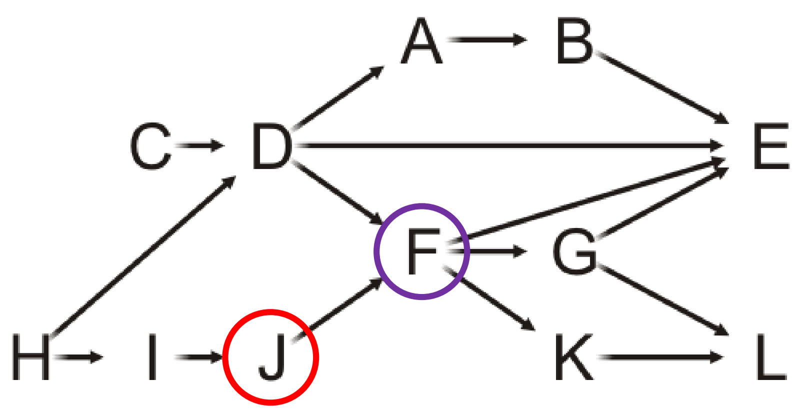
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | | | | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 1 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue

- J has one neighbor: F




| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| **F** | **1** |
| G | 1 |
| H | 0 |
| I | 0 |
| **J** | **0** |
| K | 1 |
| L | 2 |

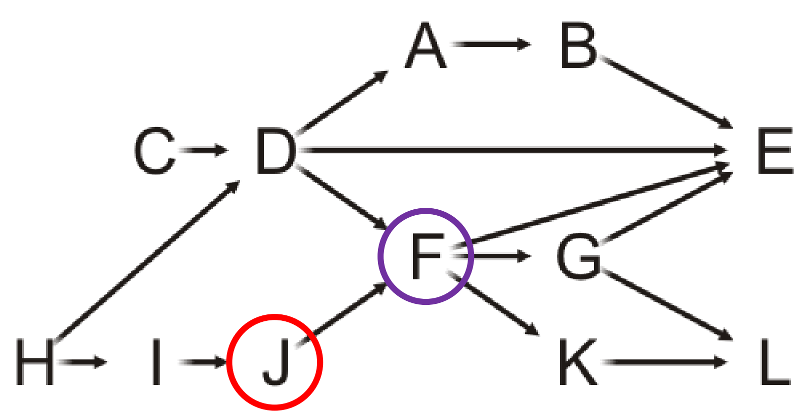
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- J has one neighbor: F
- Decrement its in-degree




Queue:

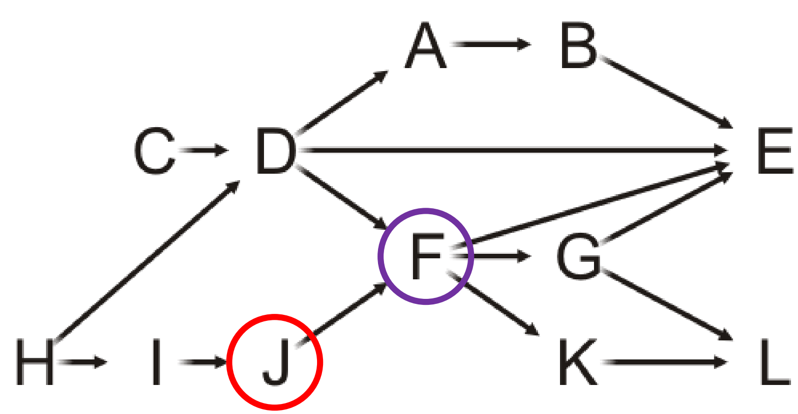
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | | | | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| **F** | **0** |
| G | 1 |
| H | 0 |
| I | 0 |
| **J** | **0** |
| K | 1 |
| L | 2 |

# Example: Kahn’s Algorithm

Pop the front of the queue

- J has one neighbor: F
- Decrement its in-degree
  - F is decremented to zero, so push it onto the queue



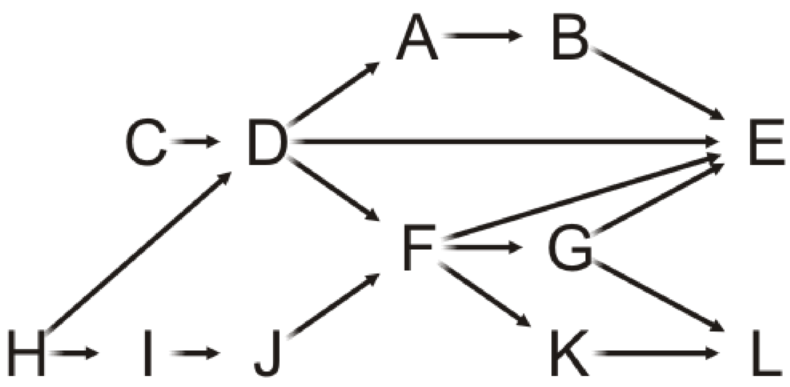

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| **F** | **0** |
| G | 1 |
| H | 0 |
| I | 0 |
| **J** | **0** |
| K | 1 |
| L | 2 |

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue




Queue:

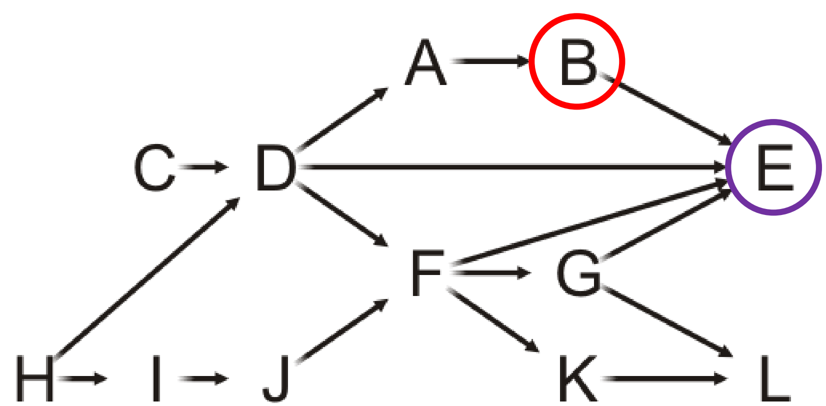
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 3 |
| F | 0 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

# Example: Kahn’s Algorithm

Pop the front of the queue

- B has one neighbor: E




| | |
|---|---|
| A | 0 |
| **B** | **0** |
| C | 0 |
| D | 0 |
| **E** | **3** |
| F | 0 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

Queue:

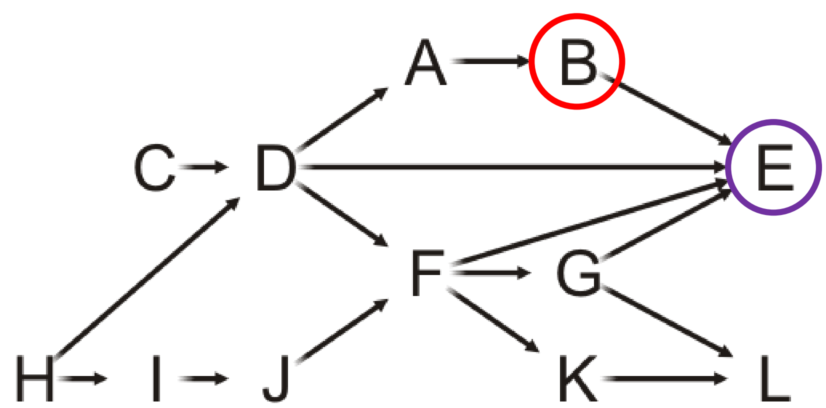
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- B has one neighbor: E
- Decrement its in-degree

| | |
|---|---|
| A | 0 |
| **B** | **0** |
| C | 0 |
| D | 0 |
| **E** | **2** |
| F | 0 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

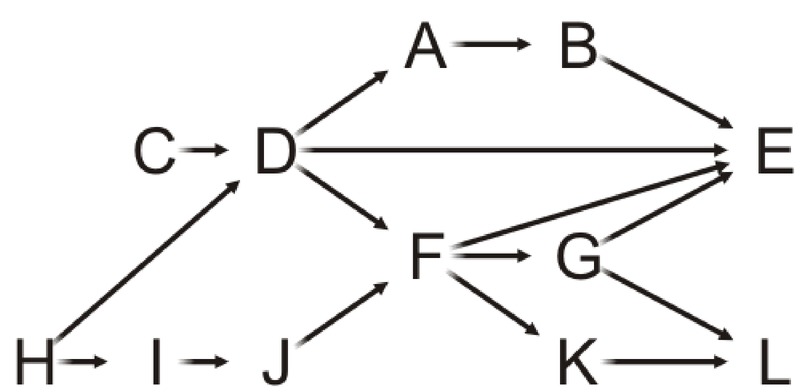



Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue




| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 2 |
| F | 0 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

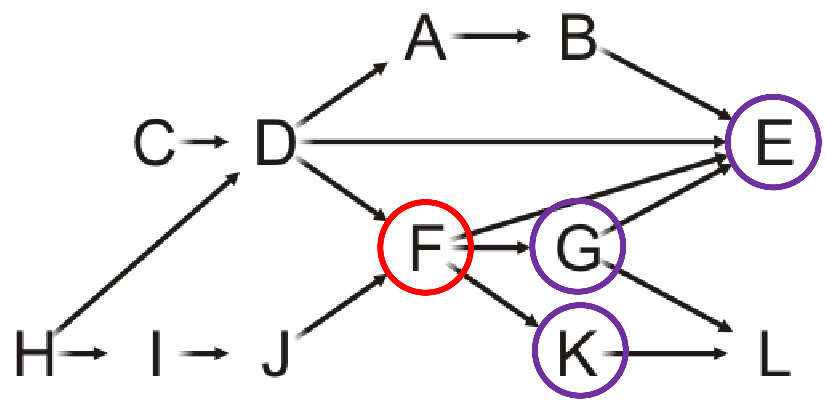
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

# Example: Kahn’s Algorithm

Pop the front of the queue

- F has three neighbors:  E, G and K




Queue:

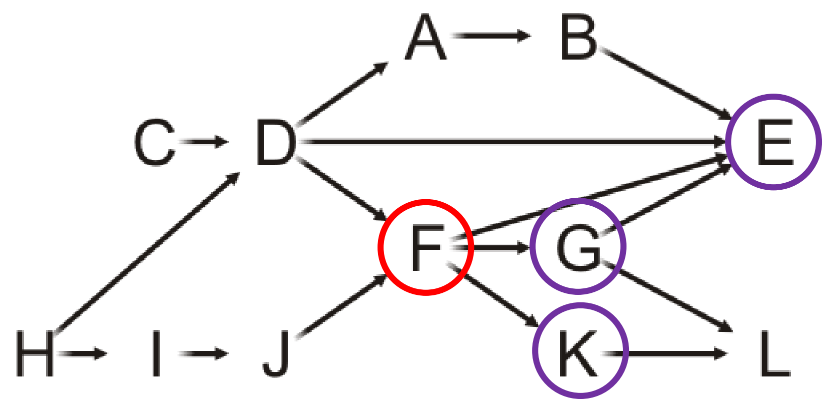
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 2 |
| F | 0 |
| G | 1 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 1 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue

- F has three neighbors: E, G and K
- Decrement their in-degrees




| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| **E** | **1** |
| **F** | **0** |
| **G** | **0** |
| H | 0 |
| I | 0 |
| J | 0 |
| **K** | **0** |
| L | 2 |

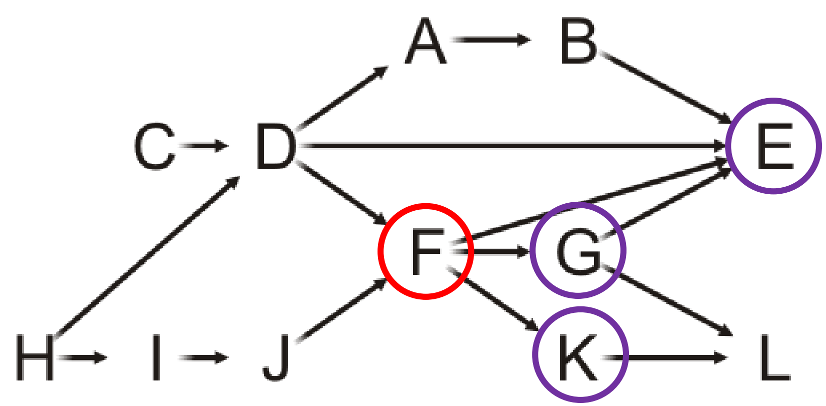
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | | | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- F has three neighbors: E, G and K
- Decrement their in-degrees
  - G and K are decremented to zero, so push them onto the queue



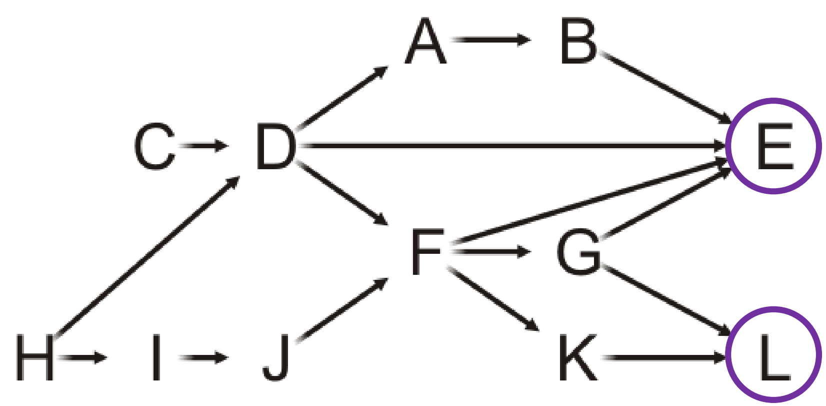

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| **E** | **1** |
| **F** | **0** |
| **G** | **0** |
| H | 0 |
| I | 0 |
| J | 0 |
| **K** | **0** |
| L | 2 |

# Example

Pop the front of the queue




Queue:

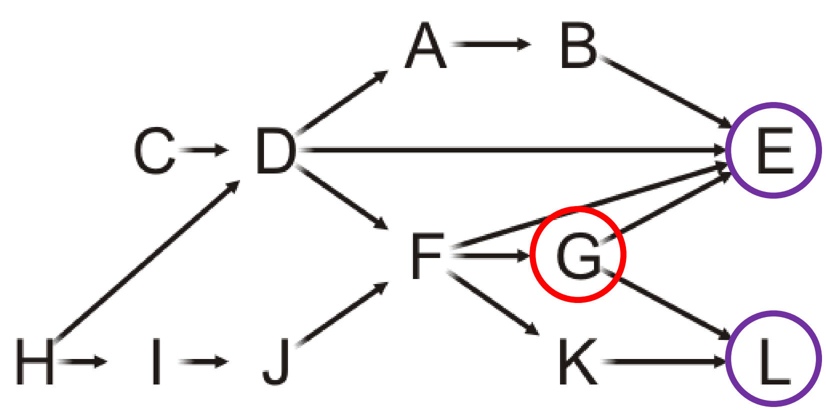
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 1 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 2 |

# Example: Kahn's Algorithm

Pop the front of the queue

- G has two neighbors: E and L




| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| **E** | **1** |
| F | 0 |
| **G** | **0** |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| **L** | **2** |

Queue:

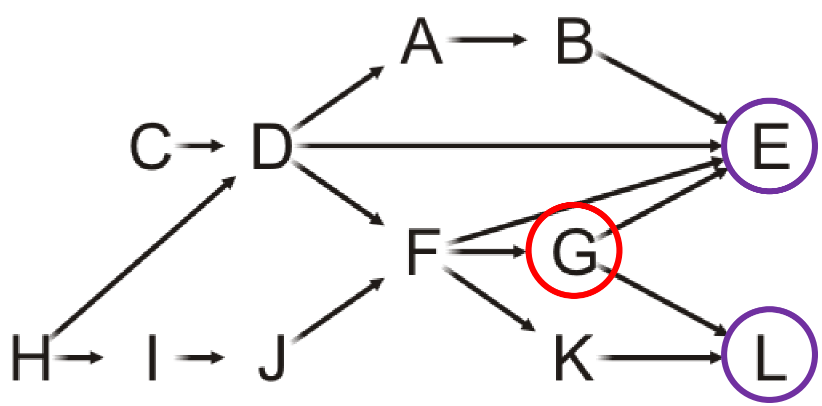
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | | |

# Example: Kahn's Algorithm

Pop the front of the queue

- G has two neighbors: E and L
- Decrement their in-degrees

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| **E** | **0** |
| F | 0 |
| **G** | **0** |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| **L** | **1** |



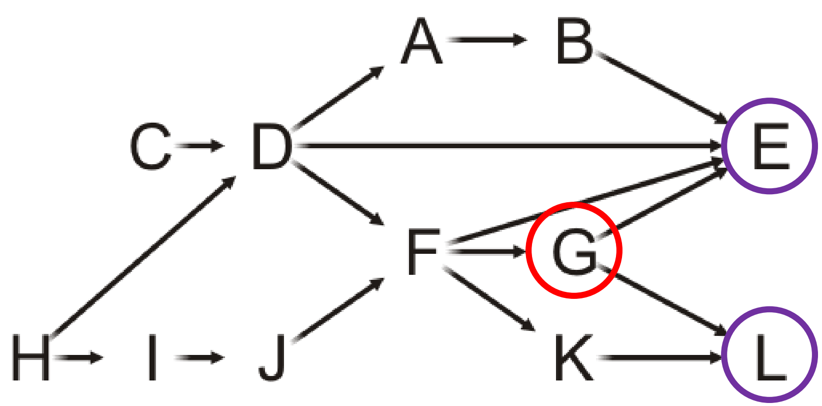

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | | |

# Example: Kahn’s Algorithm

Pop the front of the queue

- G has two neighbors: E and L
- Decrement their in-degrees
  - E is decremented to zero, so push it onto the queue



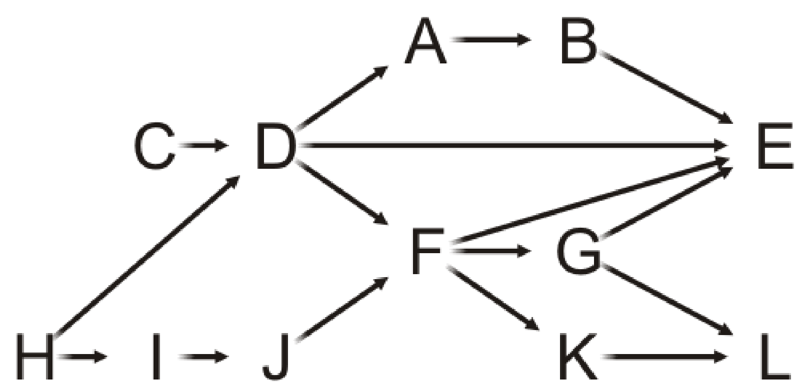

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| **E** | **0** |
| F | 0 |
| **G** | **0** |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| **L** | **1** |

# Example: Kahn's Algorithm

Pop the front of the queue




| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 1 |

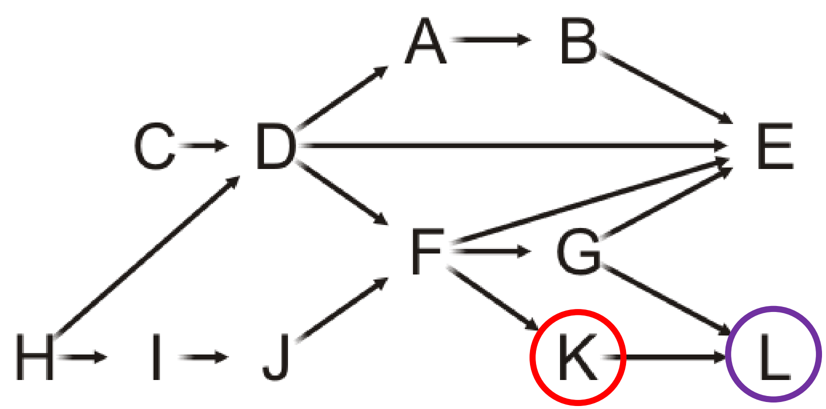
Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | |

# Example: Kahn's Algorithm

Pop the front of the queue

- K has one neighbors: L




Queue:

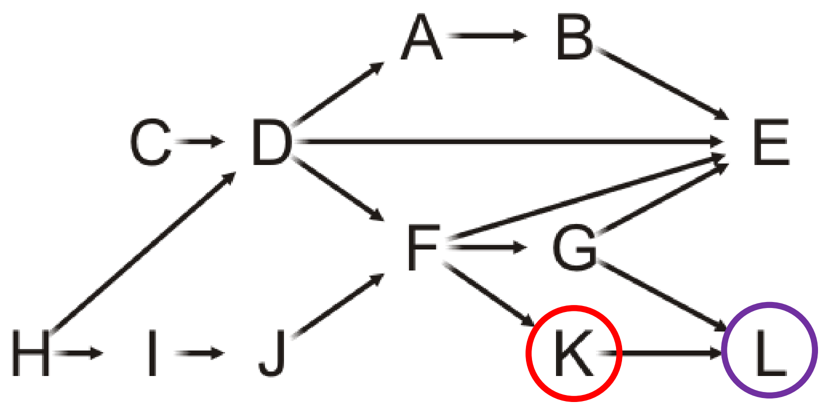
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 1 |

# Example: Kahn’s Algorithm

Pop the front of the queue

- K has one neighbors: L
- Decrement its in-degree




Queue:

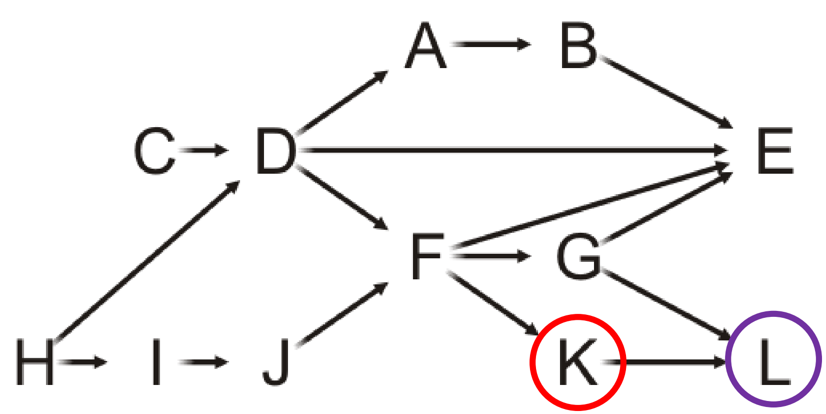
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| **K** | **0** |
| **L** | **0** |

# Example: Kahn's Algorithm

Pop the front of the queue

- K has one neighbors: L
- Decrement its in-degree
  - L is decremented to zero, so push it onto the queue




Queue:

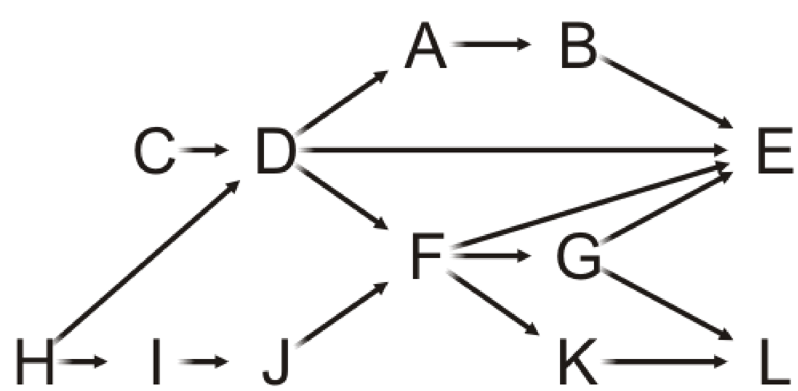
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | L |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| **K** | **0** |
| **L** | **0** |

# Example: Kahn's Algorithm

Pop the front of the queue

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 0 |

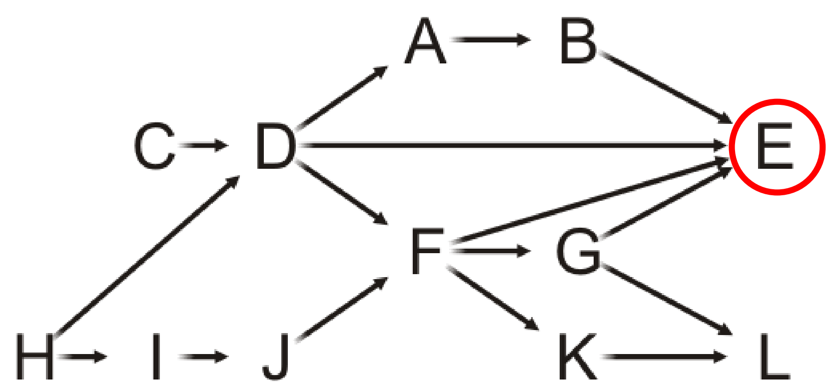



| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Queue: | C | H | D | I | A | J | B | F | G | K | E | L |

# Example: Kahn’s Algorithm

Pop the front of the queue

- E has no neighbors—it is a *sink*



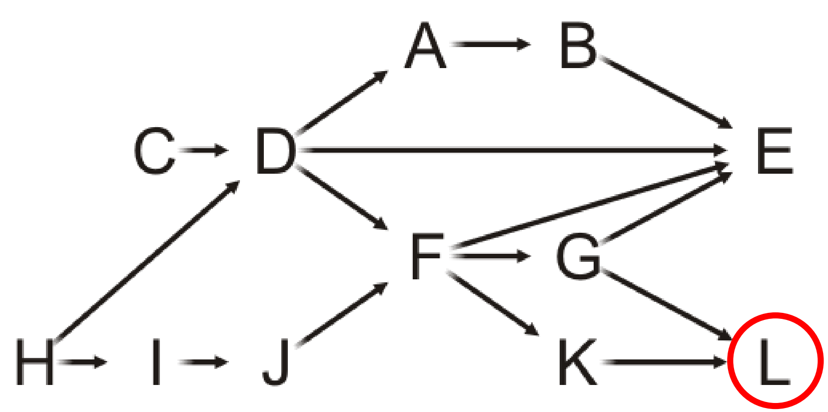

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | L |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| **E** | **0** |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 0 |

# Example: Kahn's Algorithm

Pop the front of the queue




Queue:

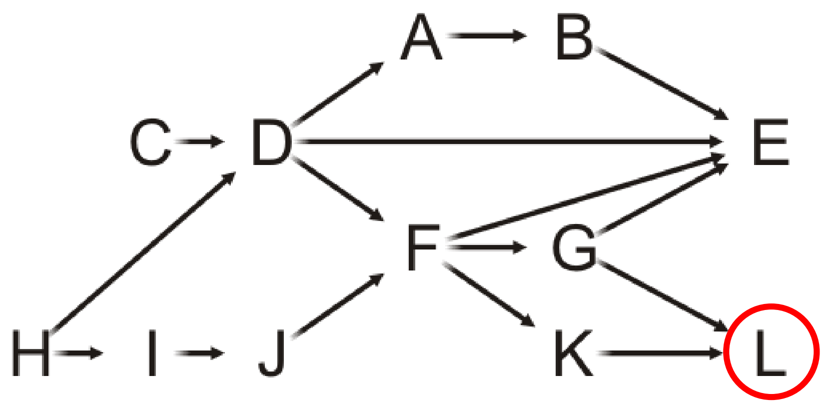
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | L |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 0 |

# Example: Kahn's Algorithm

Pop the front of the queue

- L has no neighbors—it is also a *sink*



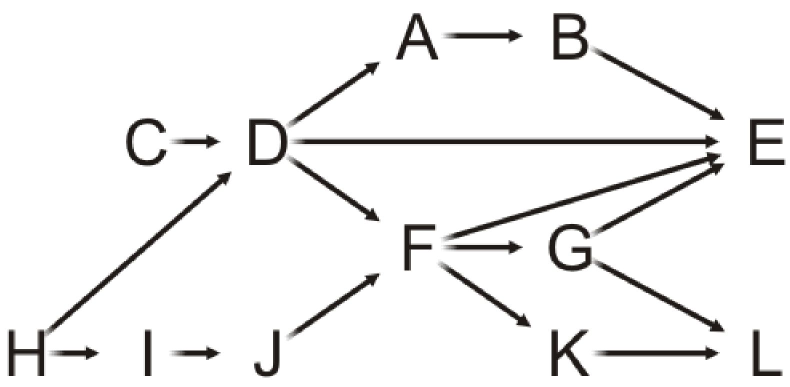

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| **L** | **0** |

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | L |

# Example: Kahn's Algorithm

The queue is empty, so we are done



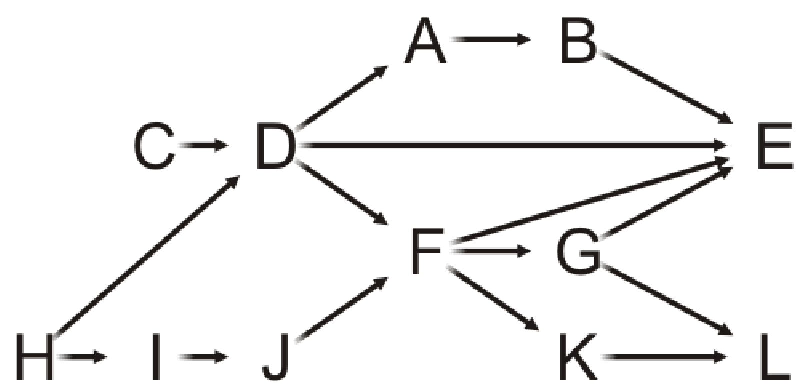

Queue:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | L |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 0 |

# Example: Kahn’s Algorithm

The array stores the topological sorting




| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | H | D | I | A | J | B | F | G | K | E | L |

| | |
|---|---|
| A | 0 |
| B | 0 |
| C | 0 |
| D | 0 |
| E | 0 |
| F | 0 |
| G | 0 |
| H | 0 |
| I | 0 |
| J | 0 |
| K | 0 |
| L | 0 |

# Runtime of Kahn's Algorithm

- **Step #1: Preparing the in-degree array**
  - Takes $\Theta(|V| + |E|)$ runtime
    - If the DAG was represented as an adjacency list
  - Takes $\Theta(|V|^2)$ runtime
    - If the DAG was represented as an adjacency matrix

- **Step #2: Keep enumerating the in-degree array**
  - Takes $\Theta(|V| + |E|)$ runtime
    - Queued vertices $|V|$ times and decrementing in-degree $|E|$ times

# Summary

In this topic, we have discussed topological sorts

- Sorting of elements in a DAG
- Kahn's Algorithm
  - A table of in-degrees
  - Select that vertex which has current in-degree zero

# References

Wikipedia, http://en.wikipedia.org/wiki/Topological_sorting

[1] Cormen, Leiserson, and Rivest, *Introduction to Algorithms*, McGraw Hill, 1990, §11.1, p.200.

[2] Weiss, Data Structures and Algorithm Analysis in C++, 3rd Ed., Addison Wesley, §9.2, p.342-5.