ECE430.217 Data Structures

# Mathematical Background

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# Outline

This topic reviews the basic mathematics required in this course:

- The ceiling and floor functions
- L'Hôpital's rule
- Logarithms
- Arithmetic and other polynomial series
  - Mathematical induction
- Geometric series
- Combinations

# Floor and ceiling functions

The ***floor*** function maps any real number $x$ onto the greatest integer less than or equal to $x$:

$$\lfloor 3.2 \rfloor = \lfloor 3 \rfloor = 3$$

$$\lfloor -5.2 \rfloor = \lfloor -6 \rfloor = -6$$

– Consider it *rounding towards negative infinity*

The ***ceiling*** function maps $x$ onto the least integer greater than or equal to $x$:

$$\lceil 3.2 \rceil = \lceil 4 \rceil = 4$$

$$\lceil -5.2 \rceil = \lceil -5 \rceil = -5$$

– Consider it *rounding towards positive infinity*

– The `cmath` library implements these as

```
double floor( double );   double ceil( double );
```

Necessary because `double` have a range just under $2^{1024}$ `long` can only represent numbers as large as $2^{63} - 1$

# L'Hôpital's rule

If you are attempting to determine

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

but both $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ , it follows

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{(1)}(n)}{g^{(1)}(n)}$$

Repeat as necessary…

Note: the $k^{\text{th}}$ derivative will always be shown as $f^{(k)}(n)$

# Logarithms

We will begin with a review of logarithms:

If $n = e^m$, we define

$$m = \ln( n )$$

It is always true that $e^{\ln(n)} = n$; however, $\ln(e^n) = n$ requires that $n$ is real

# Logarithms

Exponentials grow faster than any non-constant polynomial

$$\lim_{n \to \infty} \frac{e^n}{n^d} = \infty$$

for any $d > 0$

Thus, their inverses—logarithms—grow slower than any polynomial

$$\lim_{n \to \infty} \frac{\ln(n)}{n^d} = 0$$

# Logarithms

We have compared logarithms and polynomials

– How about $\log_2(n)$ versus $\ln(n)$ versus $\log_{10}(n)$

You have seen the formula

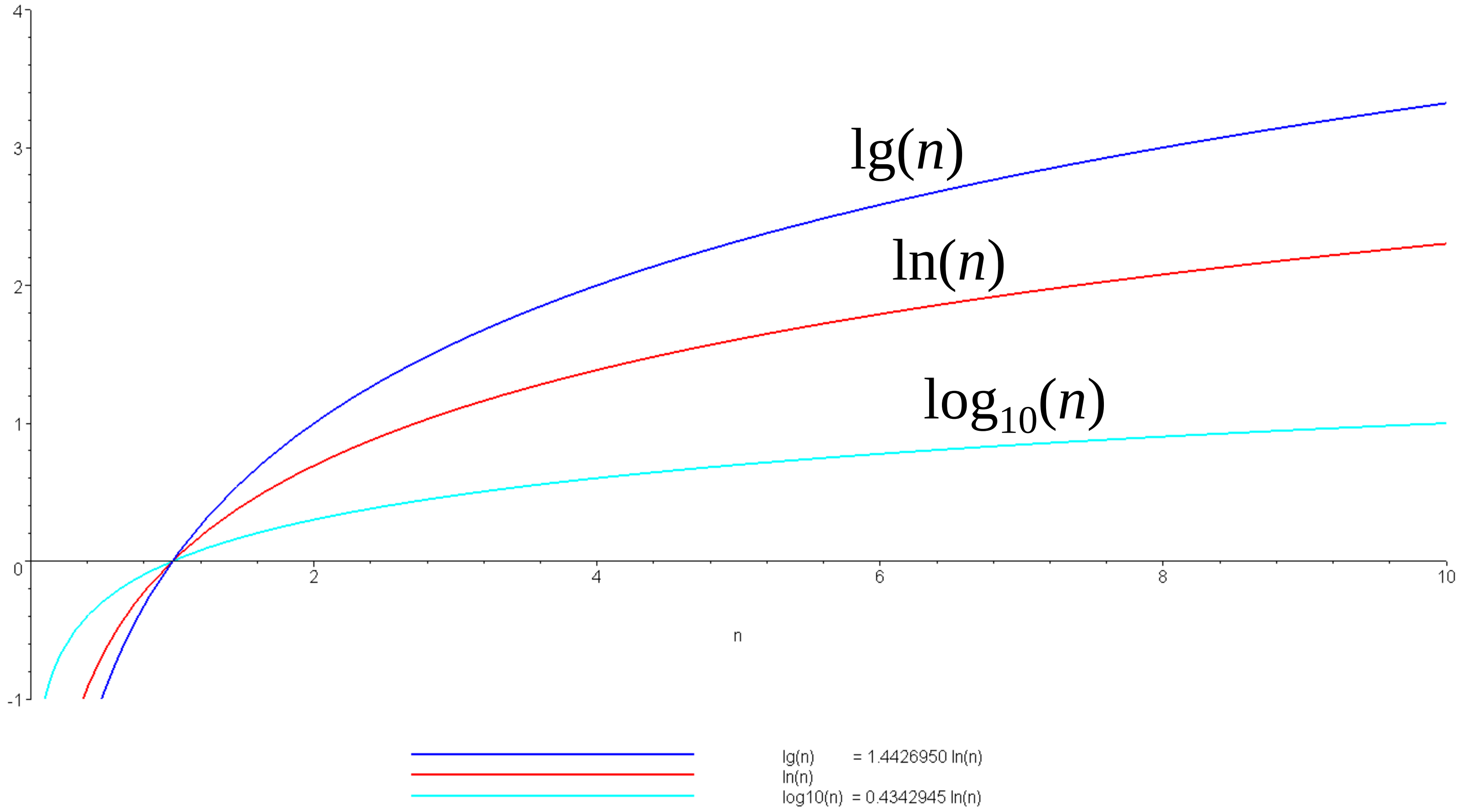
Constant

$$\log_b(n) = \frac{\ln(n)}{\ln(b)}$$

All logarithms are scalar multiples of each others

# Logarithms

A plot of $\log_2(n) = \lg(n)$, $\ln(n)$, and $\log_{10}(n)$



# Logarithms

Note: the base-2 logarithm $\log_2(n)$ is written as $\lg(n)$

It is an industry standard to implement the natural logarithm $\ln(n)$ as

```
double log( double );
```

The *common* logarithm $\log_{10}(n)$ is implemented as

```
double log10( double );
```

# Logarithms

We will repeatedly use:

$$n^{\log_b(m)} = m^{\log_b(n)},$$

a consequence of $n = b^{\log_b n}$:

$$\begin{aligned} n^{\log_b(m)} &= \left(b^{\log_b(n)}\right)^{\log_b(m)} \\ &= b^{\log_b(n)\log_b(m)} \\ &= \left(b^{\log_b(m)}\right)^{\log_b(n)} \\ &= m^{\log_b(n)} \end{aligned}$$

# Logarithms

You should also, as electrical or computer engineers be aware of the relationship:

$$\lg(2^{10}) = \lg(1024) \qquad = 10$$

$$\lg(2^{20}) = \lg(1\,048\,576) \quad = 20$$

and consequently:

$$\lg(10^{3}) = \lg(1000) \qquad \approx 10 \qquad \text{kilo}$$

$$\lg(10^{6}) = \lg(1\,000\,000) \quad \approx 20 \qquad \text{mega}$$

$$\lg(10^{9}) \qquad \approx 30 \qquad \text{giga}$$

$$\lg(10^{12}) \qquad \approx 40 \qquad \text{tera}$$

# Arithmetic series

Next we will look various series

Each term in an arithmetic series is increased by a constant value (usually 1) :

$$0+1+2+3+\cdots+n=\sum_{k=0}^{n} k=\frac{n(n+1)}{2}$$

# Arithmetic series

Proof 1: write out the series twice and add each column

$$\begin{array}{ccccccccccccccc} & 1 & + & 2 & + & 3 & + \cdots + & n-2 & + & n-1 & + & n \\ + & n & + & n-1 & + & n-2 & + \cdots + & 3 & + & 2 & + & 1 \\ \hline \end{array}$$

$$(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$= n\,(n+1)$$

Since we added the series twice, we must divide the result by $2$

# Arithmetic series

Proof 2 (by induction):

The statement is true for $n = 0$:

$$\sum_{i=0}^{0} k = 0 = \frac{0 \cdot 1}{2} = \frac{0(0+1)}{2}$$

Assume that the statement is true for an arbitrary $n$:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

# Arithmetic series

Using the assumption that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

for $n$, we must show that

$$\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

# Arithmetic series

Then, for $n + 1$, we have:

$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{i=0}^{n} k$$

By assumption, the second sum is known:

$$= (n+1) + \frac{n(n+1)}{2}$$

$$= \frac{(n+1)2 + (n+1)n}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

# Arithmetic series

The statement is true for $n = 0$ and
the truth of the statement for $n$ implies
the truth of the statement for $n + 1$.

Therefore, by the process of mathematical induction, the statement is true for all values of $n \geq 0$.

Reference: Mr. Oprendick

# Other polynomial series

We could repeat this process, after all:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

however, it is easier to see the pattern:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{n^3}{3}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \approx \frac{n^4}{4}$$

# Geometric series

The next series we will look at is the geometric series with common ratio $r$:

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

and if $|r| < 1$ then it is also true that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

# Geometric series

Elegant proof: multiply by $1 = \dfrac{1-r}{1-r}$

Telescoping series:
all but the first and last terms cancel

$$\sum_{k=0}^{n} r^k = \frac{(1-r)\sum_{k=0}^{n} r^k}{1-r}$$

$$= \frac{\sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k}{1-r}$$

$$= \frac{(1 + r + r^2 + \cdots + r^n) - (r + r^2 + \cdots + r^n + r^{n+1})}{1-r}$$

$$= \frac{1 - r^{n+1}}{1-r}$$

Ref: Bret D. Whissel, *A Derivation of Amortization*

# Geometric series

Proof by induction:

The formula is correct for $n$ = 0: $\sum_{k=0}^{0} r^k = r^0 = 1 = \frac{1-r^{0+1}}{1-r}$

Assume the formula $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ is true for an arbitrary $n$; then

$$\sum_{k=0}^{n+1} r^k = r^{n+1} + \sum_{k=0}^{n} r^k = r^{n+1} + \frac{1-r^{n+1}}{1-r} = \frac{(1-r)r^{n+1} + 1 - r^{n+1}}{1-r}$$

$$= \frac{r^{n+1} - r^{n+2} + 1 - r^{n+1}}{1-r} = \frac{1-r^{n+2}}{1-r} = \frac{1-r^{(n+1)+1}}{1-r}$$

and therefore, by the process of mathematical induction, the statement is true for all $n \geq 0$.

# Geometric series

A common geometric series will involve the ratios $r$ = ½ and $r = 2$:

$$\sum_{i=0}^{n}\left(\frac{1}{2}\right)^{i} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2-2^{-n} \qquad \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i} = 2$$

$$\sum_{k=0}^{n} 2^{k} = \frac{1-2^{n+1}}{1-2} = 2^{n+1}-1$$

# Combinations

Given $n$ distinct items, in how many ways can you choose $k$ of these?

- I.e., "In how many ways can you combine $k$ items from $n$?"
- For example, given the set $\{1, 2, 3, 4, 5\}$, I can choose three of these in any of the following ways:

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\},$$
$$\{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}$$

The number of ways such items can be chosen is written

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $\binom{n}{k}$ is read as "$n$ choose $k$"s

There is a recursive definition: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

# Combinations

The most common question we will ask:

- Given $n$ items, in how many ways can we choose two of them?
- In this case, the formula simplifies to:

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

For example, given $\{0, 1, 2, 3, 4, 5, 6\}$, we have the following 21 pairs:

$$\begin{array}{r}
\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}, \{0, 6\}, \\
\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \\
\{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \\
\{3, 4\}, \{3, 5\}, \{3, 6\}, \\
\{4, 5\}, \{4, 6\}, \\
\{5, 6\}
\end{array}$$

# Combinations

You have also seen this in expanding polynomials:

$$\left(x+y\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

For example,

$$\begin{aligned}
\left(x+y\right)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k} \\
&= \binom{4}{0} y^4 + \binom{4}{1} xy^3 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^3 y + \binom{4}{4} x^4 \\
&= y^4 + 4xy^3 + 6x^2 y^2 + 4x^3 y + x^4
\end{aligned}$$

# Summary

In this topic, we have discussed:

- A review of the necessity of quantitative analysis in engineering

We reviewed the following mathematical concepts:

- The floor and ceiling functions
- L'Hôpital's rule
- Logarithms
- Arithmetic and other polynomial series
  - Mathematical induction
- Geometric series
- Combinations

# Reference

[1] Cormen, Leiserson, and Rivest, *Introduction to Algorithms*, McGraw Hill, 1990, Chs 2-3, p.42-76.

[2] Weiss, Data Structures and Algorithm Analysis in C++, 3rd Ed., Addison Wesley, §§ 1.2-3, p.2-11.