ECE430.217 Data Structures

Open Addressing

Weiss Book Chapter 5

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Lecture slides were prepared based on Douglas W. Harder's notes (**dwharder@alumni.uwaterloo.ca).**

Outline

To handle collision, chained hash tables need additional memory allocation
- Additional memory allocation imposes non-trivial performance overheads
- Can we create a hash table without additional memory allocation?

We will deal with collisions by storing collisions in the same table
- We will define a rule, dictating where to look next

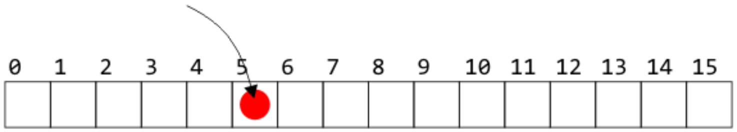
Collision Handling in Hash Tables

- Common strategies to handle collisions in hash tables
  - **Closed addressing**
    - Store all elements with hash collisions in a secondary data structures (linked list, BST, etc.)
    - Chained hash table
  - **Perfect Hashing**
    - Choose a hash function to ensure that collisions never happen (if possible)
  - **Open addressing (or closed hashing)**
    - Define a rule to locate the next bin
    - Linear probing, Quadratic probing, and double hashing

Open Addressing: Insert

Suppose an object hashes to bin 5

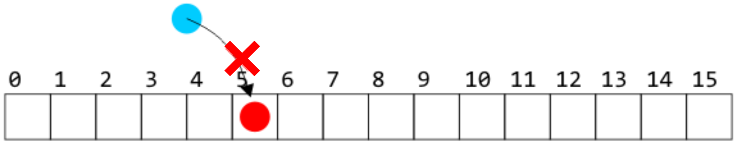
– If bin 5 is empty, we can store the object in bin 5



Open Addressing: Insert

Suppose, however, another object hashes to bin 5

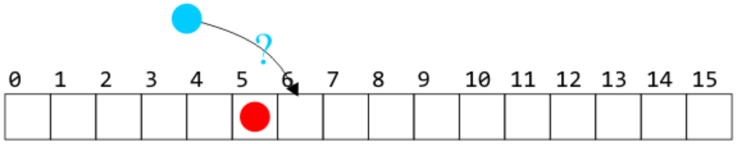
– Without a linked list, we cannot store the object in bin 5



Open Addressing: Insert

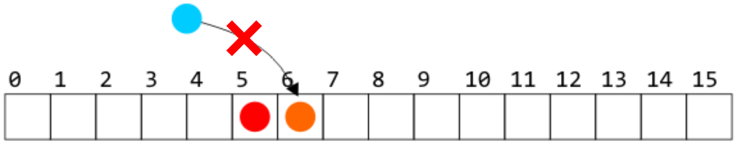
We could have a rule which says:

– Look in the next bin to see if it is occupied



Open Addressing: Insert

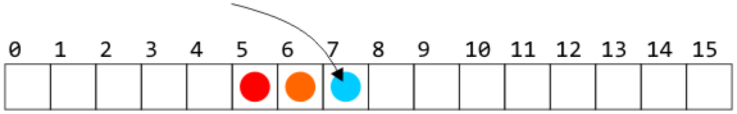
What if even the next bin is already occupied?



Open Addressing: Insert

We could then store the object in the next location

- Problem: we can only store as many objects as the array capacity
  - i.e., the load factor ≤ 1

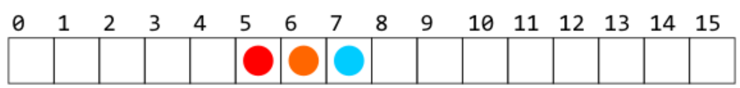


Open Addressing: Supporting Other Operations

The rule should support both **search and remove**.

Recall that our goal is $\Theta(1)$ access times

- Q. how do we avoid to access too many bins (on average)?



Open Addressing: Strategies

There are numerous strategies for defining the order in which the bins should be searched:

- **Linear probing**
- **Quadratic probing**
- Double hashing

There are many alternate strategies, as well:

- Last come, first served
  - Always place the object into the bin moving what may be there already
- Cuckoo hashing

Linear Probing

Linear Probing

The easiest method to probe the is to **search forward linearly**

Assume we are inserting into bin k:

- If bin k is empty, we occupy it
- Otherwise, check bin $k + 1$, $k + 2$, and so on, until an empty bin is found
  - If we reach the end of the array, go back to the front (bin 0)

Linear Probing

Consider a hash table with $M = 16$ bins

Given a hexadecimal number as input:

- Suppose the hash function outputs the least significant 4-bits of input
- Example: for 6B72**A**$_{16}$, the initial bin is **A**

Insertion

Insert these numbers into this initially empty hash table:

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, DBE, E9C

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

19A, 207, 3AD, 488

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, DBE, E9C

Example

Start with the first four values:

19**A**, 20**7**, 3A**D**, 48**8**

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							207	**488**		**19A**			**3AD**		

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, DBE, E9C

Example

Next we must insert 5BA

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							207	488		19A			3AD		

19A, 207, 3AD, 488, **5BA**, 680, 74C, 826, 946, ACD, B32, C8B, DBE, E9C

Example

Next we must insert 5B**A**

- Bin **A** is occupied
- We search forward for the next empty bin

0	1	2	3	4	5	6	7	8	9	**A**	B	C	D	E	F
							207	488		19A	**5BA**		3AD		

19A, 207, 3AD, 488, **5BA**, 680, 74C, 826, 946, ACD, B32, C8B, DBE, E9C

Example

Next we are adding **680, 74C, 826**

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							207	488		19A	5BA		3AD		

19A, 207, 3AD, 488, 5BA, **680, 74C, 826**, 946, ACD, B32, C8B, DBE, E9C

Example

Next we are adding 68**0**, 74**C**, 82**6**

– All the bins are empty—simply insert them

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						**826**	207	488		19A	5BA	**74C**	3AD		

19A, 207, 3AD, 488, 5BA, **680, 74C, 826**, 946, ACD, B32, C8B, DBE, E9C

Example

Next, we must insert 946

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						826	207	488		19A	5BA	74C	3AD		

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, **946**, ACD, B32, C8B, DBE, E9C

Example

Next, we must insert 94**6**

- Bin **6** is occupied
- The next empty bin is 9

0	1	2	3	4	5	**6**	7	8	9	A	B	C	D	E	F
680						826	207	488	**946**	19A	5BA	74C	3AD		

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, **946**, ACD, B32, C8B, DBE, E9C

Example

Next, we must insert ACD

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						826	207	488	946	19A	5BA	74C	3AD		

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, **ACD**, B32, C8B, DBE, E9C

Example

Next, we must insert AC**D**

- Bin **D** is occupied
- The next empty bin is E

0	1	2	3	4	5	6	7	8	9	A	B	C	**D**	E	F
680						826	207	488	946	19A	5BA	74C	3AD	**ACD**	

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, **ACD**, B32, C8B, DBE, E9C

Example

Next, we insert B32

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680						826	207	488	946	19A	5BA	74C	3AD	ACD	

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, **B32**, C8B, DBE, E9C

Example

Next, we insert B3**2**

- Bin **2** is unoccupied

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		**B32**				826	207	488	946	19A	5BA	74C	3AD	ACD	

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, **B32**, C8B, DBE, E9C

Example

Next, we insert C8B

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32				826	207	488	946	19A	5BA	74C	3AD	ACD	

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, **C8B**, DBE, E9C

Example

Next, we insert C8**B**

- Bin **B** is occupied
- The next empty bin is F

0	1	2	3	4	5	6	7	8	9	A	**B**	C	D	E	F
680		B32				826	207	488	946	19A	5BA	74C	3AD	ACD	**C8B**

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, **C8B**, DBE, E9C

Example

Next, we insert **DBE**

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680		B32				826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, **DBE**, E9C

Example

Next, we insert **DBE**

- Bin **E** is occupied
- The next empty bin is 1

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	**DBE**	B32				826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, **DBE**, E9C

Example

Finally, insert E9C

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	DBE	B32				826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, DBE, **E9C**

Example

Finally, insert E9C

- Bin **C** is occupied
- The next empty bin is 3

0	1	2	3	4	5	6	7	8	9	A	B	**C**	D	E	F
680	DBE	B32	**E9C**			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

19A, 207, 3AD, 488, 5BA, 680, 74C, 826, 946, ACD, B32, C8B, DBE, **E9C**

Example

Having completed these insertions:

- The load factor is $\lambda = 14/16 = 0.875$
- The average number of probes is $38/14 \approx 2.71$

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	DBE	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Resizing the array

To double the capacity of the array, each value must be rehashed

- Now the hash function outputs the least significant 5-bits of input
- 680, B32, ACD, 5BA, 826, 207, 488, D59 may be immediately placed

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488					ACD					B32							D59	5BA					

Resizing the array

To double the capacity of the array, each value must be rehashed

- 19A resulted in a collision

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	**1A**	1B	1C	1D	1E	1F
680						826	207	488					ACD					B32							D59	5BA	**19A**				

Resizing the array

To double the capacity of the array, each value must be rehashed

– 946 resulted in a collision

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946				ACD					B32							D59	5BA	19A				

Resizing the array

To double the capacity of the array, each value must be rehashed

- 74C fits into its bin

0	1	2	3	4	5	6	7	8	9	A	B	**C**	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946			**74C**	ACD				946	B32							D59	5BA	19A				

Resizing the array

To double the capacity of the array, each value must be rehashed

- 3AD resulted in a collision

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946			74C	ACD	3AD			946	B32							D59	5BA	19A				

Resizing the array

To double the capacity of the array, each value must be rehashed

- Both E9C and C8B fit without a collision
- The load factor is $\lambda = 14/32 = 0.4375$
- The average number of probes is $18/14 \approx 1.29$

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		**C8B**	74C	ACD	3AD			946	B32							D59	5BA	19A	**E9C**			

Searching

Testing for membership is similar to insertions:

Start at the appropriate bin, and searching forward until

1. The item is found,
2. An empty bin is found, or
3. We have traversed the entire array

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Searching

Searching for C8B

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Searching

Searching for C8**B**

- Examine bins B, C, D, E, F
- The value is found in Bin F

0	1	2	3	4	5	6	7	8	9	A	**B**	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	**C8B**

Searching

Searching for 23E

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Searching

Searching for 23E

- Search bins E, F, 0, 1, 2, 3, 4
- The last bin is empty; therefore, 23E is not in the table

0	1	2	3	4	5	6	7	8	9	A	B	C	D	**E**	F
680	D59	B32	E93	×		826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Erasing

We cannot simply remove elements from the hash table

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Erasing

We cannot simply remove elements from the hash table

- For example, consider erasing 3AD

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	D59	B32	E93			826	207	488	946	19A	5BA	74C	3AD	ACD	C8B

Erasing

We cannot simply remove elements from the hash table

- For example, consider erasing 3AD
- If we just erase it, it is now an empty bin
    - By our algorithm, we cannot find ACD, C8B and D59

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	**D59**	B32	E93			826	207	488	946	19A	5BA	74C		**ACD**	**C8B**

Erasing

Instead, **you should mark the bin "erased".**

This "erased bin" is different from an empty bin---the search should not stop at an erased bin

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
680	**D59**	B32	E93			826	207	488	946	19A	5BA	74C		**ACD**	**C8B**

Each bin may be represented with the following states:

- Occupied
- Empty
- Erased

Your "bin" positioning algorithm should be different for "search" and "insert"

Primary Clustering

We have already observed the following phenomenon:

– With more insertions, the contiguous regions (or *clusters*) get larger

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

This results in longer search times

Primary Clustering

We currently have three clusters of length four

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

There is a $5/32 \approx 16\ \%$ chance that an insertion will fill Bin A

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946		C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

There is a $5/32 \approx 16\ \%$ chance that an insertion will fill Bin A

– This causes two clusters to *coalesce* into one larger cluster of length 9

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946	**747**	C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

There is now a $11/32 \approx 34\ \%$ chance that the next insertion will increase the length of this cluster

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946	747	C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

Primary Clustering

As the cluster length increases, the probability of further increasing the length increases

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
680						826	207	488	946	747	C8B	74C	ACD	3AD			946	B32							D59	5BA	19A	E9C			

In general:

- Suppose that a cluster is of length ℓ
- An insertion either into any bin occupied by the chain or into the locations immediately before or after it will increase the length of the chain
- This gives a probability of $\dfrac{\ell + 2}{M}$

Run-time analysis

- Recall: our goal is to keep all operations **O**(1).

- Which operations should we analyze?
  - **Search**
    - **Unsuccessful search**: After probing, we failed to find a key k in the hash table
    - **Successful search**: After probing, we found a key k in the hash table
  - **Insert**
    - The runtime would be the same as **an unsuccessful search**
  - **Remove**
    - The runtime would be the same as **a successful search**

Run-time Analysis: Unsuccessful Search

- **Theorem**

  Given an linear-probing hash table with the load factor λ,
  the expected number of probes in an unsuccessful search is
  at most $1/(1-\lambda)$, assuming uniform hashing

Run-time Analysis: Unsuccessful Search

- **Proof** (details in CLRS p274)
  - In an unsuccessful search,
    - every probe (except the last) accesses an occupied bin, which does not contain the desired key.
    - The last probe accesses an empty bin
  - Let **the random variable X** be the number of probes made in an unsuccessful search
  - Let **the event A_i** be the event that an i-th probe occurs and it is to an occupied bin (which does not contain the desired key)

Run-time Analysis: Unsuccessful Search

Thus, the event $\{X \geq i\}$ is

$A_1 \cap A_2 \cap \cdots \cap A_{i-1}$.

$$Pr\{A_1 \cap \cdots \cap A_{i-1}\} = Pr(A_1) \cdot Pr(A_2 | A_1) \cdot Pr(A_3 | A_1 \cap A_2) \cdot \cdots \cdot Pr(A_{i-1} | A_1 \cap \cdots \cap A_{i-2})$$

$$Pr(A_1) = \frac{n}{M}$$

$$Pr(A_2 | A_1) = \frac{n-1}{M-1}$$

$$Pr(A_j | A_1 \cap \cdots \cap A_{j-1}) = \frac{n-(j-1)}{M-(j-1)}$$

$$Pr\{X \geq i\} = \frac{n}{M} \cdot \frac{n-1}{M-1} \cdot \cdots \cdot \frac{n-(j-2)}{M-(j-2)}$$

$$\leq \left(\frac{n}{M}\right)^{i-1} = \lambda^{i-1}$$

$$E(X) \leq \sum_{i=1}^{\infty} Pr(X \geq i) \leq \sum_{i=1}^{\infty} \lambda^{i-1}$$

$$= 1 + \lambda + \lambda^2 + \cdots = \frac{1}{1-\lambda}$$

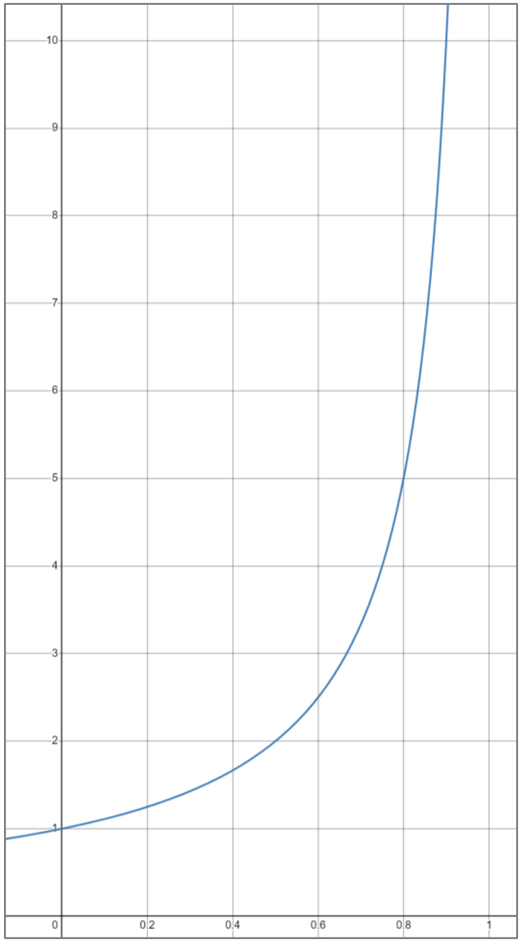
Run-time Analysis: Insertion

- **Corollary**

  Inserting an element into a linear-probing hash table with load factor λ requires **at most $1/(1-\lambda)$ probes on average**, assuming uniform hashing

- **Proof sketch**

  An unsuccessful search implies that an empty bin is found, which can be used for the insertion. So the insertion should take no more than the unsuccessful search.



Run-time Analysis: Successful Search

- **Theorem**
  - Given a linear-probing hash table, **the expected number of probes in a successful search** is at most

$$\frac{1}{\lambda}\ln\left(\frac{1}{1-\lambda}\right)$$

  - Assuming uniform hashing
  - Assuming that each key in the table is equally likely to be search for.

- **Proof sketch** (CLRS p276)
  - The successful search should take place after the insertion (w.r.t. key k)
  - The successful search would follow the same probing sequence as the insertion
  - So we take the average of the probing sequence in the insertion is the average number of successful probes

Run-time Analysis: Successful Search

- **Proof**
  - Suppose k was the (i+1)st key, which is inserted into the table
  - Then the expected number of probes is at most $\frac{1}{1-i/m}=\frac{m}{m-i}$
  - Averaging over all n keys in the table,

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{m}{m-i} = \frac{m}{n}\sum_{i=0}^{n-1}\frac{1}{m-i}$$

$$= \frac{1}{\lambda}\sum_{i=0}^{n-1}\left(\frac{1}{m}+\frac{1}{m-1}+\cdots+\frac{1}{m-n+1}\right)$$

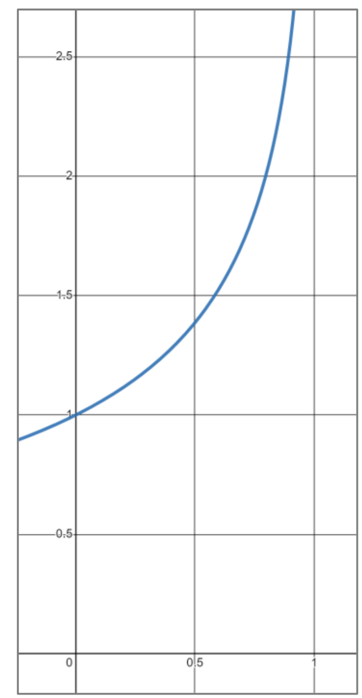
$$= \frac{1}{\lambda}\sum_{k=m-n+1}^{m}\frac{1}{k}$$

$$\leq \frac{1}{\lambda}\int_{m-n}^{m}\left(\frac{1}{k}\right)dk$$

See A.12 in CLRS

$$= \frac{1}{\lambda}\ln\frac{m}{m-n}$$

$$= \frac{1}{\lambda}\ln\frac{1}{1-\lambda}$$



Run-time analysis

The analysis shows that if we assume λ is constant, all operations are $\mathbf{O}(1)$ on average.

	Average	**Worst**
Search	**O(1)**	**O(n)**
Insert	**O(1)**	**O(n)**
Delete	**O(1)**	**O(n)**

Still the analysis implicates that as λ gets bigger, the number of probes increases.

- Q. What's the number of probes if the table is half full?
- Q. what's the number of probes if the table is 90% full?

Run-time analysis

The analysis implies:

– Choose M large enough so that we will not pass the pre-defined load factor

- This could waste memory

– Double the number of bins if the chosen load factor is reached

- Rehasing will be required

Q. Would other collision resolution methods help to reduce the number of probes?

- It won't help the asymptotic complexity, but may help for some cases

- We will cover quadratic probing next

Quadratic Probing

Primary Clustering in Linear Probing

Recall Linear probing:

- Look at bins $k, k+1, k+2, k+3, k+4, \ldots$
- Linear probing causes primary clustering
- All entries follow the same search pattern for bins:

```
int initial = hash_M(x);
for ( int k = 0; k < M; ++k ) {
    bin = (initial + k) % M;
    // ...
}
```

Description

Quadratic probing suggests moving forward by different amounts

For example,

```
int initial = hash_M(x);

for ( int k = 0; k < M; ++k ) {
    bin = (initial + k*k) % M;
}
```

Q. Will "`(initial + k*k) % M`" step through all of the bins?

Description

Problem:

- Will `initial + k*k` step through all of the bins?
- Here, the array size is 10:

```
M = 10;
initial = 5

for ( int k = 0; k <= M; ++k ) {
    std::cout << (initial + k*k) % M << ' ';
}
```

- The output is

```
5 6 9 4 1 0 1 4 9 6 5
```

- Notice the sequence repeats!

Description

Problem:

- Will `initial + k*k` step through all of the bins?
- Now the array size is 12:

```
M = 12;
initial = 5

for ( int k = 0; k <= M; ++k ) {
    std::cout << (initial + k*k) % M << ' ';
}
```

- The output is now

  5 6 9 2 9 6 5 6 9 2 9 6 5

- Notice the sequence repeats!

Best in Theory: Making M Prime

Theorem:

If the table size is $M = p$ a prime number and a quadratic probing is used, the first $p/2$ probes are distinct.

This theorem implies that at least the half of slots will be visited before the probe sequence repeats.

Best in Theory: Making M Prime

Proof by contradiction:
Suppose there is a slot, which is visited twice during the first $M/2$ probes.
Let i and j be such two visits, where $0 \le i < j \le \frac{M}{2}$.

$$(H + i^2)\% M = (H + j^2)\% M$$
$$(H + j^2) = (H + i^2) + kM$$
$$j^2 = i^2 + kM$$
$$j^2 - i^2 = kM$$
$$(j - i)(j + i) = kM$$

Because M is prime, either $(j - i)$ or $(j + i)$ should have a factor M.
In other words, either $(j - i)$ or $(j + i)$ should be divisible by M.

Case#1: $(j - i)$ is divisible by M.
From assumption, $i < j \le \frac{M}{2}$.
So $(j - i) < M$, which contradicts the assumption of case#1.

Case#2: $(j + i)$ is divisible by M.
From assumption, $i < j \le \frac{M}{2}$.
So $(j + i) < M$, which contradicts the assumption of case#2.

Best in Theory: Making M Prime

Engineering difficulties in using a prime M in practice:

- No optimized modulus operations
  - The modulus operator % is relatively slow
  - With a prime M, you cannot optimize with &, <<, or >>
- Troublesome memory management
  - Memory Fragmentation
- Doubling the number of bins is difficult:
  - You always need to find the next prime number
  - What is the next prime after 263?
    - You can't pick 2 * 263 as it's not a prime number

Generic Use

More generally, we could consider an approach like:

```
int initial = hash_M(x);

for ( int k = 0; k < M; ++k ) {
    bin = (initial + c1*k + c2*k*k) % M;
}
```

Practical Use: $M = 2^m$ with constraints

If we ensure $M = 2^m$ then choose

$$c_1 = c_2 = ½$$

```
int initial = hashM(x);

for ( int k = 0; k < M; ++k ) {
    bin = (initial + (k + k*k)/2) % M;
}
```

– Note that `k + k*k` is always even
– This guarantees that **all M entries are visited before the pattern repeats!**
  - Proof sketch: Similar to the proof when M is prime

Practical Use: $M = 2^m$ with constraints

For example:

– Use an array size of 16:

```
M = 16;
initial = 5

for ( int k = 0; k <= M; ++k ) {
    std::cout << (initial + (k + k*k)/2) % M << ' ';
}
```

– The output is now

`5 6 8 11 15 4 10 1 9 2 12 7 3 0 14 13 13`

Practical Use: $M = 2^m$ with constraints

There is an even easier means of calculating this approach

```
int bin = hash_M(x);

for ( int k = 0; k < M; ++k ) {
    bin = (bin + k) % M;
}
```

– Recall that $\frac{k^2 + k}{2} = \sum_{j=0}^{k} j$, so just keep adding the next highest value

Example

Consider a hash table with $M = 16$ bins

Given a 2-digit hexadecimal number:

- The least-significant digit is the primary hash function (bin)
- Example: for $6B7\mathbf{A}_{16}$, the initial bin is **A**

Example

Insert these numbers into this initially empty hash table

9A, 07, AD, 88, BA, 80, 4C, 26, 46, C9, 32, 7A, BF, 9C

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

9A, 07, AD, 88

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example

Start with the first four values:

9A, 07, AD, 88

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	**88**		**9A**			**AD**		

Example

Next we must insert BA

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	88		9A			AD		

Example

Next we must insert BA

– The next bin is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	88		9A	BA		AD		

Example

Next we are adding 80, 4C, 26

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
							07	88		9A	BA		AD		

Example

Next we are adding 80, 4C, 26

- All the bins are empty—simply insert them

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						**26**	07	88		9A	BA	**4C**	AD		

Example

Next, we must insert 46

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						26	07	88		9A	BA	4C	AD		

Example

Next, we must insert 46

- Bin **6** is occupied
- Bin **6 + 1 = 7** is occupied
- Bin **7 + 2 = 9** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						26	07	88	**46**	9A	BA	4C	AD		

Example

Next, we must insert C9

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80						26	07	88	46	9A	BA	4C	AD		

Example

Next, we must insert C9

- Bin **9** is occupied
- Bin **9 + 1 = A** is occupied
- Bin **A + 2 = C** is occupied
- Bin **C + 3 = F** is empty

0	1	2	3	4	5	6	7	8	**9**	A	B	C	D	E	F
80						26	07	88	46	9A	BA	4C	AD		C9

Example

Next, we insert 3**2**

– Bin **2** is unoccupied

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80		**32**				26	07	88	46	9A	BA	4C	AD		C9

Example

Next, we insert 7A

- Bin **A** is occupied
- Bins **A + 1 = B**, **B + 2 = D** and **D + 3 = 0** are occupied
- Bin **0 + 4 = 4** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80		32		**7A**		26	07	88	46	9A	BA	4C	AD		C9

Example

Next, we insert BF

- Bin **F** is occupied
- Bins **F + 1 = 0** and **0 + 2 = 2** are occupied
- Bin **2 + 3 = 5** is empty

0	1	2	3	4	5	6	7	8	9	**A**	B	C	D	E	F
80		32		7A	**BF**	26	07	88	46	9A	BA	4C	AD		C9

Example

Finally, we insert 9**C**

- Bin **C** is occupied
- Bins **C + 1 = D**, **D + 2 = F**, **F + 3 = 2**, **2 + 4 = 6** and **6 + 5 = B** are occupied
- Bin **B + 6 = 1** is empty

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	**9C**	32		7A	BF	26	07	88	46	9A	BA	4C	AD		C9

Example

Having completed these insertions:

- The load factor is $\lambda = 14/16 = 0.875$
- The average number of probes is $32/14 \approx 2.29$

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
80	9C	32		7A	BF	26	07	88	46	9A	BA	4C	AD		C9

Resizing the array

To double the capacity of the array, each value must be rehashed

- 80, 9C, 32, 7A, BF, 26, 07, 88 may be immediately placed
  - We use the least-significant five bits for the initial bin

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
80						26	07	88										32								7A		9C			BF

- If the next least-significant digit is
  - Even, use bins 0 – F
  - Odd, use bins 10 – 1F

Resizing the array

To double the capacity of the array, each value must be rehashed

- 46 results in a collision
  - We place it in bin 9

0	1	2	3	4	5	**6**	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
80						26	07	88	**46**									32								7A		9C			BF

Resizing the array

To double the capacity of the array, each value must be rehashed

- 9A results in a collision
  - We place it in bin 1B

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	**1A**	1B	1C	1D	1E	1F
80						26	07	88	46									32								7A	**9A**	9C			BF

Resizing the array

To double the capacity of the array, each value must be rehashed

- BA also results in a collision
  - We place it in bin 1D

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	**1A**	1B	1C	1D	1E	1F
80						26	07	88	46									32								7A	9A	9C	**BA**		BF

Resizing the array

To double the capacity of the array, each value must be rehashed

- 4C and AD don't cause collisions

0	1	2	3	4	5	6	7	8	9	A	B	**C**	**D**	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
80						26	07	88	46			**4C**	**AD**					32								7A	9A	9C	BA		BF

Resizing the array

To double the capacity of the array, each value must be rehashed

- Finally, C9 causes a collision
  - We place it in bin A

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
80						26	07	88	46	C9		4C	AD					32								7A	9A	9C	BA		BF

Resizing the array

To double the capacity of the array, each value must be rehashed

- The load factor is $\lambda = 14/32 = 0.4375$
- The average number of probes is $20/14 \approx 1.43$

0	1	2	3	4	5	6	7	8	**9**	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F
80						26	07	88	46	**C9**		4C	AD					32								7A	9A	9C	BA		BF

Run-time Analysis

To summarize, quadratic probing shows the same asymptotic complexity as linear probing.

	Average	**Worst**
Search	**O(1)**	**O(n)**
Insert	**O(1)**	**O(n)**
Delete	**O(1)**	**O(n)**

Secondary clustering

Advantage of quadratic probing over linear probing

- Quadratic probing avoids primary clustering

One weakness with quadratic problem

- Objects initially placed in the same bin will follow the same sequence
- It forms yet another clustering, so called **the secondary clustering**
- Q. how would you solve this problem?

References

Wikipedia, http://en.wikipedia.org/wiki/Hash_function

[1] Cormen, Leiserson, and Rivest, *Introduction to Algorithms*, McGraw Hill, 1990.

[2] Weiss, Data Structures and Algorithm Analysis in C++, 3^{rd} Ed., Addison Wesley.