Model description of 2D QG numerical model

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1 Introduction and model organisation

The 2D QG numerical model solves the PV equation and omega equation for the quasi-geostrophic streamfunction (ψ) and vertical motion (ω) in pressure coordinates. It assumes wave-like solutions for ψ and ω on the form

$$[\psi, \omega] = [\Psi(p), W(p)] \exp[i(kx - \sigma t)], \tag{1}$$

where $\Psi(p)$ and $W(p)$ are complex numbers describing the amplitude and the phase of ψ and ω, k is the zonal wavenumber, x is the zonal coordinate, σ is the wave frequency, and t is time.

The model is run from the main.ipynb file in Jupyter notebook by loading various .py files. It consists of 3 main parts:

1. Setup and definition of variables, determining:
   (a) Forcing (diabatic heating, friction, etc.)
   (b) Background baroclinicity (wind shear)
   (c) Stability
   (d) Coriolis
   (e) Model resolution and parametrisation levels
   (f) Setup for potential inclusion of tropopause and stratosphere
2. Model core: Computation of solution
3. Post-processing and model output

After determining the background state in part 1, a matrix equation $A\mathbf{x} = \sigma B\mathbf{x}$ is defined in part 2 representing the PV equation and the omega equation, where the eigenvalue σ is the wave frequency used in the wave solution in (1) and the eigenvector $\mathbf{x} = [\psi_0, \psi_1, ..., \psi_N, \omega_0, \omega_1, ..., \omega_N]$ describes Ψ and W at N+1 vertical levels. The equation set and its numerical discretisation are described in further detail in Sections 2-3. Part 2 is also described in Haualand and Spengler (2019).

In part 3, the eigenvalue σ and the eigenvector $\mathbf{x}$ are used to obtain other variables like the growth rate ($\mathrm{Im}\{\sigma\}$), meridional wind ($\partial\psi/\partial x$), and temperature ($\propto \partial\psi/\partial p$). Based on the variables, various diagnostics, such as energetic terms, may be calculated. In the end of this part of the model, the requested output may be plotted.

2 Model equations

The governing model equations are the nondimensional omega and PV equations in pressure coordinates, which were derived by (Mak, 1994). Terms associated with latent heating from Mak (1994) are marked in green, and terms associated with surface sensible heat fluxes from Mak (1998) are marked in red. Further parametrisations of surface sensible heat fluxes were tested and described in Haualand and Spengler (2020), where $d\Psi/dp$ was substituted by either $-v = -\partial\Psi/\partial x = -ik\Psi$ or W_B.

The generalised versions of the omega equation and the PV equation from Mak (1994) and Mak (1998) are:

$$\frac{d^2W}{dp^2} - Sk^2W = \mathrm{i}2\lambda k^3\Psi - \frac{\varepsilon_1 k^2}{2}h_1 W_B + \varepsilon_2 k^2 h_2 \frac{d\Psi}{dp}\bigg|_S, \tag{2}$$

$$(\overline{u}k-\sigma)\left(\frac{d}{dp}\left(\frac{1}{S}\frac{d\Psi}{dp}\right)-k^2\Psi\right)+k\frac{d}{dp}\left(\frac{\lambda}{S}\right)\Psi = -i\frac{\varepsilon_1}{2}W_B\frac{d}{dp}\left(\frac{h_1}{S}\right)+i\varepsilon_2\frac{d\Psi}{dp}\bigg|_S\frac{d}{dp}\left(\frac{h_2}{S}\right), \tag{3}$$

where $\overline{u}$ is the basic-state zonal wind, k is the zonal wavenumber, S is stratification, λ is the zonal wind shear with height, ε_1 is the latent heating intensity, h_1 is the vertical latent heating profile, W_B is the amplitude of vertical motion at a predefined cloud base level (p_B), ε_2 is the heating intensity for surface fluxes, and h_2 is the vertical heating profile for surface fluxes.

If h_1, h_2, S, and/or λ are step functions, their derivatives will be zero everywhere except at the steps. At these levels, we use modified versions of the equations by integrating the PV equation around the discontinuity. Integrating (3) from $p_c - \nu$ to $p_c + \nu$, where p_c is a critical level where at least one of the terms is discontinuous and ν is an infinitely small number, yields

$$(\overline{u}_j k - \sigma)\left(\frac{1}{S}\frac{d\Psi}{dp}\right)\Big|_{c-\nu}^{c+\nu} + k\Psi\left[\frac{\lambda}{S}\right]_{c-\nu}^{c+\nu} = -i\frac{\varepsilon_1}{2}W_B\frac{h_1}{S}\Big|_{c-\nu}^{c+\nu} + i\varepsilon_2\frac{d\Psi}{dp}\Big|_S\frac{h_2}{S}\Big|_{c-\nu}^{c+\nu}. \quad (4)$$

The boundary conditions required to solve the problem are described by nonzero vertical motion together with the thermodynamic equation:

$$W = 0, \quad \text{at } p = p_1, 1, \quad (5)$$

$$(\overline{u}k - \sigma)\frac{d\Psi}{dp} + \lambda k\Psi = 0, \quad \text{at } p = p_1, \quad (6)$$

$$(\overline{u}k - \sigma - i\varepsilon_2 h_2)\frac{d\Psi}{dp}\Big|_S + i\frac{\varepsilon_1 h_1}{2}W_B + \lambda k\Psi = 0, \quad \text{at } p = 1, \quad (7)$$

where $p = 1$ is the nondimensional pressure at the surface which corresponds to 1000 hPa, and $p = p_1$ is the top of the model domain.

When including the stratosphere, the upper boundary condition is changed to

$$\frac{d\Psi}{dp} = 0, \quad \text{at } p = p_1. \quad (8)$$

which implies vanishing temperature anomalies at the model lid such that PV anomalies at this interface are suppressed.

3 Numerical discretisation of model

The first and second order derivatives in the equation set (2)-(8) are expressed numerically using the method of finite differences. Using central and one-sided differences to express derivatives, the numerical discretisation of the omega equation in (2) is

$$\frac{W_{j-1} - 2W_j + W_{j+1}}{\Delta p^2} - Sk^2 W_j = \mathrm{i}2\lambda k^3 \Psi_j - \frac{\varepsilon_1 k^2}{2} h_{1,j} W_{j_B} + \varepsilon_2 k^2 h_{2,j} \frac{\Psi_N - \Psi_{N-1}}{\Delta p}, \tag{9}$$

where Δp is the pressure increment between two vertical levels and the subscript j denotes the vertical level of the variable. In a similar way, the numerical discretisation of the PV equation in (3) is

$$\begin{aligned}(\overline{u}_j k - \sigma)\left(\frac{1/S_{j+1} - 1/S_{j-1}}{2\Delta p}\frac{\Psi_{j+1} - \Psi_{j-1}}{2\Delta p} + \frac{1}{S_j}\frac{\Psi_{j-1} - 2\Psi_j + \Psi_{j+1}}{\Delta p^2} - k^2\Psi_j\right)& \\ + k\frac{\lambda_{j+1}/S_{j+1} - \lambda_{j-1}/S_{j-1}}{2\Delta p}\Psi_j& \\ = -i\frac{\varepsilon_1}{2} W_{j_B}\left(h_{1,j}\frac{1/S_{j+1} - 1/S_{j-1}}{2\Delta p} + \frac{1}{S_j}\frac{h_{1,j+1} - h_{1,j-1}}{2\Delta p}\right)& \\ + i\varepsilon_2 \frac{\Psi_N - \Psi_{N-1}}{\Delta p}\left(h_{2,j}\frac{1/S_{j+1} - 1/S_{j-1}}{2\Delta p} + \frac{1}{S_j}\frac{h_{2,j+1} - h_{2,j-1}}{2\Delta p}\right).&\end{aligned} \tag{10}$$

For the cases where h_1, h_2, S, and/or λ are step functions, the vertically integrated PV equation in (4) is discretised to

$$
\begin{aligned}
\left(\frac{\overline{u}_{c+\nu}+\overline{u}_c}{2}k-\sigma\right)\left(\frac{1}{S_{c+\nu}}\frac{\Psi_{c+\nu}-\Psi_c}{\Delta p}\right)-\left(\frac{\overline{u}_{c-\nu}+\overline{u}_c}{2}k-\sigma\right)\left(\frac{1}{S_{c-\nu}}\frac{\Psi_c-\Psi_{c-\nu}}{\Delta p}\right)& \\
+k\Psi_c\left(\left.\frac{\lambda}{S}\right|_{c+\nu}-\left.\frac{\lambda}{S}\right|_{c-\nu}\right)& \\
=-i\frac{\varepsilon_1}{2}W_B\left(\left.\frac{h_1}{S}\right|_{c+\nu}-\left.\frac{h_1}{S}\right|_{c-\nu}\right)& \\
+i\varepsilon_2\frac{\Psi_N-\Psi_{N-1}}{\Delta p}\left(\left.\frac{h_2}{S}\right|_{c+\nu}-\left.\frac{h_2}{S}\right|_{c-\nu}\right)&. \qquad (11)
\end{aligned}
$$

Note that some of these terms will be zero unless all discontinuity interfaces overlap. The numerical discretisation of the boundary conditions (5) - (7) is

$$W_0=W_N=0, \qquad (12)$$

$$\left(\frac{\overline{u}_0+\overline{u}_1}{2}k-\sigma\right)\frac{\Psi_1-\Psi_0}{\Delta p}+\lambda k\frac{\Psi_0+\Psi_1}{2}=0, \qquad (13)$$

$$
\begin{aligned}
\left(\frac{\overline{u}_{N-1}+\overline{u}_N}{2}k-\sigma\right)\frac{\Psi_N-\Psi_{N-1}}{\Delta p}+\lambda k\frac{\Psi_{N-1}+\Psi_N}{2}& \\
=-i\frac{\varepsilon_1 h_1}{2}W_{j_B}+i\varepsilon_2 h_2\frac{\Psi_N-\Psi_{N-1}}{\Delta p}&, \qquad (14)
\end{aligned}
$$

or, when including the stratosphere,

$$\psi_0-\psi_1=0. \qquad (15)$$

Sorting the terms in (13) and (14) yields

$$
\begin{aligned}
-\left(\frac{1}{\Delta p}\left(\frac{\overline{u}_0+\overline{u}_1}{2}k-\sigma\right)-\frac{\lambda k}{2}\right)\Psi_0& \\
+\left(\frac{1}{\Delta p}\left(\frac{\overline{u}_0+\overline{u}_1}{2}k-\sigma\right)+\frac{\lambda k}{2}\right)\Psi_1=0&, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{1}{\Delta p}\left(\frac{\overline{u}_{N-1}+\overline{u}_N}{2}k-\sigma\right)-\frac{\lambda k}{2}-\frac{i\varepsilon_2 h_2}{\Delta p}\right)\Psi_{N-1}\\
&\qquad+\left(\frac{1}{\Delta p}\left(\frac{\overline{u}_{N-1}+\overline{u}_N}{2}k-\sigma\right)+\frac{\lambda k}{2}-\frac{i\varepsilon_2 h_2}{\Delta p}\right)\Psi_N+i\frac{\varepsilon_1 h_1}{2}W_{j_B}=0. \quad (17)
\end{aligned}
$$

4 Heating and stratification step profiles

The default heating profiles of latent heating in Mak (1994) and surface sensible heat fluxes in Mak (1998) are described by the Heaviside functions, with their vertical derivatives described by the Dirac delta functions:

$$h_1(p) = H(p - p_{**}) - H(p - p_*) = \begin{cases} 1 & \text{for } p_{**} < p < p_* \\ 0 & \text{for } p < p_{**}, \quad p > p_* \end{cases}, \tag{18}$$

$$\frac{dh_1}{dp} = \delta(p - p_{**}) - \delta(p - p_*), \tag{19}$$

$$h_2(p) = H(p - p_{**}) - H(p - p_*) = \begin{cases} 1 & \text{for } p_{***} < p \leq 1 \\ 0 & \text{for } p < p_{***} \end{cases}, \tag{20}$$

$$\frac{dh_1}{dp} = \delta(p - p_{***}). \tag{21}$$

When latent cooling is included below the layer of latent heating, the heating profile in (18) and its vertical derivative in (19) are modified to

$$h_1(p) = H(p - p_{**}) - (1+\gamma)H(p - p_*) = \begin{cases} 0 & \text{for } p < p_{**} \\ 1 & \text{for } p_{**} < p < p_* \\ -\gamma & \text{for } p > p_* \end{cases}, \tag{22}$$

$$\frac{dh_1}{dp} = \delta(p - p_{**}) - (1+\gamma)\delta(p - p_*), \tag{23}$$

where γ is a factor controlling the amount of latent cooling relative to latent heating.

Furthermore, in some cases, like in Mak (1998), the vertical profile of stratification changes from a constant value dependent on the surface heat flux intensity ε_2 to another constant values at a level p_{****} in the troposphere:

$$S(p) = \begin{cases} S_1 & \text{for } p < p_{****} \\ S_2(\varepsilon_2) & \text{for } p > p_{****} \end{cases} . \tag{24}$$

5 Modified vertical profiles

In part I of the model, alternative and more realistic formulations of the vertical heating profiles are described. Some of these profiles were used by Haualand and Spengler (2019) to investigate the sensitivity to the formulation of latent heating and latent cooling.

In addition to the modified heating profiles, the vertical profiles of wind shear and stratification may be varied. For example, Haualand and Spengler (2021) used vertical profiles of λ and S that were smoothed in a region around the tropopause. In part I of the model, this smoothing around the tropopause level p_{trop} is implemented by letting λ/S be defined by a sine function that increases from $(\lambda/S)_{p=0}$ at $p_{trop} - \delta/2$ to $(\lambda/S)_{p=1}$ at $p_{trop} + \delta/2$:

$$\frac{1+\alpha}{2}A + \frac{1-\alpha}{2}A\sin(\tau(p)), \tag{25}$$

where $A \leq (\lambda/S)_{p=1}$ is the amplitude of the step, $\tau(p)$ is a vector from $-\pi/2$ to $\pi/2$ confined within the smoothing range $p_{trop} \pm \delta/2$, $\alpha = \hat{\alpha}\frac{(\lambda/S)_{p=0}}{(\lambda/S)_{p=1}} \in [-1, 1]$ is the scaling parameter, and $\hat{\alpha}$ is the offset parameter that shifts $(\lambda/S)_{p=0}$ such that the integral of $\overline{q}_y = \partial/\partial p(\lambda/S)$ around the tropopause region changes relative to the original profile of λ/S. This smoothing procedure is further explained by Haualand and Spengler (2021).

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