MTH 377/577 Convex Optimization
Assignment
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Ans 1

Say x amount of product X is produced
& y amount of product Y is produced

Max $z = 3x + 5y$

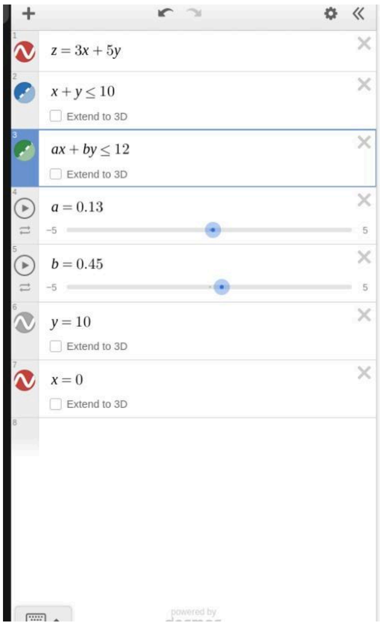
s.t
$$x + y \le 10 \quad -(1)$$
$$a_x x + a_y y \le 10 \quad -(2)$$
$$b_x x + b_y y \le 12 \quad -(3)$$
$$c_x x + c_y y \le 5 \quad -(4)$$
$$x \ge 0 \quad -(5)$$
$$y \ge 0 \quad -(6)$$
$$a_x - a_y > 0 \quad -(7)$$
$$b_y - b_x > 0 \quad -(8)$$
$$c_x = c_y \quad -(9)$$

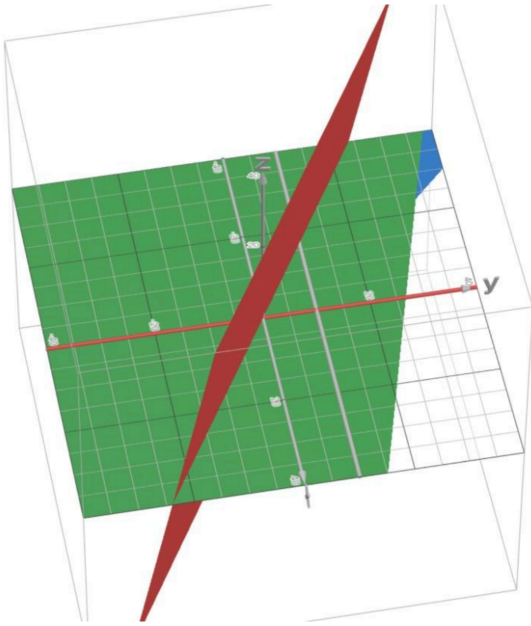
The above is the maximization problem for firm A
where $a_x \rightarrow$ skilled labour for per unit X
$a_y \rightarrow$ skilled labour for per unit Y
$b_x \rightarrow$ unskilled labour for per unit X
$b_y \rightarrow$ unskilled labour for per unit Y
$c_x \rightarrow$ managment expertise per unit X
$c_y \rightarrow$ managment expertise per unit Y

Geometrical Argument

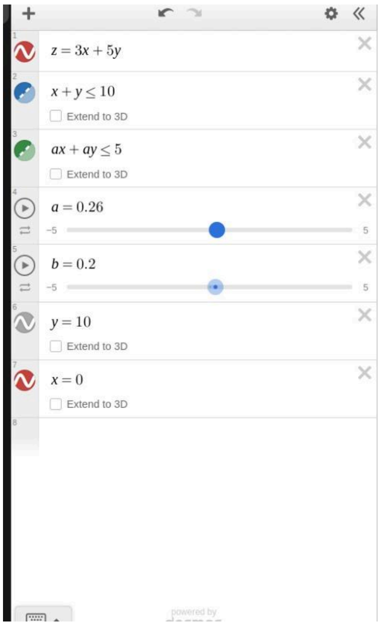
Geometrical Arguments given just to get the feel of Question and its demands

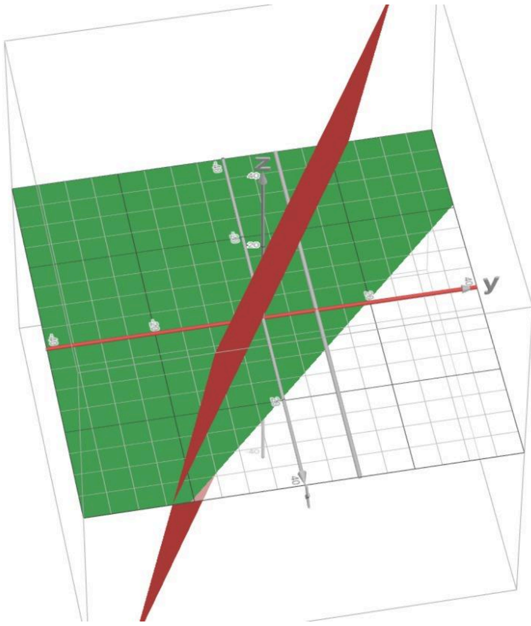
Later in this Section we have provided mathematical argument



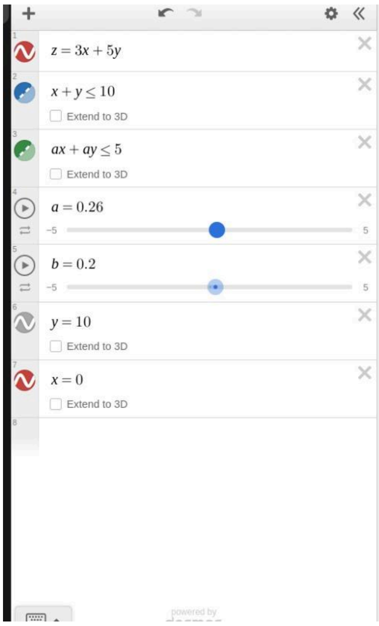


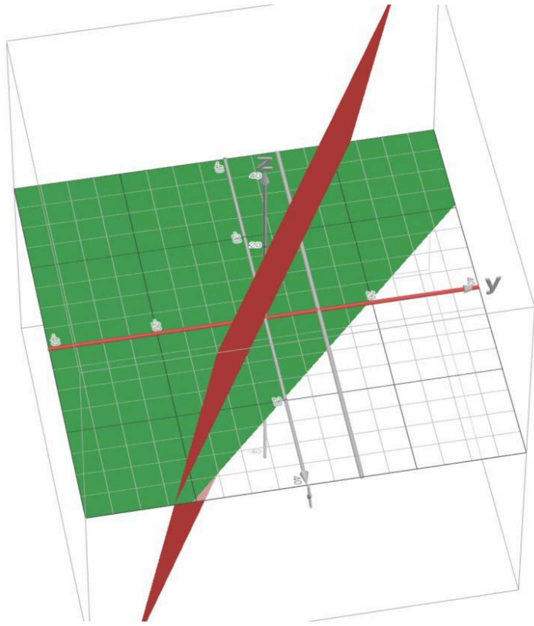
Unskilled Labor (b > a, a < 1.2): Similar to skilled labour, a constraint where unskilled labor is higher than other resources (with both limiting 1.2), leads to a situation where the optimal solution is not guaranteed for all slider values. The threshold for losing the optimal solution is slightly higher compared to skilled labor.





Skilled Labor (a > b, a < 1): When skilled labor requirements are higher than other resources (and both limits 1), the optimal solution can become infeasible at certain slider positions





Management Expertise: Under this scenario, where management expertise for both goods is equal and limits a value of 5, the optimal solution (maximizing revenue) is achievable regardless of a slider movement in the 3D space. Additionally, it's observed that cx = cy < 0.5

Mathematical Argument

Suppose our problem is only

$$\text{Max } z = 3x + 5y$$
$$\text{s.t } x + y \le 10$$

for Max we get $y = 10$ & $x = 0$

& now second half of our problem can be broken as

I
$$0\,a_x + 10\,a_y \le 10$$
$$a_x > a_y$$
— There must exist a sol^n for it for (0, 10) to be answer

Simillary

II
$$0\,b_x + 10\,b_y \le 12$$
$$b_y > b_x$$
— A sol^n must exist for (0, 10) to be sol^n

III
$$c_x 0 + c_y 10 \le 5$$
$$c_x = c_y$$
— A sol^n must exist for (0,10) to be sol^n

if I, II & III have a feasible sol^n we can argue overall (0, 10) sol^n as I, II, III ~~variab~~ are independent as variables of I do not come in II, III. variable of II do not come in I & III, variables of III do not come in I & II

I) on solving

$a_y \le 1$ & ~~$a_x > a_y$~~ $a_x > a_y$ so any value like (0.9, 0.8) will satisfy

$a_x \not> 1$, $a_y < 1$

II) ~~$10 b_y$~~ on solving

$b_y \le 1.2$ $b_y > b_x$ so any value (1, 1.1) will satisfy

$b_x < 1.2$, $b_y > 1.2$

III on solving

$c_x = c_y$, $c_y < 0.5$ so any value like (0.5, 0.5) will satisfy

$c_x, c_y < 0.5$

Eureka, (0, 10) satisfies all the equation so max

Solution

profit we will get = 50

if $X > 0$ $Y > 0$ then we will choose $X = 0.00001$

$Y = 9.99999$ & ans will tend to 49.99998

Tech Advancement effect

Due to technological advancement, if the requirement for unskilled labour force in production of X falls marginally, say for instance b_x' decreases where $b_x' < b_x$ (b_x is initial unskilled labour force unit), keeping the other parameters a_x, a_y, b_y, c_x and c_y the same, this advancement will make X more attractive to produce since it would require less of the constrained resources, potentially shifting optimal production mix towards more of X and less of Y, depending on the magnitude of the decrease in the skilled labour.

Here, the new constraint would be:

$$b_x' \cdot X + b_y \cdot Y \leq 12.$$

Dual Problem or Firm B's greed fulfilment

P.T.O

profit we will get = 50
if $X > 0$ $Y > 0$ then we will choose $X = 0.00001$
$Y = 9.99999$ & ans will tend to 49.99998

if Firm B acquires firm A it represent a dual problem because firm B would be interested in minimizing its cost while still mantaing certain production requirments & revenue target

So firm's B problem would involve minimizing cost & acquiring & operating Firm A's production
let price x_1 for every unit skilled labour
x_2 for every unit unskilled labour
x_3 for every unit managment expense
x_4 for every unit of total output for firm B

$$\min \quad 10x_1 + 12x_2 + 5x_3 + 10x_4$$

$$a_x x_1 + b_x x_2 + c_x x_3 + x_4 \geq 3$$

$$a_y x_1 + b_y x_2 + c_y x_3 + x_4 \geq 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

let's write K.K.T for it

$$10x_1 + 12x_2 + 5x_3 + 10x_4 - \lambda_1(a_x x_1 + b_x x_2 + c_x x_3 + x_4 - 3)$$
$$- \lambda_2(a_y x_1 + b_y x_2 + c_y x_3 + x_4 - 5)$$

$\frac{\partial L}{\partial x_1} = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0$ — (2)

$\frac{\partial L}{\partial x_3} = 0$ — (3)

$\frac{\partial L}{\partial x_4} = 0$ — (4)

$\lambda(a_x x_1 + b_x x_2 + c_x x_3 + x_4 - 3) = 0$ — (5)

$\lambda_2 (a_x x_1 + b_x x_2 + c_x x_3 + x_4 - 3) = 0$ — (6)

$a_x x_1 + b_x x_2 + c_x x_3 + x_4 - 3 \le 0$ — (7)

$a_y x_1 + b_y x_2 + c_y x_3 + x_4 - 5 \le 0$ — (8)

$x_1, x_2 \ge 0$ — (9)

$\lambda_2, \lambda_1 \le 0$ — (10)

On solving these 10 equation we get

$a_x > 1, \; a_y < 1$

$b_y > 1.2 \quad b_x < 1.2$

$c_x, c_y \le \frac{1}{2}$

$x_1 = 1, \; x_2 = 0, x_3 = 1, \; x_4 = 3$

minimum value of function

$10 \times 1 + 0 + 5 \times 2 + 10 \times 3 = 50$

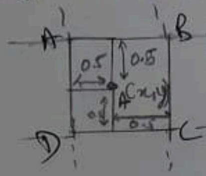
price by firm B $= 50$

Q.2

First, I would like to geometrically argue whether a hyperplane exist for which we can have error ≤ 0.5

where a error $= 0.5$ denotes square ABCD

⇒ let A (x, y) be point

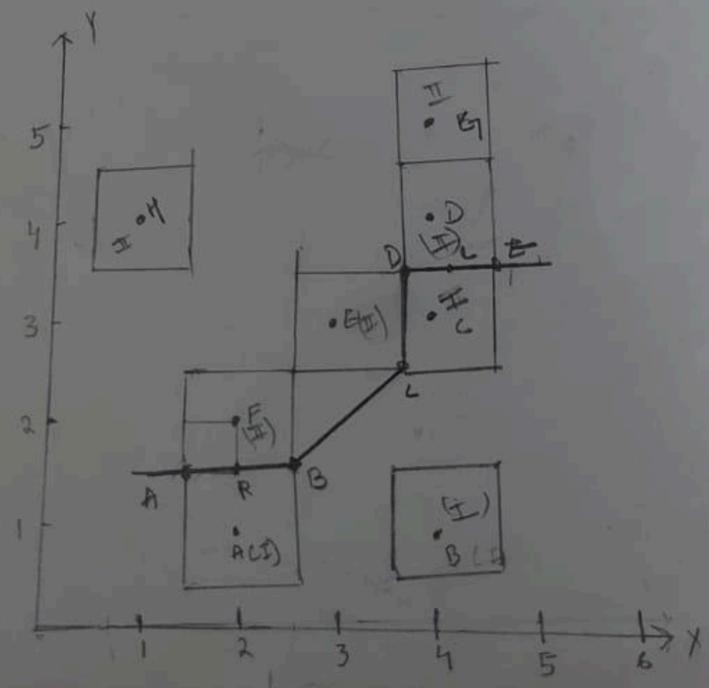


Claim: A can go 0.5 error geometrically means A can move around the square

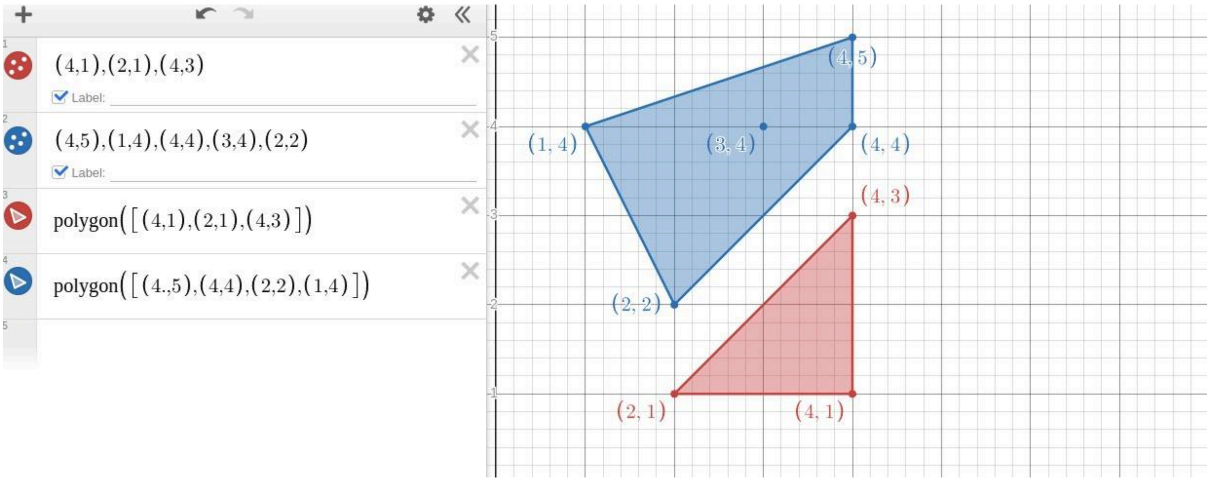
Simple proof: For the given claim if x can go from $x+0.5$ to x we will have equation of BC similarly $x-0.5$ we will have equation of AD

y can go from $(y+0.5)$ this will correspond to AB &

y can go from y to $y-0.5$ this will correspond to CD



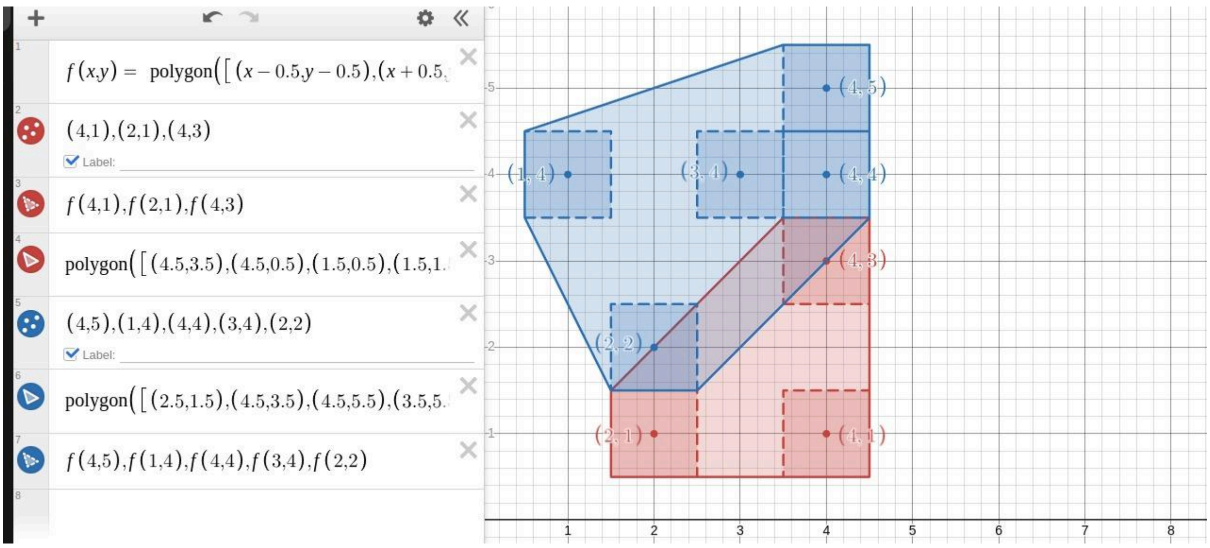
Clearly we can see there is no such hyperplane which can seperate them, we need a combination of hyperplane with constraints.



Drawing the Polygons

Two disjoint convex sets can be separated by a hyperplane. Depending on the convex sets the separation can be different.
Any line segment connecting two points within the polygon will also lie within the polygon and hence it will be convex

These 2 sets are disjoint and not even touching so there will exist infinite hyperplanes in between.



Just something we saw on pen and paper with the help of desmos hence showcasting it using technology

Two disjoint convex sets can be separated by a hyperplane. Depending on the convex sets the separation can be different.

Any line segment connecting two points within the polygon will also lie within the polygon and hence it will be convex

These 2 sets are clearly not disjoint and there will not exist any hyperplanes in between.

for $x < 2.5 \rightarrow y = 1.5$

$2.5 \leq x \leq 3.5 \rightarrow y = x-1$

$x \geq 3.5 \rightarrow y = 3.5$

Now, We need to find single hyperplane that ~~minimizes~~ allow maximum error

OR

I need to find a hyperplane that give the least error on misclassifiation

We convert into an optimisation problem

$$\text{Maximize} \min_{i=1} \sum_{i=1} d_i$$

s.t

$$\begin{cases} Ax_i + By_i + C \geq 0 & \forall (x_i, y_i) \text{ in set } S_1 \\ Ax + By_i + C \leq 0 & \forall (x_i, y_i) \text{ in set } S_2 \end{cases}$$

where $Ax + By + C$ is equation of line

Set $S_1 = \{(4,1), (2,1), (4,3)\}$

S_2 = remaining points

By putting constraints for all 8 equation formed in S_1 & S_2 we indeed get a line from point midpoint of AB & DE i.e line pass through K & L which could be also geometrically argued

equation of line $y = x + (-0.5) \Rightarrow \boxed{y + 0.5 = x}$ best line

To get the feasible error rate for 100% correct predict

We only need to look at points A, F, C & D for given line

notation used: $d(L, x) \Rightarrow$ distance of line L from point x

feasible error: $\min(d(y+0.5=x, A), d(y+0.5=x, F), d(y+0.5=x, C), d(y+0.5=x, D))$

feasible error $= \frac{1}{2\sqrt{2}} \approx 0.35 \rightarrow$ if error $= 0.3535$ we can pred with 100% accuracy

Q3

Q3.

Yes, an envy-free allocation exists for 3 agents.
It can be argued using the Selfridge-Conway Protocol and the Steinhaus protocol.
Argument using Selfridge-Conway Protocol:
Considering a cake as our divisible good:

- Let's assume that there are 3 agents. Agent 1 cuts the cake into 3 equal parts according to his valuation function f_1. Agent 2 can either pass or trim the cake.
  For instance, let's assume agent 2 trims the largest piece according to his valuation function f_2, so that he and the other agents prefer their assigned pieces as valuable as to ~~ay~~ any other assigned pieces.
  At this stage, we have Cake 1 (the trimmed part) and the rest as Cake 2.

Case 1: Division with Cake 2

Now, at the proceeding stage, if agent 3 chooses a piece of cake 2, then agent 2 can choose the trimmed cake without any envy as it is from his function f_2.
Hence, agent 1 takes the remaining third piece of cake 2 which he initially cut according to his valuation function f_1.
But, if at this stage, agent 3 chases the trimmed piece, agent 2 can then choose freely from the remaining pieces. Else, he can take the trimmed piece. Then, agent 1 takes the remaining piece of cake 2.

Here, it can be written (in short) as:
- Agent 3 (P3) chooses 1/3 of cake 1
- Agent 2 (P2) takes the trimmed piece
- Agent 1 (P1) takes the remaining of cake 1.

Case 2: Division with Cake 1

* Let's denote the agent who took the trimmed part by TP and the one who took non-trimmed part by NTP.
When we are proceeding with Cake 1 division, Agent NTP divides the cake into three equal parts and agent TP chooses first, then agent 1 and then agent NTP.
- Agent NTP divides cake 2 into 3 equal parts (in accordance of valuation funᶜ)
- Agent TP ~~d~~ chooses first, then Agent 1 and finally Agent NTP.

Hence, through this, it can be concluded that the Selfridge-Conway Protocol provides an envy-free allocation for $n=3$ agents as —

PTO

- P3 gets his preferred 1/3 of cake 1 and 1/3 of cake 2.
- P2 gets the trimmed piece of cake 1 and 1/3 of cake 2, which he values as his second-largest piece.
- P1 gets the remaining 1/3 of cake 1 and 1/3 of cake 2, which is worth exactly 1/3 to him.

Argument using Steinhaus Protocol:
Under this protocol:

1. First P1 cuts the cake into 3 parts according to his/her valuation function f_1.
2. P2 is then given the choice of either passing or labeling the pieces ~~as~~ 'not desirable', according to his valuation function f_2.
   Else, if he thinks the division is worth at least 1/3 according to f_2, then the cake is divided envy-freely.
3. If P2 didn't pass at point 2, then P3 is given the same two options that P2 had at point 2 (without considering P2 labels).
   If P3 passes then all the ~~player~~ agents choose the piece without being envious in order of P2, P3 and P1.
4. If none of P2 and P3 pass; ~~the other two pieces are reassembled~~, then P1 is given the choice to choose any of the pieces that P2 and P3 labeled as 'not desirable'. P1 doesn't have any envy due to this bcz all the pieces are according to f_1.

As a result, for further division, the other two pieces are reassembled, and P2 & P3 use [the cut and choose protocol] to divide the reassembled piece. The cut and choose Protocol in step 4 leads to division and updating of the pieces that will ensure the division of the product envy-freely for agent 3.

— x —

If our argument for n=3 agents is extending to n=4 agents under the some constraints then it always doesn't guarantee an envy-free solution.

BUT,
if we suppose that we have a bounded protocol which gives a specified agent i and an unallocated piece of cake returns a partial envy-free allocation st i dominates 2 other agents,
then we can extend this into a 4-Agent Envy-Free ~~bounde~~ solution using Core protocol and 4-agent protocol extension (say).

Core Protocol

Initial Division: let Agent P4 divides the unallocated cake into 4 equally preferred pieces.

Trimming Process: Agents P1, P2 and P3 are then asked to trim the left side of their top two preferred pieces to equalize their value with their third most preferred piece. This results in six trims across the four pieces.

- Agents who trim a segment the most are entitled to that segment up to their trim point, achieving an envy-free allocation.
- To ensure at least two agents receive complete pieces, in specific cases, one or two agents may be asked to further trim their most preferred segment to match its value with their second preferred segment, ignoring their previous trims.

4-Agent Protocol Extension: If we start by identifying an agent P1, who dominates two other agents, let's say P3 and P4.

- If agent P2 also dominates P3 and P4, proceed by allowing P3 and P4 to divide the remaining cake using ~~cho~~ divide and choose method.
- If P2 doesn't dominate P3 and P4 but dominates P1 and one of agents P3 or P4 (let's say P4):
  - If P3 dominates P4, allocate all the remaining cake to P4 since everyone dominates P4.
  - If P3 doesn't dominate P4 but instead dominates P1 and P2, let P4 cut the remaining cake into ~~four~~ another equally preferred pieces.
  - Then, agents P1, P2 and P3 pick their most preferred remaining piece in the same manner as defined for n=3 in selfridge protocol.

Proof of Envy-Free Division for n=4

The core protocol and the 4-agent extension together guarantee a partial envy-free allocation as follows:

- The core protocol results in a partial envy-free allocation where the cutter (agent 4)(P4) and at least one another agent recieve complete pieces.

PTO

- Agent Dominance: By identifying and leveraging dominant agents in the 4-agent extension, we ensure that at least one agent dominates two others, facilitating the allocation process.
- In cases, where additional dominance is not established, the Divide and choose method ensures an envy-free allocation among the agents involved.
- The allocation ~~press~~ process is further refined through sequential choice, where agents pick their preferred pieces in a defined order, ensuring fairness and envy-freeness.

Hence, through this it can be claimed that envy-free division is possible for $n=4$ agents.

References used

1.https://www.desmos.com/calculator
2.https://www.desmos.com/3d
3.https://arxiv.org/pdf/1508.05143.pdf
4. Additional resources and class notes uploaded on classroom