INF5620
Compulsory exercise 4

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Exercise 8

$$-u''(x) = 2, \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = 1$$

Finite element method:

We start out by multiplying the differential equation by a test function v

$$u''v = -2v,$$

and do integration by parts on the left hand side:

$$\int_\Omega u''v dx = [u'v]_0^1 - \int_0^1 u'v'dx = -\int u'v'dx.$$

The first term gives no contribution to the integral since u is known at the boundaries and the test function v must vanish.

The equation is now on *weak from*,

$$\int_\Omega u'v'dx = \int_\Omega 2vdx$$

and we must find u in $V = span\{\phi_0, \phi_1, .., \phi_N\}$ such the equation above holds for all v.

Using *P1 elements* and dividing our domain with nodes, $x = x_0, x_1, .., x_N$ (N elements) u takes the form:

$$u(x) = \sum_{j=0}^{N} c_j \phi_j(x)$$

All the computations can be done in the same *reference element* $\tilde{x} \in [-1, 1]$ through the mapping

$$x = \frac{1}{2}(x_i + x_{i+1}) + \frac{1}{2}(x_{i+1} - x_i)\tilde{x}$$

with the two linear basis functions

$$\phi_a(\tilde{x}) = \frac{1}{2}(1 - \tilde{x})$$
$$\phi_b(\tilde{x}) = \frac{1}{2}(1 + \tilde{x})$$

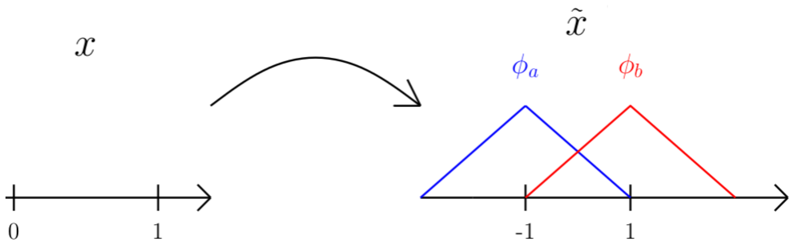

Figure 1: Illustration of the mapping and the reference element.

The entries in the element matrix and vector can then be computed as follows:

$$\begin{aligned} A^e_{ij} &= \int_{x_i}^{x_{i+1}} \phi_i'(x)\phi_{i+1}'(x)dx = \int_{-1}^{1} \frac{d\phi_a}{d\tilde{x}}\frac{d\tilde{x}}{dx}\frac{d\phi_b}{d\tilde{x}}\frac{d\tilde{x}}{dx}\frac{dx}{d\tilde{x}}d\tilde{x} \\ &= \int_{-1}^{1} \frac{d\phi_a}{d\tilde{x}}\frac{d\phi_b}{d\tilde{x}} 2\frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2}d\tilde{x} = -\frac{1}{(x_{i+1} - x_i)} \\ &= A^e_{ji} \end{aligned}$$

$$\begin{aligned} A^e_{ii} &= \int_{x_i}^{x_i} \phi_i'(x)\phi_i'(x)dx = \int_{-1}^{1} \frac{d\phi_a}{d\tilde{x}}\frac{d\tilde{x}}{dx}\frac{d\phi_a}{d\tilde{x}}\frac{d\tilde{x}}{dx}\frac{dx}{d\tilde{x}}d\tilde{x} \\ &= \int_{-1}^{1} \frac{d\phi_a}{d\tilde{x}}\frac{d\phi_a}{d\tilde{x}} 2\frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2}d\tilde{x} = \frac{1}{(x_{i+1} - x_i)} \\ &= A^e_{jj} \end{aligned}$$

$$\begin{aligned} b^e_i &= \int_{x_i}^{x_i} 2\phi_i(x)dx = 2\int_{-1}^{1} \phi_a(x)\frac{dx}{d\tilde{x}}d\tilde{x} = x_{i+1} - x_i \\ &= b^e_j \end{aligned}$$

If let $h = x_{i+1} - x_i$ the full linear system for one element can be written as:

$$A^e = \begin{pmatrix} 1/h_e & -1/h_e \\ -1/h_e & 1/h_e \end{pmatrix}, \quad b^e = \begin{pmatrix} h_e \\ h_e \end{pmatrix}.$$

Gathering all the contributions from all the elements, $e = 0, 1, 2, .., N-1$ in the physical domain and denoting the width of element e_i by $h_i = x_{i+1} - x_i$, $i = 0, 1, .., N-1$ we get the full system :

$$\begin{pmatrix} 1/h_0 & -1/h_0 & 0 & 0 & \dots & 0 \\ -1/h_0 & 1/h_0 + 1/h_1 & -1/h_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \ddots & \dots \\ 0 & \dots & \dots & \dots & \dots & -1/h_{N-1} \\ 0 & \dots & \dots & \dots & -1/h_{N-1} & 1/h_{N-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \dots \\ \dots \\ \dots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} h_0 \\ h_0 + h_1 \\ \dots \\ \dots \\ \dots \\ h_{N-1} \end{pmatrix}$$

These equation can be written in terms of:

$$-c_{i-1}\left(1/h_{i-1}\right) + c_i\left(1/h_{i-1} + 1/h_i\right) - c_{i+1}\left(1/h_i\right) = h_{i-1} + h_i$$

<u>Finite difference method:</u>

We discreteize the equation according to

$$[D_x D_x u]_i = -2$$

Writing out the left hand side we get:

$$[D_x D_x u]_i = [D_x \tilde{U}]_i = \frac{\tilde{U}_{i+1/2} - \tilde{U}_{i-1/2}}{x_{i+1/2} - x_{i+1/2}} = \frac{\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}}}{x_{i+1/2} - x_{i+1/2}}$$

Now using $x_{i+1/2} - x_{i+1/2} = \frac{1}{2}(x_{i+1} - x_i) + \frac{1}{2}(x_i - x_{i-1}) = \frac{1}{2}(h_i + h_{i-1})$ the expression above becomes

$$u''(x_i) \simeq [D_x D_x u]_i = \frac{2}{h_i + h_{i-1}}\left(\frac{u_{i+1} - u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}}\right)$$

Inserting this into the original ODE and rearranging we get:

$$-u_{i-1}\left(1/h_{i-1}\right) + u_i\left(1/h_{i-1} + 1/h_i\right) - u_{i+1}\left(1/h_i\right) = h_{i-1} + h_i$$

By letting $u_i = c_i$ we obtain exactly the same result as for finite element method.