INF5620
Mandatory assignment 3

Krister Stræte Karlsen

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$\boxed{\text{Ex. 2}}$

$$u''(x) = 1 \;, \quad x \in (0,1) \;, \quad u(0) = u'(1) = 0$$

a)
$$\left.\begin{aligned} u(x) &= \tfrac{1}{2}x^2 + C_1 x + C_2 \\ u(0) &= C_2 = 0 \\ u'(1) &= 1 + C_1 = 0 \;\Rightarrow\; C_1 = -1 \end{aligned}\right\} \; u(x) = \tfrac{1}{2}x^2 - x$$

b) Function space: $\Omega = \{\psi_0, \psi_1, \dots, \psi_N\}$

$\psi_i = \sin\left((2i+1)\frac{\pi x}{2}\right)$, so

$u(x) = \sum c_j \psi_j$

$$\boxed{\begin{aligned} R &= u''(x) + 1 \\ &= \sum c_j \psi_j''(x) + 1 \end{aligned}}$$

Least-squares:

We want to determine $c_0, c_1, \dots, c_N$ such that

$$||R|| = (R, R) = \int_\Omega R^2 \, dx$$

is minimized. Differentiating with respect to the coefficients we get

$$\frac{\partial}{\partial c_i}\left(\int_\Omega R^2 dx\right) = 2\int_\Omega R \frac{\partial R}{\partial c_i} = 0 \;\Leftrightarrow\; \left(R, \frac{\partial R}{\partial c_i}\right) = 0 \qquad (*)$$

Inserting the residual:

$$\frac{\partial R}{\partial c_i} = \frac{\partial}{\partial c_i}\left(\sum_{j=0}^{N} c_j \psi_j''(x) + 1\right) = \psi_i''(x)$$

We can then write $(*)$ as:

$$\left(\sum_{j=0}^{N} c_j \psi_j(x) + 1 \,,\; \psi_i''(x)\right) \quad \text{or}$$

$$\sum_j (\psi_i'', \psi_j'') c_j = -(1, \psi_i'')$$

$$\psi_i = \sin\left((2i+1)\frac{\pi x}{2}\right)$$

and as a matrix system

$$\sum_{j=0}^{N} A_{ij} c_j = b_i \quad , \quad i = 0,1,\dots,N \qquad (\square)$$

where

$$A_{ij} = (\psi_i'', \psi_j'') = \int_\Omega \psi_i'' \psi_j'' \, dx \quad \text{and} \quad b_i = -\int_0^1 \psi_i'' \, dx$$

Computing the integral for A_{ij}

$$\left((2i+1)\frac{\pi}{2}\right)^2 \left((2j+1)\frac{\pi}{2}\right)^2 \int_0^1 \sin\left((2i+1)\frac{\pi x}{2}\right) \sin\left((2j+1)\frac{\pi x}{2}\right) dx = \begin{cases} \frac{1}{2}\left((2i+1)\frac{\pi}{2}\right)^2\left((2j+1)\frac{\pi}{2}\right)^2, & i=j \\ 0, & i \neq j \end{cases}$$

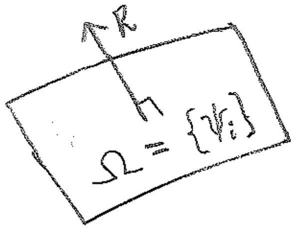
For b_i

$$\left((2i+1)\frac{\pi}{2}\right)^2 \int_0^1 \sin\left((2i+1)\frac{\pi}{2}x\right) dx = -\frac{2}{\cancel{\pi(2i+1)}}\left(\frac{(2i+1)\pi}{2}\right)^{\cancel{2}} = -(2i+1)\frac{\pi}{2}$$

Solving $(\square)$ for c_i we obtain: $\boxed{c_i = \frac{b_i}{A_{ii}} = -\frac{16}{\pi^3(2i+1)^3}}$

Galerkin method

We want the residual to be orthogonal to the space spanned by $\{\psi_i\}$, $i = 0,1,2,\dots,N$. That means



$$(R, \psi_i) = 0 \quad \text{for } i = 0,1,\dots,N$$

$$\left(\sum_j c_j \psi_j'' + 1, \psi_i\right) = 0$$

Which can be written as

$$\int_\Omega \left(\sum_j c_j \psi_j'' + 1\right)\psi_i \, dx = \int_\Omega \psi_i \sum_j c_j \psi_j'' \, dx + \int_\Omega \psi_i \, dx = 0$$

or

$$\sum_j (\psi_j'', \psi_i) c_j = -(1, \psi_i)$$

If we now multiply both sides of the equation by $\left((2i+1)\frac{\pi}{2}\right)^2$ we arrive at exactly the same linear system, $(\square)$, as with the metod of least squres. Hence the coefficients must be the same.

Rate of decay for coefficients:

$$\frac{C_i}{C_{i-1}} = \frac{(2i+1)^3}{(2(i-1)+1)^3} = \frac{(2i+1)^3}{(2i-1)^3}$$

To avoid negative ratios one can just move the indices up by 2.

Approximation with only one function (Ψ)

$$\left.\begin{array}{l} \Omega = \Psi_0 = \sin(\frac{\pi}{2}x) \\ C_0 = -\frac{16}{\pi^3} \end{array}\right\} \quad u(x) \approx v(x) = -\frac{16}{\pi^3}\sin(\frac{\pi}{2}x)$$

Error: $|u(1) - v(1)| = \underline{0{,}0160}$ Not bad.

c) See attachment.

c)

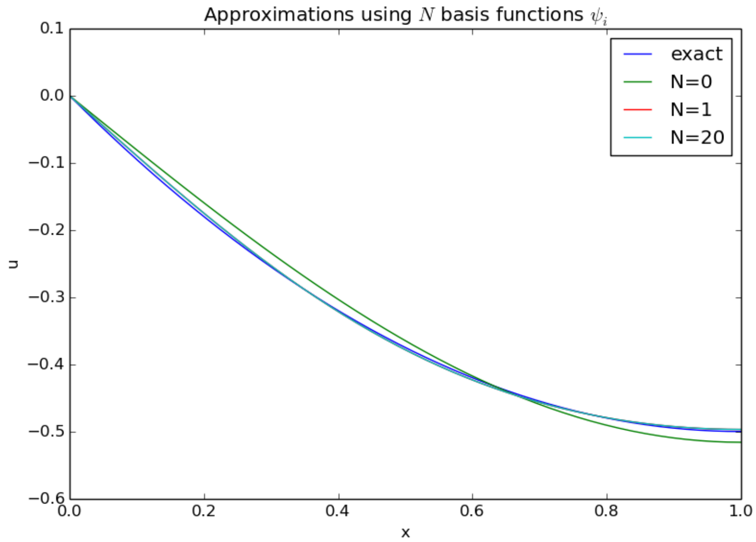


FIGURE 1. Plot of the approximations using N basis functions against exact solution.

task_c.py

```
from numpy import *
from matplotlib.pyplot import *

c = lambda i: -16.0/(pi*(2*i+1))**3
psi = lambda i,x: sin((2*i+1)*(pi/2.0)*x)

def u(x,N):
    u_accumulate = 0
    for i in range(0,N+1):
        u_accumulate += c(i)*psi(i,x)
    return u_accumulate

x, u0, u1, u20 = linspace(0,1,50)
u_exact = 0.5*x**2 - x

for i in range(0,50):
    u0[i] = u(x[i],0)
    u1[i] = u(x[i],1)
    u20[i] = u(x[i],1)

plot(x,u_exact,label='exact')
....
```

d) Now using $\psi_i = \sin\left((i+1)\frac{\pi}{2}x\right)$ with Galerkin:

Again we want

$$(R, \psi_i) = 0 \quad , \quad i = 0, 1, \ldots, N$$

or

$$\sum_j \left(\psi_j'', \psi_i\right) c_j = -\left(1, \psi_i\right)$$

As a matrix system:

$$A_{ij} = \int_\Omega \psi_j'' \psi_i \, dx = \left((i+1)\frac{\pi}{2}\right)^2 \int_0^1 \sin\left((i+1)\frac{\pi}{2}x\right) \sin\left((j+1)\frac{\pi}{2}x\right) dx = \begin{cases} \frac{1}{2}\left((i+1)\frac{\pi}{2}\right)^2, & i = j \\ 0, & i \neq j \end{cases}$$

$$b_i = -\int_\Omega \psi_i \, dx = \int_0^1 \sin\left((i+1)\frac{\pi}{2}x\right) dx = -\frac{2}{\pi(i+1)}\left(\cos\left(\frac{i+1}{2}\pi\right) - 1\right)$$

$$b_i = \begin{cases} 0 \quad \text{when } \frac{i+1}{2} = 2, 4, 6, \ldots \\ -\frac{4}{\pi(i+1)} \quad \text{when } \frac{i+1}{2} = 1, 3, 5 \ldots \\ \frac{-2}{\pi(i+1)} \quad \text{when } \frac{i+1}{2} = \pm\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \end{cases}$$

Which means that

$$c_i = \frac{b_i}{A_{ii}} = \begin{cases} 0 \quad , \quad \frac{i+1}{2} = 2, 4, 6, \ldots \\ -\frac{32}{\pi^3 (i+1)^3} \quad , \quad \frac{i+1}{2} = 1, 3, 5, \ldots \\ -\frac{16}{\pi^3 (i+1)^3} \quad , \quad \frac{i+1}{2} = \pm\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \end{cases}$$

e) $u''(x) = 1, \quad x \in (0,2), \quad u(0) = u(2) = 0$

For this problem we arrive at the same integrals as in d), but over different limits.

$$A_{ij} = \int_\Omega \psi_j'' \psi_i \, dx = \left((i+1)\frac{\pi}{2}\right)^2 \int_0^2 \sin\left((i+1)\frac{\pi}{2}x\right) \sin\left((j+1)\frac{\pi}{2}\right) dx = \begin{cases} \left((i+1)\frac{\pi}{2}\right)^2 & i = j \\ 0 & i \neq j \end{cases}$$

$$b_i = -\int_\Omega \psi_i \, dx = \int_0^2 \sin\left((i+1)\frac{\pi}{2}x\right) dx = \frac{-2}{\pi(i+1)}\left(\cos\left((i+1)\pi\right) - 1\right) = \begin{cases} 0 & i: \text{odd} \\ -\frac{4}{\pi(i+1)} & i: \text{even} \end{cases}$$

and

$$C_i = \frac{b_i}{A_{ii}} = \begin{cases} 0, & i: \text{odd} \\ -\frac{16}{\pi^3 (i+1)^3} \end{cases}$$

Ex 5

$$-u''(x) = 1 \ , x \in (0,1), \quad u(0) = u'(0) = 0$$

Method 1

Divide the domain into elements:

$$\Omega = \Omega^{(0)} \cup \Omega^{(1)} \ , \ \Omega^{(0)} = [0, \tfrac{1}{2}] \ , \ \Omega^{(1)} = [\tfrac{1}{2}, 1]$$

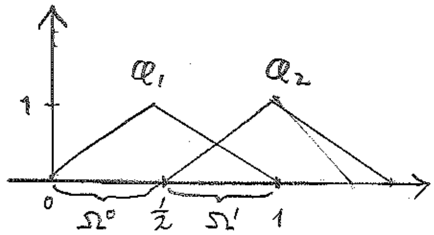


Fig 5.1

We can exclude $x=0$ from the computations without any problems since $u(0)=0$, and use a basis function $\varphi_1(x)$ over the nodes x_0, x_1, x_2 (which is zero at x_0).

This gives the following approximation:

$$u(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)$$

$$\varphi_i = \begin{cases} 0, & x < x_{i-1} \\ 2(x - x_{i-1}), & x_{i-1} \le x < x_i \\ 1 - 2(x - x_i), & x_i \le x \le x_{i+1} \\ 0, & x \ge x_{i+1} \end{cases}$$

Using Galerkin method:

$$\left(c_1 \varphi_1'' + c_2 \varphi_2'' + 1, \ \varphi_i\right) = \left(u'' + 1, v\right) = 0$$

or

$$\left(u'', v\right) = -\left(1, v\right)$$

$$\int_\Omega u'' v \, dx = \left[\overset{0}{u'} v\right]_0^1 - \int_0^1 u' v' \, dx = -\int_0^1 u' v' \, dx$$

As a matrix system: $A\bar{c} = \bar{b}$

$$\begin{bmatrix} \int_\Omega \varphi_1' \varphi_1' \, dx & \int_\Omega \varphi_2' \varphi_1' \, dx \\ \int_\Omega \varphi_1' \varphi_2' \, dx & \int_\Omega \varphi_2' \varphi_2' \, dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\int_\Omega \varphi_1 \, dx \\ -\int_\Omega \varphi_2 \, dx \end{bmatrix}$$

Computing matrix entries A_{ij}

$$\int_\Omega \varphi_1' \varphi_1' \, dx = 2 \int_0^{1/2} 2 \cdot 2 \, dx = 4 \ , \quad \int_\Omega \varphi_1' \varphi_2' \, dx = \int_{1/2}^1 -2 \cdot 2 \, dx = -2$$

$$\int_\Omega \varphi_2' \varphi_2' \, dx = \int_{1/2}^1 2 \cdot 2 \, dx = 2$$

5. continued...

Computing $\bar{b}$

$$\left.\begin{aligned}\int_\Omega \varphi_1\,dx &= \tfrac{1}{2}\\ \int_\Omega \varphi_2\,dx &= \tfrac{1}{4}\end{aligned}\right\}$$ Obvious by looking at Fig S.1.

Now the full system reads:

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix} \Rightarrow \begin{aligned} C_1 &= -\tfrac{3}{8} \\ C_2 &= -\tfrac{1}{2} \end{aligned}$$

$$U(x) = -\tfrac{3}{8}\varphi_1(x) - \tfrac{1}{2}\varphi_2(x)$$

Method 2

The domain is divided in the same way, but the approximation of the solution is now:

$$U(x) = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3$$

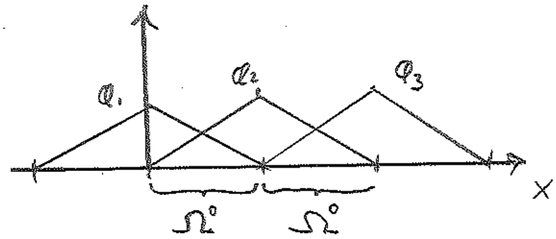


Fig. S.2

Using Galerkin Method:

$$(u'', v) = -(1, v)$$

$$\int_\Omega u''v\,dx = [u'v]_0^1 - \int_0^1 u'v'\,dx = -\int_0^1 u'v'\,dx$$

We can ignore $[u'v]_0^1$ because inserting the BC into the linear system will make it irrelevant anyway.

5 continued ..

The matrix system is now:

$$\begin{bmatrix} \int_\Omega \varphi_1'\varphi_1' dx & \int_\Omega \varphi_1'\varphi_2' dx & \int_\Omega \varphi_1'\varphi_3' dx \\ \int_\Omega \varphi_2'\varphi_1' dx & \int_\Omega \varphi_2'\varphi_2' dx & \int \varphi_2'\varphi_3' dx \\ \int_\Omega \varphi_3'\varphi_1' dx & \int_\Omega \varphi_3'\varphi_2' dx & \int_\Omega \varphi_3'\varphi_3' dx \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} u(0) \\ -\int_\Omega \varphi_2 \, dx \\ -\int_\Omega \varphi_3 \, dx \end{bmatrix}$$

Looking at the system and Figure 5.2 we realize this is very symmetric and similar to the one in method 1.

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix} \Rightarrow \begin{aligned} C_1 &= 0 \\ C_2 &= -\frac{3}{8} \\ C_3 &= -\frac{1}{2} \end{aligned}$$

$$u(x) = -\frac{3}{8}\varphi_2(x) - \frac{1}{2}\varphi_3(x)$$