Applications of finite element methods in biomechanics

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December 1, 2016

1 Stokes flow in zebrafish

Since the 1960s, the zebrafish has become increasingly important to scientific research. It has many characteristics that make it a valuable model for studying human genetics and disease. It was the first vertebrate to be cloned and is particularly notable for its regenerative abilities. Zebrafish have a similar genetic structure to humans. They share 70 per cent of genes with us and they are cheaper to maintain than mice. The zebrafish adult is about 2.5 cm to 4 cm long.

To study the effect of different drugs being able to model the blood flow is important. For instance, if the drug actually never reaches the infected cells a potentially effective drug might be considered ineffective on wrong the wrong basis.

The blood velocities in a zebrafish are low thus using *Stokes flow* as a model is a fair approximation.

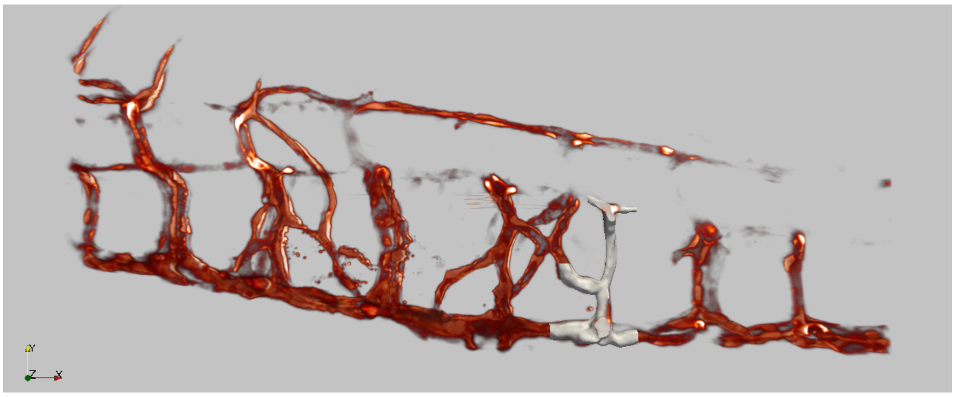

Figure 1: The circulatory system of a zebrafish where a small part is meshed.

1.1 Mathematical formulation

For for geometries with lots of cells using $P_1 - P_1$ formulations saves a lot of time and memory, and to even be able to run simulations on your own computer with the `zebrafish.xml` mesh such a formulation is needed.

Find $u, p \in W$, $W = V \times Q$ such that

$$a((u,p),(v,q)) = L((v,q)) \quad \forall \quad v,q \in W$$

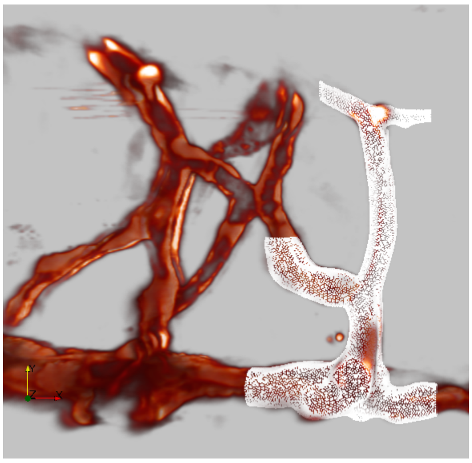

Figure 2: Zooming in to see the meshed region.

where

$$a((u,p),(v,q)) = \int_\Omega \nabla u \cdot \nabla v - (\nabla \cdot v)p + (\nabla \cdot u)q + \epsilon \nabla q \cdot \nabla p \, dx$$
$$L((v,q)) = \int_\Omega (v + \epsilon \nabla q) \cdot f \, dx$$

Here $\epsilon = \beta h^2$ and β is some number and h is the mesh cell size.

Boundary conditions

$$u = 0 \quad on \quad \partial\Omega_{\text{ no-slip}}$$
$$\sigma \cdot \mathbf{n} = p_i \mathbf{n} \quad on \quad \partial\Omega_{\text{ opening(i)}}$$

2 Squeezing a postdoc's brain

We would very much like to squeeze postdoc Erika Lindström's brain. Since she has refused to let us do this with our hands in her office, we must do this on a computer using her brain as our computational domain. The brain will be deformed as a result of the squeezing and to capture this effect we will use a *linear elastic* model.

A mesh of Erika's brain can be found in the git repository: https://github.com/krikarls/fun-with-fem.

The brain is not clamped in the skull, but in a sence floating around. This means that we must employ *neumann boundary conditions* on the entire boundary. As we know, there are no unique solution to such a problem since all *rigid motions* satify the equation. So in order to obtain a unique solution we must remove all rigid motions. All the possible rigid motions in 3D are: translations in x, y, z-direction and rotations around the corresponding axes. Thus six in total.

An example using *FEniCS* on how to remove these can be found in the same repository as the brain.

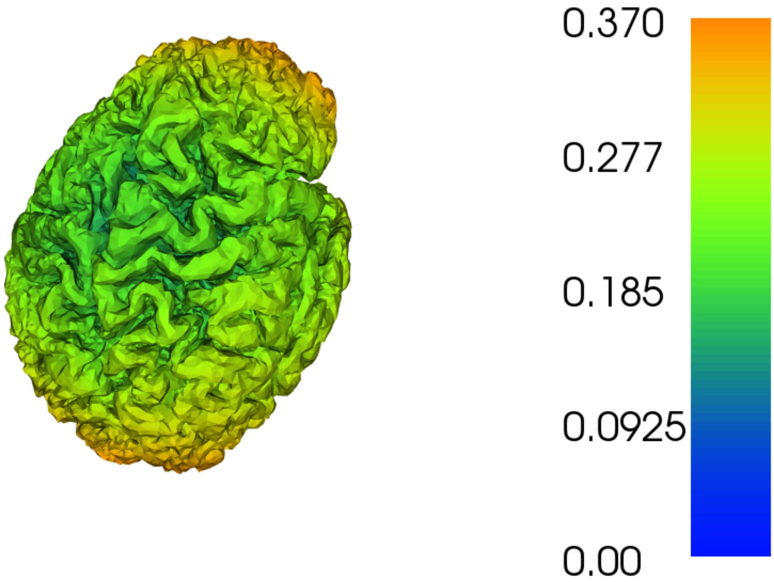

Figure 3: Numerical solution using FEniCS. Displacement measured in mm.

2.1 Mathematical formulation

Find u such that

$$\int_\Omega 2\mu(\epsilon(u) : \epsilon(v)) + \lambda(\nabla \cdot u)(\nabla \cdot v)\, dx = \int_\Omega f \cdot v\, dx \quad \forall v \in V$$

Boundary conditions

$$\sigma \cdot \mathbf{n} = p\mathbf{n} \quad on \quad \partial\Omega$$