Home task #6. Sequences and Induction

1. a) $c_i = \frac{(-1)^i}{3^i}$, for all integers $i \geq 0$

first four terms: $i = 0; 1; 2; 3$:

$c_0 = 1 \quad c_1 = \frac{-1}{3} \quad c_2 = \frac{1}{9} \quad c_3 = -\frac{1}{27}$

b) $e_n = \left[\frac{n}{2}\right] \cdot 2$, for $n \geq 0 \quad e_0 = 0 \; e_1 = 0 \; e_2 = 2 \; e_3 = 2$

c) $f_n = \left[\frac{n}{4}\right] 4$, for $n \geq 1 \quad f_1 = 0 \quad f_2 = 0 \quad f_3 = 0 \quad f_4 = 4$

2. a) $\sum_{k=1}^{5} (k+1) = 2+3+4+5+6 = 20$ b) $\sum_{k=-1}^{1} (k^2+3) = 4+3+4 = 11$

c) $\sum_{m=0}^{3} \frac{1}{2^m} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} = 1\frac{7}{8}$ d) $\sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1}\right) =$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{9} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{11}\right) = 1 - \frac{1}{11} = \frac{10}{11}$

3. a) $\prod_{k=2}^{4} k^2 = 4 \cdot 9 \cdot 16 = 456$ b) $\prod_{i=0}^{4} (-1)^i = 1 \cdot (-1) \cdot 1 \cdot (-1) \cdot 1 = 1$

$\sim (j-1)(j+1)$, where $i = j-1$

c) $\prod_{k=2}^{2} \left(1 - \frac{1}{k}\right) = \frac{1}{2}$ d) $\prod_{i=2}^{5} \frac{i(i+2)}{(i-1)(i+1)} = \frac{8}{3} \cdot \frac{15}{8} \cdot \frac{24}{15} \cdot \frac{35}{24} = \frac{35}{3} = 11\frac{2}{3}$

4. a) $\sum_{i=1}^{k} i^3 + (k+1)^3 = \sum_{i=1}^{k+1} i^3$ b) $\sum_{k=1}^{m} \frac{k}{k+1} + \frac{m+1}{m+2} = \sum_{k=1}^{m+1} \frac{k}{k+1}$

c) $\sum_{m=0}^{n} (m+1) 2^m + (n+2) 2^{n+1} = \sum_{m=0}^{n+1} (m+1) 2^m$ d)

d) $2 \cdot \sum_{k=1}^{n} (3k^2+4) + 5 \cdot \sum_{k=1}^{n} (2k^2-1) = \sum_{k=1}^{n} \left[2 \cdot (3k^2+4) + 5 \cdot (2k^2-1)\right] = \sum_{k=1}^{n} (16k^2+3)$

e) $\left(\prod_{k=1}^{n} \frac{k}{k+1}\right)\left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right) = \prod_{k=1}^{n} \left[\frac{k}{k+1} \cdot \frac{k+1}{k+2}\right] = \prod_{k=1}^{n} \frac{k}{k+2}$

5. a) $\sum_{i=1}^{7} (-1)^{i+1} \cdot i^2$ b) $\prod_{i=2}^{4} (i^2 - 1)$ c) $\sum_{i=2}^{6} \frac{(-1)^i \cdot i}{(i+1)(i+2)}$

d) $\prod_{i=1}^{4} (1 - t^i)$

6. a) $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} = \left| \begin{matrix} j = i-1 \\ i = j+1 \end{matrix} \right\} = \sum_{j=0}^{n-2} \frac{j+1}{(n-j-1)^2}$

b) $\prod_{i=n}^{2n} \frac{n-i+1}{n+i} = \left\{ \begin{matrix} j = i-1 \\ i = j+1 \end{matrix} \right\} = \prod_{j=n-1}^{2n-1} \frac{n-j}{n+j+1}$

7. $4^3 + 4^4 + 4^5 + \ldots + 4^n = \frac{4(4^n - 16)}{3}$ for all $n \geq 3$, $n \in \mathbb{Z}$

base case: $n = 3$

$\sum_{i=3}^{n} 4^i = \underline{4^3} \qquad \frac{4(4^3-16)}{3} = \frac{4^3(4-1)}{3} = \underline{4^3} \qquad \frac{4 \cdot (4^n - 4^2)}{3} = \frac{4^3(4^{n-2}-1)}{3}$

so its work for base case

assume its true for $n = k$:

$\sum_{i=3}^{k} 4^i = \frac{4(4^k - 16)}{3} \quad \left(\text{or } \frac{4^3(4^{k-2}-1)}{3}\right) \quad (1)$

then if its also true for $n = k+1$, then its true for $n \in \mathbb{Z}$

$\sum_{i=3}^{k+1} 4^i = \frac{4(4^{(k+1)} - 16)}{3}$ or $\sum_{i=3}^{k+1} 4^i = \sum_{i=3}^{k} 4^i + 4^{k+1}$, so:

$\frac{4(4^k - 16)}{3} + 4^{k+1} = \frac{4(4^{(k+1)} - 16)}{3}$

$\frac{4^3(4^{k-2}-1) + 3 \cdot 4^3 \cdot 4^{k-2}}{3} = \frac{4^3(4^{k-1}-1)}{3}$

$\frac{4^3((1+3) \cdot 4^{k-2} - 1)}{3} = \frac{4^3(4^{k-1}-1)}{3}$

$\frac{4^3(4^{k-1}-1)}{3} = \frac{4^3(4^{k-1}-1)}{3}$ Q.E.D

8. $\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2}\right) = \frac{1}{(2n+2)!}$, for all integer $n \geq 1$

consider base case $n = 1$:

$\prod_{i=0}^{1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2}\right) = \left(\frac{1}{1} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{3} \cdot \frac{1}{4}\right) = \underline{\frac{1}{4!}} \qquad \frac{1}{(2n+2)!} = \underline{\frac{1}{4!}}$

assume its true for $n = k$, $k \in \mathbb{Z}$

$\prod_{i=0}^{k} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2}\right) = \frac{1}{(2k+2)!}$

then if its also right for $n = k+1$, then its true for $n \in \mathbb{Z}$

$\prod_{i=0}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2}\right) = \prod_{i=0}^{k} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2}\right) \cdot \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2}\right)$

$\frac{1}{(2(k+1)+2)!} = \frac{1}{(2k+2)!} \cdot \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2}\right)$

$\frac{1}{(2(k+1)+2)!} = \frac{1}{(2k+2)!} \cdot \frac{1}{2k+3} \cdot \frac{1}{2k+4}$

$\frac{1}{(2(k+1)+2)!} = \frac{1}{(2k+4)!}$ or $\frac{1}{(2(k+1)+2)!} = \frac{1}{(2(k+1)+2)!}$ Q.E.D