Homework DM Functions

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1

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$. Define $g : X \to Y$ as follows: $g(1) = a$, $g(2) = a$, $g(3) = a$, $g(4) = d$

1. Draw an arrow diagram for g.
2. Let $A = \{2, 3\}$, $C = \{a\}$, $D = \{b, c\}$. Find $g(A)$, $g(X)$, $g^{-1}(C)$, $g^{-1}(D)$, $g^{-1}(Y)$

2

Let X and Y be any sets, $A \subset X$, $B \subset X$, $C \subset Y$, $D \subset Y$.
Is the following formula:

1. $F(A \cap B) \subseteq F(A) \cap F(B)$
2. $F(F^{-1}(C)) \subseteq C$

true for all functions F from X to Y? Justify your answer.

3

Prove that if p is a prime number and n is an integer, $n \geq 1$, then $\phi(p^n) = p^n - p^{n-1}$ where ϕ is an Euler phi function.

4

Define $F : Z^+ \times Z^+ \to Z^+$ and $G : Z^+ \times Z^+ \to Z^+$ as follows:
For all $(n, m) \in Z^+ \times Z^+$

$$F(n, m) = 3^n 5^m \quad G(n, m) = 3^n 6^m$$

1. Prove or disprove that F and G are one-to-one functions.
2. Prove or disprove that F and G are onto functions.

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Suppose $f : X \to Y$ and $g : Y \to Z$ are both one-to-one and onto. Prove that $(f \circ g)^{-1}$ exists and that $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.

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Suppose $F : X \to Y$ is one-to-one.

1. Prove that $\forall A \subseteq X, \quad F^{-1}(F(A)) = A$
2. Prove that $\forall A_1 \subseteq X, \forall A_2 \subseteq X, \quad F(A_1 \cap A_2) = F(A_1) \cap F(A_2)$

7

Given $(a)_x \in Z$ (x is a base of number a), provide a function $F(a_1, a_2, ...)$, where a_i - is the $i-th$ digit of a number a, such that:
if $F(a_1, a_2, ...) \bmod n = 0$ then $a \bmod n = 0$

1. $n = 9$, $x = 10$
2. $n = 11$, $x = 10$
3. $n = 101$, $x = 100$

8

A factorial $n!$, $n \in Z$ can be decomposed into its canonical form:

$$n! = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot ... \cdot m^{a_i} \cdot ...$$

where p_i, m are prime numbers and $m < n$. For example:

$$20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$$

Your task is to write a script with a function $F(n, m)$ that calculates a_i, which is a power of a prime factor m. It should work fast and with relatively big numbers (for example n = 1000, so you should count the a_i value in 1000!).