Discrete Mathematics

Inductions-II

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“Induction is a process of inference it proceeds from the known to the unknown!”

- John Stuart Mill -

Let P be some property. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

$$P(0) \wedge \forall k[P(k) \Rightarrow P(k+1)] \Rightarrow \forall n\, P(n)$$

- Typically, a proof by induction will not explicitly state *P(n)*.

- Rather, the proof will describe P(n) implicitly and leave it to the reader to fill in the details.

- Provided that there is sufficient detail to determine.

    - What P(n) is;

    - That P(0) is true, and that

    - Whenever P(k) is true, P(k+1) is true,

- The proof is usually valid.

Can be proved using the “well-ordering principle”!

Variants of Induction

Induction starting at *0* [*Remember that it could be 1 as well, all depends on the definition of natural numbers*]

To prove that $P(n)$ is true for all natural numbers greater than or equal to 0

- Show that $P(0)$ is true.
- Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
- Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Variants of Induction

Induction starting at *m* (Starting Induction Later)

To prove that *P(n)* is true for all natural numbers **greater than or equal to *m***:

- Show that *P(m)* is true.
- Show that for any $k \geq m$, that if *P(k)* is true, then *P(k+1)* is true.
- Conclude *P(n)* holds for all natural numbers greater than or equal to *m*.

Starting Induction Later

Theorem: For all $n \epsilon N$, where $n \geq 5$, $n^2 < 2^n$.

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Proof: By Induction:

Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$.

We will prove that $P(n)$ holds for all $n \epsilon N$.

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Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction:

Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$.

We will prove that $P(n)$ holds for all $n \in N$.

From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$.

This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n\text{-}5)$ then implies that $n^2 < 2^n$.

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For our base case,

We prove $P(5)$, that $5^2 < 2^5$.

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For the inductive step assume for some $k \epsilon N$ that $P(k)$ holds and $(k+5)^2 < 2^{(k+5)}$.

We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

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$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$

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$< 2^{k+5} +2(k+5)+5$ (Since 1<5)

{Remember that it might have taken the creator of this proof quite some time and creativity to come up with this. Same technique is used in the rest of the proof}

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Thus $((k+5)+1)^2 < 2^{((k+5)+1)}$, so $P(k+1)$ holds.

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Important Bits

This proof is interesting for a few reasons.

- First, it shows that we can use induction to reason about properties of numbers larger than a certain size.

- Second, it shows a style of proof that we have not seen before. To prove an inequality holds between two quantities, we can often expand out the inequality across multiple steps, at each point showing one smaller piece of the inequality. Since inequalities are transitive, the net result of these inequalities gives us the result that we want.

Practice

Prove the following theorem using Induction.

Theorem: For all $n \in N$, where $n \geq 4$, $2^n < n!$.

Strong Induction

Normal induction, and **induction starting at *m***, enable us to prove a variety of useful results.

We will now turn to an even more powerful form of induction called **strong induction**

- It has numerous applications within computer science, from the analysis of algorithms to the understanding of the structure of numbers themselves.

Strong Induction – Cont.

Similar to **ordinary mathematical induction** in that it is a technique for establishing the truth of a sequence of statements about natural numbers.

Also, a proof by **strong mathematical induction** consists of a basis step and an inductive step.

Strong Induction – Cont.

However, the basis step may contain proofs for several initial values.

And, in the inductive step the truth of the predicate $P(n)$ is assumed not just for one value of n but for all values through k, and then the truth of $P(k + 1)$ is proved.

Strong Induction – Cont.

Theorem: Let *P(n)* be a property that applies to natural numbers. If the following are true:

- *P(0)* is true.
- For any $k \in N$, if *P(0)*, *P(1)*, ... , *P(k)* are true, then *P(k+1)* is true.

Then for any $k \in N$, *P(n)* is true.

Strong Induction – Cont.

In other words,

- In a regular induction, we would assume that $P(k)$ is true, then use it to show that $P(k + 1)$ is true.

- In strong induction, we assume that all of $P(0)$, $P(1)$, ..., and $P(k)$ are true, then use this to show that $P(k+1)$ is true.

Why Strong Induction

The following example will illustrate how strong induction can help us prove a result that cannot easily be proved using the principle of mathematical induction.

Example

Suppose we can reach the first and second steps of an infinite ladder, and we know that if we can reach a step, then we can reach two steps higher.

Can we prove that we can reach every step of this ladder?

Example

Attempt through Normal Induction:

Basis Step *P(1)*:

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To complete the inductive step, we need to show that if we assume that we can reach the k^{th} step of the ladder, then we can show that we can reach the $(k+1)^{st}$ step of the ladder.

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Basis Step *P(1)*:

The basis step of such a proof holds; here it simply verifies that we can reach the first step.

Inductive Step:

The inductive hypothesis is the statement that we can reach the kth step of the ladder.

To complete the inductive step, we need to show that if we assume the inductive hypothesis for the positive integer k, if we assume that we can reach the k^{th} step of the ladder, then we can show that we can reach the $(k+1)^{st}$ step of the ladder.

However, there is no obvious way to complete this inductive step because we do not know from the given information that we can reach the $(k+1)^{st}$ step from the k^{th} step. After all, **we only know that if we can reach a step we can reach the step two higher**.

Example

Attempt through Strong Induction:

Basis Step: The basis step is the same as before; it simply verifies that we can reach the first step.

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Example

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We can complete the inductive step by noting that as long as $k \geq 2$, **we can reach the $(k+1)^{st}$ step of the ladder from the $(k-1)^{st}$ step** because we know we can climb two steps from a step we can already reach, and because $k\text{-}1 \leq k$, by inductive hypothesis we can reach the $(k\text{-}1)^{st}$ step.

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We can complete the inductive step by noting that as long as $k \geq 2$, **we can reach the $(k+1)^{st}$ step of the ladder from the $(k-1)^{st}$ step** because we know we can climb two steps from a step we can already reach, and because $k-1 \leq k$, by inductive hypothesis we can reach the $(k-1)^{st}$ step.

This completes the inductive step and finishes the proof by strong induction.

Example

Attempt through Strong Induction:

Basis Step: The basis step is the same as before; it simply verifies that we can reach the first step.

Inductive Step: The inductive hypothesis states that we can reach each of the first k steps.

To complete the inductive step, we need to show that if we assume that the inductive hypothesis is true, that is, if we can reach each of the first k steps, then we can reach the $(k+1)^{st}$ step.

We already know that we can reach the second step of the ladder. We can complete the inductive step by noting that as long as $k \geq 2$, we can reach the $(k+1)^{st}$ step of the ladder from the $(k+1)^{st}$ step because we know we can climb two steps from a step we can already reach, and because $k\text{-}1 \leq k$, by inductive hypothesis we can reach the $(k\text{-}1)^{st}$ step.

This completes the inductive step and finishes the proof by strong induction.

Example

Prove: Any integer greater than 1 is divisible by a prime number.

Solution:

The idea for the inductive step is this: If a given integer greater than 1 is not itself prime, then it is a product of two smaller positive integers, each of which is greater than 1.

Since you are assuming that each of these smaller integers is divisible by a prime number, by transitivity of divisibility, those prime numbers also divide the integer you started with.

Example

Proof (by strong mathematical induction):

Let the property $P(n)$ be the sentence

n is divisible by a prime number.

Basis Step: Show that $P(2)$ is true:

To establish $P(2)$, we must show that

2 is divisible by a prime number.

But this is true because 2 is divisible by 2 and 2 is a prime number.

Example

Inductive Step: Show that for all integers $k \geq 2$, if $P(i)$ is true for all integers i from 2 through k, then $P(k + 1)$ is also true:

[*in other words*]

Let k be any integer with $k \geq 2$ and suppose that

$$\left\{ \begin{array}{c} i \text{ is divisible by a prime number for all integers} \\ i \text{ from } 2 \text{ through } k. \end{array} \right.$$

$\boxed{\leftarrow \text{Inductive Hypothesis}}$

We must show that

$k + 1$ is divisible by a prime number.

$\boxed{\leftarrow P(k+1)}$

Example

Case 1 (k + 1 is prime): In this case $k + 1$ is divisible by a prime number, namely itself.

Case 2 (k + 1 is not prime): In this case $k + 1 = ab$ where a and b are integers with $1 < a < k + 1$ and $1 < b < k + 1$.

Thus, in particular, $2 \leq a \leq k$, and so by inductive hypothesis, a is divisible by a prime number p.

In addition because $k + 1 = ab$, we have that $k + 1$ is divisible by a.

Example

Hence, since $k + 1$ is divisible by a and a is divisible by p, by transitivity of divisibility, $k + 1$ is divisible by the prime number p.

Therefore, regardless of whether $k + 1$ is prime or not, it is divisible by a prime number *[as was to be shown]*.

[Since we have proved both the basis and the inductive step of the strong mathematical induction, we conclude that the given statement is true.]

Example

Which amounts of money can be formed using just two dollar bills and five dollar bills? Justify your answer by formulating a proof!

How would you solve this problem?

Recurrence Relations

- We started our journey of “mathematical induction” by introducing sequences. Let us end by telling you that sequences can be represented in many ways.
  - One of such ways is called a “**recurrence relation**”.

Definition:

A **recurrence relation** for sequence $a_0, a_1, a_2, a_3, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$. The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, a_3, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, a_2, a_3, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on k.

Example

n people are invited to a party. If every person speaks to every other person on the phone, how many calls will be there?

Example

n people are invited to a party. If every person speaks to every other person on the phone, how many calls will be there?

NO. of People	1	2	3	4	5
NO. of calls	0	1	3	6	10

So, we have $0, 1, 2, 3, 6, 10, \ldots$

$$a_1 = 0$$

$$a_k = a_{k-1} + (k-1) \quad \text{where } k \geq 2$$

Which is the recurrence relation for the above sequence.