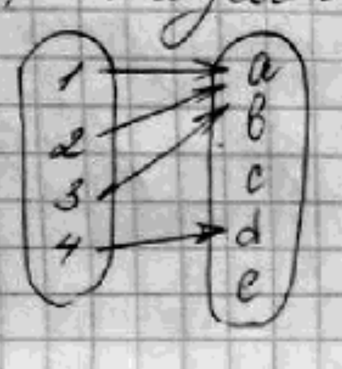
Homework # 8 Krikun G.

(1.) Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$
define $g: X \to Y$ as follows $g(1) = a$, $g(2) = a$
$g(3) = a$, $g(4) = d$

1. diagram:



2. Let $A = \{2, 3\}$, $C = \{a\}$, $D = \{b, c\}$

$g(A) = g(\{2, 3\}) = \{a, a\}$

$g(X) = g(\{1, 2, 3, 4\}) = \{a, d\}$ or $\{a, a, a, d\}$?

$g^{-1}(C) = g^{-1}(\{a\}) = \{1, 2, 3\}$

$g^{-1}(D) = g^{-1}(\{b, c\}) = \emptyset$

$g^{-1}(Y) = g^{-1}(\{a, b, c, d, e\}) = \{\{1, 2, 3\}, \emptyset, \emptyset, \{4\}, \emptyset\}$

(2.) Let X and Y be any sets, $A \subset X$, $B \subset X$, $C \subset Y$, $D \subset Y$
Is the following formula, true for all $F: X \to Y$?

(1.) $F(A \cap B) \subseteq F(A) \cap F(B)$ true because
for some x from A and B $\{\forall x : x \in A \cap B\} \Rightarrow$
$^*x \in A$ and $x \in B$ $F(x) \in F(A)$ and $F(x) \in F(B) \Rightarrow F(x) \in F(A) \cap F(B)$
$\Rightarrow F(A \cap B) \subseteq F(A) \cap F(B)$, actually $F(A \cap B) \equiv F(A) \cap F(B)$

(2.) $F(F^{-1}(C)) \subseteq C$ true because by definition
of F^{-1}, $F: X \to Y \Rightarrow F^{-1}: Y \to X$, then
$F(F^{-1}(C))$ means that for $F^{-1}: C \to Z$ $F: (Z) \to C$
and $F(F^{-1}(C)) \subseteq C$ actually $F(F^{-1}(C)) = C$

(3) Prove that if p is a prime number and n - integer, $n \geq 1$ then $\phi(p^n) = p^n - p^{n-1}$ where ϕ is an Euler phi function.
(is an arithmetic function that counts the positive integers less than or equal to n that are relatively prime to n.)

(2.) Let X and Y be any sets, $A \subset X$, $B \subset X$, $C \subset Y$, $D \subset Y$
Is the following formula:

1. $F[A \cap B] \subseteq F[A] \cap F[B]$
consider $x \in A$:
$F(x) \in F[A]$, as soon as $F[A] = \{f(x) / x \in A\}$
consider $x \in B$:
$F(x) \in F[B]$, as soon as $F[B] = \{f(x) / x \in B\}$
consider $x \in A$ and $x \in B$: $(x \in A \cap B)$
$F(x) \in F[A \cap B]$, as soon as $F[A \cap B] = \{f(x) / x \in (A \cap B)\}$
therefore
$F[A \cap B] \equiv F[A] \cap F[B]$ it covers in
$F[A \cap B] \subseteq F[A] \cap F[B]$.

2. $F[F^{-1}[C]] \subseteq C$
consider $F^{-1}[C]$:
$F^{-1}[C] = \{x \in A / F(x) \in C\}$ by definition
consider $F(x)$, $x \in A$:
$F(x) \in C$
therefore
$F[F^{-1}[C]] \subseteq C$

"Relatively prime" means that gcd (great common divisor) equal 1.

If p - prime then $\phi(p) = p$
As soon as p - prime then $p^n = \overbrace{p \cdot p \cdots p}^{n\text{-times}}$ has and
It means that p^n has great common divisour not equal to 1 with $\underbrace{p, 2p, 3p \ldots p^{n-1} p}_{p^{n-1}}$,
so $\phi(p^n) = p^n - p^{n-1}$

(3.) Define $F: Z^+ \times Z^+ \to Z^+$ and $G: Z^+ \times Z^+ \to Z^+$
as follows:
For all $(n, m) \in Z^+ \times Z^+$

$F(n, m) = 3^n 5^m$ $G(n, m) = 3^n 6^m$

1. Prove or disprove that F and G are one-to-one functions
Assume $n_1, n_2, m_1, m_2 \in Z^+$ and $n_1 \neq n_2$, $m_1 \neq m_2$

- If ~~this~~ function F is n't one-to-one it means that $F(n_1, m_1) = F(n_2, m_2)$ for assuming n's and m's
Thus
$3^{n_1} 5^{m_1} = 3^{n_2} 5^{m_2}$ or $\underbrace{3 \cdot 3 \cdot \ldots \cdot 3}_{n_1} \cdot \underbrace{5 \cdot 5 \cdot \ldots \cdot 5}_{m_1} = \underbrace{3 \cdot \ldots \cdot 3}_{n_2} \cdot \underbrace{5 \cdot \ldots \cdot 5}_{m_2}$
as soon as 3 and 5 is prime numbers then they cannot be represent as product of prime number (by fundamental theorem of arithmetic), thus
$n_1 = n_2$ and $m_1 = m_2$ ← this is contradiction with our base statement, so function F ~~aren't~~ is one-to-one.

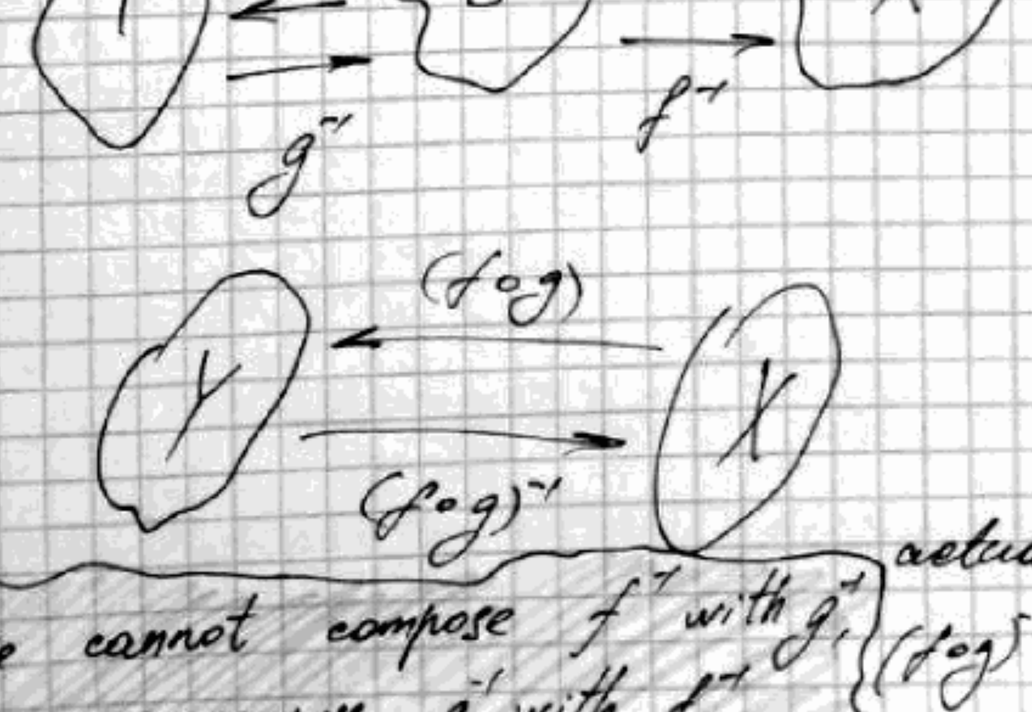
- If function G is n't one-to-one it means that $G(n_1, m_1) = G(n_2, m_2)$ for assuming n's and m's
Thus
$3^{n_1} 6^{m_1} = 3^{n_2} 6^{m_2}$ or $\underbrace{3 \cdot 3 \ldots 3}_{n_1} \underbrace{6 \cdot 6 \ldots 6}_{m_1} = \underbrace{3 \cdot 3 \ldots 3}_{n_2} \underbrace{6 \cdot 6 \ldots 6}_{m_2}$
it can be represent as:
$\underbrace{3 \cdot 3 \cdot \ldots \cdot 3}_{n_1} \cdot \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{m_1} \underbrace{3 \cdot 3 \cdot 3 \cdots 3}_{m_1} = \underbrace{3 \cdot 3 \cdot \ldots \cdot 3}_{n_2} \cdot \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m_2} \cdot \underbrace{3 \cdot 3 \cdot \ldots \cdot 3}_{m_2}$
it means that (by fundamental theorem of arithmetic)
$\begin{cases} n_1 + m_1 = n_2 + m_2 \\ m_1 = m_2 \end{cases} \Longleftrightarrow \begin{cases} n_1 = n_2 \\ m_1 = m_2 \end{cases}$ ← contradiction
Thus $G(n, m)$ - one-to-one function

2. Prove or disprove that F and G are onto functions
For onto functions:
for every $z \in Z^+$ there are some $(n, m) \in Z^+ \times Z^+$ such that $F(n, m) = z$, but for $z = 1$, there aren't $(n, m) \in Z^+ \times Z^+$ ~~sat~~ such that $3^n 5^m = 1$, thus function F - ~~aren't~~ is n't onto function.
also for G - isn't onto function.

(4.) Suppose $f: Z \to Y$ and $g: X \to Z$ are both one-to-one and onto functions (Bijection). Prove that $(f \circ g)^{-1}$ exists and that $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$

1. Consider functions f and g as soon as they both are one-to-one and onto functions as given, then composition $(f \circ g)$ also one-to-one and onto function.
* by properties of composition functions
Thus there are exists inverse function $(f \circ g)^{-1}$.
(Composition $(f \circ g)$ exists because given domains and codomains are suitable)

2. Consider functions f, g, $(f \circ g)$, $(f \circ g)^{-1}$, g^{-1}, f^{-1}



$f: Z \to Y$
$f^{-1}: Y \to Z$
$g: X \to Z$
~~$f: Z \to Y$~~
$g^{-1}: Z \to X$
$f \circ g: X \to Y$
$(f \circ g)^{-1}: Y \to X$

* we cannot compose f^{-1} with g^{-1}, but we can compose g^{-1} with f^{-1} because domain and codomain suitable.

actually $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$